MODULE-2 : Classification of Linear PDE of 2^{nd} Order.

DATE: / / PAGE:

(I) If $u = F(x,y)$. Then for PDE :-

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right) = 0} \quad -(1)$$

where A, B, C are constants or some function of x and y.

1. If $B^2 - 4AC < 0$, The eqn (1) is Elliptical.
2. If $B^2 - 4AC = 0$, Then eqn (1) is Parabolic.
3. If $B^2 - 4AC > 0$, Then eqn (1) is hyperbolic.

Q1. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0 \quad — (1)$$

$u = F(x,t)$

$A = 1,\ B = 1,\ C = 1$

Now $B^2 - 4AC = (1)^2 - 4(1)(1)$

$= -3$

$\Rightarrow -3 < 0$

given equation is elliptical in nature

2011, 16, 18

Q2. Show that the equation $Z_{xx} + 2x Z_{xy} + (1-y^2) Z_{yy} = 0$ is elliptic for value of x & y in the region $x^2 + y^2 < 1$, Parabolic on the boundary & hyperbolic outside this region.

$Z_{xx} + 2x Z_{xy} + (1-y)^2 Z_{yy} = 0$

$$\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} = 0 \quad — (1)$$

DATE: / / PAGE:

$A = 1,\ B = 2x,\ C = (1-y^2)$

Now $B^2 - 4AC = (2x)^2 - 4(1)(1-y^2)$

$= 4x^2 - 4 + 4y^2$

$$\boxed{B^2 - 4AC = 4x^2 + 4y^2 - 4} \quad — (2)$$

$\Rightarrow$ from eq (2) for Parabolic :- $B^2 - 4AC = 0$

$4x^2 + 4y^2 - 4 = 0$

$4x^2 + 4y^2 = 4$

$\cancel{4}(x^2+y^2) = \cancel{4}$

eqn of circle with centre at origin and $r = 1$. $\leftarrow \boxed{x^2 + y^2 = 1}$

i.e nature of curve is Parabolic at the boundary.

$\Rightarrow$ from eq (2) for Elliptic $\Rightarrow$ $B^2 - 4AC < 0$

$4x^2 + 4y^2 - 4 < 0$

$4(x^2+y^2) < 4$

inside the circle, nature of curve is elliptic. $\leftarrow \boxed{x^2 + y^2 < 1}$ OR

within the circle (region), curve is elliptic.

$\Rightarrow$ from eq (2) for hyperbolic $\Rightarrow$ $B^2 - 4AC > 0$

$\boxed{x^2 + y^2 > 1}$

Therefore, nature of curve is hyperbolic if x, y lies outside the curve (region)

2012

Q3. $U_{xx} + 3U_{xy} + U_{yy} = 0$

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1,\ B = 3,\ C = 1$ $\Rightarrow B^2 - 4AC$

$\Rightarrow (3)^2 - 4(1)(1) = 9 - 4$

$\Rightarrow 5 > 0$

$\Rightarrow$ curve is hyperbolic

2016, 17

Q4. Classify the equation ?

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} - 2z = 0 \quad (1)$$

$A = (1-x^2)$ $\quad B = -2xy$ $\quad C = (1-y^2)$

Now, $B^2 - 4AC = (-2xy)^2 - 4(1-x^2)(1-y^2)$

$= 4x^2y^2 - 4(1-y^2-x^2+x^2y^2)$

$= \cancel{4x^2y^2} - 4 + 4y^2 + 4x^2 - \cancel{4x^2y^2}$

$$\boxed{B^2 - 4AC = 4x^2 + 4y^2 - 4} \quad (2)$$

$\Rightarrow$ If $B^2 - 4AC < 0 \Rightarrow 4x^2 + 4y^2 - 4 < 0$

$4(x^2+y^2) < 4$

$\boxed{x^2+y^2 < 1}$

If x, & y lies inside the circle then nature of curve is elliptical.

$\Rightarrow$ If $B^2 - 4AC = 0 \Rightarrow 4x^2 + 4y^2 - 4 = 0$

$4(x^2+y^2) = 4$

$\boxed{x^2+y^2 = 1}$

If x, y lies at the boundary, then curve is Parabolic

$\Rightarrow$ If $B^2 - 4AC > 0 \Rightarrow 4x^2 + 4y^2 - 4 > 0$.

$4(x^2+y^2) > 4$

$\boxed{x^2+y^2 > 1}$

If x, y lies outside the (circle region), curve is hyperbolic

2012

Q5. Classify the following (i)

i) $u_{xx} + 3u_{xy} + 2u_{yy} = 0$

$A = 1,\ B = 3,\ C = 2$

$B^2 - 4AC = (3)^2 - 4(1)(2)$

$= 9 - 8$

$= 1 > 0$

$\Rightarrow$ hyperbolic

ii) $y^2 r - 2xys + x^2 t = \left(\frac{y^2}{z}\right)p + \left(\frac{x^2}{y}\right)q$

$$y^2\left(\frac{\partial^2 z}{\partial x^2}\right) - 2xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) + x^2\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{y^2}{z}\right)p - \left(\frac{x^2}{y}\right)q = 0$$

$A = y^2,\ B = -2xy,\ C = x^2$

$B^2 - 4AC = (-2xy)^2 - 4(y^2)(x^2)$

$= \cancel{4x^2y^2} - \cancel{4y^2x^2}$

$= 0$

$\Rightarrow$ Parabolic.

Q.

$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$	$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$
$c^2\left(\frac{\partial^2 u}{\partial x^2}\right) - \frac{\partial^2 u}{\partial t^2} = 0$	$c^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial t} = 0$	$A = 1,\ C = 1,\ B = 0$
$A = c^2,\ C = -1$	$A = 0,\ B = 0,\ C = c^2$	$B^2 - 4AC$
$B^2 - 4AC = 0 - 4(c^2)(-1)$	$B^2 - 4AC$	$0 - 4(1)(1)$
$= 4c^2 > 0$	$= 0$	$\boxed{-4 < 0}$
$\boxed{c^2 > 0}$	$\Rightarrow$ Parabolic	$\Rightarrow$ elliptic.
$\Rightarrow$ hyperbolic.		

V. Imp

(II) Method of Separation of Variables

Q. Using method of separation of variation solve

(i) $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad ; z = z(x,y)$

let $Z = X(x) \cdot Y(y)$ —— (1)

where X is funt^c of 'x' only and Y is funt^c of 'y' only.

Now $\frac{\partial z}{\partial x} = Y\frac{\partial X}{\partial x}$,

$$\frac{\partial^2 z}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

and $\frac{\partial z}{\partial y} = X\frac{\partial Y}{\partial y}$

putting the above values in given (1) eqn :-

$$Y\frac{\partial^2 X}{\partial x^2} - 2Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 0$$

dividing whole eqn by XY

$$\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2}\right) - \left(\frac{2}{X}\frac{\partial X}{\partial x}\right) + \left(\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = 0$$

or $\left(\frac{1}{X}\frac{\partial^2 X}{\partial x^2} - \frac{2}{X}\frac{\partial X}{\partial x}\right) = \left(-\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = -p^2$ (say) —— (2)

now from eq (2) considering Ist & IIIrd part

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} - \frac{2}{X}\frac{\partial X}{\partial x} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x} = -p^2 X$$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} - 2\frac{\partial X}{\partial x} + p^2 X = 0$$

AE $\Rightarrow$ $m^2 - 2m + p^2 = 0$

$$m = \frac{2 \pm \sqrt{4-4p^2}}{2}$$

$$m = 1 \pm \sqrt{1-p^2}$$

$$m = 1+\sqrt{1-p^2} \ \& \ 1-\sqrt{1-p^2}$$

Ordinary Solution

$X = C_1 e^{m_1 x} + C_2 e^{m_2 x} \ ; Y = C_3 e^{m_3 y} + C_4 e^{m_4 y}$

$\therefore \boxed{X = C_1 e^{(1+\sqrt{1-p^2})x} + C_2 e^{(1-\sqrt{1-p^2})x}}$ —— (3)

now from eq (2) considering IInd & IIIrd part

$$-\frac{1}{Y}\frac{\partial Y}{\partial y} = -p^2$$

$$\frac{\partial Y}{\partial y} = p^2 Y$$

$$\Rightarrow \frac{\partial Y}{\partial y} - p^2 Y = 0$$

AE $\Rightarrow$ $m - p^2 = 0$

$$m = p^2$$

ordinary soln $\Rightarrow$ $\boxed{Y = C_3 e^{p^2 y}}$ —— (4)

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(from eq ③ & ④)

Soln of given eqn :-

$$Z = X(x) \cdot Y(y)$$

$$\boxed{Z = \left\{ C_1 e^{(1+\sqrt{1-p^2})x} + C_2 e^{(1-\sqrt{1-p^2})x} \right\} \left\{ C_3 e^{p^2 y} \right\}}$$

10 mark

Q. 2013, 17

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \quad \begin{cases} \text{given that } u(x,t) = 0 \\ \quad \text{at } t = 0. \\ \text{and } \frac{\partial u}{\partial t} = 0 \text{ at } x = 0 \end{cases}$$

let $U = X(x) \cdot T(t)$ — ①

Diff eq ① partially w.r.t 'x' and then by 't'.

$$\frac{\partial u}{\partial t} = X \frac{\partial T}{\partial t} \Rightarrow \frac{\partial^2 u}{\partial x \partial t} = \left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial T}{\partial t}\right) \quad \text{— ②}$$

Using eq ② in given eqn

$$\frac{\partial X}{\partial x} \frac{\partial T}{\partial t} = e^{-t} \cos x$$

$$\frac{1}{e^{-t}}\left(\frac{\partial T}{\partial t}\right) = \frac{\cos x}{\left(\frac{\partial X}{\partial x}\right)}$$

$$e^{t}\left(\frac{\partial T}{\partial t}\right) = \frac{\cos x}{\left(\frac{\partial X}{\partial x}\right)} = -p^2 \text{ (say)} \quad \text{— ③}$$

now $e^{t}\left(\frac{\partial T}{\partial t}\right) = -p^2$

$$\partial T = \left(\frac{-p^2}{e^t}\right) \partial t$$

$$\partial T = -p^2 e^{-t} \partial t$$

now integrating

$$T = -p^2 \frac{e^{-t}}{-1} + C_1$$

$$\boxed{T = p^2 e^{-t} + C_1} \quad \text{— ④}$$

Now, $\frac{\cos x}{\left(\frac{\partial X}{\partial x}\right)} = -p^2$

$$\frac{\partial X}{\partial x} = -\left(\frac{\cos x}{p^2}\right)$$

$$\partial X = -\left(\frac{\cos x}{p^2}\right) \partial x$$

Now integrating

$$X = -\frac{1}{p^2}(\sin x) + C_2$$

$$\boxed{X = -\frac{\sin x}{p^2} + C_2} \quad \text{— ⑤}$$

Putting Value of 4th and 5th eq in 1st eqn

$$\boxed{U = \left(-\frac{\sin x}{p^2} + C_2\right)\left(p^2 e^{-t} + C_1\right)} \quad \text{— ⑥}$$

diff 6th eqn partially w.r.t 't'

$$\frac{\partial U}{\partial t} = \left(-\frac{\sin x}{p^2} + C_2\right)\left(-p^2 e^{-t} + C_1\right) \quad \text{— ⑦}$$

$\Rightarrow$ using the condition at $x = 0$, $\frac{\partial u}{\partial t} = 0$

So, from eq ④

$$0 = \left(-\frac{\sin 0}{p^2} + c_2\right)\left(-p^2 e^{-t}\right)$$

$$0 = c_2\left(-p^2 e^{-t}\right)$$

$$\Rightarrow \boxed{c_2 = 0}$$

and using $u(x,t) = 0$ at $t = 0$

so from eq ⑥ we get

$$0 = \left(-\frac{\sin x}{p^2} + \cancel{c_2}\right)\left(p^2 e^{-t} + c_1\right) \quad \boxed{t=0}$$

$$0 = \left(\frac{-\sin x}{p^2}\right)\left(p^2 e^{-0} + c_1\right)$$

$$0 = \left(\frac{-\sin x}{p^2}\right)\left(p^2 + c_1\right)$$

$$0 = p^2 + c_1$$

$$\Rightarrow \boxed{c_1 = -p^2}$$

Now putting c_1 & c_2 in eq ⑥ we get

$$U = \left(-\frac{\sin x}{p^2} + 0\right)\left(p^2 e^{-t} - p^2\right)$$

$$U = \left(\frac{-\sin x}{p^2}\right) p^2 \left(e^{-t} - 1\right)$$

$$U = -\cancel{p^2}\left(\frac{\sin x}{-\cancel{p^2}}\right)(1 - e^{-t})$$

$$\boxed{U = \sin x\,(1 - e^{-t})}$$

2013 Q3 ★ $U_{xx} = U_y + 2u$, given $u(0,y) = 0$, $\frac{\partial u}{\partial x} = 1 + e^{-3y}$ at $x = 0$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u \quad \text{— ①}$$

Let $U = X \cdot Y$ — ②

$\frac{\partial U}{\partial x} = Y\frac{\partial X}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$

and $\frac{\partial U}{\partial y} = X\frac{\partial Y}{\partial y}$

now substituting above values in eq ①

$$Y\frac{\partial^2 X}{\partial x^2} = X\frac{\partial Y}{\partial y} + 2(XY)$$

$$Y\frac{\partial^2 X}{\partial x^2} = X\left(\frac{\partial Y}{\partial y} + 2Y\right)$$

multiplying whole eqn by $\frac{1}{XY}$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\left(\frac{\partial Y}{\partial y} + 2Y\right)$$

$$\Rightarrow \frac{1}{X}\left(\frac{\partial^2 X}{\partial x^2}\right) = \frac{1}{Y}\left(\frac{\partial Y}{\partial y}\right) + 2 = -p^2 \text{ (say)} \quad \text{— ③}$$

from eq ③ taking I^{st} and III^{rd} term

$$\frac{1}{X}\left(\frac{\partial^2 X}{\partial x^2}\right) = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} = -p^2 X$$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

AE $\Rightarrow$ $m^2 + p^2 = 0$

$$m^2 = -p^2$$

$$m = \sqrt{-p^2}$$

$$m = \pm pi$$

$$m = 0 \pm ip$$

$$\boxed{m = \alpha \pm i\beta}$$

CF $\Rightarrow$ $C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$

$\therefore$ CF $\Rightarrow$ $C_1 e^{0x} \cos px + C_2 e^{0x} \sin px$

$$\boxed{X = c_1 \cos px + C_2 \sin px} \quad \text{—— (4)}$$

considering II^{nd} and III^{rd} term

$$\frac{1}{Y}\left(\frac{\partial Y}{\partial y}\right) + 2 = -p^2$$

$$\frac{\partial Y}{\partial y} + 2Y = -p^2 Y$$

$$\frac{\partial Y}{\partial y} + 2Y + p^2 Y = 0$$

$$\frac{\partial Y}{\partial y} + Y(2+p^2) = 0$$

$$\frac{\partial Y}{\partial y} = -(p^2+2)Y$$

$$\Rightarrow \frac{\partial Y}{Y} = -(p^2+2)\, dy$$

now integrating

$$\log Y = -(p^2+2)y + \log C_3$$

$$\log Y - \log C_3 = -(p^2+2)y$$

$$\log\left(\frac{Y}{C_3}\right) = -(p^2+2)y$$

$$\left(\frac{Y}{C_3}\right) = e^{-(p^2+2)y}$$

$$\boxed{Y = C_3 e^{-(p^2+2)y}} \quad \text{—— (5)}$$

Substituting value of eq (4) & (5) in eq (2), we get

$$\boxed{U(x,y) = \left(C_1 \cos px + C_2 \sin px\right)\left(C_3 e^{-(p^2+2)y}\right)} \quad \text{— (6)}$$

given that $u(0,y) = 0$ at $x = 0$

$\therefore$ from eq (6)

$$0 = \left(C_1 \underbrace{\cos 0}_{1} + C_2 \underbrace{\sin 0}_{0}\right)\left(C_3 e^{-(p^2+2)y}\right)$$

$$0 = C_1 \left(C_3 e^{-(p^2+2)y}\right)$$

$$\Rightarrow \boxed{C_1 = 0}$$

now diff^{n} eq (6), we get

$$\frac{\partial u}{\partial x} = \left(-C_1 p \sin px + pC_2 \cos px\right)\left(C_3 e^{-(p^2+2)y}\right) \quad \text{—— (7)}$$

given $\frac{\partial u}{\partial x} = (1 + e^{-3y})$ at $x = 0$.

$$\therefore \quad (1+e^{-3y}) = \left(-c_1 p \underset{0}{\sin 0} + p c_2 \underset{1}{\cos 0}\right)\left(c_3 e^{-(p^2+2)y}\right)$$

$$(1+e^{-3y}) = p c_2 \left(c_3 e^{-(p^2+2)y}\right)$$

$$1+e^{-3y} = c_2 c_3 \, p \, e^{-(p^2+2)y}$$

$$\boxed{c_2 c_3 = \frac{1+e^{-3y}}{p e^{-(p^2+2)y}}} \quad \text{——— (8)}$$

from eq (6) using $c_1 = 0$

$$u = (0 + c_2 \sin px) \cdot c_3 e^{-(p^2+2)y}$$

$$u = (c_2 c_3) \cdot \sin px \cdot e^{-(p^2+2)y}$$

from (8)th

$$u = \frac{1+e^{-3y}}{p \cancel{e^{-(p^2+2)y}}} \times \sin px \times \left(\cancel{e^{-(p^2+2)y}}\right)$$

Soln.

$$\boxed{u = \frac{(1+e^{-3y}) \sin px}{p}}$$

Q4. 2017

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0; \text{ given } u(x,0) = 4e^{-x}$$

Let $\quad U = X \cdot Y$ ——— (1)

$$\frac{\partial U}{\partial x} = Y\frac{\partial X}{\partial x} \quad \text{and} \quad \frac{\partial U}{\partial y} = X\frac{\partial Y}{\partial y}$$

$$3\left(Y\frac{\partial X}{\partial x}\right) + 2\left(X\frac{\partial Y}{\partial y}\right) = 0$$

$$3\left(\frac{1}{X}\frac{\partial X}{\partial x}\right) = -2\left(\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = -p^2 \text{ (say)} \text{ ——— (2)}$$

considering Ist and IIIrd term

$$3\left(\frac{1}{X}\frac{\partial X}{\partial x}\right) = -p^2$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = -\frac{p^2}{3}$$

$$\cancel{\frac{\partial X}{\partial x} + p^2 X = 0}$$

$$\Rightarrow \quad \frac{1}{X} dX = -\frac{p^2}{3} dx$$

integrating

$$\log X = -\frac{p^2}{3}x + \log c_1$$

$$\log X - \log c_1 = -\frac{p^2}{3}x$$

$$\log\left(\frac{X}{c_1}\right) = -\frac{p^2}{3}x$$

$$\frac{X}{c_1} = e^{-p^2/3 \, x}$$

$$\boxed{X = c_1 e^{\left(-\frac{p^2}{3}\right)x}} \text{ ——— (3)}$$

considering IInd & IIIrd

$$-2\left(\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = -p^2$$

$$\frac{1}{Y}\frac{\partial Y}{\partial y} = \frac{p^2}{2}$$

$$\frac{1}{Y}\partial Y = \frac{p^2}{2}\partial y$$

integrating

$$\log Y = \frac{p^2}{2}y + \log c_2$$

$$\log\left(\frac{Y}{c_2}\right) = \frac{p^2}{2}y$$

$$\left(\frac{Y}{c_2}\right) = e^{\frac{p^2}{2}y}$$

$$\boxed{Y = c_2 e^{\frac{p^2}{2}y}} \text{ ——— (4)}$$

$$\therefore \quad \boxed{U = c_1 e^{-\frac{p^2}{3}x} \cdot c_2 e^{\frac{p^2}{2}y}} \text{ ——— (5)}$$

given that $u(x,0) = 4e^{-x}$ at $y = 0$

$$4e^{-x} = c_1 e^{-\frac{p^2}{3}x} \cdot c_2 e^{\frac{p^2}{2}0}$$

$$4e^{-x} = c_1 e^{-\frac{p^2 x}{3}} \cdot c_2$$

$$4e^{-x} = c_1 c_2 e^{-\frac{p^2 x}{3}}$$

$$\boxed{c_1 c_2 = \frac{4e^{-x}}{e^{-\frac{p^2 x}{3}}}}$$

from eq 6

$$U = c_1 c_2 e^{-\frac{p^2 x}{3}} \cdot e^{\frac{p^2 y}{2}}$$

$$U = \frac{4e^{-x}}{\cancel{e^{-\frac{p^2 x}{3}}}} \times \cancel{e^{-\frac{p^2 x}{3}}} \times e^{\frac{p^2 y}{2}}$$

$$\boxed{U = 4e^{-x} \cdot e^{\frac{p^2 y}{2}}}$$

2018

Q $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ given $u(0,y) = 4e^{-y} - e^{-5y}$

Let $U = X \cdot Y$

$$\frac{\partial U}{\partial x} = Y\left(\frac{\partial X}{\partial x}\right) \text{ and } \frac{\partial U}{\partial y} = X\left(\frac{\partial Y}{\partial y}\right)$$

$$4\left(Y\frac{\partial X}{\partial x}\right) + \left(X\frac{\partial Y}{\partial y}\right) = 3(XY)$$

$$4\left(\frac{1}{X}\frac{\partial X}{\partial x}\right) + \left(\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = 3$$

$$4\left(\frac{1}{X}\frac{\partial X}{\partial x}\right) = 3 - \left(\frac{1}{Y}\frac{\partial Y}{\partial y}\right) = -p^2 \text{ (say)}$$

$$3 - \frac{1}{Y}\frac{\partial Y}{\partial y} = -p^2$$

$$\cancel{-}\left(\frac{1}{Y}\frac{\partial Y}{\partial y} - 3\right) = \cancel{-}p^2$$

$$\frac{1}{Y}\frac{\partial Y}{\partial y} = p^2 + 3$$

$$\frac{1}{Y}\partial Y = (p^2+3)\partial y$$

integration

$$\log Y = (p^2+3)y + \log c_1$$

$$\log\left(\frac{Y}{c_1}\right) = (p^2+3)y$$

$$\frac{Y}{c_1} = e^{(p^2+3)y}$$

$$\boxed{Y = c_1 e^{(p^2+3)y}} \quad \text{— (2)}$$

$$4\frac{1}{X}\frac{\partial X}{\partial x} = -p^2$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = -\frac{p^2}{4}$$

$$\frac{1}{X}\partial X = -\frac{p^2}{4}\partial x$$

integration

$$\log X = -\frac{p^2}{4}x + \log c_2$$

$$\log X - \log c_2 = -\frac{p^2 x}{4}$$

$$\log\left(\frac{X}{c_2}\right) = -\frac{p^2 x}{4}$$

$$\frac{X}{c_2} = e^{-\frac{p^2 x}{4}}$$

$$\boxed{X = c_2 \cdot e^{-\frac{p^2 x}{4}}} \quad (3)$$

$$\boxed{U = c_2 e^{-\frac{p^2 x}{4}} \cdot c_1 e^{(p^2+3)y}} \quad \text{—— (4)}$$

given that $U(0,y) = 4e^{-y} - e^{-5y}$ at $x = 0$

$$4e^{-y} - e^{-5y} = c_2 e^{-\frac{p^2 0}{4}} \cdot c_1 e^{(p^2+3)y}$$

$$4e^{-y} - e^{-5y} = c_2 c_1 e^{(p^2+3)y}$$

$$\boxed{c_1 c_2 = \frac{4e^{-y} - e^{-5y}}{e^{(p^2+3)y}}} \quad \text{—— (5)}$$

Substituting in eq (4)th

$$U = C_1 C_2 e^{-\frac{p^2 x}{4}} \cdot e^{(p^2+3)y}$$

$$U = \frac{4e^{-y} - e^{-5y}}{\cancel{e^{(p^2+3)y}}} \times e^{-\frac{p^2 x}{4}} \times \cancel{e^{(p^2+3)y}}$$

$$\boxed{U = \left(4e^{-y} - e^{-5y}\right) e^{-\frac{p^2 x}{4}}}$$

2018 Q

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} - 2u \quad \text{— (1)} \quad \text{given } u(x,0) = 10e^{-x} - 6e^{-4x}$$

Let $U = T \cdot X$ — (2)

$$\frac{\partial U}{\partial T} = X \frac{\partial T}{\partial t} \quad \text{and} \quad \frac{\partial U}{\partial x} = T \frac{\partial X}{\partial x}$$

$$X \frac{\partial T}{\partial t} = T \frac{\partial X}{\partial x} - 2XT$$

$$\frac{1}{T} \frac{\partial T}{\partial t} = \frac{1}{X} \frac{\partial X}{\partial x} - 2 = -p^2 \text{ (say)}$$

$$\frac{1}{T} \frac{\partial T}{\partial t} = -p^2$$

$$\frac{\partial T}{T} = -p^2 \partial t$$

integration

$$\log T = -p^2 t + \log C_1$$

$$\log\left(\frac{T}{C_1}\right) = -p^2 t$$

$$\boxed{T = C_1 \cdot e^{-p^2 t}} \quad \text{— (3)}$$

$$\frac{1}{X} \frac{\partial X}{\partial x} - 2 = -p^2$$

$$\cancel{-}\left(2 - \frac{1}{X} \frac{\partial X}{\partial x}\right) = \cancel{-}p^2$$

$$2 - p^2 = \frac{1}{X} \frac{\partial X}{\partial x}$$

$$(2-p^2)\, \partial x = \frac{\partial X}{X}$$

integration $\log_{C_2} + (2-p^2)x = \log X$

$$\log X - \log C_2 = (2-p^2)x$$

$$\log\left(\frac{X}{C_2}\right) = (2-p^2)x$$

$$\boxed{X = C_2 \cdot e^{(2-p^2)x}} \quad \text{— (4)}$$

$$\boxed{U = C_2 e^{(2-p^2)x} \cdot C_1 e^{-p^2 t}} \quad \text{— (5)}$$

given $U(x,0) = 10e^{-x} - 6e^{-4x}$ at $t = 0$

$$U = C_1 C_2 e^{(2-p^2)x} \cdot e^{-p^2 0}$$

$$U = C_1 C_2 e^{(2-p^2)x}$$

$$10e^{-x} - 6e^{-4x} = C_1 C_2 e^{(2-p^2)x}$$

$$\boxed{C_1 C_2 = \frac{10e^{-x} - 6e^{-4x}}{e^{(2-p^2)x}}} \quad \text{— (6)}$$

Substituting in eq (5)

$$U = C_1 C_2 \, e^{(2-p^2)x} \, e^{-p^2 t}$$

$$U = \frac{10e^{-x} - 6e^{-4x}}{\cancel{e^{(2-p^2)x}}} \times \cancel{e^{(2-p^2)x}} \times e^{-p^2 t}$$

$$\boxed{U = (10e^{-x} - 6e^{-4x}) e^{-p^2 t}}$$

2017,15

Q. $2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 5z = 0$ —(1), $z(0,y) = 2e^{-5}$

$Z = X \cdot Y$ —(2)

$$2\left(Y\frac{\partial X}{\partial x}\right) + 3\left(X\frac{\partial Y}{\partial y}\right) + 5(XY) = 0$$

$$\frac{2}{X}\frac{\partial X}{\partial x} + \frac{3}{Y}\frac{\partial Y}{\partial y} + 5 = 0$$

$$\frac{2}{X}\frac{\partial X}{\partial x} = -\left(5 + \frac{3}{Y}\frac{\partial Y}{\partial y}\right) = -p^2 \text{ (say)}$$

$$\frac{2}{X}\frac{\partial X}{\partial x} = -p^2$$

$$\frac{1}{X}\partial X = -\frac{p^2}{2}\partial x$$

$$\log X = -\frac{p^2}{2}x + \log C_1$$

$$\frac{X}{C_1} = e^{-\frac{p^2}{2}x}$$

$$\boxed{X = C_1 e^{-\frac{p^2 x}{2}}}$$

$$\cancel{-}\left(5 + \frac{3}{Y}\frac{\partial Y}{\partial y}\right) = \cancel{-}p^2$$

$$\frac{1}{Y}\partial Y = \frac{(p^2-5)}{3}\partial y$$

$$\log Y = \frac{p^2-5}{3}y + \log C_2$$

$$\boxed{Y = C_2 \cdot e^{\frac{(p^2-5)}{3}y}}$$

$$\boxed{Z = C_1 C_2 \, e^{-\frac{p^2 x}{2}} \, e^{\frac{(p^2-5)}{3}y}}$$

given $Z(0,y) = 2e^{-5}$ at $x = 0$

$$2e^{-5} = C_1 C_2 \, e^{\left(\frac{p^2-5}{3}\right)y}$$

$$\boxed{C_1 C_2 = \frac{2e^{-5}}{e^{\frac{(p^2-5)}{3}y}}}$$

$$Z = \frac{2e^{-5}}{\cancel{e^{\frac{(p^2-5)}{3}y}}} \times e^{-\frac{p^2 x}{2}} \times \cancel{e^{\frac{(p^2-5)}{3}y}}$$

$$Z = 2e^{-5} \cdot e^{-\frac{p^2 x}{2}}$$

(10 marks) Impt → (either soln of 1-D eqn / Numerical on 1-D eqn)

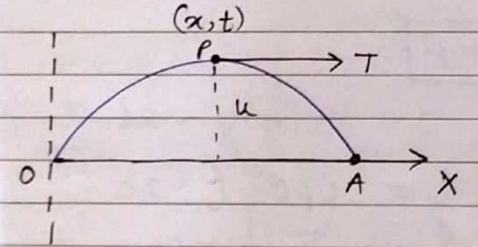
DATE: / / PAGE:

(III) **One-Dimensional Wave Equation** (Vibration of a stretched string)

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Above eqn is One-Dim: wave equation where $c^2 = \frac{T}{m}$ (tension of string) / (mass of string)

→ x is the direction of propogation of wave.
→ $u(x,t)$ is displacement
→ 't' is time



Solution of 1-D wave eqn

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, given initial condition (or boundary condn)

$u(x,t) = 0$ at $x=0$, $u=0$, $x=l$
and $u = f(x)$ at $t=0$
$\frac{\partial u}{\partial t} = 0$, at $t=0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $U = X \cdot T$ —— (2)

DATE: / / PAGE:

diffn eq (2) w.r.t 't' & 'x' twice we get :-

$$\frac{\partial^2 u}{\partial t^2} = X \frac{\partial^2 T}{\partial t^2} \quad \& \quad \frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

from eq (1) $\quad X \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$

$$\left(\frac{1}{c^2 T}\right) \frac{\partial^2 T}{\partial t^2} = \left(\frac{1}{X}\right) \frac{\partial^2 X}{\partial x^2} = -p^2 \text{ (say)} \quad \text{—— (3)}$$

from 1st and 3rd term, we get

$$\frac{1}{c^2 T} \frac{\partial^2 T}{\partial t^2} = -p^2$$

$$\frac{\partial^2 T}{\partial t^2} = -p^2 c^2 T$$

$$\frac{\partial^2 T}{\partial t^2} + p^2 c^2 T = 0$$

AE ⇒ $m^2 + p^2 c^2 = 0$

$m^2 = -p^2 c^2$

$m = \sqrt{-p^2 c^2}$

$m = \pm ipc$

$m = 0 \pm ipc$

CF ⇒ $c_1 e^{0 \cdot x} \cos pct + c_2 e^{0x} \sin pct$

$$\boxed{T = c_1 \cos pct + c_2 \sin pct} \quad \text{—— (4)}$$

DATE: / /
PAGE:

from last two terms, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} = -p^2 X$$

$$\frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

AE $\Rightarrow$ $m^2 + p^2 = 0$

$m = \pm ip$

$$X = C_3 e^{0x}\cos px + C_4 e^{0x}\sin px$$

$$\boxed{X = C_3\cos px + C_4\sin px} \quad — (5)$$

from eq (2), we get

(6) — $$\boxed{U = (C_3\cos px + C_4\sin px)(C_1\cos cpt + C_2\sin cpt)}$$

given $u=0$, $x=0$, $u(x,t)=0$ at $x=l$

$$0 = (C_3\cos 0 + C_4\sin 0)(C_1\cos cpt + C_2\sin cpt)$$

$$0 = C_3(C_1\cos cpt + C_2\sin pt)$$

$$\boxed{C_3 = 0}$$

DATE: / /
PAGE:

again from eq (6), we get

using $u=0$, $x=l$.

(7) — $$0 = (C_3\cos pl + C_4\sin pl)(C_1\cos cpt + C_2\sin ctp)$$

as we know $C_3 = 0$

$\therefore$ $$0 = C_4\sin pl(C_1\cos cpt + C_2\sin cpt)$$

$0 = C_4\sin pl \Rightarrow \sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$$\boxed{p = \frac{n\pi}{l}}$$

now putting $C_3 = 0$ & $p = \frac{n\pi}{l}$ in eq (6)

$$U = C_4\sin\left(\frac{n\pi}{l}\right)x\left[C_1\cos c\left(\frac{n\pi}{l}\right)t + C_2\sin c\left(\frac{n\pi}{l}\right)t\right]$$

OR

(8) — $$U = C_4\left[C_1\cos\left(\frac{n\pi ct}{l}\right) + C_2\sin\left(\frac{n\pi ct}{l}\right)\right]\cdot\sin\left(\frac{n\pi x}{l}\right)$$

from eq (8)

$$U = \left[C_1C_4\cos\left(\frac{n\pi ct}{l}\right) + C_2C_4\sin\left(\frac{n\pi ct}{l}\right)\right]\cdot\sin\left(\frac{n\pi x}{l}\right)$$

replacing $C_1C_4 = a_n$ & $C_2C_4 = b_n$

$$U = \left[a_n\cos\left(\frac{n\pi ct}{l}\right) + b_n\sin\left(\frac{n\pi ct}{l}\right)\right]\cdot\sin\left(\frac{n\pi x}{l}\right)$$

DATE: / / PAGE:

for different values of n, we get

$$U = \sum_{n=1}^{\infty} \left[\left\{ a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right) \right\} \cdot \sin\left(\frac{n\pi x}{l}\right) \right] \quad (9)$$

—x—x—x—x—x—x—

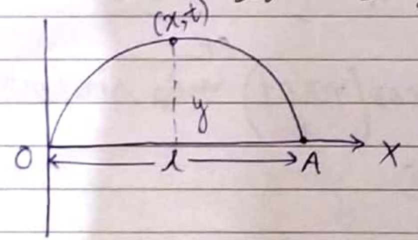
vvv imp

2010, 11, 12, 17, 18, 19

Numerical based on 1-D wave eqn.

Q A string is stretched and fixed between two points 'l' apart. Motion is started by displacing the string in the form $y = A\sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t=0$. Show that the displacement of any point at a distance 'x' from one end at time 't' is given by

$$y(x,t) = A\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$$



We know that stretched string follow one-dimensional wave equation.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial t^2} \quad (1)$$

$$\because y = y(x,t)$$

DATE: / / PAGE:

Since string is streched between two points $O(0,0)$ and $A(l,0)$ hence displacement is zero at end points.

i.e $y(0,t)=0$ at $x=0$ and $y(l,t)=0$ at $x=l$ — (2)

Now string is released from rest, hence

velocity $\frac{\partial y}{\partial t} = 0$ at $t=0$ — (3)

and also initial displacement at $t=0$ is

$$y = A\sin\left(\frac{\pi x}{l}\right) \quad \text{at } t=0 \quad (4)$$

Now, let $y = XT$ — (5)

diff^n 5th w.r.t. x & t twice we get

$$\frac{\partial^2 y}{\partial t^2} = X\frac{\partial^2 T}{\partial t^2}, \quad \frac{\partial^2 y}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

Now eqn (1), we get

$$X\frac{\partial^2 T}{\partial t^2} = c^2 T\frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{T}\frac{\partial^2 T}{\partial t^2} = \frac{c^2}{X}\frac{\partial^2 X}{\partial x^2}$$

$$\left(\frac{1}{c^2 T}\right)\frac{\partial^2 T}{\partial t^2} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2 \text{ (say)} \quad (6)$$

from (1)st & Last term

$$\left(\frac{1}{c^2 T}\right)\frac{\partial^2 T}{\partial t^2} = -p^2$$

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$$\frac{\partial^2 T}{\partial t^2} = -p^2C^2T$$

$$\frac{\partial^2 T}{\partial t^2} + p^2C^2T = 0$$

AE $\Rightarrow$ $m^2 + p^2C^2T = 0$

$$m^2 = -p^2C^2$$

$$\boxed{m = \pm ipc}$$

$$T = C_1 e^{0t}\cos pct + C_2 e^{0t}\sin pct$$

$$\boxed{T\ C_1\cos pct + C_2\sin pct} \quad \text{—— (7)}$$

now from last two eqn, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} = -p^2 X$$

$$\frac{\partial^2 X}{\partial x^2} + p^2X = 0$$

AE $\Rightarrow$ $m^2 + p^2 = 0$

$$m^2 = -p^2$$

$$\boxed{m = \pm ip}$$

$$\therefore \boxed{X = C_3\cos px + C_4\sin px} \quad \text{—— (8)}$$

Now putting the value of 'X' & 'T' in eqn (5)

general soln $\rightarrow$ $$\boxed{y = (C_3\cos px + C_4\sin px)(C_1\cos pct + C_2\sin pct)}$$

—— (9)

Now using condition (ii) $y=0$ at $x=0$ and $x=l$

from (9)th $0 = (C_3\underbrace{\cos 0}_{1} + C_4\underbrace{\sin 0}_{0})(C_1\cos pct + C_2\sin pct)$

$$\boxed{0 = C_3(C_1\cos pct + C_2\sin pct)}$$

$$\Rightarrow C_3 = 0$$

Now at $x=l$, $y=0$ also, then from (9) we get

$$0 = (C_3\cos pl + C_4\sin pl)(C_1\cos cpt + C_2\sin cpt)$$

as $C_3 = 0$

$$0 = (0 + C_4\sin pl)(C_1\cos cpt + C_2\sin cpt)$$

$C_4\sin pl = 0 \Rightarrow \sin pl = 0$

$\sin pl = \sin n\pi$

$pl = n\pi$

$\boxed{p = \frac{n\pi}{l}}$

$\sin n\pi = 0$, $n = 1, 2, 3 \dots$

Now, from eqn (9)th, putting $C_3 = 0$, $p = \frac{n\pi}{l}$, we get

$$y = C_4\sin\left(\frac{n\pi x}{l}\right)\left[C_1\cos\left(\frac{n\pi ct}{l}\right) + C_2\sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (10)}$$

Now $$\frac{\partial y}{\partial t} = \left[-C_1C_4\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi ct}{l}\right) + C_2C_4\left(\frac{n\pi c}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)\right] \times \sin\left(\frac{n\pi x}{l}\right)$$

$$\frac{\partial y}{\partial t} = \left(\frac{n\pi c}{l}\right)\left[-C_1C_4\sin\left(\frac{n\pi ct}{l}\right) + C_2C_4\cos\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (11)}$$

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PAGE

Now, using cond^n (iii) & (iv) (3) (4), we get at $t=0$, $\frac{\partial y}{\partial t}=0$

, from (xi) we get

$$0 = C_2C_4\left[\cos\left(\frac{n\pi ct}{l}\right)\cdot\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$C_2C_4 = 0$$

$$\Rightarrow \boxed{C_2 = 0}$$

From (10), we get

$$y = \left[C_1C_4\cos\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right) \quad — (7)$$

at $t=0$, $y = A\sin\frac{\pi x}{l}$ (given)

from (7) we get

$$A\sin\left(\frac{\pi x}{l}\right) = \left[C_1C_4\cos 0\right]\sin\left(\frac{n\pi x}{l}\right)$$

for $n=1$

$$A\sin\left(\frac{\pi x}{l}\right) = C_1C_4\sin\left(\frac{\pi x}{l}\right)$$

$$\Rightarrow \boxed{C_1C_4 = A} \quad (\text{from } n=1)$$

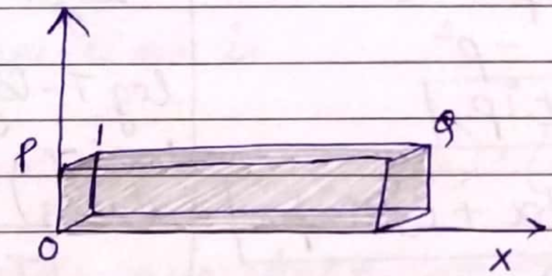
Now from (7)th we get the final sol^n

Ans. $\boxed{y = A\cos\left(\frac{\pi ct}{l}\right)\sin\left(\frac{\pi x}{l}\right)}$

(IV) One-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solution of 1-D heat equation by Separation of Variable method.



$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

Let $\quad U = X \cdot T \quad$ —— (2)

from eqn (2) by difft w.r.t 'x' & 't' we get

$$\frac{\partial u}{\partial x} = T\frac{\partial X}{\partial x} \quad ; \quad \frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}$$

putting values in eq (2)

$$X\frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$\frac{1}{c^2 T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2 (\text{say}) \quad \text{—— (3)}$$

Page No. ____ Date ____

from eq (3)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} = -p^2 X$$

$$\frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

AE $\Rightarrow$ $m^2 + p^2 = 0$

$m^2 = -p^2$

$\boxed{m = \pm ip}$

$$\boxed{X = C_1 \cos px + C_2 \sin px} \quad \text{--- (4)}$$

$$\frac{1}{c^2 T}\frac{\partial T}{\partial t} = -p^2$$

$$\frac{\partial T}{\partial t} = -p^2 c^2 T$$

$$\int \frac{1}{T}\partial T = \int -p^2 c^2 \partial t$$

$$\log T = -p^2c^2t + \log C_3$$

$$\log T - \log C_3 = -p^2c^2t$$

$$\log\left(\frac{T}{C_3}\right) = -p^2c^2t$$

$$\boxed{T = C_3 e^{-p^2c^2t}} \quad \text{--- (5)}$$

Substituting eq (4) & (5) in eq (2)

general soln $\rightarrow$ $\boxed{U = (C_1 \cos px + C_2 \sin px) C_3 e^{-p^2c^2t}}$

of One dimension heat Equation.

* If 10 marks question then also solve the general soln

Page No. ____ Date ____

2015.

Q A rod of 'l' length insulated sides is at a uniform temp 'u_0' initially. Its ends are suddenly cooled at $0°c$ and are kept at that temp. Find the expression for temp $u(x, t)$

The one dim heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{--- (1)}$$

and its solution is

$$u = (C_1 \cos px + C_2 \sin px) \cdot (C_3 e^{-p^2c^2t}) \quad \text{--- (2)}$$

As the ends are kept at $0°c$ i.e temp

$u = 0$ at both ends $x = 0$ & $x = l$ --- (3)

$u=0$ at $x=0$; $u=0$ at $x=l$

and at $t = 0$, $u = u_0$ (initial tempt) --- (4)

Putting $x = 0$ in eq (2), we get

$$0 = (C_1 \underbrace{\cos 0}_{1} + C_2 \underbrace{\sin 0}_{0}) (C_3 e^{-p^2c^2t})$$

$$0 = C_1 C_3 e^{-p^2c^2t}$$

$$\Rightarrow \boxed{C_1 = 0}$$

$\therefore$ from eq (2) $u = C_2 \sin px \cdot C_3 . e^{-p^2c^2t}$

$$\boxed{u = C_2 C_3 \sin px \cdot e^{-p^2c^2t}} \quad \text{--- (5)}$$

Now using $x = l$ then $u = 0$ in eq 5

$$0 = C_2 C_3 \sin pl \cdot e^{-c^2p^2t}$$

$$\Rightarrow \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}} \quad \text{for } n = 1, 2, 3 \ldots$$

from eq 5 $\boxed{u = C_2 C_3 \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-\frac{c^2n^2\pi^2 t}{l^2}}}$ — 6

~~[illegible]~~ now replacing $C_2 C_3 = b_n$ and for $n = 1, 2, 3 \ldots$

$$\boxed{u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-\frac{c^2n^2\pi^2 t}{l^2}}} \quad \text{— 6}$$

Now at $t = 0$, $u = u_0$ from eq 4 we get

$$u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-\frac{c^2n^2\pi^2 \cancel{0}}{l^2}}$$

$e^0 = 1$

$$\boxed{u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{— 7}$$

from 6 & 7 the general solution is

$$u = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-\left(\frac{c^2n^2\pi^2 t}{l^2}\right)}$$

along with $u_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ at $t = 0$.

* bar → 1D head questn

Impt

Q. 2015, 17

Find the temp in a bar of length 2 whose ends are kept at zero and lateral surface insulated if initial temperature is $\left(\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}\right)$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{— 1}$$

and the general solution of eq 1 is

$$U = (C_1 \cos px + C_2 \sin px)\left(C_3 e^{-c^2p^2t}\right) \quad \text{— 2}$$

→ as we know $U = 0$ at $x = 0$ and $x = 2$

∴ from 2nd equation

$$0 = (C_1 \underbrace{\cos 0}_{1} + C_2 \underbrace{\sin 0}_{0})\left(C_3 e^{-c^2p^2t}\right)$$

$$0 = C_1 C_3 \cdot e^{-c^2p^2t}$$

$$\Rightarrow \boxed{C_1 = 0}$$

Now from eq ②

$$U = C_2 \sin px \cdot C_3 e^{-c^2p^2t}$$

$$U = C_2 C_3 \sin px \cdot e^{-c^2p^2t} \quad \text{—— (3)}$$

now as $u = 0$ at $x = 2$

$$\therefore \quad 0 = C_2 C_3 \sin 2p \cdot e^{-c^2p^2t}$$

$$\Rightarrow \quad \sin 2p = \sin n\pi$$

$$2p = n\pi$$

$$\boxed{p = \frac{n\pi}{2}}$$

Now in eq ③, we get

$$U = C_2 C_3 \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{c^2n^2\pi^2t}{4}\right)} \quad \text{—— (4)}$$

now replacing C_2C_3 by b_n

$$U = \sum_{\infty}^{n=1} b_n \sin\left(\frac{n\pi x}{2}\right) \cdot e^{-\left(\frac{c^2n^2\pi^2t}{4}\right)} \quad \text{—— (5)}$$

$\Longrightarrow$ As it is given that at $t = 0$, $U = \sin\frac{\pi x}{2} + 3\sin\left(\frac{5\pi x}{2}\right)$

So from eq ⑤

$$\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = \sum_{\infty}^{n=1} b_n \sin\left(\frac{n\pi x}{2}\right) \cdot \underset{1}{e^{0}}$$

$$\sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = b_1 \sin\frac{\pi x}{2} + b_5 \sin\left(\frac{5\pi x}{2}\right)$$

by comparing

$b_1 = 1$ and $b_5 = 3$ and $b_2, b_3, b_4 \ldots$ are all zeroes.

Now from eq ⑤, we get

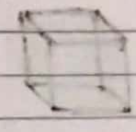
$$U = \sum_{\infty}^{n=1} b_n \sin\left(\frac{n\pi x}{2}\right) \cdot e^{-\left(\frac{c^2n^2\pi^2t}{4}\right)}$$

$$U = b_1 \sin\left(\frac{n\pi x}{2}\right) e^{-c^2n^2\pi^2t/4} + b_2 \sin\left(\frac{n\pi x}{2}\right) \cdot e^{-c^2n^2\pi^2t/4}$$

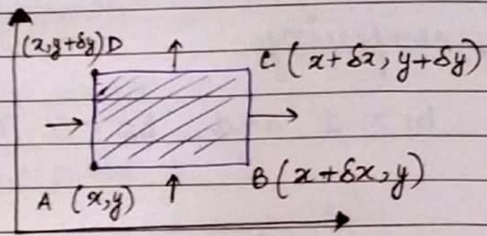
$$+ \ldots + b_5 \sin\left(\frac{n\pi x}{2}\right) e^{-c^2n^2\pi^2t/4}$$

$$\boxed{U = 1 \cdot \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{c^2n^2\pi^2t}{4}} + 3\sin\left(\frac{n\pi x}{2}\right) e^{-\frac{c^2n^2\pi^2t}{4}}}$$

This is the required temperature.



(V) Two-Dimensional Heat Equation



When heat flow in a metal sheet, then heat flow is called Two-dimensional.

⟹ Equation of 2-Dim Heat flow

$$\boxed{\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)} \quad ; u = u(x,y,t)$$

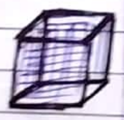
$t \to$ time , $c^2 \to$ constant

In steady state i.e $\frac{\partial u}{\partial t} = 0$ when temp. 'u' is independent of time , the we get

$$0 = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \quad ; u = u(x,y)$$

This equation is called Laplace Eqn of 2-Dim



$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0} \quad ; u = u(x,y,z)$$

Laplace Eqn of 3-Dimension

2011

Q. Solve the Laplace eqn of 2-dimension.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

given equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — (1)

Let $U = X \cdot Y$ — (2)

differenciating eqn (2) w.r.t x & y twice

$$\frac{\partial u}{\partial x} = Y\frac{\partial X}{\partial x} \quad ; \quad \frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = X\frac{\partial Y}{\partial y} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2}$$

substituting above values in eqn (1), we get

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} = -X\frac{\partial^2 Y}{\partial y^2}$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial y^2}\right) = -p^2 \text{ (say)} \quad \text{— (3)}$$

from eq (3)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -p^2$$

$$\frac{\partial^2 X}{\partial x^2} = -p^2 X$$

$$\frac{\partial^2 X}{\partial x^2} + p^2 X = 0$$

AE $\Rightarrow$ $m^2 + p^2 = 0$

$$\boxed{m = \pm ip}$$

$$\boxed{X = c_1 \cos px + c_2 \sin px} \quad \text{--- (4)}$$

Now from $\frac{1}{Y}\left(\frac{\partial^2 Y}{\partial y^2}\right) = -p^2$

$$\frac{\partial^2 Y}{\partial y^2} = Yp^2$$

$$\frac{\partial^2 Y}{\partial y^2} - Yp^2 = 0$$

AE $\Rightarrow$ $m^2 - p^2 = 0$

$$m^2 = p^2$$

$$\boxed{m = \pm p}$$

$$\boxed{Y = c_3 e^{-py} + c_4 e^{py}} \quad \text{--- (5)}$$

From eq (4) & (5) and eq (2), we get

$$\boxed{U = (c_1 \cos px + c_2 \sin px)(c_3 e^{-py} + c_4 e^{py})} \quad \text{--- (6)}$$

This is the general soln of Laplace 2-D eqn.

2017

Q Use separation of variable method to solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the boundary ~~equation~~ conditions :-

$u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin\left(\frac{n\pi x}{l}\right)$

given equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ --- (1)

Also given boundary conditions

$$\left.\begin{aligned} u(x,y) &= 0 \quad \text{at } x = 0 \\ u(x,y) &= 0 \quad \text{at } x = l \\ u(x,y) &= 0 \quad \text{at } y = 0 \\ u(x,y) &= \sin\left(\frac{n\pi x}{l}\right) \quad \text{at } y = a \end{aligned}\right\} \text{--- (2)}$$

As we know that general soln of Laplace eqn of 2-Dim is

$$U = (c_1 \cos px + c_2 \sin px) \cdot (c_3 e^{-py} + c_4 e^{py}) \quad \text{--- (3)}$$

using $x = 0$, then $u(x,y) = 0$ in eq (3) we get

$$0 = (c_1 \underbrace{\cos 0}_{1} + c_2 \underbrace{\sin 0}_{0}) \cdot (c_3 e^{-py} + c_4 e^{py})$$

$$0 = c_1 (c_3 e^{-py} + c_4 e^{py})$$

$$\Rightarrow \boxed{c_1 = 0}$$

using $x = l$, $u = 0$ in eq (3)

also putting $C_1 = 0$ in eq (3) we get

$$0 = (C_1 \cos pl + C_2 \sin pl)(C_3 e^{-py} + C_4 e^{py})$$

$$0 = C_2 \sin pl \left(C_3 e^{-py} + C_4 e^{py}\right) \text{ --- (4)}$$

$$0 = C_2 \sin pl$$

$$\Rightarrow \sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = \frac{n\pi}{l}}$$

substituting value of p and C_1 in eq (3)

$$U = \left[0 + C_2 \sin\left(\frac{n\pi x}{l}\right)\right] \cdot \left[C_3 e^{-\frac{n\pi y}{l}} + C_4 e^{\frac{n\pi y}{l}}\right]$$

$$U = C_2 \sin\left(\frac{n\pi x}{l}\right) \cdot \left[C_3 e^{-\frac{n\pi y}{l}} + C_4 e^{\frac{n\pi y}{l}}\right] \text{ --- (4)}$$

now using $y = 0$ and $u = 0$ in eq (4)

$$0 = C_2 \sin\left(\frac{n\pi x}{l}\right)\left[C_3 e^0 + C_4 e^0\right]$$

$$0 = C_2 \sin\left(\frac{n\pi x}{l}\right)(C_3 + C_4)$$

$$0 = C_3 + C_4$$

$$\rightarrow \boxed{C_4 = -C_3}$$

now putting $C_4 = -C_3$ in eq (4)

$$U = C_2 \sin\left(\frac{n\pi x}{l}\right)\left[C_3 e^{-\frac{n\pi y}{l}} - C_3 e^{\frac{n\pi y}{l}}\right]$$

$$U = C_2 C_3 \sin\left(\frac{n\pi x}{l}\right)\left[e^{-n\pi y/l} - e^{n\pi y/l}\right] \text{ --- (5)}$$

Now let $C_2 C_3 = b_n$

$$U = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)\left[e^{-n\pi y/l} - e^{n\pi y/l}\right] \text{ --- (6)}$$

now $y = a$ and $u = \sin\left(\frac{n\pi x}{l}\right)$

in eq (6)th we get

$$\sin\left(\frac{n\pi x}{l}\right) = b_n \sin\left(\frac{n\pi x}{l}\right)\left[e^{-\frac{n\pi a}{l}} - e^{\frac{n\pi a}{l}}\right]$$

$$b_n = \frac{\cancel{\sin\left(\frac{n\pi x}{l}\right)}}{\cancel{\sin\left(\frac{n\pi x}{l}\right)} \times \left(e^{-\frac{n\pi a}{l}} - e^{\frac{n\pi a}{l}}\right)}$$

$$b_n = \frac{1}{\left(e^{-\frac{n\pi a}{l}} - e^{\frac{n\pi a}{l}}\right)}$$

now

$$b_n = \frac{1}{-\left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right)}$$

as $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$e^{\theta} - e^{-\theta} = 2\sinh\theta$

$$\boxed{b_n = \frac{-1}{2\sinh\left(\frac{n\pi a}{l}\right)}}$$

putting bn in eq (6) we get

$$U = \sum_{n=1}^{\infty} \frac{-1}{2\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot \left(e^{-\frac{n\pi y}{l}} - e^{\frac{n\pi y}{l}}\right)$$

$$U = \sum_{n=1}^{\infty} \frac{\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)}{2\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$U = \sum_{n=1}^{\infty} \frac{2\sinh\left(\frac{n\pi y}{l}\right)}{2\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin\left(\frac{n\pi y}{l}\right)$$

Soln

$$\boxed{U = \sum_{n=1}^{\infty} \frac{\sinh\left(\frac{n\pi y}{l}\right)}{\sinh\left(\frac{n\pi a}{l}\right)} \cdot \sin\left(\frac{n\pi y}{l}\right)}$$

$x - 0$

2018

Numerical on Laplace Eqn.

Q. A rectangular plate with insulated surfaces of 8 cm wide and so long compare to width (maybe considered infinite) without any error. If temperature along one-short edge $y=0$ is $u(x,0) = 100 \sin\frac{\pi x}{8}$, $0<x<8$

while the two long edges $x=0$ and $x=8$ as well as other short edge are kept at $0^\circ C$, show that the steady state temperature at any point of the plate is given by

$$u(x,y) = 100\, e^{-\left(\frac{\pi y}{8}\right)} \sin\left(\frac{\pi x}{8}\right)$$

As the Laplace equation in the steady state is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{— (1)}$$

And the given boundary conditions are

$$\left.\begin{aligned} u(x,y) &= 0 \text{ at } x = 0 \\ u(x,y) &= 0 \text{ at } x = 8 \\ u(x,y) &= 0 \text{ at } y \to \infty \\ u(x,y) &= 100\, e^{-\left(\frac{\pi y}{8}\right)} \sin\left(\frac{\pi x}{8}\right) \text{ at } y = 0 \end{aligned}\right\} \text{— (2)}$$

We know the general solution of the Laplace 2-Dim equ is as follow

$$U = (C_1 \cos px + C_2 \sin px)(C_3 e^{-py} + C_4 e^{py}) \quad \text{— (3)}$$

from eq (2) putting $x = 0$ at $u = 0$ in eq (3)

$$0 = (C_1 \underset{1}{\cos 0} + C_2 \underset{0}{\sin 0})(C_3 e^{-py} + C_4 e^{py})$$

$$0 = C_1 (C_3 e^{-py} + C_4 e^{py})$$

$$\Rightarrow \boxed{C_1 = 0}$$

Now using $x = 8$ at $u = 0$ in eq (3)

$$0 = (\underset{0}{\cancel{C_1}} \cos 8p + C_2 \sin 8p)(C_3 e^{-py} + C_4 e^{py})$$

$$0 = C_2 \sin 8p (C_3 e^{-py} + C_4 e^{py})$$

$$0 = C_2 \sin 8p \Rightarrow \sin 8p = \sin n\pi$$

$$8p = n\pi$$

$$\boxed{p = \frac{n\pi}{8}}$$

now putting $C_1 = 0$ & 'p' in eq (3)

$$U = C_2 \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cdot \left(C_3 e^{-\frac{n\pi y}{8}} + C_4 e^{\frac{n\pi y}{8}}\right) \quad \text{--- (4)}$$

now putting $y \to \infty$ at $u = 0$ in eq (4)

$$U = C_2 \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \lim_{y\to\infty} \left(C_3 e^{-\frac{n\pi y}{8}} + C_4 e^{\frac{n\pi y}{8}}\right)$$

as $e^{-\infty} = 0$, $e^{\infty} = \infty$

$$U = C_2 \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cancel{\lim} (C_4 e^{\infty})$$

This will exist only if $\boxed{C_4 = 0}$

Now putting $C_4 = 0$ in eq (4)

$$U = C_2 \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cdot C_3 e^{-\frac{n\pi y}{8}}$$

$$U = C_2 C_3 \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cdot e^{-\frac{n\pi y}{8}}$$

let $b_n = C_2 C_3$

$$U = \sum_{n=1}^{\infty} b_n \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cdot e^{-\frac{n\pi y}{8}} \quad \text{--- (5)}$$

Now using $u = 100 \sin\left(\frac{\pi x}{8}\right)$ at $y = 0$ in eq (5)

$$100 \sin\left(\frac{\pi x}{8}\right) = \sum_{n=1}^{\infty} b_n \overset{\sin}{\cancel{\cos}}\left(\frac{n\pi x}{8}\right) \cdot \underset{1}{e^{-\frac{n\pi 0}{8}}}$$

$$100 \sin\left(\frac{\pi x}{8}\right) = b_1 \overset{\sin}{\cancel{\cos}}\left(\frac{\pi x}{8}\right) + b_2 \overset{\sin}{\cancel{\cos}}\left(\frac{2\pi x}{8}\right) + \cdots \quad \text{--- (6)}$$

comparing the coffecient of both the sides we get

$$\Rightarrow b_1 = 100 , b_2 = b_3 = b_4 \cdots = 0$$

from eq (5) we get

Soln. $$\boxed{u(x,y) = 100 \overset{\sin}{\cancel{\cos}}\left(\frac{\pi x}{8}\right) e^{-\left(\frac{\pi y}{8}\right)}}$$ Required temperate at $p(x,y)$

2016

Q. Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in x-y plane with
$u(x,0) = u(x,b) = 0$, $u(0,y) = 0$ and
$u(a,y) = f(y)$ parallel to y-axis.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{---} \ (1)$$

~~And the general solution is~~ Let $U = X \cdot Y$ — (2)

~~...~~ now differenciating eq (2) twice w.r.t x & y.

$$\frac{\partial^2 u}{\partial x^2} = Y \frac{\partial^2 X}{\partial x^2} \ ; \ \frac{\partial^2 u}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2}$$

putting above values in eq (1)

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow \quad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -p^2 \ (say) \quad \text{---} \ (3)$$

from last two terms

$$\cancel{-}\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \cancel{-}p^2$$

$$\frac{\partial^2 X}{\partial x^2} - p^2 X = 0$$

AE $\Rightarrow$ $m^2 - p^2 = 0 \Rightarrow m = \pm p$

$$\boxed{X = C_1 e^{-px} + C_2 e^{px}} \quad \text{---} \ (4)$$

from I[st] and last term

$$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = -p^2$$

$$\frac{\partial^2 Y}{\partial y^2} = -p^2 Y$$

$$\frac{\partial^2 Y}{\partial y^2} + p^2 Y = 0 \Rightarrow m^2 + p^2 = 0$$

$$m^2 = \pm ip$$

$$\boxed{Y = C_3 \cos py + C_4 \sin py} \quad \text{---} \ (5)$$

putting (4) & (5) in eq (2)

$$\boxed{U = (C_1 e^{-px} + C_2 e^{px}) \cdot (C_3 \cos py + C_4 \sin py)} \quad \text{---} \ (6)$$

now at $y = 0$, $u = 0$ putting in eq (6)

$$0 = (C_1 e^{-px} + C_2 e^{px}) (C_3 \underbrace{\cos 0}_{1} + C_4 \underbrace{\sin 0}_{0})$$

$$0 = C_3 (C_1 e^{-px} + C_2 e^{px})$$

$$\Rightarrow \boxed{C_3 = 0}$$

now at $y = b$, $u = 0$ putting in eq (6)

$$0 = (C_1 e^{-px} + C_2 e^{px}) (\underbrace{C_3}_{0} \cos pb + C_4 \sin bp)$$

$$0 = (C_1 e^{-px} + C_2 e^{px}) C_4 \sin bp$$

$$\Rightarrow C_4 \sin by = 0 \Rightarrow \sin bp = \sin n\pi$$

$$\boxed{p = \frac{n\pi}{b}}$$

now at $x=0$, $u=0$ using in eq (6)

$$0 = \left(c_1 e^{-0p} + c_2 e^{0p}\right)\left(\underset{\swarrow 0}{\cancel{c_3}}\cos py + c_4 \sin py\right)$$

$$0 = (c_1 + c_2)(c_4 \sin py)$$

$$\Rightarrow \boxed{c_1 = -c_2}$$

in eq (6):

$$u = \left(c_1 e^{-\frac{n\pi x}{b}} - c_1 e^{\frac{n\pi x}{b}}\right)\left(\underset{\swarrow 0}{\cancel{c_3}}\cos\left(\frac{n\pi y}{b}\right) + c_4 \sin\left(\frac{n\pi y}{b}\right)\right)$$

$$u = c_1 c_4 \left(e^{-\frac{n\pi x}{b}} - e^{\frac{n\pi x}{b}}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Let $c_1 c_4 = b_n$

$$u = \sum_{n=1}^{\infty} b_n \left(e^{-\frac{n\pi x}{b}} - e^{\frac{n\pi x}{b}}\right) \sin\left(\frac{n\pi y}{b}\right) \quad \text{--- (7)}$$

now at $x = a$, $u = f(y)$ pallel to y-axis

$$f(y) = \sum_{n=1}^{\infty} b_n \left(e^{-\frac{n\pi a}{b}} - e^{\frac{n\pi a}{b}}\right) \sin\left(\frac{n\pi y}{b}\right).$$

Since L.H.S is independent of n

$$f(y) = b_n \sin\left(\frac{n\pi y}{b}\right) \cdot \left(e^{-\frac{n\pi a}{b}} - e^{\frac{n\pi a}{b}}\right)$$

Multiply & dividing by 2

$$f(y) = 2 b_n \sin\left(\frac{n\pi y}{b}\right) \left(\frac{e^{-\frac{n\pi a}{b}} - e^{\frac{n\pi a}{b}}}{2}\right)$$

$$f(y) = -2 b_n \sin\left(\frac{n\pi y}{b}\right) \left(\frac{e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}}{2}\right)$$

$$f(y) = -2 b_n \sin\left(\frac{n\pi y}{b}\right) \times \sinh\left(\frac{n\pi a}{b}\right)$$

$$\boxed{b_n = \frac{-f(y)}{2 \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)}} \quad \text{--- (8)}$$

Now from eq (8) and (7)

$$u = \sum_{n=1}^{\infty} \left[\frac{-f(y)}{2 \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)}\right] \left(-\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)\right) \sin\left(\frac{n\pi y}{b}\right)$$

$$u = \sum_{n=1}^{\infty} \left[\frac{f(y)}{\cancel{2}\cancel{\sin\left(\frac{n\pi y}{b}\right)} \sinh\left(\frac{n\pi a}{b}\right)}\right] \times \cancel{2} \sinh\left(\frac{n\pi x}{b}\right) \times \cancel{\sin\left(\frac{n\pi y}{b}\right)}$$

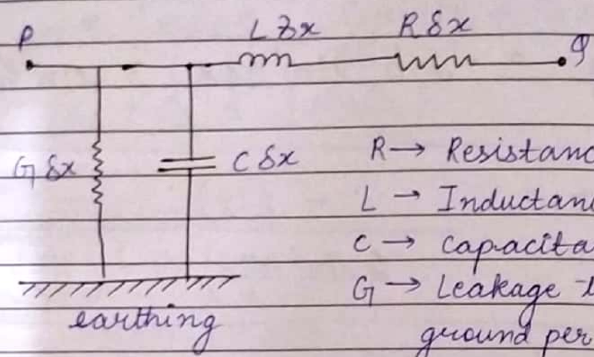
$$u = \sum_{n=1}^{\infty} \frac{f(y) \sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)}$$

soln $\Rightarrow$

$$\boxed{u = f(y) \sum_{n=1}^{\infty} \frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)}}$$

2 marks.

(VI) Equation of Transmission lines



R → Resistance
L → Inductance
C → capacitance
G → Leakage to the ground per unit length

⇒ **Telephone Equations**

in terms of voltage
$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (RC+LG)\frac{\partial V}{\partial t} + RGV \quad —(1)$$

in terms of current
$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (LG+RC)\frac{\partial I}{\partial t} + RGI \quad —(2)$$

Remark

(1) If $L = G = 0$, then equation (1) & (2)

$$\frac{\partial^2 V}{\partial x^2} = RC\frac{\partial V}{\partial t} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t} \quad —(3)$$

which are known as **Telegraph Equation**

(2) If $R = G = 0$, then equation (1) & (2)

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} \quad —(4)$$

These equations are known as **Radio-Equation**

(3) If L and C are negligible, i.e $L = C = 0$, then equation (1) and (2)

$$\frac{\partial^2 V}{\partial x^2} = RGV \quad \text{and} \quad \frac{\partial^2 I}{\partial x^2} = RGI \quad —(5)$$

Eqn (5)th are the equations of **Submarine-Cable.**

Numerical

2012 Q.