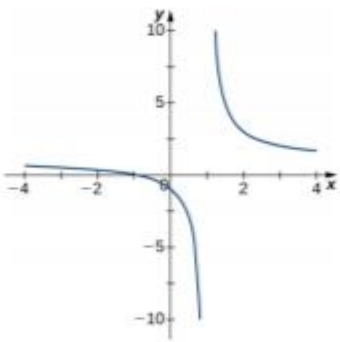
Math G180
Blank Lecture Notes
Chapter 4 – Sections 4.6 and 4.8

4.6 | Limits at Infinity and Asymptotes

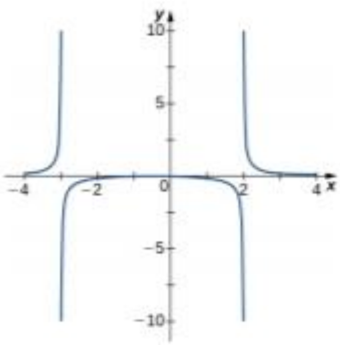
4.6 EXERCISES

For the following exercises, examine the graphs. Identify where the vertical asymptotes are located.

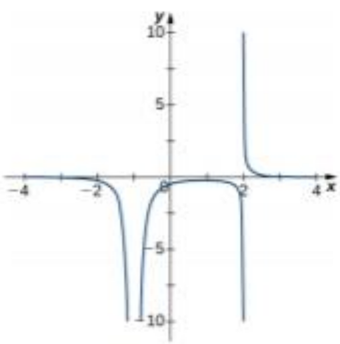
251.



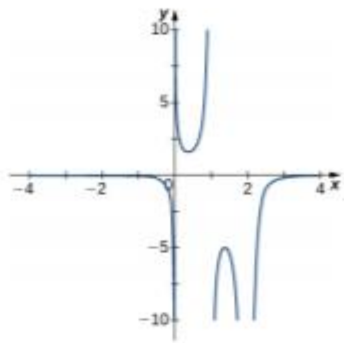
252.



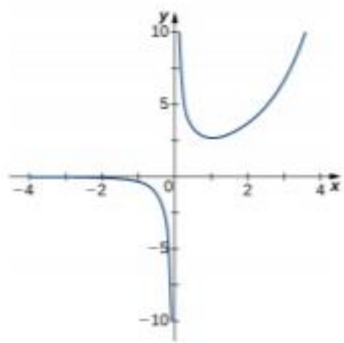
253.



254.



255.



For the following functions $f(x)$, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

256. $f(x) = \frac{x+1}{x^2+5x+4}$, $a = -1$

257. $f(x) = \frac{x}{x-2}$, $a = 2$

258. $f(x) = (x+2)^{3/2}$, $a = -2$

259. $f(x) = (x-1)^{-1/3}$, $a = 1$

260. $f(x) = 1 + x^{-2/5}$, $a = 1$

For the following exercises, evaluate the limit.

261. $\lim_{x \to \infty} \frac{1}{3x+6}$

262. $\lim_{x \to \infty} \frac{2x-5}{4x}$

263. $\lim_{x \to \infty} \frac{x^2-2x+5}{x+2}$

264. $\lim_{x \to -\infty} \frac{3x^3-2x}{x^2+2x+8}$

265. $\lim_{x \to -\infty} \frac{x^4-4x^3+1}{2-2x^2-7x^4}$

266. $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2+1}}$

267. $\lim_{x \to -\infty} \frac{\sqrt{4x^2-1}}{x+2}$

268. $\lim_{x \to \infty} \frac{4x}{\sqrt{x^2-1}}$

269. $\lim_{x \to -\infty} \frac{4x}{\sqrt{x^2-1}}$

270. $\lim_{x \to \infty} \frac{2\sqrt{x}}{x-\sqrt{x}+1}$

For the following exercises, find the horizontal and vertical asymptotes.

271. $f(x) = x - \frac{9}{x}$

272. $f(x) = \frac{1}{1-x^2}$

273. $f(x) = \frac{x^3}{4-x^2}$

274. $f(x) = \frac{x^2+3}{x^2+1}$

275. $f(x) = \sin(x)\sin(2x)$

276. $f(x) = \cos x + \cos(3x) + \cos(5x)$

277. $f(x) = \frac{x\sin(x)}{x^2-1}$

278. $f(x) = \frac{x}{\sin(x)}$

279. $f(x) = \frac{1}{x^3+x^2}$

280. $f(x) = \frac{1}{x-1} - 2x$

281. $f(x) = \frac{x^3+1}{x^3-1}$

282. $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

283. $f(x) = x - \sin x$

284. $f(x) = \frac{1}{x} - \sqrt{x}$

For the following exercises, construct a function $f(x)$ that has the given asymptotes.

285. $x = 1$ and $y = 2$

286. $x = 1$ and $y = 0$

287. $y = 4,\ x = -1$

288. $x = 0$

For the following exercises, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

289. **[T]** $f(x) = \frac{1}{x+10}$

290. **[T]** $f(x) = \frac{x+1}{x^2+7x+6}$

291. **[T]** $\lim_{x \to -\infty} x^2 + 10x + 25$

292. **[T]** $\lim_{x \to -\infty} \frac{x+2}{x^2+7x+6}$

293. **[T]** $\lim_{x \to \infty} \frac{3x+2}{x+5}$

For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

294. $y = 3x^2 + 2x + 4$

295. $y = x^3 - 3x^2 + 4$

296. $y = \frac{2x+1}{x^2+6x+5}$

297. $y = \frac{x^3+4x^2+3x}{3x+9}$

4.8 | L'Hôpital's Rule

Theorem 4.12: L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty$.

Theorem 4.13: L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or

Example 4.38

Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x \to 0} \frac{1 - \cos x}{x}$

b. $\lim\limits_{x \to 1} \frac{\sin(\pi x)}{\ln x}$

c. $\lim\limits_{x \to \infty} \frac{e^{1/x} - 1}{1/x}$

d. $\lim\limits_{x \to 0} \frac{\sin x - x}{x^2}$

Example 4.39

Applying L'Hôpital's Rule (∞/∞ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x \to \infty} \frac{3x + 5}{2x + 1}$

379. $\lim\limits_{x \to 1} \dfrac{x-1}{\ln x}$

380. $\lim\limits_{x \to 0} (x+1)^{1/x}$

381. $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - \sqrt[3]{x}}{x-1}$

382. $\lim\limits_{x \to 0^+} x^{2x}$

383. $\lim\limits_{x \to \infty} x \sin\left(\frac{1}{x}\right)$

384. $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x^2}$

385. $\lim\limits_{x \to 0^+} x \ln\left(x^4\right)$

386. $\lim\limits_{x \to \infty} (x - e^x)$

387. $\lim\limits_{x \to \infty} x^2 e^{-x}$

391. $\lim_{x \to \infty} xe^{1/x}$

392. $\lim_{x \to 0^+} x^{1/\cos x}$

393. $\lim_{x \to 0^+} x^{1/x}$

394. $\lim_{x \to 0^-} \left(1 - \frac{1}{x}\right)^x$

395. $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x$

For the following exercises, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

396. **[T]** $\lim_{x \to 0} \frac{e^x - 1}{x}$

397. **[T]** $\lim_{x \to 0} x\sin\left(\frac{1}{x}\right)$

398. **[T]** $\lim_{x \to 1} \frac{x - 1}{1 - \cos(\pi x)}$

399. **[T]** $\lim_{x \to 1} \frac{e^{(x-1)} - 1}{x - 1}$