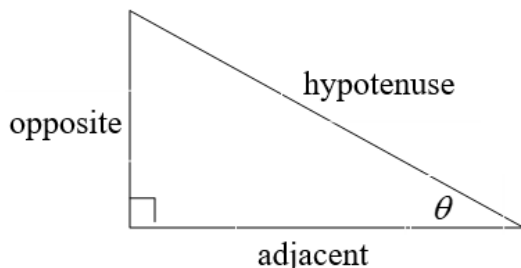


Definition of the Trig Functions

Right triangle definition

For this definition we assume that

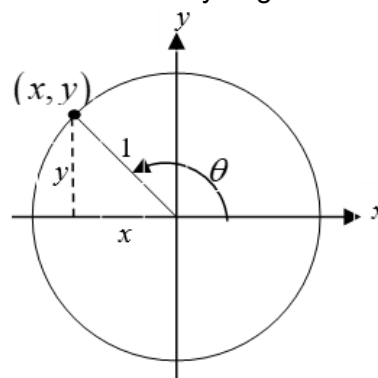
$$0 < \theta < \frac{\pi}{2} \text{ or } 0^\circ < \theta < 90^\circ.$$



$$\begin{aligned} \sin(\theta) &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc(\theta) &= \frac{\text{hypotenuse}}{\text{opposite}} \\ \cos(\theta) &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec(\theta) &= \frac{\text{hypotenuse}}{\text{adjacent}} \\ \tan(\theta) &= \frac{\text{opposite}}{\text{adjacent}} & \cot(\theta) &= \frac{\text{adjacent}}{\text{opposite}} \end{aligned}$$

Unit Circle Definition

For this definition θ is any angle.



$$\begin{aligned} \sin(\theta) &= \frac{y}{1} = y & \csc(\theta) &= \frac{1}{y} \\ \cos(\theta) &= \frac{x}{1} = x & \sec(\theta) &= \frac{1}{x} \\ \tan(\theta) &= \frac{y}{x} & \cot(\theta) &= \frac{x}{y} \end{aligned}$$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin(\theta)$, θ can be any angle

$\cos(\theta)$, θ can be any angle

$\tan(\theta)$, $\theta \neq \left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm 1, \pm 2, \dots$

$\csc(\theta)$, $\theta \neq n\pi$, $n = 0, \pm 1, \pm 2, \dots$

$\sec(\theta)$, $\theta \neq \left(n + \frac{1}{2}\right)\pi$, $n = 0, \pm 1, \pm 2, \dots$

$\cot(\theta)$, $\theta \neq n\pi$, $n = 0, \pm 1, \pm 2, \dots$

Period

The period of a function is the number, T , such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\begin{aligned} \sin(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \cos(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \tan(\omega\theta) &\rightarrow T = \frac{\pi}{\omega} \\ \csc(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \sec(\omega\theta) &\rightarrow T = \frac{2\pi}{\omega} \\ \cot(\omega\theta) &\rightarrow T = \frac{\pi}{\omega} \end{aligned}$$

Range

The range is all possible values to get out of the function.

$$\begin{aligned} -1 &\leq \sin(\theta) \leq 1 & -1 &\leq \cos(\theta) \leq 1 \\ -\infty &< \tan(\theta) < \infty & -\infty &< \cot(\theta) < \infty \\ \sec(\theta) &\geq 1 \text{ and } \sec(\theta) \leq -1 & \csc(\theta) &\geq 1 \text{ and } \csc(\theta) \leq -1 \end{aligned}$$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Reciprocal Identities

$$\begin{aligned} \csc(\theta) &= \frac{1}{\sin(\theta)} & \sin(\theta) &= \frac{1}{\csc(\theta)} \\ \sec(\theta) &= \frac{1}{\cos(\theta)} & \cos(\theta) &= \frac{1}{\sec(\theta)} \\ \cot(\theta) &= \frac{1}{\tan(\theta)} & \tan(\theta) &= \frac{1}{\cot(\theta)} \end{aligned}$$

Pythagorean Identities

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \tan^2(\theta) + 1 &= \sec^2(\theta) \\ 1 + \cot^2(\theta) &= \csc^2(\theta) \end{aligned}$$

Even/Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin(\theta) & \csc(-\theta) &= -\csc(\theta) \\ \cos(-\theta) &= \cos(\theta) & \sec(-\theta) &= \sec(\theta) \\ \tan(-\theta) &= -\tan(\theta) & \cot(-\theta) &= -\cot(\theta) \end{aligned}$$

Periodic Formulas

If n is an integer then,

$$\begin{aligned} \sin(\theta + 2\pi n) &= \sin(\theta) & \csc(\theta + 2\pi n) &= \csc(\theta) \\ \cos(\theta + 2\pi n) &= \cos(\theta) & \sec(\theta + 2\pi n) &= \sec(\theta) \\ \tan(\theta + \pi n) &= \tan(\theta) & \cot(\theta + \pi n) &= \cot(\theta) \end{aligned}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Double Angle Formulas

$$\begin{aligned} \sin(2\theta) &= 2 \sin(\theta) \cos(\theta) \\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2 \cos^2(\theta) - 1 \\ &= 1 - 2 \sin^2(\theta) \\ \tan(2\theta) &= \frac{2 \tan(\theta)}{1 - \tan^2(\theta)} \end{aligned}$$

Half Angle Formulas

$$\begin{aligned} \sin\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 - \cos(\theta)}{2}} \\ \cos\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 + \cos(\theta)}{2}} \\ \tan\left(\frac{\theta}{2}\right) &= \pm \sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}} \end{aligned}$$

Half Angle Formulas (alternate form)

$$\begin{aligned} \sin^2(\theta) &= \frac{1}{2} (1 - \cos(2\theta)) & \tan^2(\theta) &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \\ \cos^2(\theta) &= \frac{1}{2} (1 + \cos(2\theta)) \end{aligned}$$

Sum and Difference Formulas

$$\begin{aligned} \sin(\alpha \pm \beta) &= \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta) \\ \cos(\alpha \pm \beta) &= \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) \\ \tan(\alpha \pm \beta) &= \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \tan(\beta)} \end{aligned}$$

Product to Sum Formulas

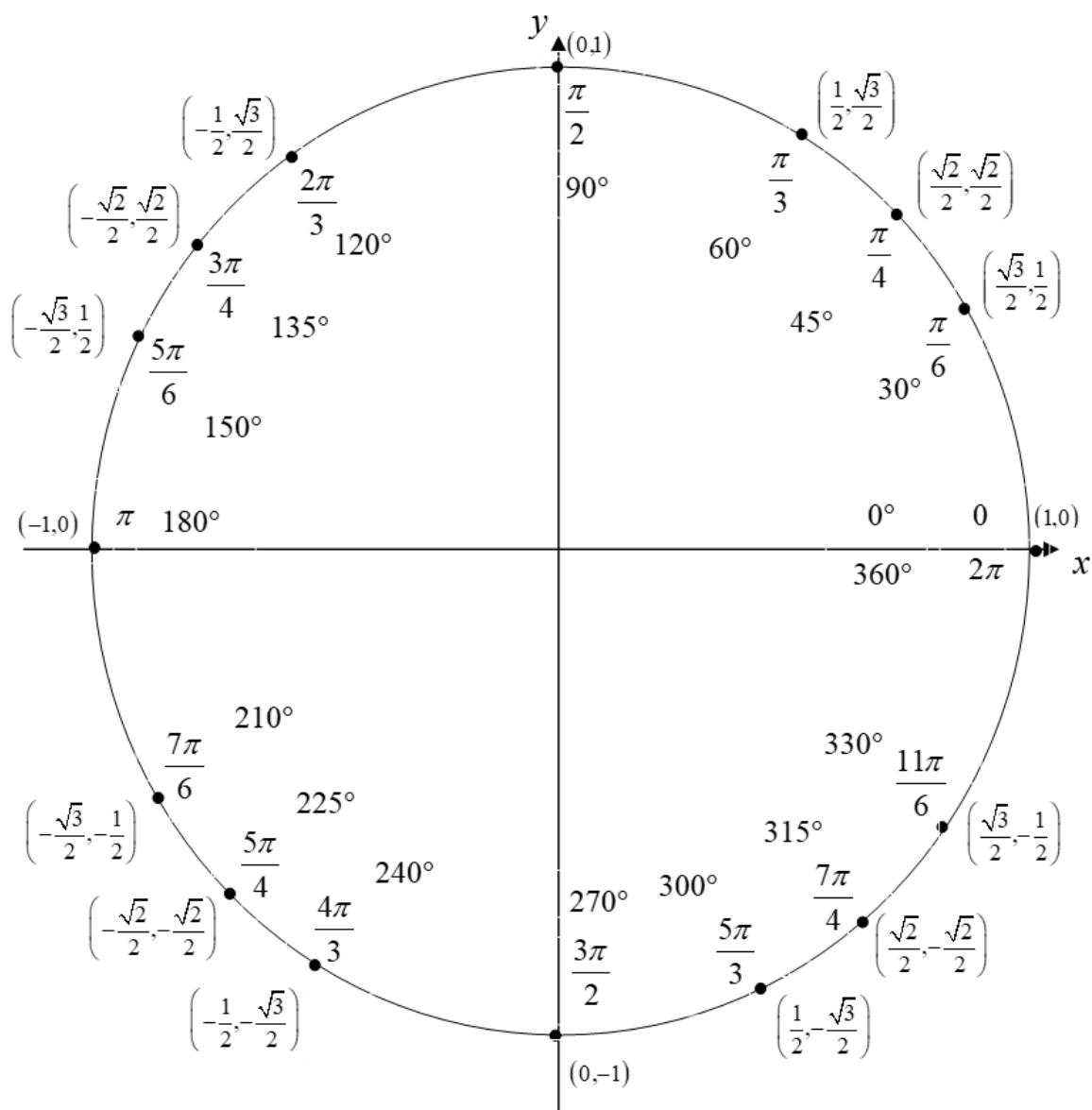
$$\begin{aligned} \sin(\alpha) \sin(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\ \cos(\alpha) \cos(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)] \\ \sin(\alpha) \cos(\beta) &= \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)] \\ \cos(\alpha) \sin(\beta) &= \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)] \end{aligned}$$

Sum to Product Formulas

$$\begin{aligned} \sin(\alpha) + \sin(\beta) &= 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \\ \sin(\alpha) - \sin(\beta) &= 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \\ \cos(\alpha) + \cos(\beta) &= 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \\ \cos(\alpha) - \cos(\beta) &= -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \end{aligned}$$

Cofunction Formulas

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos(\theta) & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin(\theta) \\ \csc\left(\frac{\pi}{2} - \theta\right) &= \sec(\theta) & \sec\left(\frac{\pi}{2} - \theta\right) &= \csc(\theta) \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot(\theta) & \cot\left(\frac{\pi}{2} - \theta\right) &= \tan(\theta) \end{aligned}$$



For any ordered pair on the unit circle (x, y) : $\cos(\theta) = x$ and $\sin(\theta) = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \quad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Inverse Trig Functions**Definition**

$y = \sin^{-1}(x)$ is equivalent to $x = \sin(y)$

$y = \cos^{-1}(x)$ is equivalent to $x = \cos(y)$

$y = \tan^{-1}(x)$ is equivalent to $x = \tan(y)$

Inverse Properties

$$\cos(\cos^{-1}(x)) = x \quad \cos^{-1}(\cos(\theta)) = \theta$$

$$\sin(\sin^{-1}(x)) = x \quad \sin^{-1}(\sin(\theta)) = \theta$$

$$\tan(\tan^{-1}(x)) = x \quad \tan^{-1}(\tan(\theta)) = \theta$$

Domain and Range

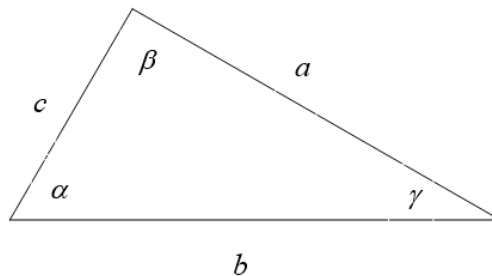
| Function | Domain | Range |
|--------------------|------------------------|--|
| $y = \sin^{-1}(x)$ | $-1 \leq x \leq 1$ | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ |
| $y = \cos^{-1}(x)$ | $-1 \leq x \leq 1$ | $0 \leq y \leq \pi$ |
| $y = \tan^{-1}(x)$ | $-\infty < x < \infty$ | $-\frac{\pi}{2} < y < \frac{\pi}{2}$ |

Alternate Notation

$$\sin^{-1}(x) = \arcsin(x)$$

$$\cos^{-1}(x) = \arccos(x)$$

$$\tan^{-1}(x) = \arctan(x)$$

Law of Sines, Cosines and Tangents**Law of Sines**

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos(\alpha)$$

$$b^2 = a^2 + c^2 - 2ac \cos(\beta)$$

$$c^2 = a^2 + b^2 - 2ab \cos(\gamma)$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{1}{2}(\alpha-\beta)\right)}{\tan\left(\frac{1}{2}(\alpha+\beta)\right)}$$

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{1}{2}(\beta-\gamma)\right)}{\tan\left(\frac{1}{2}(\beta+\gamma)\right)}$$

$$\frac{a-c}{a+c} = \frac{\tan\left(\frac{1}{2}(\alpha-\gamma)\right)}{\tan\left(\frac{1}{2}(\alpha+\gamma)\right)}$$

Mollweide's Formula

$$\frac{a+b}{c} = \frac{\cos\left(\frac{1}{2}(\alpha-\beta)\right)}{\sin\left(\frac{1}{2}\gamma\right)}$$