Math G180
Blank Lecture Notes
Chapter 5 – Sections 5.6 and 5.7

5.6 | Integrals Involving Exponential and Logarithmic Functions

Learning Objectives

5.6.1 Integrate functions involving exponential functions.
5.6.2 Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\begin{aligned} \int e^x\,dx &= e^x + C \\ \int a^x\,dx &= \frac{a^x}{\ln a} + C \end{aligned} \tag{5.21}$$

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

$$\begin{aligned} \int x^{-1}\,dx &= \ln|x| + C \\ \int \ln x\,dx &= x\ln x - x + C = x(\ln x - 1) + C \\ \int \log_a x\,dx &= \frac{x}{\ln a}(\ln x - 1) + C \end{aligned} \tag{5.22}$$

5.6 EXERCISES

In the following exercises, compute each indefinite integral.

320. $\int e^{2x}dx$

321. $\int e^{-3x}dx$

322. $\int 2^{x}dx$

323. $\int 3^{-x}dx$

324. $\int \frac{1}{2x}dx$

325. $\int \frac{2}{x}dx$

326. $\int \frac{1}{x^2}dx$

327. $\int \frac{1}{\sqrt{x}}dx$

In the following exercises, find each indefinite integral by using appropriate substitutions.

328. $\int \frac{\ln x}{x}dx$

329. $\int \frac{dx}{x(\ln x)^2}$

330. $\int \frac{dx}{x\ln x}\ (x > 1)$

331. $\int \frac{dx}{x\ln x\ln(\ln x)}$

332. $\int \tan\theta\, d\theta$

333. $\int \frac{\cos x - x\sin x}{x\cos x}dx$

334. $\int \frac{\ln(\sin x)}{\tan x}dx$

335. $\int \ln(\cos x)\tan x dx$

336. $\int xe^{-x^2}dx$

337. $\int x^2e^{-x^3}dx$

338. $\int e^{\sin x}\cos x dx$

339. $\int e^{\tan x}\sec^2 x dx$

340. $\int e^{\ln x}\frac{dx}{x}$

341. $\int \frac{e^{\ln(1-t)}}{1-t}dt$

In the following exercises, verify by differentiation that $\int \ln x\, dx = x(\ln x - 1) + C,$ then use appropriate changes of variables to compute the integral.

342. $\int \ln x dx$ (*Hint*: $\int \ln x dx = \frac{1}{2}\int x\ln\left(x^2\right)dx$)

343. $\int x^2\ln^2 x\, dx$

344. $\int \frac{\ln x}{x^2}dx$ (*Hint*: Set $u = \frac{1}{x}$.)

345. $\int \frac{\ln x}{\sqrt{x}}dx$ (*Hint*: Set $u = \sqrt{x}$.)

346. Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

347. Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x,$ and evaluate the integral.

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

348. $\int \tan(2x)dx$

349. $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)}dx$

350. $\int \frac{x\sin\left(x^2\right)}{\cos\left(x^2\right)}dx$

351. $\int x\csc\left(x^2\right)dx$

353. $\int \ln(\csc x)\cot x dx$

354. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

355. $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

356. $\int_0^{\pi/4} \tan x\, dx$

357. $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

358. $\int_{\pi/6}^{\pi/2} \csc x dx$

359. $\int_{\pi/4}^{\pi/3} \cot x dx$

In the following exercises, integrate using the indicated substitution.

360. $\int \frac{x}{x - 100} dx;\ u = x - 100$

361. $\int \frac{y - 1}{y + 1} dy;\ u = y + 1$

362. $\int \frac{1 - x^2}{3x - x^3} dx;\ u = 3x - x^3$

363. $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx;\ u = \sin x - \cos x$

364. $\int e^{2x}\sqrt{1 - e^{2x}} dx;\ u = e^{2x}$

365. $\int \ln(x)\frac{\sqrt{1 - (\ln x)^2}}{x} dx;\ u = \ln x$

5.7 | Integrals Resulting in Inverse Trigonometric Functions

Learning Objectives

5.7.1 Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from **Functions and Graphs** that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in **Derivatives**, we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1.
$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{|a|} + C \tag{5.23}$$

2.
$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{5.24}$$

3.
$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{u}{|a|} + C \tag{5.25}$$

Example 5.49

Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x\Big|_0^{\frac{1}{2}} \\ &= \sin^{-1}\frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6}.\end{aligned}$$

Example 5.52

Finding an Antiderivative Involving the Inverse Tangent Function

Find an antiderivative of $\int \frac{1}{1+4x^2}dx$.

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1} u + C$. So we use substitution, letting $u = 2x$, then $du = 2dx$ and $1/2du = dx$. Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(2x) + C.$$

5.42 Use substitution to find the antiderivative $\int \frac{dx}{25+4x^2}$.

Example 5.53

Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9+x^2}dx$.

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

5.7 EXERCISES

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

391. $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

392. $\int_{-1/2}^{1/2} \frac{dx}{1-x^2}$

393. $\int_{\sqrt{3}}^{1} \frac{dx}{1-x^2}$

394. $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

395. $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

396. $\int_1^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

397. $\int \frac{dx}{\sqrt{9-x^2}}$

398. $\int \frac{dx}{\sqrt{1-16x^2}}$

399. $\int \frac{dx}{9+x^2}$

400. $\int \frac{dx}{25+16x^2}$

401. $\int \frac{dx}{|x|\sqrt{x^2-9}}$

402. $\int \frac{dx}{|x|\sqrt{4x^2-16}}$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{|a|} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{|a|}\sec^{-1}\frac{u}{|a|} + C$$

In the following exercises, compute each integral using appropriate substitutions.

423. $\int \frac{e^t}{\sqrt{1 - e^{2t}}} dt$

424. $\int \frac{e^t}{1 + e^{2t}} dt$

425. $\int \frac{dt}{t\sqrt{1 - \ln^2 t}}$