Math G180
Blank Lecture Notes
Chapter 4 – Section 4.1

4.1 | Related Rates

Learning Objectives

4.1.1 Express changing quantities in terms of derivatives.
4.1.2 Find relationships among the derivatives in a given problem.
4.1.3 Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example 4.1

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\ \text{cm}^3/\text{sec}$ (Figure 4.2). How fast is the radius increasing when the radius is $3\ \text{cm}$?

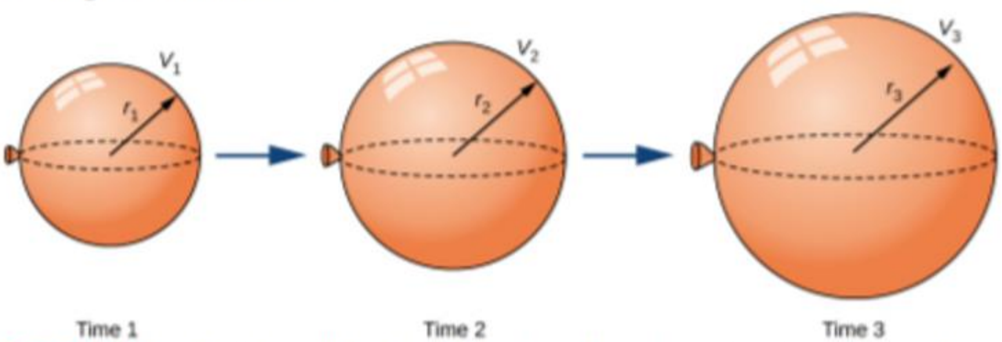


Figure 4.2 As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution

The volume of a sphere of radius r centimeters is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

$$V(t) = \frac{4}{3}\pi[r(t)]^3 \text{ cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

$$V'(t) = 4\pi[r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V'(t) = 2 \text{ cm}^3\text{/sec}$. Therefore,

$$2\text{cm}^3\text{/sec} = \left(4\pi[r(t)]^2 \text{ cm}^2\right)\cdot(r'(t)\text{cm/s}),$$

which implies

$$r'(t) = \frac{1}{2\pi[r(t)]^2}\text{cm/sec.}$$

When the radius $r = 3$ cm,

$$r'(t) = \frac{1}{18\pi}\text{cm/sec.}$$

4.1 What is the instantaneous rate of change of the radius when $r = 6$ cm?

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

4.1 EXERCISES

For the following exercises, find the quantities for the given equation.

1. Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

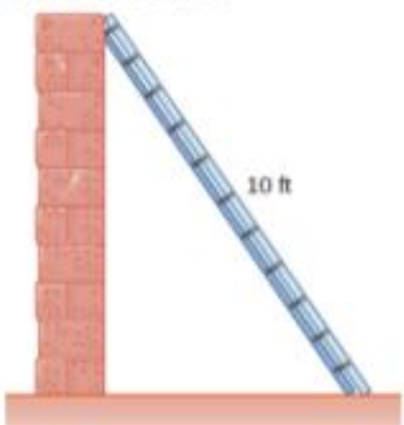
2. Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$.

3. Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

For the following exercises, sketch the situation if necessary and used related rates to solve for the quantities.

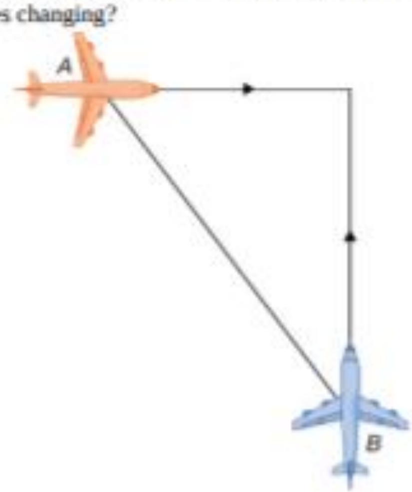
4. **[T]** If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5\,\Omega$/min and R_2 decreases at a rate of 1.1Ω/min, at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega$?

5. A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



6. A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

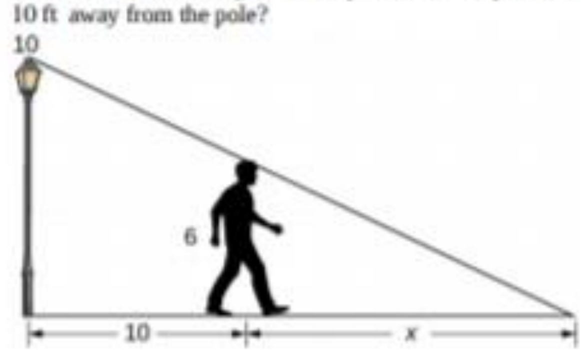
7. Two airplanes are flying in the air at the same height: airplane A is flying east at 250 mi/h and airplane B is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane A and 40 miles north of airplane B, at what rate is the distance between the airplanes changing?



8. You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled 4 mi, at what rate is the distance between you changing?

9. Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant 55 mph, find the rate at which the distance between the buses is changing when they are 13 mi apart, heading toward each other.

10. A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



11. Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

13. Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

14. A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

15. Using the previous problem, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

For the following exercises, draw and label diagrams to help solve the related-rates problems.

16. The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

17. The volume of a cube decreases at a rate of 10 m^3/s. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

18. The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

19. The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

20. The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

21. The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

22. The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

23. A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.