Math G180
Blank Lecture Notes
Chapter 3 – Section 3.7,3.9

3.7 | Derivatives of Inverse Functions

Learning Objectives

3.7.1 Calculate the derivative of an inverse function.

3.7.2 Recognize the derivatives of the standard inverse trigonometric functions.

In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. Figure 3.28 shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $\left(a,\, f^{-1}(a)\right)$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $\left(f^{-1}\right)'(a) = \frac{p}{q}$. This point corresponds to a point $\left(f^{-1}(a),\, a\right)$ on the graph of $f(x)$ having a tangent line with a slope of $f'\left(f^{-1}(a)\right) = \frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a, then it must be the case that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

Theorem 3.11: Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Example 3.60

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

$$g'(x) = -\frac{2}{x^2}.$$

Theorem 3.13: Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}} \tag{3.22}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}} \tag{3.23}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+(x)^2} \tag{3.24}$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+(x)^2} \tag{3.25}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{(x)^2-1}} \tag{3.26}$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{(x)^2-1}} \tag{3.27}$$

Example 3.65

Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}\left(x^2\right)$.

Solution

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into **Equation 3.24**, we obtain

$$f'(x) = \frac{1}{1+\left(x^2\right)^2} \cdot (2x).$$

Simplifying, we have

$$f'(x) = \frac{2x}{1+x^4}.$$

Example 3.66

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution

By applying the product rule, we have

$$h'(x) = 2x\sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2.$$

3.46 Find the derivative of $h(x) = \cos^{-1}(3x-1)$.

Example 3.67

Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

$$v(t) = s'(t) = \frac{1}{1+\left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

$$v(t) = -\frac{1}{t^2+1}.$$

Thus, $v(1) = -\frac{1}{2}$.

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For the following exercises, use the functions $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

264. $f(x) = 6x - 1,\ x = -2$

265. $f(x) = 2x^3 - 3,\ x = 1$

b. find the equation of the tangent line to the graph of f^{-1} at the indicated point.

274. $f(x) = \frac{4}{1+x^2}$, $P(2, 1)$

275. $f(x) = \sqrt{x-4}$, $P(2, 8)$

For each of the following functions, find $\left(f^{-1}\right)'(a)$.

268. $f(x) = x^2 + 3x + 2$, $x \geq -\frac{3}{2}$, $a = 2$

269. $f(x) = x^3 + 2x + 3$, $a = 0$

270. $f(x) = x + \sqrt{x}$, $a = 2$

271. $f(x) = x - \frac{2}{x}$, $x < 0$, $a = 1$

272. $f(x) = x + \sin x$, $a = 0$

273. $f(x) = \tan x + 3x^2$, $a = 0$

For each of the given functions $y = f(x)$,

a. find the slope of the tangent line to its inverse function f^{-1} at the indicated point P, and

3.9 | Derivatives of Exponential and Logarithmic Functions

Theorem 3.16: Derivatives of General Exponential and Logarithmic Functions

Let $b > 0,\ b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x,$ then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{3.32}$$

More generally, if $h(x) = \log_b(g(x)),$ then for all values of x for which $g(x) > 0,$

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{3.33}$$

ii. If $y = b^x,$ then

$$\frac{dy}{dx} = b^x \ln b. \tag{3.34}$$

More generally, if $h(x) = b^{g(x)},$ then

$$h'(x) = b^{g(x)} g'(x) \ln b. \tag{3.35}$$

Example 3.79

Applying Derivative Formulas

Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}.$

Example 3.80

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$$\ln y = \ln\left(2x^4+1\right)^{\tan x}$$ Step 1. Take the natural logarithm of both sides.

$$\ln y = \tan x \ln\left(2x^4+1\right)$$ Step 2. Expand using properties of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x$$ Step 3. Differentiate both sides. Use the product rule on the right.

$$\frac{dy}{dx} = y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x\right)$$ Step 4. Multiply by y on both sides.

$$\frac{dy}{dx} = \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x\right)$$ Step 5. Substitute $y = \left(2x^4+1\right)^{\tan x}$.

Example 3.82

Using Logarithmic Differentiation

Find the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$$\ln y = \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$$ Step 1. Take the natural logarithm of both sides.

$$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln \sin x$$ Step 2. Expand using properties of logarithms.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x}$$ Step 3. Differentiate both sides.

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$ Step 4. Multiply by y on both sides.

$$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right)$$ Step 5. Substitute $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

3.9 EXERCISES

For the following exercises, find $f'(x)$ for each function.

331. $f(x) = x^2 e^x$

332. $f(x) = \frac{e^{-x}}{x}$

333. $f(x) = e^{x^3 \ln x}$

334. $f(x) = \sqrt{e^{2x} + 2x}$

335. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

336. $f(x) = \frac{10^x}{\ln 10}$

337. $f(x) = 2^{4x} + 4x^2$

338. $f(x) = 3^{\sin 3x}$

339. $f(x) = x^{\pi} \cdot \pi^x$

340. $f(x) = \ln\left(4x^3 + x\right)$

341. $f(x) = \ln\sqrt{5x - 7}$

342. $f(x) = x^2 \ln 9x$

343. $f(x) = \log(\sec x)$

344. $f(x) = \log_7\left(6x^4 + 3\right)^5$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

347. $y = (\sin 2x)^{4x}$

348. $y = (\ln x)^{\ln x}$

349. $y = x^{\log_2 x}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

351. $y = x^{\cot x}$

352. $y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where $x = -1$. Graph both the function and the tangent line.

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

357. Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.
b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.