Math G180
Blank Lecture Notes
Chapter 3 – Section 3.9

3.9 | Derivatives of Exponential and Logarithmic Functions

Theorem 3.16: Derivatives of General Exponential and Logarithmic Functions

Let $b > 0,\ b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If, $y = \log_b x,$ then

$$\frac{dy}{dx} = \frac{1}{x \ln b}. \tag{3.32}$$

More generally, if $h(x) = \log_b(g(x)),$ then for all values of x for which $g(x) > 0,$

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{3.33}$$

ii. If $y = b^x,$ then

$$\frac{dy}{dx} = b^x \ln b. \tag{3.34}$$

More generally, if $h(x) = b^{g(x)},$ then

$$h'(x) = b^{g(x)} g'(x) \ln b. \tag{3.35}$$

Example 3.79

Applying Derivative Formulas

Find the derivative of $h(x) = \dfrac{3^x}{3^x + 2}.$

Example 3.80

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 3.81

Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4 + 1\right)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$$\begin{array}{rcll}
\ln y & = & \ln\left(2x^4+1\right)^{\tan x} & \text{Step 1. Take the natural logarithm of both sides.} \\
\ln y & = & \tan x \ln\left(2x^4+1\right) & \text{Step 2. Expand using properties of logarithms.} \\
\frac{1}{y}\frac{dy}{dx} & = & \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x & \text{Step 3. Differentiate both sides. Use the product rule on the right.} \\
\frac{dy}{dx} & = & y\cdot\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x\right) & \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} & = & \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1}\cdot \tan x\right) & \text{Step 5. Substitute } y = \left(2x^4+1\right)^{\tan x}.
\end{array}$$

Example 3.82

Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$$\begin{array}{rcll}
\ln y & = & \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x} & \text{Step 1. Take the natural logarithm of both sides.} \\
\ln y & = & \ln x + \frac{1}{2}\ln(2x+1) - x\ln e - 3\ln \sin x & \text{Step 2. Expand using properties of logarithms.} \\
\frac{1}{y}\frac{dy}{dx} & = & \frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos x}{\sin x} & \text{Step 3. Differentiate both sides.} \\
\frac{dy}{dx} & = & y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) & \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} & = & \frac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\cot x\right) & \text{Step 5. Substitute } y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}.
\end{array}$$

3.9 EXERCISES

For the following exercises, find $f'(x)$ for each function.

331. $f(x) = x^2 e^x$

332. $f(x) = \frac{e^{-x}}{x}$

333. $f(x) = e^{x^3 \ln x}$

334. $f(x) = \sqrt{e^{2x} + 2x}$

335. $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

336. $f(x) = \frac{10^x}{\ln 10}$

337. $f(x) = 2^{4x} + 4x^2$

338. $f(x) = 3^{\sin 3x}$

339. $f(x) = x^{\pi} \cdot \pi^x$

340. $f(x) = \ln\left(4x^3 + x\right)$

341. $f(x) = \ln\sqrt{5x - 7}$

342. $f(x) = x^2 \ln 9x$

343. $f(x) = \log(\sec x)$

344. $f(x) = \log_7\left(6x^4 + 3\right)^5$

345. $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

346. $y = x^{\sqrt{x}}$

347. $y = (\sin 2x)^{4x}$

348. $y = (\ln x)^{\ln x}$

349. $y = x^{\log_2 x}$

350. $y = \left(x^2 - 1\right)^{\ln x}$

351. $y = x^{\cot x}$

352. $y = \frac{x + 11}{\sqrt[3]{x^2 - 4}}$

353. $y = x^{-1/2}\left(x^2 + 3\right)^{2/3}(3x - 4)^4$

354. **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{\left(x^2 - 1\right)}$ at the point where $x = -1$. Graph both the function and the tangent line.

355. **[T]** Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

356. **[T]** Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

357. Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.
b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.