Math G180
Blank Lecture Notes
Chapter 3 – Section 3.1-3.3

3.1 Defining the Derivative

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in Figure 3.2. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.

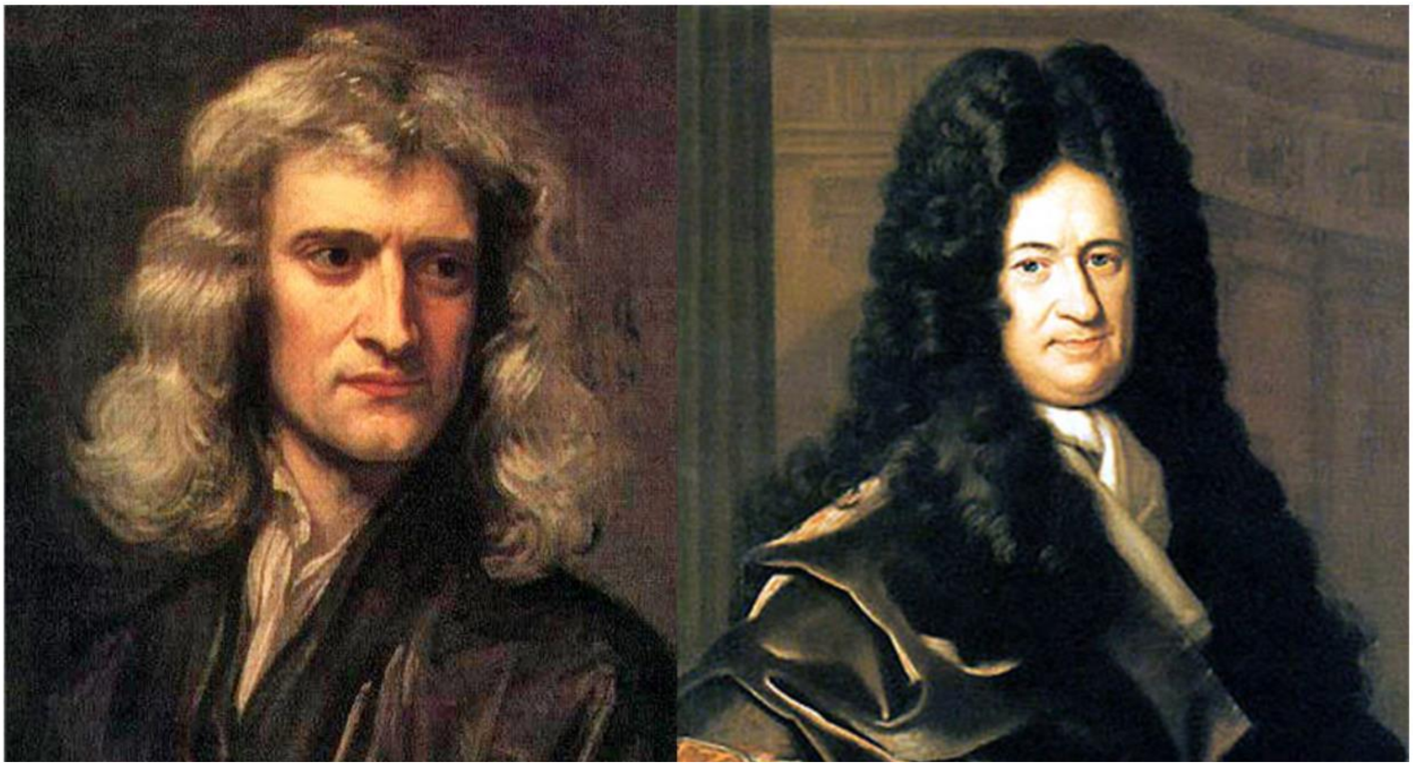

Figure 3.2 Newton and Leibniz are credited with developing calculus independently.

3.2 The Derivative as a Function

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

DEFINITION

Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{3.9}$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

3.3 Differentiation Rules

Selected Differentiation Formulas

Definition of the Derivative	$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$		
Constant Rule	$(c)' = 0$, c is a constant		
Derivative of x	$(x)' = 1$		
Chain Rule	$(u)' = 1 \cdot u'$	or	$(u)' = 1 \cdot \frac{du}{dx}$
Power Rule	$(x^n)' = n \cdot x^{n-1}$		
	$(u^n)' = n \cdot u^{n-1} \cdot u'$	or	$(u^n)' = n \cdot u^{n-1} \cdot \frac{du}{dx}$
Product Rule	$(u \cdot v)' = u' \cdot v + v' \cdot u$	or	$\frac{d}{dx}(u \cdot v) = v \cdot \frac{du}{dx} + u \cdot \frac{dv}{dx}$
Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - v' \cdot u}{v^2}$	or	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$
Exponential Functions	$(e^x)' = e^x$		
	$(e^u)' = e^u \cdot u'$	or	$\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$
Natural Logarithm Functions	$(\ln x)' = \frac{1}{x}$		
	$(\ln u)' = \frac{1}{u} \cdot u'$	or	$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$
Trigonometric Functions	$(\sin x)' = \cos x$		
	$(\sin u)' = \cos u \cdot u'$	or	$\frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$
	$(\cos x)' = -\sin x$		
	$(\cos u)' = -\sin u \cdot u'$	or	$\frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

$$f''(x),\ f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x)$$

$$y''(x), y'''(x), y^{(4)}(x), \ldots, y^{(n)}(x)$$

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.