Math G180
Blank Lecture Notes
Chapter 5 – Sections 5.1 and 5.2

5.1 | Approximating Areas

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i,$$

which is much more compact.

Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is

Example 5.1

Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5.$

b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\begin{aligned}\sum_{i=1}^{5} 3^i &= 3 + 3^2 + 3^3 + 3^4 + 3^5 \\ &= 363.\end{aligned}$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}.$

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

1.

$$\sum_{i=1}^{n} c = nc \tag{5.1}$$

2.

$$\sum_{i=1}^{n} ca_i = c \sum_{i=1}^{n} a_i \tag{5.2}$$

3.

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \tag{5.3}$$

4.

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \tag{5.4}$$

5.

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \tag{5.5}$$

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Example 5.2

Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.

b. The sum of the terms $\left(i^3 - i^2\right)$ for $i = 1, 2, 3, 4, 5, 6.$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200} (i-3)^2 &= \sum_{i=1}^{200} \left(i^2 - 6i + 9\right) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2{,}686{,}700 - 120{,}600 + 1800 \\
&= 2{,}567{,}900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}
\sum_{i=1}^{6} \left(i^3 - i^2\right) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\
&= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\
&= \frac{1764}{4} - \frac{546}{6} \\
&= 350
\end{aligned}$$

Example 5.3

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution

Using the formula, we have

$$\begin{aligned}\sum_{i=0}^{10} i^3 &= \frac{(10)^2 (10+1)^2}{4} \\ &= \frac{100(121)}{4} \\ &= 3025.\end{aligned}$$

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (**Figure 5.2**).

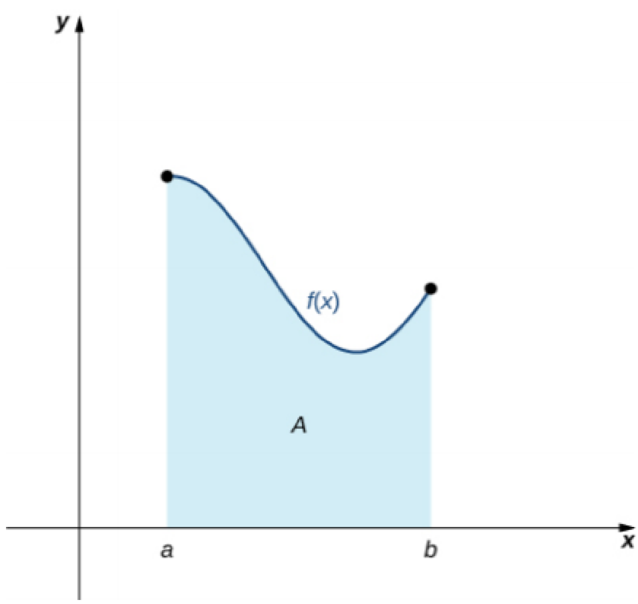


Figure 5.2 An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 5.3). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x \tag{5.6}$$

$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

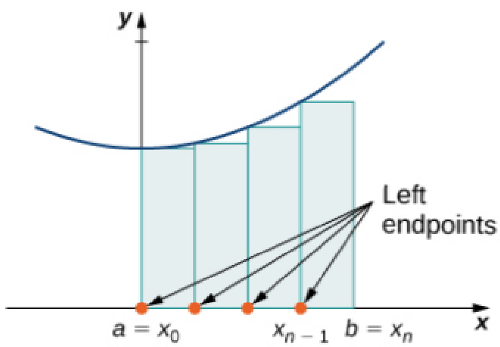


Figure 5.3 In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \tag{5.7}$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 5.4).

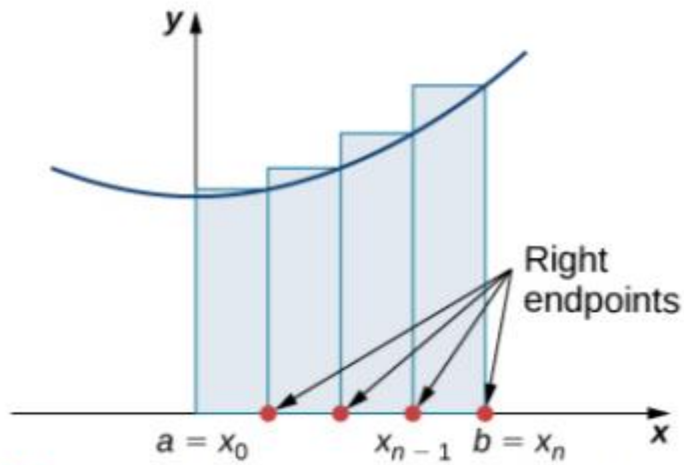


Figure 5.4 In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in **Figure 5.3**.

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375. \end{aligned}$$

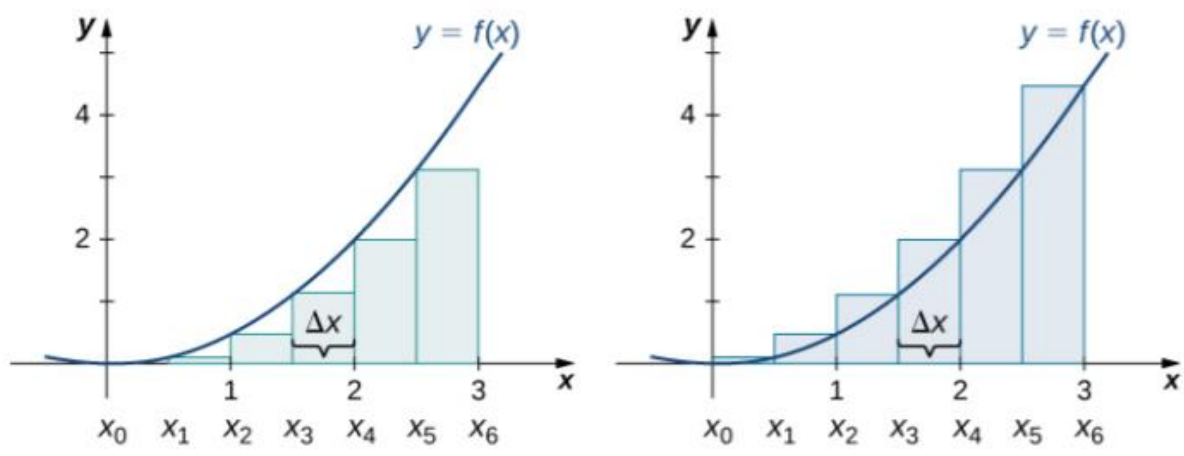


Figure 5.5 Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

$$
\begin{aligned}
A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\
&= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\
&= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\
&= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\
&= 5.6875.
\end{aligned}
$$

Example 5.4

Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 5.6. Using a left-endpoint approximation, the heights are $f(0) = 0,\ f(0.5) = 0.25,\ f(1) = 1,\ f(1.5) = 2.25.$ Then,

$$
\begin{aligned}
L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\
&= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\
&= 1.75.
\end{aligned}
$$

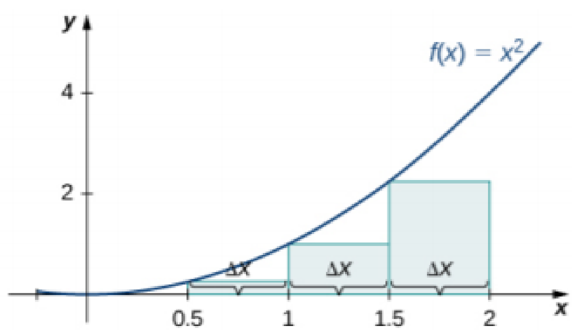


Figure 5.6 The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 5.7. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$
\begin{aligned}
R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\
&= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\
&= 3.75.
\end{aligned}
$$

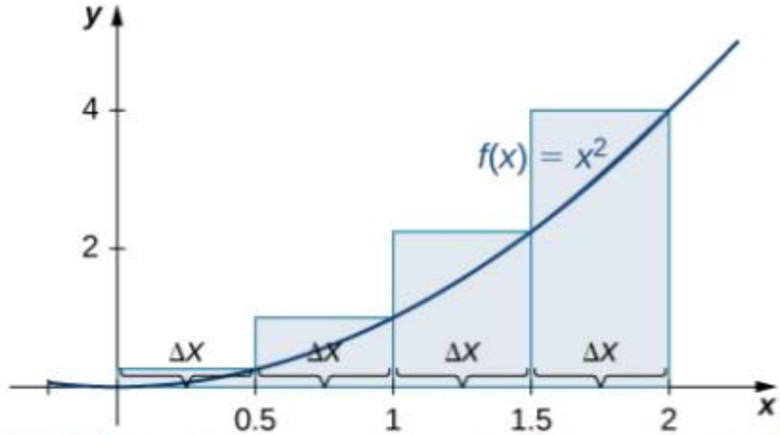


Figure 5.7 The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75; the right-endpoint approximation is 3.75.

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example 5.5

Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25], [1.25, 1.5], [1.5, 1.75], [1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, **Figure 5.14** shows that a lower sum is obtained by using the right endpoints.

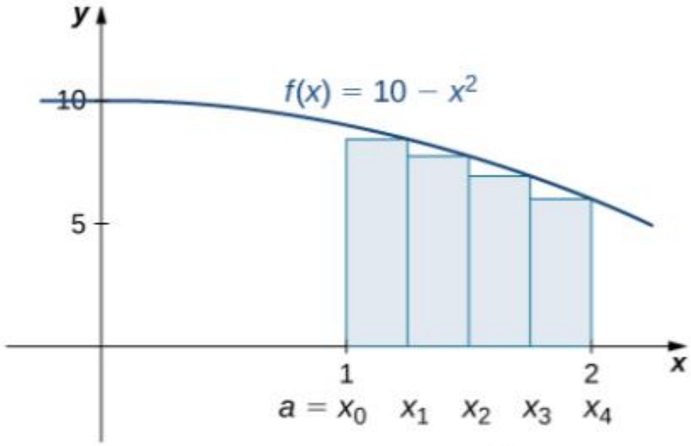


Figure 5.14 The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}\left(10 - x^2\right)(0.25) &= 0.25\left[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2\right] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28.\end{aligned}$$

The area of 7.28 is a lower sum and an underestimate.

5.1 EXERCISES

1. State whether the given sums are equal or unequal.

   a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

   b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

   c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

   d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} \left(k^2-k\right)$

In the following exercises, use the rules for sums of powers of integers to compute the sums.

2. $\sum_{i=5}^{10} i$

3. $\sum_{i=5}^{10} i^2$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. In the following exercises, compute the sums.

4. $\sum_{i=1}^{100} (a_i + b_i)$

5. $\sum_{i=1}^{100} (a_i - b_i)$

6. $\sum_{i=1}^{100} (3a_i - 4b_i)$

7. $\sum_{i=1}^{100} (5a_i + 4b_i)$

In the following exercises, use summation properties and formulas to rewrite and evaluate the sums.

8. $\sum_{k=1}^{20} 100\left(k^2-5k+1\right)$

9. $\sum_{j=1}^{50} \left(j^2-2j\right)$

10. $\sum_{j=11}^{20} \left(j^2-10j\right)$

11. $\sum_{k=1}^{25} \left[(2k)^2-100k\right]$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

12. L_4 for $f(x) = \frac{1}{x-1}$ on $[2, 3]$

13. R_4 for $g(x) = \cos(\pi x)$ on $[0, 1]$

14. L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

15. R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

16. R_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

17. L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

18. R_4 for x^2-2x+1 on $[0, 2]$

19. L_8 for x^2-2x+1 on $[0, 2]$

20. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = (2-|x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of f.

21. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = (3-|3-x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of f.

22. Compute the left and right Riemann sums—L_4 and R_4, respectively—for $f(x) = \sqrt{4-x^2}$ on $[-2, 2]$ and compare their values.

23. Compute the left and right Riemann sums—L_6 and R_6, respectively—for $f(x) = \sqrt{9-(x-3)^2}$ on $[0, 6]$ and compare their values.

Express the following endpoint sums in sigma notation but do not evaluate them.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

5.2 | The Definite Integral

Learning Objectives

5.2.1 State the definition of the definite integral.
5.2.2 Explain the terms integrand, limits of integration, and variable of integration.
5.2.3 Explain when a function is integrable.
5.2.4 Describe the relationship between the definite integral and net area.
5.2.5 Use geometry and the properties of definite integrals to evaluate them.
5.2.6 Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of *f* from *a* to *b* is given by

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x, \tag{5.8}$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

Theorem 5.1: Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then *f* is integrable on $[a, b]$.

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (**Figure 5.19**). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x dx = A_1 - A_2 = 9 - 9 = 0.$$

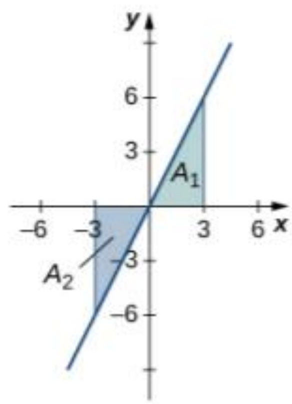


Figure 5.19 The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Definition

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies *above* the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies *below* the axis. Then, the **net signed area** between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)dx = A_1 - A_2.$$

The **total area** between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|dx = A_1 + A_2.$$

Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (**Figure 5.22**).

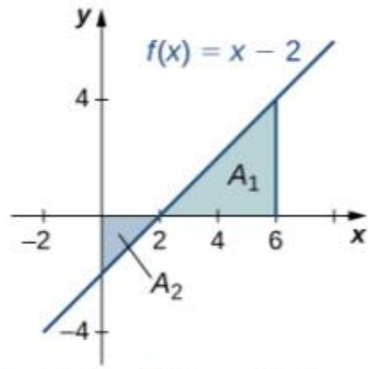


Figure 5.22 The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10.$$

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)dx = 0 \tag{5.9}$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \tag{5.10}$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx \tag{5.11}$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx \tag{5.12}$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)dx = c\int_a^b f(x) \tag{5.13}$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \tag{5.14}$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or $\mathbf{f_{ave}}$) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example 5.14

Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution

First, graph the function on the stated interval, as shown in **Figure 5.25**.

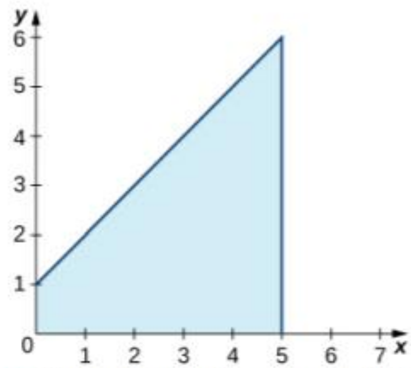


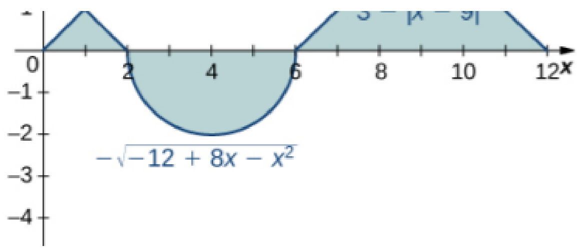
Figure 5.25 The graph shows the area under the function $f(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where *h* represents height, and *a* and *b* represent the two parallel sides. Then,

$$\begin{aligned}\int_0^5 x+1dx &= \frac{1}{2}h(a+b)\\ &= \frac{1}{2}\cdot 5\cdot(1+6)\\ &= \frac{35}{2}.\end{aligned}$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x+1dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}.$$



In the following exercises, evaluate the integral using area formulas.

76. $\int_0^3 (3-x)dx$

77. $\int_2^3 (3-x)dx$

78. $\int_{-3}^3 (3-|x|)dx$

79. $\int_0^6 (3-|x-3|)dx$

80. $\int_{-2}^2 \sqrt{4-x^2}dx$

81. $\int_1^5 \sqrt{4-(x-3)^2}dx$

82. $\int_0^{12} \sqrt{36-(x-6)^2}dx$

Suppose that $\int_0^4 f(x)dx = 5$ and $\int_0^2 f(x)dx = -3$, and $\int_0^4 g(x)dx = -1$ and $\int_0^2 g(x)dx = 2$. In the following exercises, compute the integrals.

88. $\int_0^4 (f(x)+g(x))dx$

89. $\int_2^4 (f(x)+g(x))dx$

90. $\int_0^2 (f(x)-g(x))dx$

91. $\int_2^4 (f(x)-g(x))dx$

92. $\int_0^2 (3f(x)-4g(x))dx$

93. $\int_2^4 (4f(x)-3g(x))dx$

In the following exercises, use the identity $\int_{-A}^A f(x)dx = \int_{-A}^0 f(x)dx + \int_0^A f(x)dx$ to compute the integrals.

94. $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2}dt$ (*Hint*: $\sin(-t) = -\sin(t)$)

96. $\int_1 (2 - x)dx$ (*Hint:* Look at the graph of *f*.)

97. $\int_2^4 (x - 3)^3\,dx$ (*Hint:* Look at the graph of *f*.)

In the following exercises, given that $\int_0^1 x dx = \frac{1}{2}$, $\int_0^1 x^2\,dx = \frac{1}{3}$, and $\int_0^1 x^3\,dx = \frac{1}{4}$, compute the integrals.

98. $\int_0^1 \left(1 + x + x^2 + x^3\right)dx$

99. $\int_0^1 \left(1 - x + x^2 - x^3\right)dx$

100. $\int_0^1 (1 - x)^2\,dx$

101. $\int_0^1 (1 - 2x)^3\,dx$

102. $\int_0^1 \left(6x - \frac{4}{3}x^2\right)dx$

103. $\int_0^1 \left(7 - 5x^3\right)dx$

In the following exercises, find the average value f_{ave} of f between a and b, and find a point c, where $f(c) = f_{\text{ave}}$.

110. $f(x) = x^2$, $a = -1$, $b = 1$

111. $f(x) = x^5$, $a = -1$, $b = 1$

112. $f(x) = \sqrt{4 - x^2}$, $a = 0$, $b = 2$

113. $f(x) = (3 - |x|)$, $a = -3$, $b = 3$

114. $f(x) = \sin x$, $a = 0$, $b = 2\pi$

115. $f(x) = \cos x$, $a = 0$, $b = 2\pi$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100}. How does your answer compare with the exact given answer?

116. **[T]** $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

117. **[T]** $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e} - 1)$.

118. **[T]** $y = \tan x$ over the interval $\left[0, \frac{\pi}{4}\right]$; the exact

Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or $\mathbf{f_{ave}}$) on $[a, b]$ is given by

$$f_{\text{ave}} = \frac{1}{b - a}\int_a^b f(x)dx.$$