Math G180
Blank Lecture Notes
Chapter 3 – Section 3.8

3.8 | Implicit Differentiation

Learning Objectives

3.8.1 Find the derivative of a complicated function by using implicit differentiation.
3.8.2 Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example 3.68

Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Example 3.70

Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Example 3.69

Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3,$ find $\frac{dy}{dx}.$

Example 3.71

Finding a Tangent Line to a Circle

Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Example 3.72

Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ (**Figure 3.32**). This curve is known as the folium (or leaf) of Descartes.

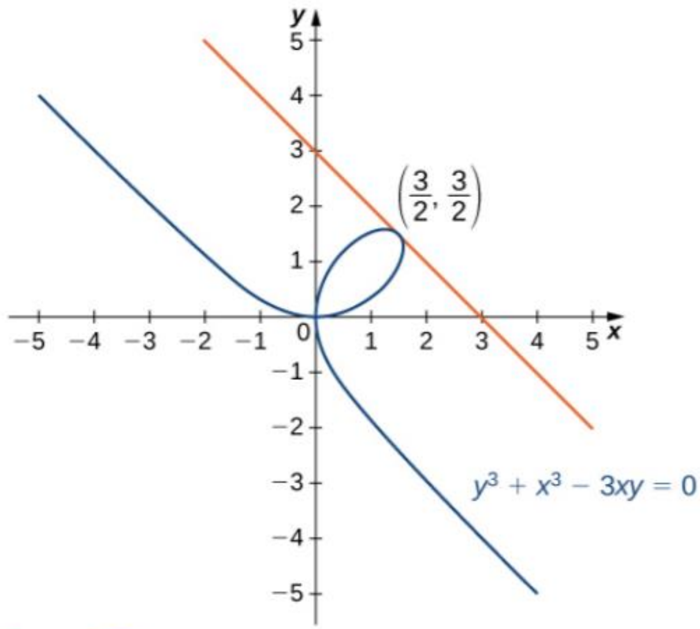


Figure 3.32 Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

3.8 EXERCISES

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

300. $x^2 - y^2 = 4$

301. $6x^2 + 3y^2 = 12$

302. $x^2 y = y - 7$

303. $3x^3 + 9xy^2 = 5x^3$

304. $xy - \cos(xy) = 1$

305. $y\sqrt{x+4} = xy + 8$

306. $-xy - 2 = \frac{x}{7}$

307. $y\sin(xy) = y^2 + 2$

308. $(xy)^2 + 3x = y^2$

309. $x^3 y + xy^3 = -8$

For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

310. **[T]** $x^4 y - xy^3 = -2,\ (-1, -1)$

311. **[T]** $x^2 y^2 + 5xy = 14,\ (2, 1)$

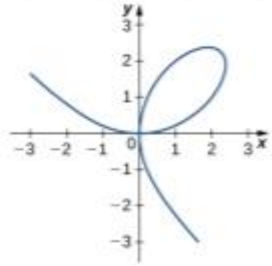
312. **[T]** $\tan(xy) = y,\ \left(\frac{\pi}{4}, 1\right)$

313. **[T]** $xy^2 + \sin(\pi y) - 2x^2 = 10,\ (2, -3)$

314. **[T]** $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y,\ (1, 2)$

315. **[T]** $xy + \sin(x) = 1,\ \left(\frac{\pi}{2}, 0\right)$

316. **[T]** The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.
b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

317. For the equation $x^2 + 2xy - 3y^2 = 0$,
a. Find the equation of the normal to the tangent line at the point $(1, 1)$.
b. At what other point does the normal line in a. intersect the graph of the equation?

318. Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

319. For the equation $x^2 + xy + y^2 = 7$,
a. Find the x-intercept(s).
b. Find the slope of the tangent line(s) at the x-intercept(s).
c. What does the value(s) in b. indicate about the tangent line(s)?

320. Find the equation of the tangent line to the graph of the equation $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

321. Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

322. Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.