**Math G180**
**Blank Lecture Notes**
**Chapter 5 – Sections 5.3 and 5.4**

# 5.3 | The Fundamental Theorem of Calculus

## Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1**, is stated here. Part 1 establishes the relationship between differentiation and integration.

### Theorem 5.4: Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt, \tag{5.16}$$

then $F'(x) = f(x)$ over $[a, b]$.

## Example 5.17

### Finding a Derivative with the Fundamental Theorem of Calculus

Use the **Fundamental Theorem of Calculus, Part 1** to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt.$$

### Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3 + 1}.$$

## Example 5.18

### Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t\,dt$. Find $F'(x)$.

### Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t\,dt$. Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$\begin{aligned} F'(x) &= \sin(u(x))\frac{du}{dx} \\ &= \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{\sin\sqrt{x}}{2\sqrt{x}}. \end{aligned}$$

## Example 5.19

### Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3\,dt$. Find $F'(x)$.

### Solution

We have $F(x) = \int_x^{2x} t^3\,dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3\,dt \\ &= \int_x^0 t^3\,dt + \int_0^{2x} t^3\,dt \\ &= -\int_0^x t^3\,dt + \int_0^{2x} t^3\,dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3\,dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned}\frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3\,dt\right] \\ &= (u(x))^3\frac{du}{dx} \\ &= (2x)^3\cdot 2 \\ &= 16x^3.\end{aligned}$$

Thus,

$$\begin{aligned}F'(x) &= \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right] \\ &= -x^3 + 16x^3 \\ &= 15x^3.\end{aligned}$$

## Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of $n$ rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

### Theorem 5.5: The Fundamental Theorem of Calculus, Part 2

If $f$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a). \tag{5.17}$$

## 5.3 EXERCISES

144. Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

145. Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

146. To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

147. Set $F(x) = \int_1^x (1-t)dt$. Find $F'(2)$ and the average value of $F'$ over $[1, 2]$.

In the following exercises, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

148. $\frac{d}{dx}\int_1^x e^{-t^2}\,dt$

149. $\frac{d}{dx}\int_1^x e^{\cos t}\,dt$

150. $\frac{d}{dx}\int_3^x \sqrt{9-y^2}\,dy$

151. $\frac{d}{dx}\int_4^x \frac{ds}{\sqrt{16-s^2}}$

152. $\frac{d}{dx}\int_x^{2x} t\,dt$

153. $\frac{d}{dx}\int_0^{\sqrt{x}} t\,dt$

177. $\int_{1/4}^4 \left(x^2 - \frac{1}{x^2}\right)dx$

178. $\int_1^2 \frac{2}{x^3}dx$

179. $\int_1^4 \frac{1}{2\sqrt{x}}dx$

180. $\int_1^4 \frac{2-\sqrt{t}}{t^2}dt$

181. $\int_1^{16} \frac{dt}{t^{1/4}}$

182. $\int_0^{2\pi} \cos\theta\, d\theta$

183. $\int_0^{\pi/2} \sin\theta\, d\theta$

# 5.4 | Integration Formulas and the Net Change Theorem

Displacement $\quad$ Velocity $\quad$ Acceleration

$s(t)$ $\quad$ $s'(t) = v(t)$ $\quad$ $v'(t) = a(t)$

$s(t) = \int v(t)\,dt$ $\quad$ $v(t) = \int a(t)\,dt$

# 5.4 EXERCISES

Use basic integration formulas to compute the following antiderivatives or definite integrals.

207. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

208. $\int \left(e^{2x} - \frac{1}{2}e^{x/2}\right) dx$

209. $\int \frac{dx}{2x}$

210. $\int \frac{x-1}{x^2} dx$

211. $\int_0^{\pi} (\sin x - \cos x) dx$

220. Write an integral that quantifies the increase in the volume of a cube when the side length doubles from $s$ unit to $2s$ units and evaluate the integral.

221. Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from $R$ unit to $2R$ units and evaluate the integral.

222. Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from $R$ unit to $2R$ units and evaluate the integral.

223. Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \le t \le 2$ (in meters per second). Find the displacement at time $t$ and the total distance traveled up to $t = 2$.

224. Suppose that a particle moves along a straight line