5.5 | Substitution

Theorem 5.7: Substitution with Indefinite Integrals

Let $u = g(x)$, , where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)dx &= \int f(u)du \\ &= F(u) + C \\ &= F(g(x)) + C.\end{aligned} \tag{5.19}$$

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

*If we cannot use the integral formulas, we will use U-substitution!

step ① Choose u

step ② Find $du = \ldots dx$

(derivative)

step ③ Substitute all x's and dx in terms of u and du.

step ④ Integrate (by using formulas)

step ⑤ Replace u in terms of x.

Kyle's Note: Choose the part as <u>u</u>, where you see the derivative of it inside the integral.

$\boxed{\text{Note: We cannot have both } x, dx\text{'s with } u\text{'s}, du\text{'s in the same integral.}}$

Guidelines to choose u:

1. The denominator.
2. The part under the $\sqrt{\boxed{}} \rightarrow u = \boxed{}$
3. The base of an exponent. $\boxed{}^{10} \rightarrow u = \boxed{}$
4. The exponent of an $e^{\boxed{}} \rightarrow u = \boxed{}$
5. The angle of a trig. function.
6. A trig. function.

Example 5.30

Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative $\int 6x\left(3x^2+4\right)^4 dx$.

Example 5.31

Using Substitution with Alteration

Use substitution to find $\int z\sqrt{z^2 - 5}dz$.

Example 5.32

Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t} dt$.

Example 5.33

Finding an Antiderivative Using *u*-Substitution

Use substitution to find the antiderivative $\int \frac{x}{\sqrt{x-1}}dx$.

In the following exercises, use a suitable change of variables to determine the indefinite integral.

271. $\int x(1-x)^{99}\,dx$

272. $\int t\left(1-t^2\right)^{10}\,dt$

273. $\int (11x-7)^{-3}\,dx$

274. $\int (7x-11)^4\,dx$

275. $\int \cos^3\theta \sin\theta\, d\theta$

276. $\int \sin^7\theta \cos\theta\, d\theta$

277. $\int \cos^2(\pi t)\sin(\pi t)\,dt$

Selected Integral Formulas

INTEGRATION PROPERTIES

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0 \text{ and } \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

COMMON INTEGRALS

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{a^2+u^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,dx = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Definition

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)dx,$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

$$\int 2x dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the *family of antiderivatives of* $2x$. **Figure 4.85** shows a graph of this family of antiderivatives.

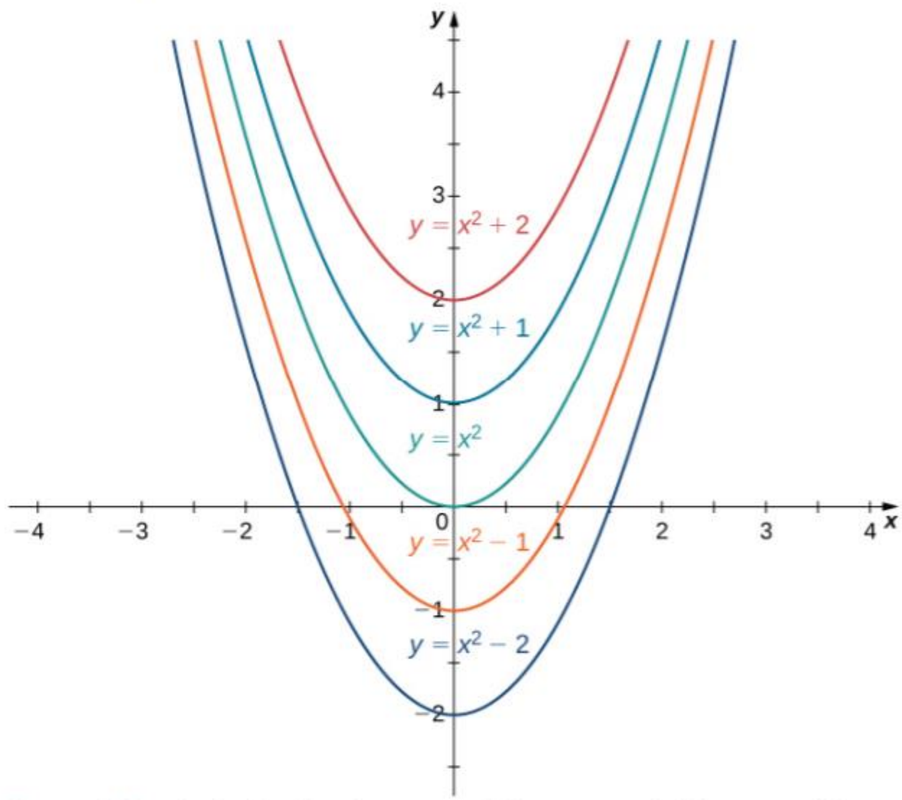

Figure 4.85 The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

Example 4.50

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \cos x$

d. $f(x) = e^x$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int kx^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dn = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
$\frac{d}{dx}\left(\sec^{-1}\lvert x\rvert\right) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1}\lvert x\rvert + C$

Table 4.13 Integration Formulas

Evaluate each of the following indefinite integrals:

a. $\int (5x^3 - 7x^2 + 3x + 4)dx$

b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x}dx$

c. $\int \frac{4}{1 + x^2}dx$

d. $\int \tan x \cos x \, dx$

Example 4.53

Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x,\ y(0) = 5.$$

4.10 EXERCISES

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

465. $F(x) = 5x^3 + 2x^2 + 3x + 1,\ f(x) = 15x^2 + 4x + 3$

466. $F(x) = x^2 + 4x + 1,\ f(x) = 2x + 4$

467. $F(x) = x^2 e^x,\ f(x) = e^x\left(x^2 + 2x\right)$

468. $F(x) = \cos x,\ f(x) = -\sin x$

469. $F(x) = e^x,\ f(x) = e^x$

For the following exercises, find the antiderivative of the function.

470. $f(x) = \frac{1}{x^2} + x$

471. $f(x) = e^x - 3x^2 + \sin x$

472. $f(x) = e^x + 3x - x^2$

473. $f(x) = x - 1 + 4\sin(2x)$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

474. $f(x) = 5x^4 + 4x^5$

475. $f(x) = x + 12x^2$

476. $f(x) = \frac{1}{\sqrt{x}}$

477. $f(x) = (\sqrt{x})^3$

478. $f(x) = x^{1/3} + (2x)^{1/3}$

479. $f(x) = \frac{x^{1/3}}{x^{2/3}}$

480. $f(x) = 2\sin(x) + \sin(2x)$

481. $f(x) = \sec^2(x) + 1$

482. $f(x) = \sin x \cos x$

483. $f(x) = \sin^2(x)\cos(x)$

484. $f(x) = 0$

485. $f(x) = \frac{1}{2}\csc^2(x) + \frac{1}{x^2}$

486. $f(x) = \csc x \cot x + 3x$

487. $f(x) = 4\csc x \cot x - \sec x \tan x$

488. $f(x) = 8\sec x(\sec x - 4\tan x)$

489. $f(x) = \frac{1}{2}e^{-4x} + \sin x$

For the following exercises, evaluate the integral.

490. $\int(-1)dx$

491. $\int \sin x\, dx$

492. $\int(4x + \sqrt{x})dx$

493. $\int \frac{3x^2 + 2}{x^2} dx$

494. $\int(\sec x \tan x + 4x)dx$

495. $\int\left(4\sqrt{x} + \sqrt[4]{x}\right)dx$

496. $\int\left(x^{-1/3} - x^{2/3}\right)dx$

497. $\int \frac{14x^3 + 2x + 1}{x^3} dx$

498. $\int(e^x + e^{-x})dx$

For the following exercises, solve the initial value problem.

499. $f'(x) = x^{-3},\ f(1) = 1$

500. $f'(x) = \sqrt{x} + x^2,\ f(0) = 2$

501. $f'(x) = \cos x + \sec^2(x),\ f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$

502. $f'(x) = x^3 - 8x^2 + 16x + 1,\ f(0) = 0$