Math G180
Blank Lecture Notes
Chapter 4 – Sections 4.7

4.7 | Applied Optimization Problems

Example 4.32

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (**Figure 4.62**). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

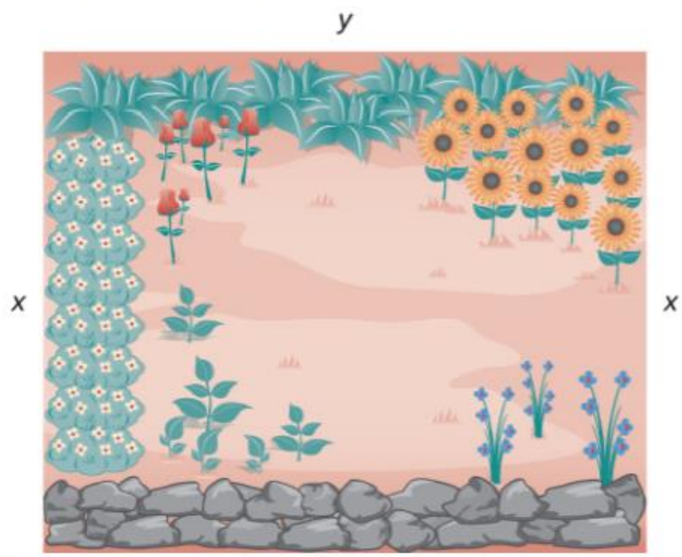


Figure 4.62 We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.33

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure 4.64). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

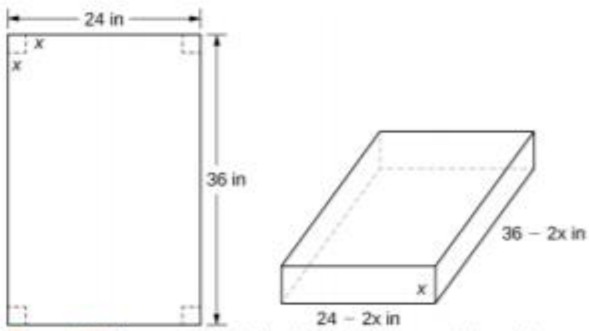


Figure 4.64 A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H$, where L, W, and H are the length, width, and height, respectively.

Step 4: From Figure 4.64, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let's examine Figure 4.64. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825\text{ in.}^3$ as shown in the following graph.

4.7 EXERCISES

For the following exercises, answer by proof, counterexample, or explanation.

311. When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

312. Why do you need to check the endpoints for optimization problems?

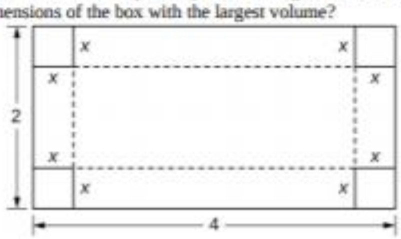
313. *True or False*. For every continuous nonlinear function, you can find the value x that maximizes the function.

314. *True or False*. For every continuous nonconstant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

315. To carry a suitcase on an airplane, the length $+\text{width}+$ height of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right)h\right)^2$. What height allows you to have the largest volume?

316. You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



317. Find the positive integer that minimizes the sum of the number and its reciprocal.

318. Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

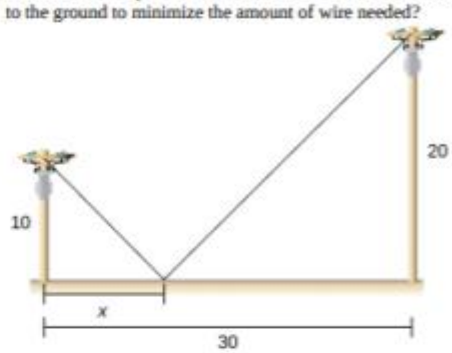
For the following exercises, consider the construction of a pen to enclose an area.

319. You have 400 ft of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

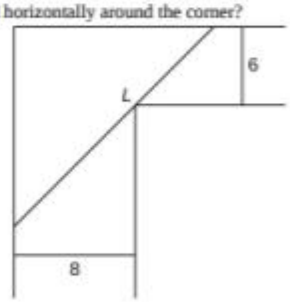
320. You have 800 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

321. You need to construct a fence around an area of 1600 ft. What are the dimensions of the rectangular pen to minimize the amount of material needed?

322. Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



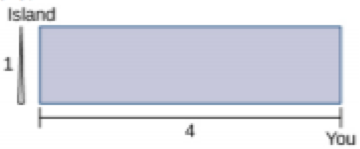
323. **[T]** You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



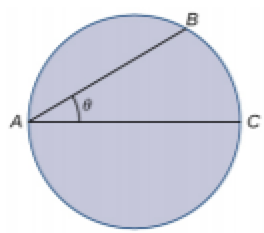
324. A patient's pulse measures 70 bpm, 80 bpm, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

325. In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

326. You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For the following problems, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w=3v$.



327. Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

328. Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

329. A truck uses gas as $g(v)=av+\frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

For the following exercises, consider a limousine that gets $m(v)=\frac{(120-2v)}{5}$ mi/gal at speed v, the chauffeur costs \$15/h, and gas is \$3.5/gal.

330. Find the cost per mile at speed v.

331. Find the cheapest driving speed.

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R(x)=ax$ and costs $C(x)=b+cx+dx^2$, where x represents the number of pizzas.

332. Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

333. Assume that $R(x)=10x$ and $C(x)=2x+x^2$. How many pizzas sold maximizes the profit?

334. Assume that $R(x)=15x$, and $C(x)=60+3x+\frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

335. Choose x to maximize the sum of their areas.

336. Choose x to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers x and y such that $x+y=10$. Maximize and minimize the quantities.

337. xy

338. x^2y^2

339. $y-\frac{1}{x}$

340. x^2-y

For the following exercises, draw the given optimization problem and solve.

341. Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

342. Find the volume of the largest right cone that fits in a sphere of radius 1.

343. Find the area of the largest rectangle that fits into the triangle with sides $x=0$, $y=0$ and $\frac{x}{4}+\frac{y}{6}=1$.

344. Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

345. Find the dimensions of the closed cylinder volume $V=16\pi$ that has the least amount of surface area.

346. Find the dimensions of a right cone with surface area $S=4\pi$ that has the largest volume.