2D Unsteady Vortex Panel Method

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1 METHODOLOGY

Under the assumptions of inviscid, irrotational flows, the velocity field can be expressed as

$$\boldsymbol{\nabla} \times \boldsymbol{V} = 0 \tag{1}$$

This indicates that V is a conservative vector field, which is expressed as the gradient of a scalar function, Φ, known as the velocity potential function. This implies that

$$\boldsymbol{V} = \boldsymbol{\nabla}\Phi \tag{2}$$

Under the further assumptions of incompressible flow, equation (2) simplifies to a linear homogeneous differential equation known as Laplace equation.

$$\boldsymbol{\nabla}.\boldsymbol{V} = \nabla^2\Phi = 0 \tag{3}$$

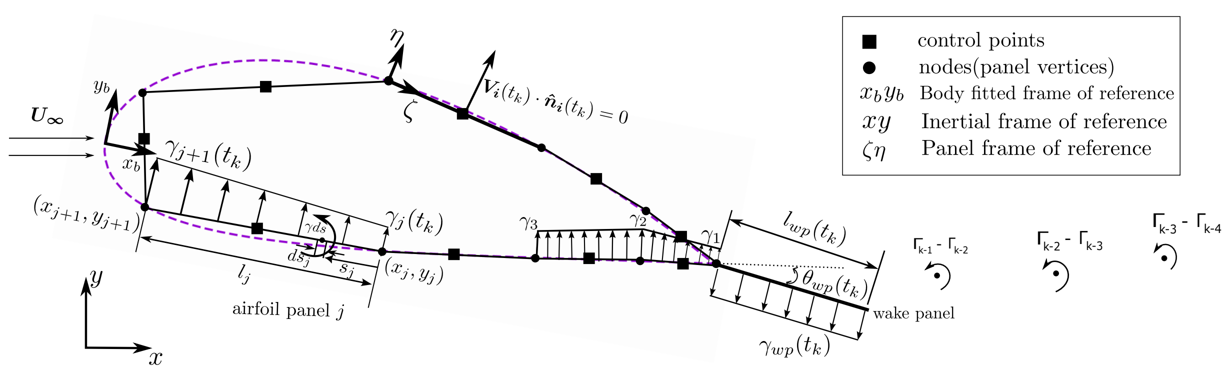

Figure 1: **A schematic diagram of the panel discretization on an airfoil and wake modeling at time instant t_k in the inertial frame of reference xy.**

The linearity of the Laplace equation allows the solution to be constructed by distributing elementary solutions along the boundaries of the geometry. Since Laplace equation is a second order differential equation, two boundary conditions are required to solve the problem: zero normal flow boundary condition at the surface of the geometry and far-field boundary conditions. The singular elementary solutions automatically fulfill the far-field boundary conditions by having velocity fields that die out as $\boldsymbol{r} \to \infty$. Therefore, solving a fluid dynamics problem is reduced to determining the appropriate distribution of singularity elements along the known boundaries while ensuring the boundary conditions are met. This is the reason why potential flow methods offer faster computations since unknown quantities need to be determined only at the surface of the geometries, unlike in high-fidelity simulations, where they are determined throughout the entire fluid volume. However, it is to be noted that for an airfoil at a given angle of attack, there are infinitely many valid theoretical solutions, corresponding to infinite choices of the total circulation around the airfoil, Γ. As a result, potential flow theory alone cannot uniquely determine the generated lift force. A closure condition must be provided to fix the correct amount of circulation. This closure is provided by the well established Kutta condition that allows the selection of a correct solution among many possible ones. Once the strengths of these singularity distributions are determined, flow-field can be easily solved. From there onwards, pressure distribution can be calculated from the unsteady Bernoulli equation, which is the algebraic version of the Euler's equation. Consequently, lift and drag forces can be estimated by integrating the pressure pressure distribution obtained on the surface. Another crucial aspect to note is that the Laplace equation does not explicitly include time dependent terms. In unsteady flow scenarios, time dependency will be introduced through no penetration boundary condition and unsteady Bernoulli equation [1].

The first key step is selecting the type of singularity element distribution for modeling the airfoil geometry and the wake. In the present study, we implemented linearly varying vortex strength as the singularity distribution to model the flow behavior. This distribution is piecewise continuous at the panel corners. The airfoil contour is represented by $n-1$ number of flat panels arranged along its surface, where n is the number of nodes or panel vertices. The panel coordinates were obtained using cosine clustering. At a given time instant, the latest wake is modelled by a constant strength vortex panel shed from the trailing-edge, which then travels downstream with the local induced velocity and is treated as an equivalent strength point vortex in the subsequent time steps. In the present study, the unsteady equations are solved in the inertial frame of reference. It is assumed that the airfoil is performing unsteady motion in a steady stream of oncoming flow oriented along the x axis. Figure 1 illustrates the body fixed frame x_b, y_b with the origin at the airfoil's leading edge and inertial or fixed frame is represented as x, y. The equations when solved in body fitted frame involves less computational time as the influence coefficient matrix remains unaltered. However, an additional effort is required for the transfer of equations from the inertial to body fitted frame and will be pursued in the future.

Following the discretization of the geometry, the next crucial step is to establish the system of linear algebraic equations. This process begins with the formulation of influence coefficient matrix which basically depends on the geometry of the airfoil and chosen singularity element. For the unsteady analysis, there are $n+3$ unknowns, $\gamma_j(t_k)$ for $j = 1:n$, representing the n strengths of the nodal bound vortices at k^{th} time step. The remaining unknowns comprise of $\gamma_{wp}(t_k), l_{wp}(t_k), \theta_{wp}(t_k)$ which signifies the strength, length and orientation of the newly shed wake panel, respectively. These $n+3$ fundamental set of equations can be formulated as follows,

(i) The $n-1$ conditions of zero normal flow are enforced by applying the outer Neumann normal velocity boundary condition at the midpoint (control point or collocation point) of each panel.

$$[(\boldsymbol{V}_{bound} + \boldsymbol{V}_{wake} + \boldsymbol{V}_{kin})_i \cdot \hat{\boldsymbol{n}}_i]_{t_k} = 0\,, (i = 1, .., n-1) \tag{4}$$

Eq. (4) represents the normal component of the total velocity induced at the i^{th} control point at time instant t_k. To analyze this expression in detail, we will break down each term.

The first term, $\boldsymbol{V}_{bound}$, refers to the velocity induced at the i^{th} control point due to the vortex panels on the airfoil, and it is expressed as follows:

$$\begin{Bmatrix} V_{x_i} \\ V_{y_i} \end{Bmatrix}_{\mathrm{B}} = \sum_{j=1}^{n-1} [\boldsymbol{P}]_{ji} \begin{Bmatrix} \gamma_j \\ \gamma_{j+1} \end{Bmatrix} \tag{5}$$

where $[\boldsymbol{P}]_{ji}$ is the 2×2 panel coefficient matrix, which describes the coefficients of the velocity induced by j^{th} panel on i^{th} control point. The expression for the $[\boldsymbol{P}]_{ji}$ is derived from Warren Phillips [2]. One thing to be noted is that in this unsteady model, $[\boldsymbol{P}]_{ji}$ is time-dependent because the problem is solved in inertial frame of reference. Consequently, the panel unit normals $\hat{\boldsymbol{n}}(t_k)$ and tangents $\hat{\boldsymbol{t}}(t_k)$ are also time dependent and need to be updated at each time instant.

The velocity contribution of the wake $\boldsymbol{V}_{wake}$, can be segregated as $\boldsymbol{V}_{wp}$ and $\boldsymbol{V}_v$, which defines the velocity induced by the wake panel shed in the current time step and the velocity induced by the point vortices shed in previous time steps, respectively. Their expressions are given as follows:

$$\begin{Bmatrix} V_{x_i} \\ V_{y_i} \end{Bmatrix}_{wp} = \gamma_{wp}(t_k)[\boldsymbol{P}]_{wp\,i} \tag{6}$$

$$\begin{Bmatrix} V_{x_i} \\ V_{y_i} \end{Bmatrix}_{v} = \sum_{m=1}^{k-1} \Gamma_m [\boldsymbol{B}]_{mi} \tag{7}$$

Influence of airfoil panels — Influence of wake panel

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & a_{1\,wp} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n} & a_{n-1\,wp} \\
1 & 0 & \cdots & 1 & a_{n\,wp} \\
0.5l_1 & 0.5(l_1+l_2) & \cdots & 0.5(l_{n-1}) & l_{wp}
\end{bmatrix}
\begin{bmatrix}
\gamma_1 \\ \vdots \\ \gamma_{n-1} \\ \gamma_n \\ \gamma_{wp}
\end{bmatrix}
=
\begin{bmatrix}
-\left(\sum_{m=1}^{k-1}\Gamma_m \boldsymbol{B}_{mi} + \boldsymbol{V}_{kin_i}\right)\cdot \hat{\boldsymbol{n}}_i\Big|_{i=1} \\
\vdots \\
-\left(\sum_{m=1}^{k-1}\Gamma_m \boldsymbol{B}_{mi} + \boldsymbol{V}_{kin_i}\right)\cdot \hat{\boldsymbol{n}}_i\Big|_{i=n-1} \\
0 \\
\Gamma_{\text{bound}}(t-\Delta t)
\end{bmatrix}
$$

Figure 2: **Equations formulated to determine the set of unknowns at a time instant t_k.**

In equation 7, $[\boldsymbol{B}]_{mi}$ denotes the 2×2 coefficient matrix that includes the terms for the velocity induced by m^{th} point vortex in the wake at i^{th} control point.

Finally, $\boldsymbol{V}_{kin}$ refers to the kinematic velocity, which represents the velocity due to the relative motion of the body with respect to the fluid. It is expressed in the following manner:

$$\boldsymbol{V}_{kin} = [\boldsymbol{V}_\infty - \boldsymbol{v}(t_k) - \boldsymbol{\omega}(t_k) \times \boldsymbol{r}(t_k)] \tag{8}$$

In this expression, $\boldsymbol{V}_\infty$ is the steady freestream velocity. The terms $\boldsymbol{v}(t_k)$ and $\boldsymbol{\omega}(t_k) \times \boldsymbol{r}(t_k)$ accounts for the unsteady translational and rotational velocities of the body.

ii) The n^{th} equation is derived from the conditon of zero vorticity at the trailing edge (Kutta Conditon) to achieve smooth outflow at the trailing edge,

$$\Gamma_{\text{TE}}(t_k) = 0 \tag{9}$$

$$\gamma_1(t_k) + \gamma_n(t_k) + \gamma_{wp}(t_k) = 0 \tag{10}$$

iii) The $(n+1)^{th}$ equation is obtained from Kelvin circulation theorem, which states that the total circulation in the flow must be conserved,

$$\frac{D\Gamma}{Dt} = 0 \tag{11}$$

Accordingly, the total circulation from the previous time step is equated to that of the current time step around the airfoil and the wake region to obtain the following relation:

$$\Gamma_{\text{B}}(t_k) + \gamma_{wp}(t_k) l_{wp}(t_k) = \Gamma_{\text{B}}(t_{k-1}) \tag{12}$$

where Γ_{B} indicates the bound circulation around the airfoil and its expression is defined as:

$$\Gamma_{\text{B}} = \int_{\text{s}=0}^{\text{L}} \gamma(s)\, ds = \frac{1}{2}\sum_{i=1}^{l-1}(\gamma_i + \gamma_{i+1}) l_i \tag{13}$$

where L is the perimeter of the airfoil, ds is the differential element along the contour of the airfoil and l_i indicates the length of i^{th} panel.

iv) To determine the coefficient term $[\boldsymbol{P}]_{wp\,i}$ in equation 6, the values of the length (l_{wp}) and orientation (θ_{wp}) of the trailing-edge wake panel need to be known. This constitutes the remaining two equations, and their expressions are provided as follows:

$$l_{wp}(t_k) = [u_{wp}^2(t_k) + v_{wp}^2(t_k)]^{\frac{1}{2}}[\Delta t] \tag{14}$$

$$\tan\theta_{wp}(t_k) = \frac{v_{wp}(t_k)}{u_{wp}(t_k)} \tag{15}$$

where u_{wp} and v_{wp} are the x and y components of the total velocity induced at the midpoint of the wake panel, excluding the effect of the panel on itself. Equation 14 and equation 15 are obtained by assuming that the wake panel is shed along the direction of the local velocity vector at the airfoil's trailing-edge. Figure 2 illustrates the fundamental set of $(n+1)$ linear equations, comprising zero normal flow boundary condition, Kutta Condition and Kelvin circulation theorem. The non-linearity in the system arises due to the evaluation of l_{wp} and θ_{wp} and Newton Raphson method has been employed to solve the non-linear simultaneous system of equations. The iterative procedure adopted for the solution is highlighted in the following paragraph.

At k^{th} time step (t_k), the guess values of l_{wp} and θ_{wp} are considered, which are the converged values obtained at t_{k-1}. Thereafter, the $(n+1)$ linear system of equations is solved to determine the unknowns. On determining the unknowns, $u_{wp}(t_k)$ and $v_{wp}(t_k)$ can be calculated and substituted in equations 14 and 15 to find the new values of $l_{wp}(t_k)$ and $\theta_{wp}(t_k)$. The procedure is repeated until $l_{wp}(t_k)$ and $\theta_{wp}(t_k)$ have converged to the desired accuracy. The bound vorticity strengths and the newly shed wake vortex panel strengths are then updated based on the converged values of $l_{wp}(t_k)$ and $\theta_{wp}(t_k)$.

The velocity distribution on the airfoil surface is obtained from equations 5,6,7 and 8 after the vorticity strengths of the panels have been calculated. Consequently, the pressure distribution is calculated from the unsteady Bernoulli equation as given below

$$C_{p_i} = 1 - \frac{V_{s_i}^2}{U_\infty^2} - \frac{2}{U_\infty^2}\frac{\partial \Phi_i}{\partial t} \quad (i = 1, ..., n-1) \tag{16}$$

where C_{p_i} is the pressure coefficient at i^{th} control point, V_{s_i} is the total induced velocity and Φ_i denotes the velocity potential at i^{th} control point. $\partial\Phi/\partial t$ is approximated by first order backward difference scheme,

$$\left.\frac{\partial \Phi}{\partial t}\right|_{t_k} = \frac{\Phi(t_k) - \Phi(t_{k-1})}{\Delta t} \tag{17}$$

The potential function Φ is obtained by integrating the velocity field in the freestream direction, starting from upstream of the airfoil to the leading edge and then around the airfoil surface.
After evaluating the pressure distribution, force coefficients are obtained by integrating it along the airfoil surface and is expressed as:

$$C_{F_x}(t_k) = -\int_{\mathrm{s}=0}^{\mathrm{L}} C_p(s)\hat{\boldsymbol{n}}(s)\cdot\hat{\boldsymbol{i}}\, ds \tag{18}$$

$$C_{F_y}(t_k) = -\int_{\mathrm{s}=0}^{\mathrm{L}} C_p(s)\hat{\boldsymbol{n}}(s)\cdot\hat{\boldsymbol{j}}\, ds \tag{19}$$

Following the computation of the aerodynamic loads, the wake needs to be updated. The most recent shed wake panel is convected downstream as a point vortex at t_{k+1} with the velocity induced at its mid-point at t_k and the updation rule is as follows,

$$x(t_{k+1}) = x_{\mathrm{TE}}(t_k) + \frac{1}{2}l_{wp}(t_k)\cos\theta_{wp}(t_k) + u_{wp}(t_k)\Delta t \tag{20}$$

$$y(t_{k+1}) = y_{\mathrm{TE}}(t_k) + \frac{1}{2}l_{wp}(t_k)\sin\theta_{wp}(t_k) + v_{wp}(t_k)\Delta t \tag{21}$$

Other concentrated wake vortices shed previously in the wake are updated according to the following vector equation,

$$\begin{aligned}\boldsymbol{r_v}(t_{k+1}) = \boldsymbol{r_v}(t_k) + \boldsymbol{V}_\infty\,\Delta t + \sum_{j=1}^{n-1}[\boldsymbol{P}]_{jv}\left\{\gamma_j(t_k)\gamma_{j+1}(t_k)\right\}\Delta t \\ + \gamma_{wp}(t_k)\,[\boldsymbol{P}]_{wp\,v}\,\Delta t + \left[\sum_{m=1}^{k-1}\Gamma_m\,[\boldsymbol{B}]_{mv} \text{ if } (v \neq m)\right]\Delta t, \text{ where } (v = 1, \ldots, k-1)\end{aligned} \tag{22}$$

where $\boldsymbol{r_v}$ is the position vector of the discrete wake vortices in inertial frame.

The above explained procedure is iterated at each time step to obtain the time varying output parameters. This numerical framework has been validated with benchmark results for the following two cases,
i) An impulsively started NACA 0012 airfoil at an angle of attack of 5° in a steady oncoming freestream flow, and
ii) NACA 0012 airfoil performing pitching and plunging motion in a steady oncoming freestream flow.

2 RESULTS AND DISCUSSION

2.1 Impulsively started NACA 0012 airfoil

This section presents the results of an impulsively initiated flow over a NACA 0012 airfoil at an angle of attack of 5°. An impulsively started flow refers to a scenario where there is no flow at $t = 0$, but a freestream velocity is suddenly applied at $t = 0^+$. In the present model, the airfoil is discretized into $n = 100$ panels and a time step of $\Delta t = 16c/nV_\infty$ was chosen for the simulation. Figure 3 illustrates the variation of the ratio of unsteady to steady lift (Kutta Joukowsky theorem) with time. The figure indicates that the unsteady lift asymptotically approaches its steady state value as time progresses. This behavior occurs because the change in bound circulation around the airfoil is gradually approaching towards zero as the overall strength of the wake accumulated in the downstream becomes equal and opposite to the bound circulation of airfoil, i.e. $\Gamma \to \Gamma_{Kutta}$.

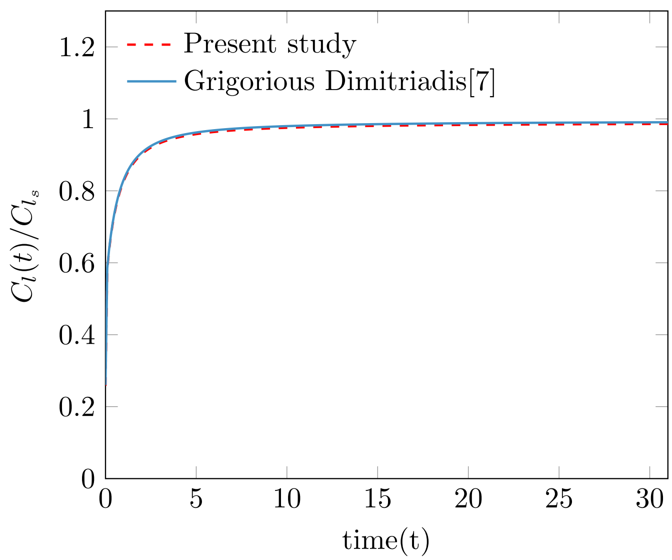

Figure 3: **Temporal behavior of lift coefficient on an impulsively started NACA 0012 airfoil at an angle of attack of 5°.**

2.2 Sinusoidal plunging and pitching

The next test case considers a NACA 0012 airfoil undergoing a sinusoidal pitching and plunging motion at a reduced frequency of $k = \omega c/U_\infty = 0.2$. In the current validation,the rotation is assumed to occur at a distance of one-third of the chord length from the airfoil's leading edge. In figure (4), x_f denotes this distance.

Figure (4) demonstrates the kinematics at a time instant t_k, with $h(t)$ and $\alpha(t)$ representing the plunging and pitching displacements, respectively. These kinematic equations are sourced from Grigorios Dimitriadis [3], against which the results are validated, and are given as follows:

$$\begin{aligned} h(t) &= h_1 \sin(\omega t + \phi_h) \\ \alpha(t) &= \alpha_1 \sin(\omega t + \phi_\alpha) \end{aligned} \tag{23}$$

Here, $h_1 = 0.25c$ represents the plunge amplitude, with c being the chord length. The pitch amplitude α_1 is calculated using the following expression:

$$\alpha_1 = 15^\circ - \tan^{-1}\left(\frac{2kh_1}{c}\right) \tag{24}$$

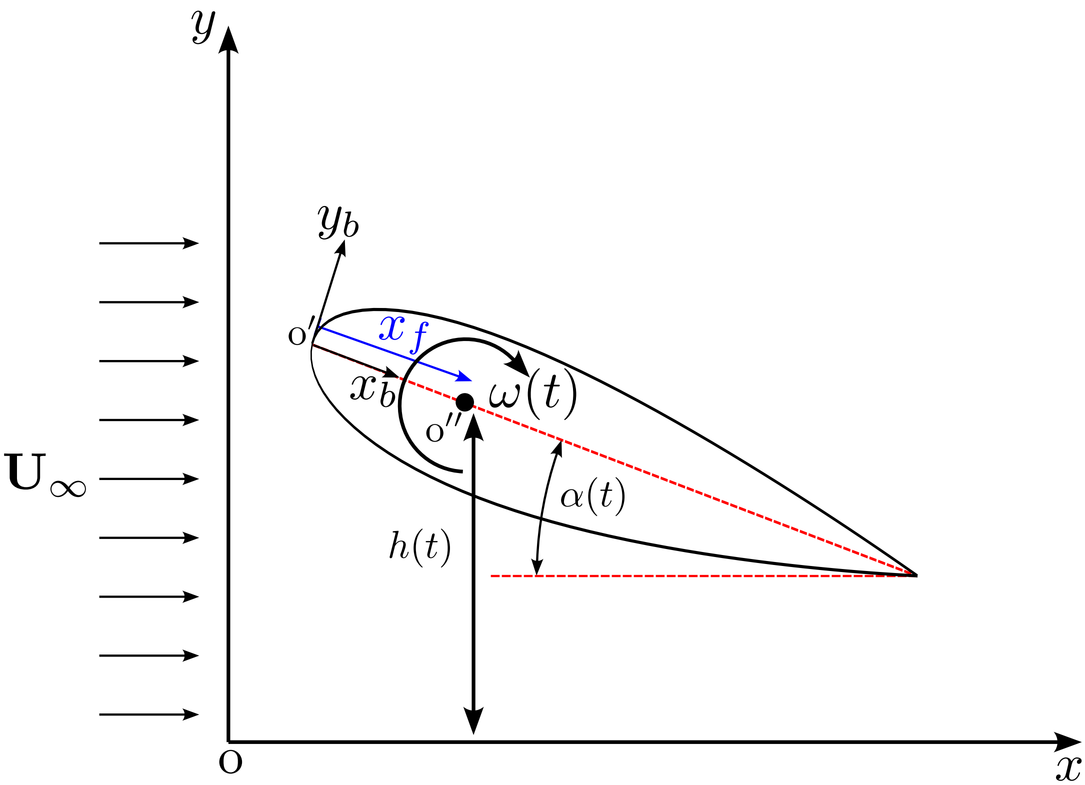

Figure 4: **Schematic representation of kinematics at time instant t_k**

ϕ_h and ϕ_α denote the phase angles for plunging and pitching, respectively. The phase difference between them is considered to be 90°, with ϕ_h set to 0°.

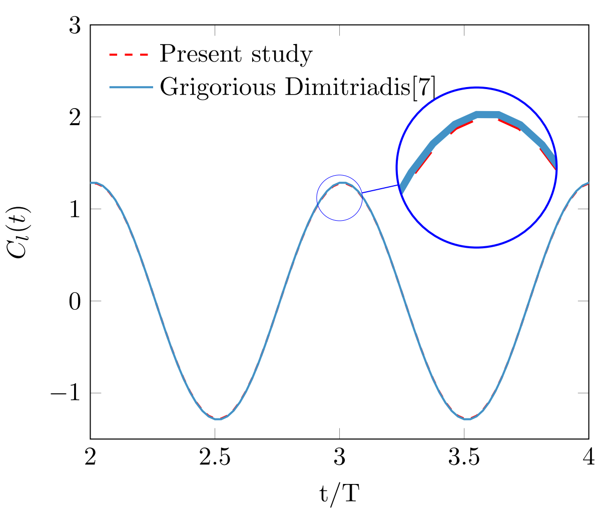

Figure 5: **Time-dependent lift coefficient**

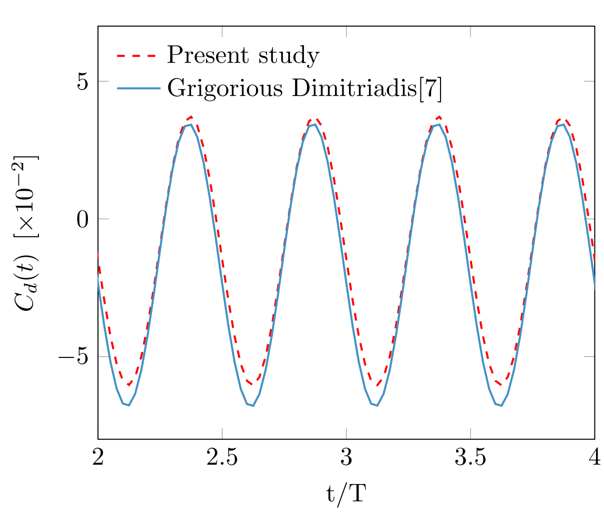

Figure 6: **Time-dependent drag coefficient**

Figures 5 and 6 show the instantaneous lift and the drag behavior, during the second to fourth stroke after quasi-steady state was achieved. The discrete time interval for this simulation was considered to be $\Delta t U_\infty / c = 0.157/k$. The results reveal that the curves exhibit oscillatory behaviour, which is attributed to the oscillatory nature of the motion. Figure 7 demonstrates the wake structure formed behind the pitching and plunging airfoil after 8 cycles.

Ignoring the transient effects of the first cycle,both the lift and the drag results demonstrates a repetitive nature. It can be visualised that during the pitch up, a counterclockwise vortex is shed from trailing edge followed by a clockwise vortex during the pitch down period.In the standard wake model adopted in this study, the concentrated counter- rotating vortices tend to fragment into smaller groups of vortices. However, this wake phenomenon is not due to viscous mixing but is a result of numerical instability involved in the wake modeling approach.

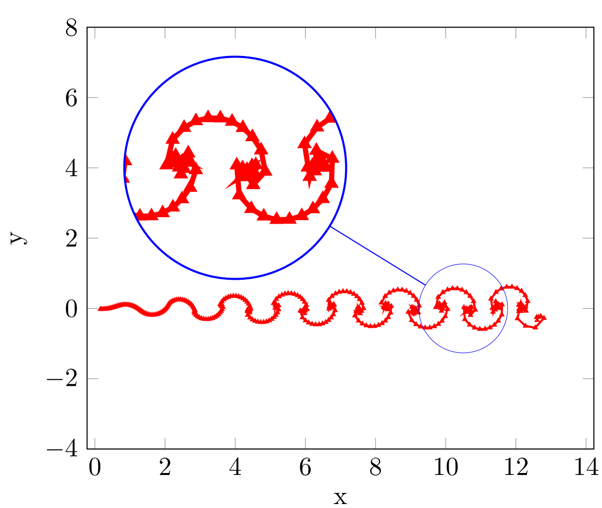

Figure 7: **Pictorial representation of the wake formed behind a NACA 0012 airfoil undergoing plunging and pitching oscillations at a reduced frequency value of 0.2**

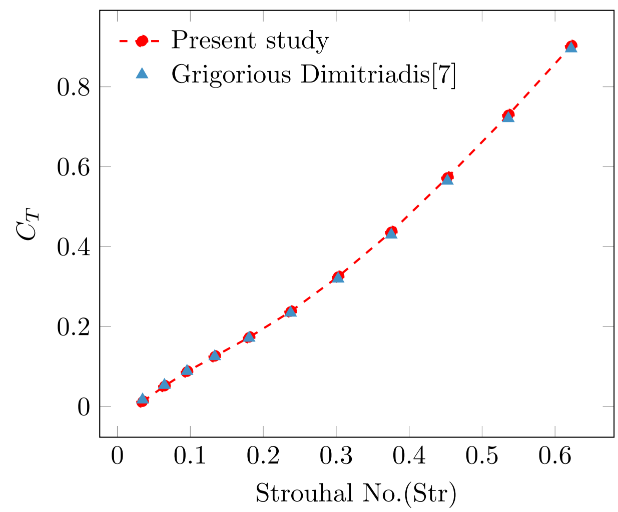

Figure 8: **Thrust Coefficient variation with Strouhal No. for reduced frequencies $k = 0.2$ to $k = 2.2$**

Figure 8 plots the mean thrust coefficient as a function of Strouhal number. The mean thrust is obtained from the following relation,

$$C_T = -\frac{1}{T}\int_0^T C_d(t)\,dt \tag{25}$$

The mean thrust coefficient is thus the average value of - $C_d(t)$ over one complete cycle with period T. For this curve, the average was taken over last cycle. Figure (8) shows that the C_T value is positive. This observation is consistent with Figure 6 where, C_d varies from almost -0.06 to 0.04 during a specific cycle apart from the transient effects of the initial cycle.

NOMENCLATURE

$[\boldsymbol{P}]$	influence coefficient matrix due to a vortex panel
$[\boldsymbol{B}]$	influence coefficient matrix due to a point vortex.
α	angle of attack
c	chord length
C_p	pressure coefficient
C_{F_x},C_{F_y}	force coefficients along x and y directions
C_T	mean thrust coefficient
k	reduced frequency
l	length
θ	orientation
$\boldsymbol{r}$	position vector
$\boldsymbol{V}_\infty$	freestream velocity
$\boldsymbol{V}_{kin}$	kinematic velocity
$\hat{\boldsymbol{n}}$	unit normal vector
Φ	velocity potential
Δt	increment in time
Γ	Circulation
γ	vorticity strength
t_k	time step instance
T	time period
x, y	cartesian coordinates in the inertial frame of reference
x_b,y_b	cartesian coordinates in body fixed frame of reference

SUBSCRIPTS

B	bound vortex
i,j	index of airfoil surface elements
s	airfoil surface
k	time step counter
v	concentrated wake vortex point
wp	wake panel

References

[1] Katz, J. & Plotkin, A. *Low-speed aerodynamics*, vol. 13 (Cambridge university press, 2001).

[2] Phillips, W. F. *Mechanics of flight* (John Wiley & Sons, 2004).

[3] Dimitriadis, G. *Unsteady Aerodynamics: Potential and Vortex Methods* (John Wiley & Sons, 2023).