Math G180
Blank Lecture Notes
Chapter 2 – Section 2.5

2.5 | The Precise Definition of a Limit

Learning Objectives

2.5.1 Describe the epsilon-delta definition of a limit.
2.5.2 Apply the epsilon-delta definition to find the limit of a function.
2.5.3 Describe the epsilon-delta definitions of one-sided limits and infinite limits.
2.5.4 Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points *a* and *b* on a number line is given by $|a-b|$.

- The statement $|f(x)-L| < \varepsilon$ may be interpreted as: *The distance between $f(x)$ and L is less than ε.*
- The statement $0 < |x-a| < \delta$ may be interpreted as: $x \neq a$ *and the distance between x and a is less than δ.*

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x)-L| < \varepsilon$ is equivalent to the statement $L-\varepsilon < f(x) < L+\varepsilon$.
- The statement $0 < |x-a| < \delta$ is equivalent to the statement $a-\delta < x < a+\delta$ and $x \neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Definition

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing *a*. Let *L* be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x-a| < \delta$, then $|f(x)-L| < \varepsilon$.

Definition	Translation
1. For every $\varepsilon > 0,$	1. For every positive distance ε from *L*,
2. there exists a $\delta > 0,$	2. There is a positive distance δ from *a*,
3. such that	3. such that
4. if $0 < \lvert x - a \rvert < \delta,$ then $\lvert f(x) - L \rvert < \varepsilon.$	4. if *x* is closer than δ to *a* and $x \neq a,$ then $f(x)$ is closer than ε to *L*.

Table 2.9 Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. **Figure 2.39** shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x),$ a number a, and a limit L at a. Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a, then the value of $f(x)$ is within ε of the limit L.

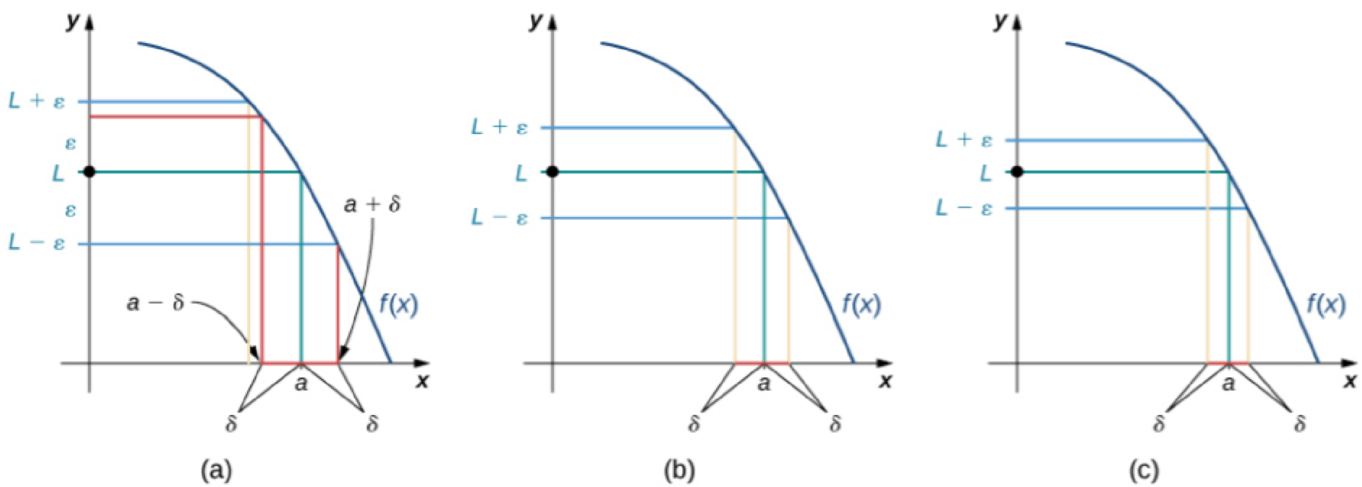


Figure 2.39 These graphs show possible values of δ, given successively smaller choices of ε.

Example 2.40

Proving a Statement about a Limit

Complete the proof that $\lim_{x \to -1} (4x + 1) = -3$ by filling in the blanks.

Let _____.

Choose $\delta =$ _______.

Assume $0 < |x -$ _______$| < \delta$.

Thus, $|$________ $-$ ________$| =$ ______________________________________ε.

Solution

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose $\delta =$ _______.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta$_______$\varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

Section 2.5 Exercises

In the following exercises, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

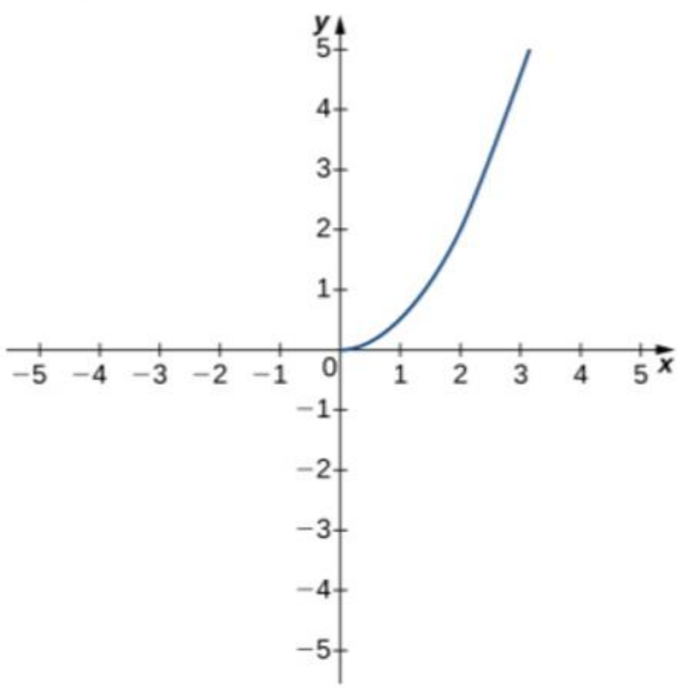
176. $\lim\limits_{x \to a} f(x) = N$

177. $\lim\limits_{t \to b} g(t) = M$

178. $\lim\limits_{x \to c} h(x) = L$

179. $\lim\limits_{x \to a} \varphi(x) = A$

The following graph of the function f satisfies $\lim\limits_{x \to 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



180. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

181. If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.