Math G180
Blank Lecture Notes
Chapter 2 – Section 2.2-2.4

2.2 | The Limit of a Function

Definition

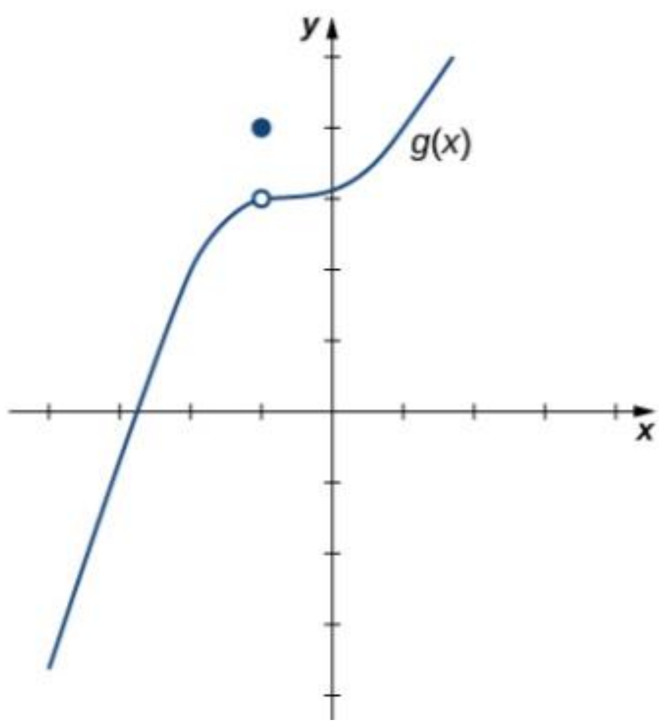
Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \quad (2.3)$$

Example 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in **Figure 2.15**, evaluate $\lim_{x \to -1} g(x)$.



Theorem 2.1: Two Important Limits

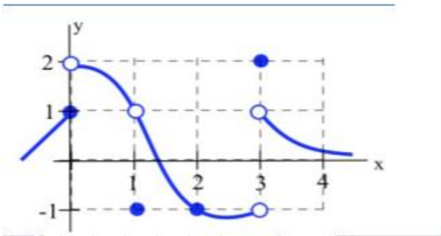
Let a be a real number and c be a constant.

i. $$\lim_{x \to a} x = a \qquad (2.4)$$

ii. $$\lim_{x \to a} c = c \qquad (2.5)$$

Definition

We define two types of **one-sided limits**.



Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (c, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L. \quad (2.6)$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L. \quad (2.7)$$

Example 2.8

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again Table 2.6. Observe that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

x	$f(x) = x + 1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table 2.6

Table of Functional Values for $f(x) = \begin{cases} x+1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. **Figure 2.18** shows a graph of $f(x)$ and reinforces our conclusion about these limits.

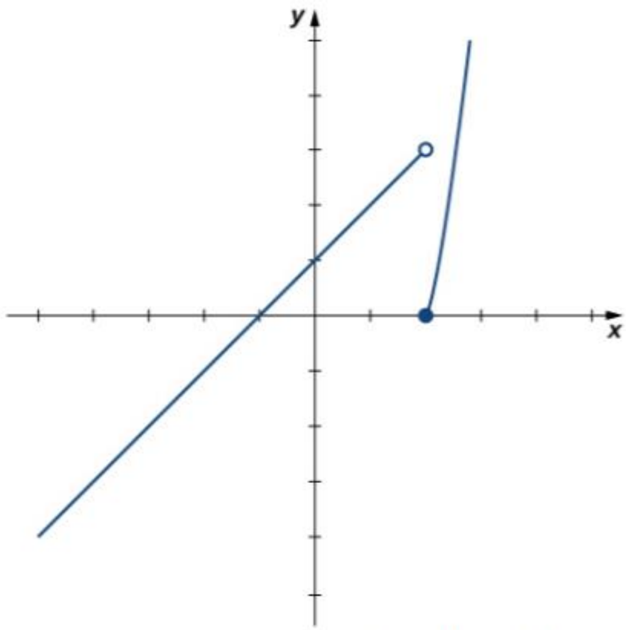


Figure 2.18 The graph of $f(x) = \begin{cases} x+1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Theorem 2.2: Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty. \tag{2.8}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty. \tag{2.9}$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty. \tag{2.10}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty. \tag{2.11}$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a.

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty. \tag{2.12}$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty. \tag{2.13}$$

Theorem 2.3: Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Definition

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

$$\begin{aligned} \lim_{x \to a^-} f(x) &= +\infty \text{ or } -\infty \\ \lim_{x \to a^+} f(x) &= +\infty \text{ or } -\infty \\ &\text{or} \\ \lim_{x \to a} f(x) &= +\infty \text{ or } -\infty \end{aligned}$$

Example 2.11

Behavior of a Function at Different Points

Use the graph of $f(x)$ in **Figure 2.21** to determine each of the following values:

a. $\lim_{x \to -4^-} f(x)$; $\lim_{x \to -4^+} f(x)$; $\lim_{x \to -4} f(x)$; $f(-4)$

b. $\lim_{x \to -2^-} f(x)$; $\lim_{x \to -2^+} f(x)$; $\lim_{x \to -2} f(x)$; $f(-2)$

c. $\lim_{x \to 1^-} f(x)$; $\lim_{x \to 1^+} f(x)$; $\lim_{x \to 1} f(x)$; $f(1)$

d. $\lim_{x \to 3^-} f(x)$; $\lim_{x \to 3^+} f(x)$; $\lim_{x \to 3} f(x)$; $f(3)$

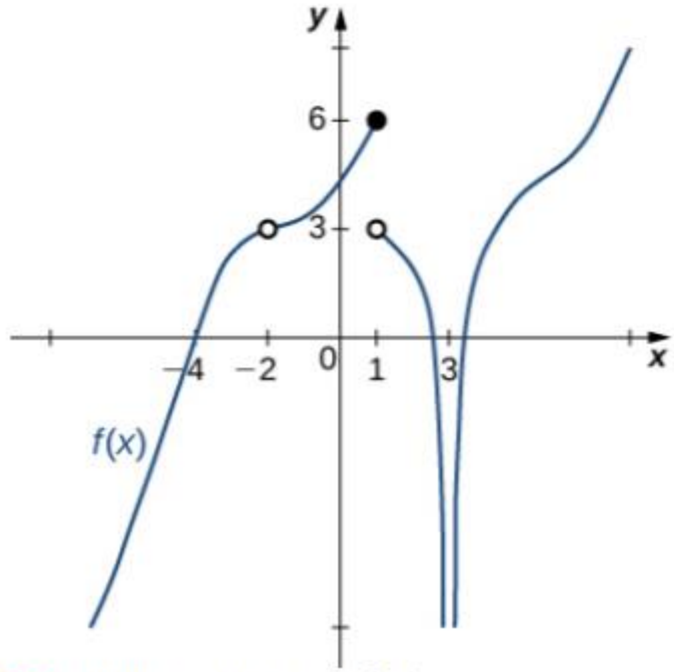
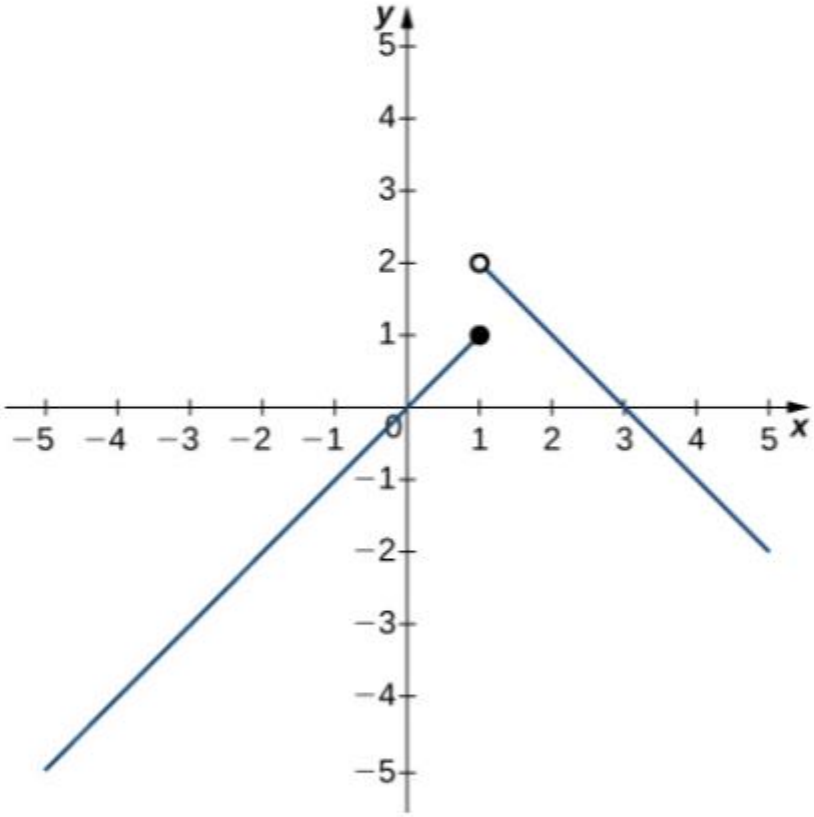


Figure 2.21 The graph shows $f(x)$.

In the following exercises, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



50. $\lim_{x \to 1^-} f(x)$

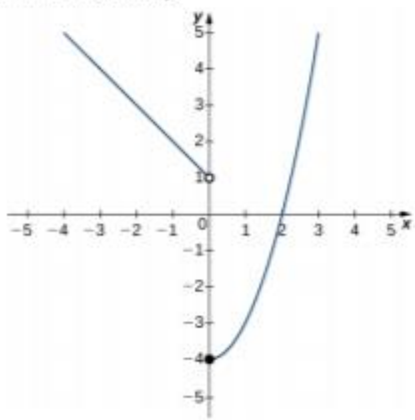
51. $\lim_{x \to 1^+} f(x)$

52. $\lim_{x \to 1} f(x)$

53. $\lim_{x \to 2} f(x)$

54. $f(1)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



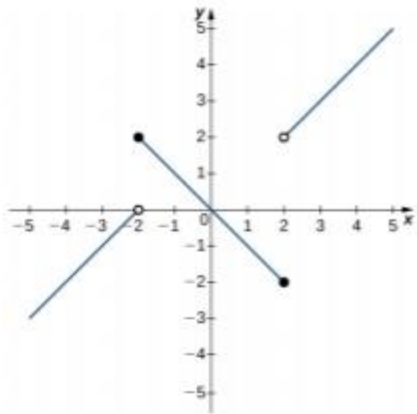
55. $\lim_{x \to 0^-} f(x)$

56. $\lim_{x \to 0^+} f(x)$

57. $\lim_{x \to 0} f(x)$

58. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



59. $\lim_{x \to -2^-} f(x)$

60. $\lim_{x \to -2^+} f(x)$

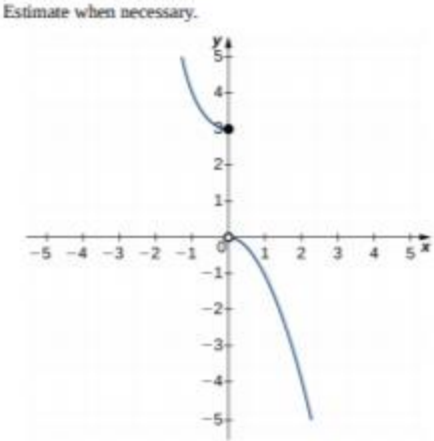
61. $\lim_{x \to -2} f(x)$

62. $\lim_{x \to 2^-} f(x)$

63. $\lim_{x \to 2^+} f(x)$

64. $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.

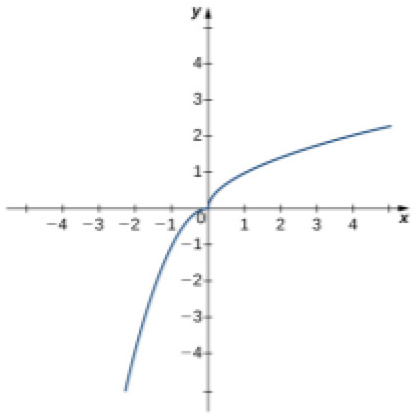


65. $\lim_{x \to 0^-} g(x)$

66. $\lim\limits_{x \to 0^+} g(x)$

67. $\lim\limits_{x \to 0} g(x)$

In the following exercises, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.

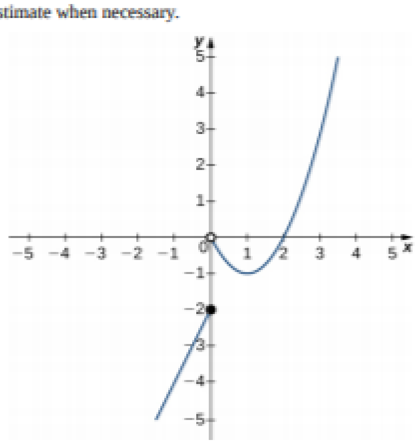


68. $\lim\limits_{x \to 0^-} h(x)$

69. $\lim\limits_{x \to 0^+} h(x)$

70. $\lim\limits_{x \to 0} h(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



71. $\lim\limits_{x \to 0^-} f(x)$

72. $\lim\limits_{x \to 0^+} f(x)$

73. $\lim\limits_{x \to 0} f(x)$

74. $\lim\limits_{x \to 1} f(x)$

75. $\lim\limits_{x \to 2} f(x)$

In the following exercises, sketch the graph of a function with the given properties.

76. $\lim\limits_{x \to 2} f(x) = 1$, $\lim\limits_{x \to 4^-} f(x) = 3$, $\lim\limits_{x \to 4^+} f(x) = 6$, $f(4)$ is not defined.

77. $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to -1^-} f(x) = -\infty$, $\lim\limits_{x \to -1^+} f(x) = \infty$, $\lim\limits_{x \to 0} f(x) = f(0)$, $f(0) = 1$, $\lim\limits_{x \to \infty} f(x) = -\infty$

78. $\lim\limits_{x \to -\infty} f(x) = 2$, $\lim\limits_{x \to 3^-} f(x) = -\infty$, $\lim\limits_{x \to 3^+} f(x) = \infty$, $\lim\limits_{x \to \infty} f(x) = 2$, $f(0) = \frac{-1}{3}$

79. $\lim\limits_{x \to -\infty} f(x) = 2$, $\lim\limits_{x \to -2} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = 2$, $f(0) = 0$

80. $\lim\limits_{x \to -\infty} f(x) = 0$, $\lim\limits_{x \to -1^-} f(x) = \infty$, $\lim\limits_{x \to -1^+} f(x) = -\infty$, $f(0) = -1$, $\lim\limits_{x \to 1^-} f(x) = -\infty$, $\lim\limits_{x \to 1^+} f(x) = \infty$, $\lim\limits_{x \to \infty} f(x) = 0$

2.3 | The Limit Laws

Theorem 2.5: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

Difference law for limits: $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

Product law for limits: $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \to a}(f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.

Root law for limits: $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

Example 2.14

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \to -3}(4x + 2)$.

Example 2.15

Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim_{x \to 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Theorem 2.6: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x \to a} p(x) = p(a)$$

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

Example 2.17

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \to 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Example 2.22

Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x - 3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure 2.27 illustrates this idea.

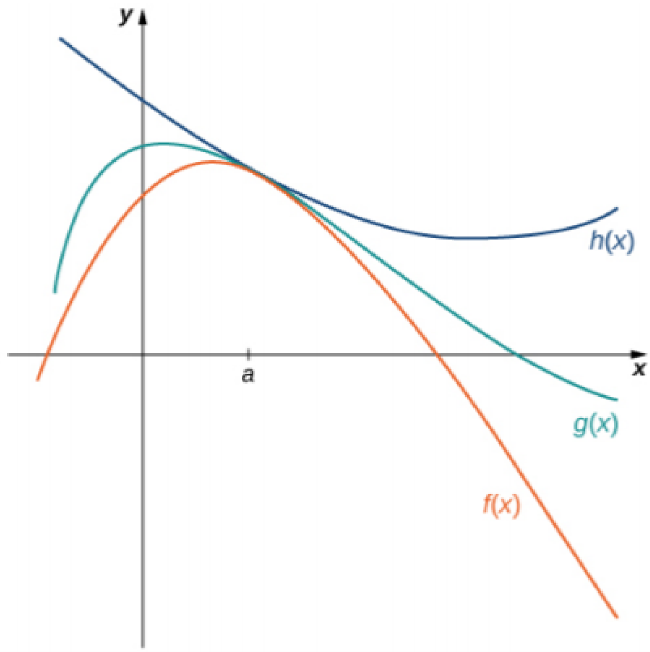


Figure 2.27 The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Theorem 2.7: The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

Example 2.24

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution

Because $-1 \leq \cos x \leq 1$ for all x, we have $-|x| \leq x \cos x \leq |x|$. Since $\lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x|$, from the squeeze theorem, we obtain $\lim_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x \cos x$, and $h(x) = |x|$ are shown in **Figure 2.28**.

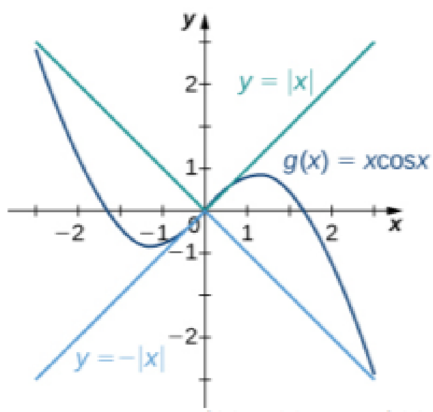


Figure 2.28 The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1.$$

$$\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0.$$

2.3 EXERCISES

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

83. $\lim_{x \to 0} \left(4x^2 - 2x + 3\right)$

84. $\lim_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

85. $\lim_{x \to -2} \sqrt{x^2 - 6x + 3}$

86. $\lim_{x \to -1} (9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.

87. $\lim_{x \to 7} x^2$

88. $\lim_{x \to -2} \left(4x^2 - 1\right)$

89. $\lim_{x \to 0} \frac{1}{1 + \sin x}$

90. $\lim_{x \to 2} e^{2x - x^2}$

91. $\lim_{x \to 1} \frac{2 - 7x}{x + 6}$

92. $\lim_{x \to 3} \ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form 0/0. Then, evaluate the limit.

93. $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$

94. $\lim_{x \to 2} \frac{x - 2}{x^2 - 2x}$

95. $\lim_{x \to 6} \frac{3x - 18}{2x - 12}$

96. $\lim_{h \to 0} \frac{(1 + h)^2 - 1}{h}$

97. $\lim_{t \to 9} \frac{t - 9}{\sqrt{t} - 3}$

98. $\lim_{h \to 0} \frac{\frac{1}{a + h} - \frac{1}{a}}{h}$, where a is a non-zero real-valued constant

99. $\lim_{\theta \to \pi} \frac{\sin\theta}{\tan\theta}$

100. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$

101. $\lim_{x \to 1/2} \frac{2x^2 + 3x - 2}{2x - 1}$

102. $\lim_{x \to -3} \frac{\sqrt{x + 4} - 1}{x + 3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of Example 2.23 to simplify the function to help determine the limit.

103. $\lim_{x \to -2^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

104. $\lim_{x \to -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

105. $\lim_{x \to 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

106. $\lim_{x \to 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

In the following exercises, assume that $\lim_{x \to 6} f(x) = 4$, $\lim_{x \to 6} g(x) = 9$, and $\lim_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

107. $\lim_{x \to 6} 2 f(x) g(x)$

108. $\lim_{x \to 6} \frac{g(x) - 1}{f(x)}$

109. $\lim_{x \to 6} \left(f(x) + \frac{1}{3} g(x)\right)$

110. $\lim_{x \to 6} \frac{[h(x)]^3}{2}$

111. $\lim_{x \to 6} \sqrt{g(x) - f(x)}$

112. $\lim_{x \to 6} x \cdot h(x)$

2.4 | Continuity

Definition

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

Types of Discontinuities

As we have seen in Example 2.26 and Example 2.27, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure 2.37 illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

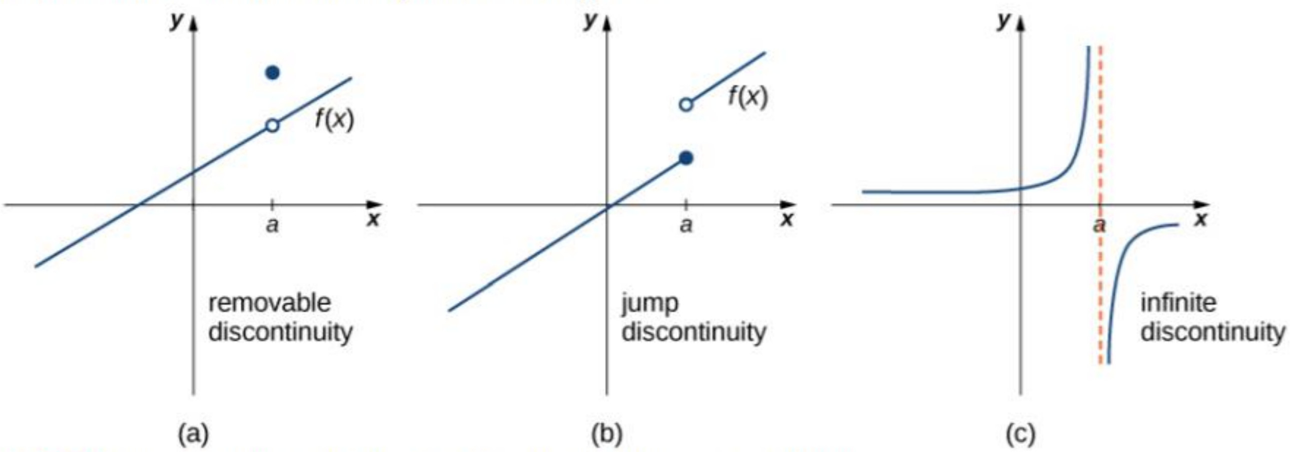


Figure 2.37 Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x \to a} f(x)$ exists. (Note: When we state that $\lim_{x \to a} f(x)$ exists, we mean that $\lim_{x \to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$. (Note: When we state that $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Theorem 2.11: The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.38.

2.4 EXERCISES

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

131. $f(x) = \frac{1}{\sqrt{x}}$

132. $f(x) = \frac{2}{x^2 + 1}$

133. $f(x) = \frac{x}{x^2 - x}$

134. $g(t) = t^{-1} + 1$

135. $f(x) = \frac{5}{e^x - 2}$

136. $f(x) = \frac{|x - 2|}{x - 2}$

137. $H(x) = \tan 2x$

138. $f(t) = \frac{t + 3}{t^2 + 5t + 6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

139. $f(x)\frac{2x^2 - 5x + 3}{x - 1}$ at $x = 1$

140. $h(\theta) = \frac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

141. $g(u) = \begin{cases} \frac{6u^2 + u - 2}{2u - 1} & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}$, at $u = \frac{1}{2}$

142. $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

143. $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$, at $x = 0$

144. $f(x) = \begin{cases} x\sin(x) \text{ if } x \leq \pi \\ x\tan(x) \text{ if } x > \pi \end{cases}$, at $x = \pi$

In the following exercises, find the value(s) of k that makes each function continuous over the given interval.

145. $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$