Math G180
Blank Lecture Notes
Chapter 4 – Sections 4.2 and 4.4

4.2 | Linear Approximations and Differentials

Learning Objectives

4.2.1 Describe the linear approximation to a function at a point.

4.2.2 Write the linearization of a given function.

4.2.3 Draw a graph that illustrates the use of differentials to approximate the change in a quantity.

4.2.4 Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials **Introduction to Power Series and Functions (http://cnx.org/content/m53760/latest/)** .

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a.$ Recall that the tangent line to the graph of f at a is given by the equation

$$y = f(a) + f'(a)(x - a).$$

In general, for a differentiable function $f,$ the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near $a.$ Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \qquad (4.1)$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a.$ This function L is also known as the **linearization** of f at $x = a.$

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9.$

Example 4.5

Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}.$

Solution

Since we are looking for the linear approximation at $x = 9,$ using **Equation 4.1** we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9).$

$$f(x) = \sqrt{x} \quad \Rightarrow \quad f(9) = \sqrt{9} = 3$$
$$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

Therefore, the linear approximation is given by **Figure 4.8**.

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

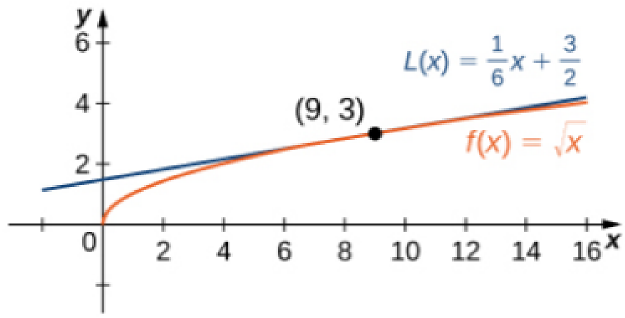


Figure 4.8 The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

4.2 EXERCISES

46. What is the linear approximation for any generic linear function $y = mx + b$?

47. Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

48. Explain why the linear approximation becomes less accurate as you increase the distance between x and a. Use a graph to prove your argument.

49. When is the linear approximation exact?

For the following exercises, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

50. $f(x) = x + x^4,\ a = 0$

51. $f(x) = \frac{1}{x},\ a = 2$

52. $f(x) = \tan x,\ a = \frac{\pi}{4}$

53. $f(x) = \sin x,\ a = \frac{\pi}{2}$

54. $f(x) = x \sin x,\ a = 2\pi$

55. $f(x) = \sin^2 x,\ a = 0$

For the following exercises, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a, and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

56. **[T]** $(2.001)^6$

57. **[T]** $\sin(0.02)$

58. **[T]** $\cos(0.03)$

59. **[T]** $(15.99)^{1/4}$

60. **[T]** $\frac{1}{[illegible]}$

62. **[T]** $(1.01)^3$

63. **[T]** $\cos(0.01)$

64. **[T]** $(\sin(0.01))^2$

65. **[T]** $(1.01)^{-3}$

66. **[T]** $\left(1 + \frac{1}{10}\right)^{10}$

67. **[T]** $\sqrt{8.99}$

For the following exercises, find the differential of the function.

68. $y = 3x^4 + x^2 - 2x + 1$

69. $y = x \cos x$

70. $y = \sqrt{1 + x}$

71. $y = \frac{x^2 + 2}{x - 1}$

For the following exercises, find the differential and evaluate for the given x and dx.

72. $y = 3x^2 - x + 6,\quad x = 2,\quad dx = 0.1$

73. $y = \frac{1}{x + 1},\quad x = 1,\quad dx = 0.25$

74. $y = \tan x,\quad x = 0,\quad dx = \frac{\pi}{10}$

75. $y = \frac{3x^2 + 2}{\sqrt{x + 1}},\quad x = 0,\quad dx = 0.1$

76. $y = \frac{\sin(2x)}{x},\quad x = \pi,\quad dx = 0.25$

77. $y = x^3 + 2x + \frac{1}{x},\quad x = 1,\quad dx = 0.05$

4.4 | The Mean Value Theorem

Learning Objectives

4.4.1 Explain the meaning of Rolle's theorem.
4.4.2 Describe the significance of the Mean Value Theorem.
4.4.3 State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure 4.21 illustrates this theorem.

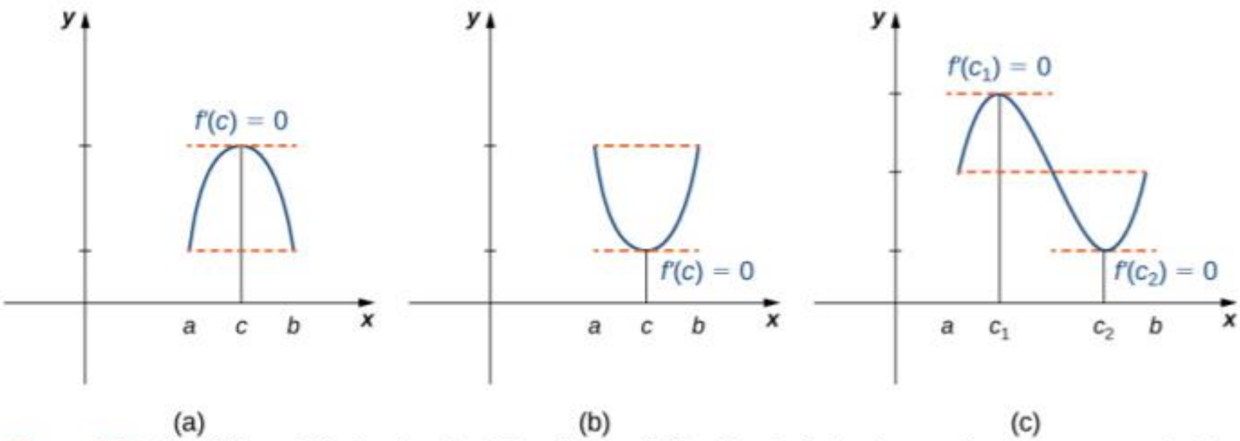


Figure 4.21 If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Theorem 4.4: Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (**Figure 4.25**). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

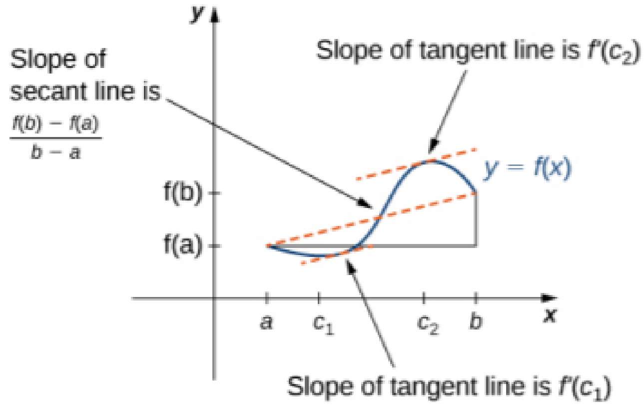


Figure 4.25 The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Theorem 4.5: Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4.4 EXERCISES

148. Why do you need continuity to apply the Mean Value Theorem? Construct a counterexample.

149. Why do you need differentiability to apply the Mean Value Theorem? Find a counterexample.

150. When are Rolle's theorem and the Mean Value Theorem equivalent?

151. If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

152. $y = \sin(\pi x)$

153. $y = \frac{1}{x^3}$

154. $y = \sqrt{4 - x^2}$

155. $y = \sqrt{x^2 - 4}$

156. $y = \ln(3x - 5)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b-a) = f(b) - f(a)$.

157. **[T]** $y = 3x^3 + 2x + 1$ over $[-1, 1]$

158. **[T]** $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

159. **[T]** $y = x^2\cos(\pi x)$ over $[-2, 2]$

160. **[T]** $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

For the following exercises, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$.

161. $f(x) = x^3$

162. $f(x) = \sin(\pi x)$

163. $f(x) = \cos(2\pi x)$

164. $f(x) = 1 + x + x^2$

165. $f(x) = (x-1)^{10}$

166. $f(x) = (x-1)^9$

For the following exercises, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

167. $f(x) = \left|x - \frac{1}{2}\right|$

168. $f(x) = \frac{1}{x^2}$

169. $f(x) = \sqrt{|x|}$

170. $f(x) = \lfloor x \rfloor$ (*Hint*: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to x.)

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

171. $y = e^x$ over $[0, 1]$

172. $y = \ln(2x + 3)$ over $\left[-\frac{3}{2}, 0\right]$

173. $f(x) = \tan(2\pi x)$ over $[0, 2]$

174. $y = \sqrt{9 - x^2}$ over $[-3, 3]$

175. $y = \frac{1}{|x+1|}$ over $[0, 3]$

176. $y = x^3 + 2x + 1$ over $[0, 6]$

177. $y = \frac{x^2 + 3x + 2}{x}$ over $[-1, 1]$

178. $y = \frac{x}{\sin(\pi x) + 1}$ over $[0, 1]$

179. $y = \ln(x + 1)$ over $[0, e - 1]$

180. $y = x\sin(\pi x)$ over $[0, 2]$

<u>Graphing Steps</u>

Sketch the graph of $f(x) = x^3 - 9x$ using the following guidelines:

a) Domain:

b) Intercepts:

c) Asymptotes:

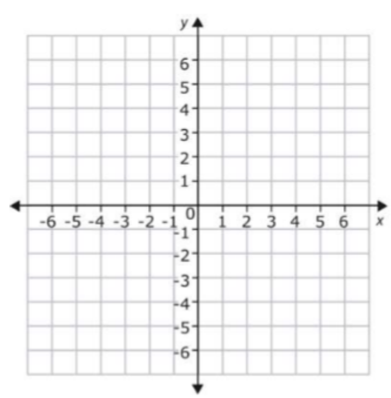
d) Increasing and decreasing intervals:

e) Local Maximum and Minimum Points (and Values):

f) Concave up and concave down intervals:

g) Inflection points:

h) Graph:



1

Sketch the graph of $f(x) = x^3 - 4x$ using the following guidelines:

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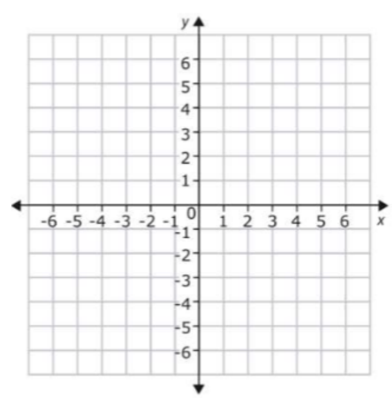

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

129. $y = x^2 + 4x + 5$

130. $y = x^3 - 12x$

131. $y = 3x^4 + 8x^3 - 18x^2$

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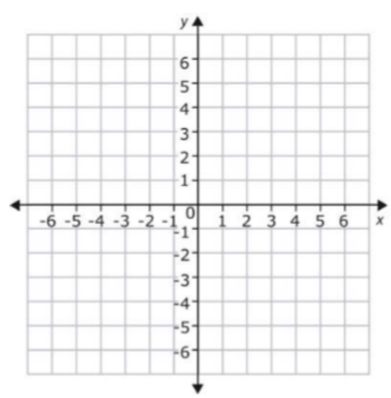

1

For the following exercises, determine

a. intervals where f is increasing or decreasing,

b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and

d. the inflection points of f.

224. $f(x) = x^2 - 6x$

225. $f(x) = x^3 - 6x^2$

226. $f(x) = x^4 - 6x^3$

227. $f(x) = x^{11} - 6x^{10}$

228. $f(x) = x + x^2 - x^3$

229. $f(x) = x^2 + x + 1$

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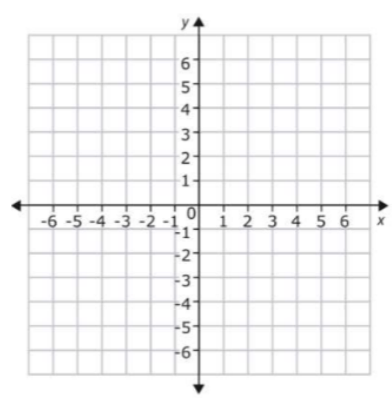

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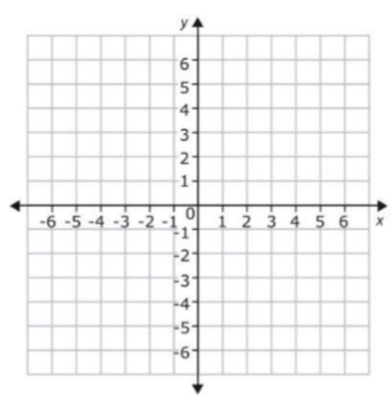

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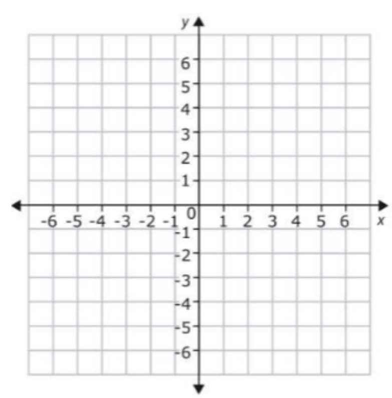

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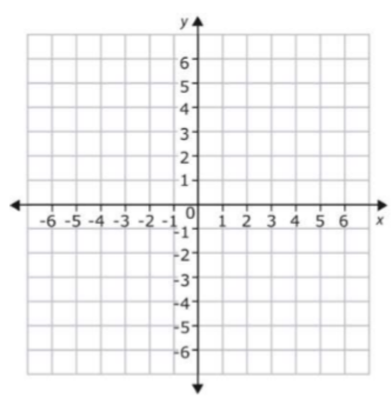

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228. $f(x) = x + x^2 - x^3$

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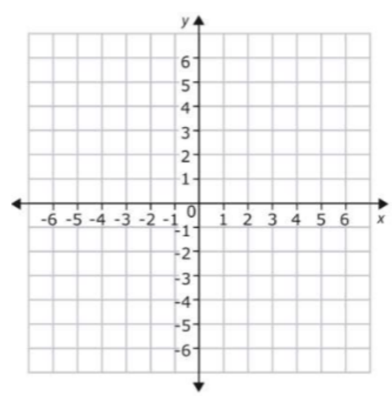

2