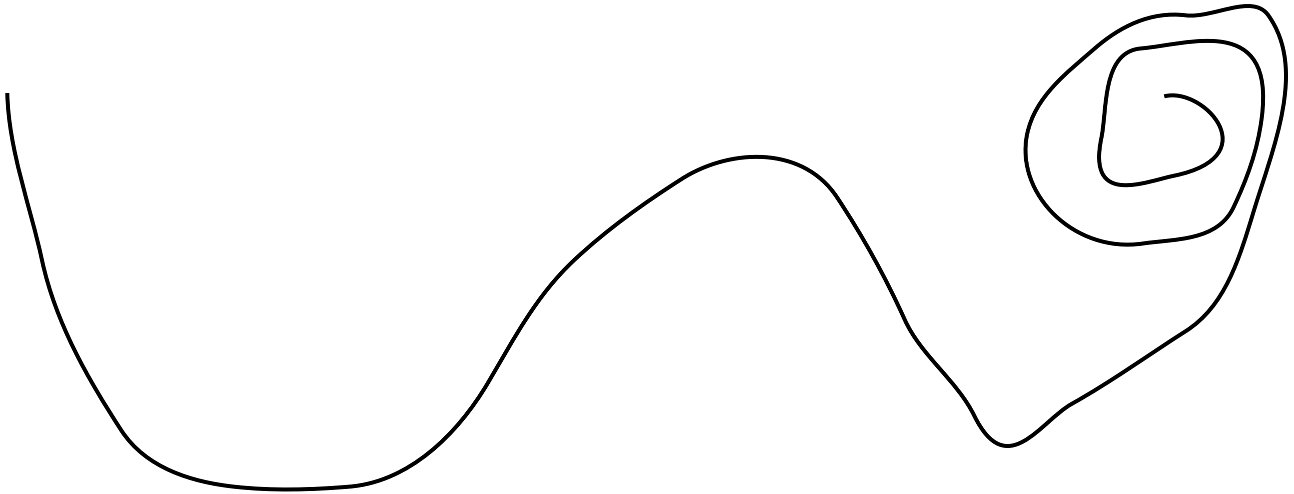
Integration

Thomas Torsney-Weir

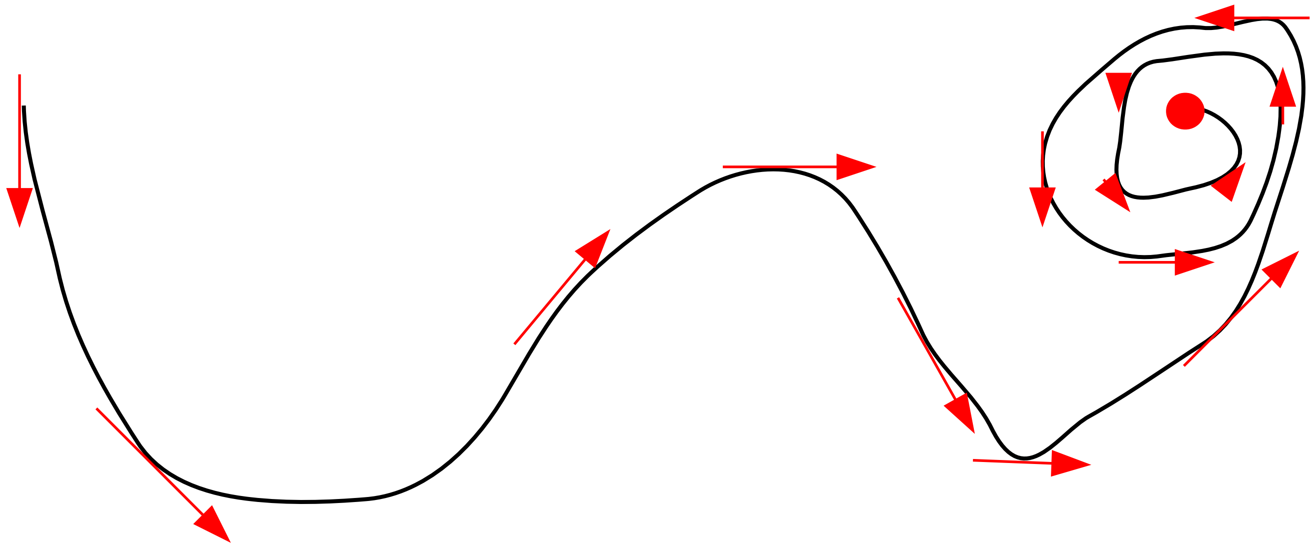
Streamline

Streamline: a curve that is everywhere tangent to the flow



Streamline

Streamline: a curve that is everywhere tangent to the flow



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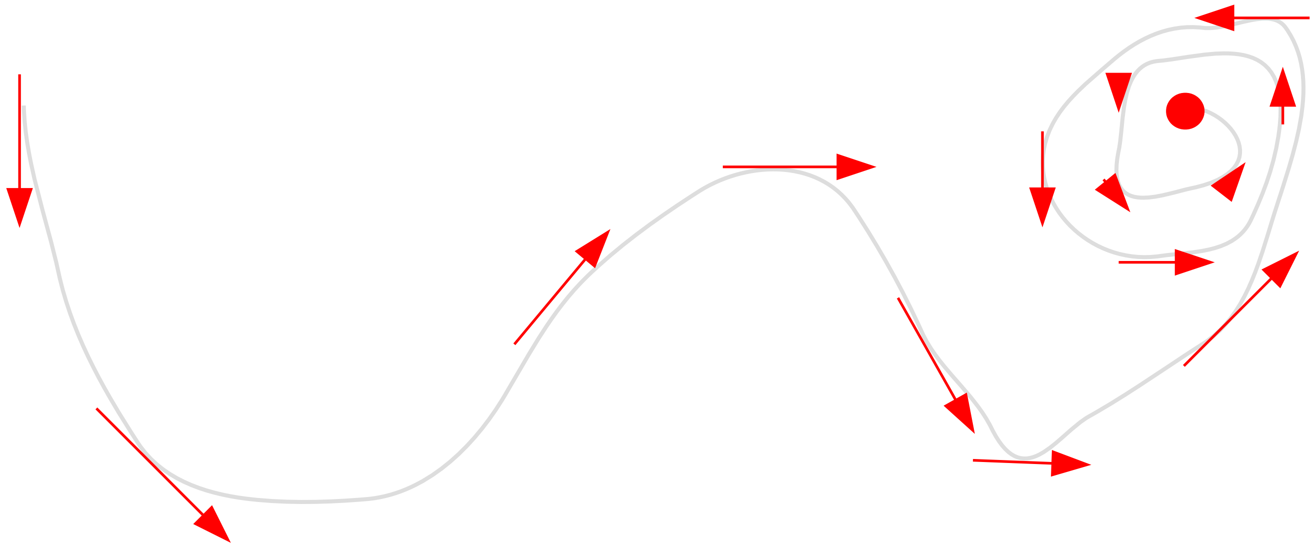

Streamline creation

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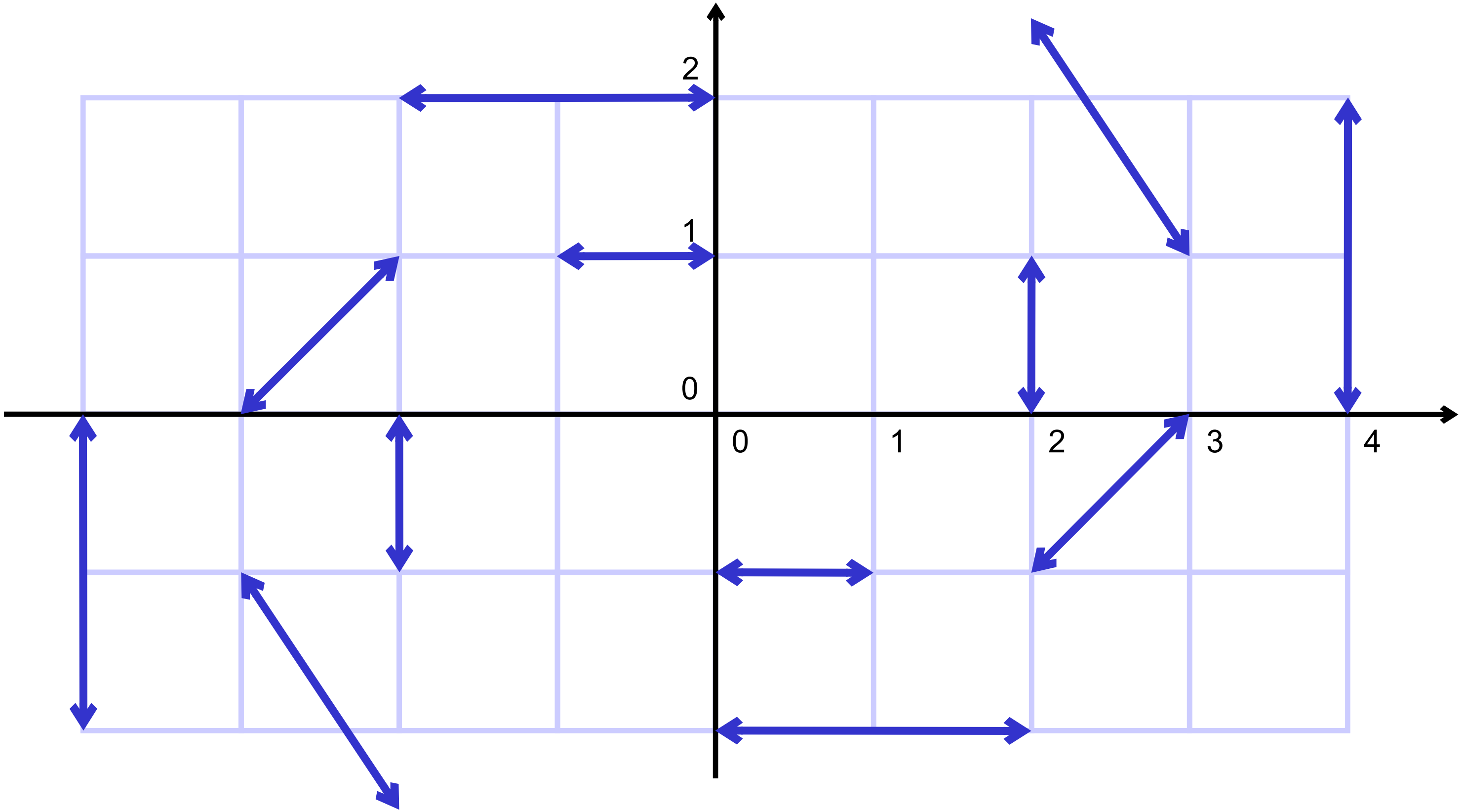

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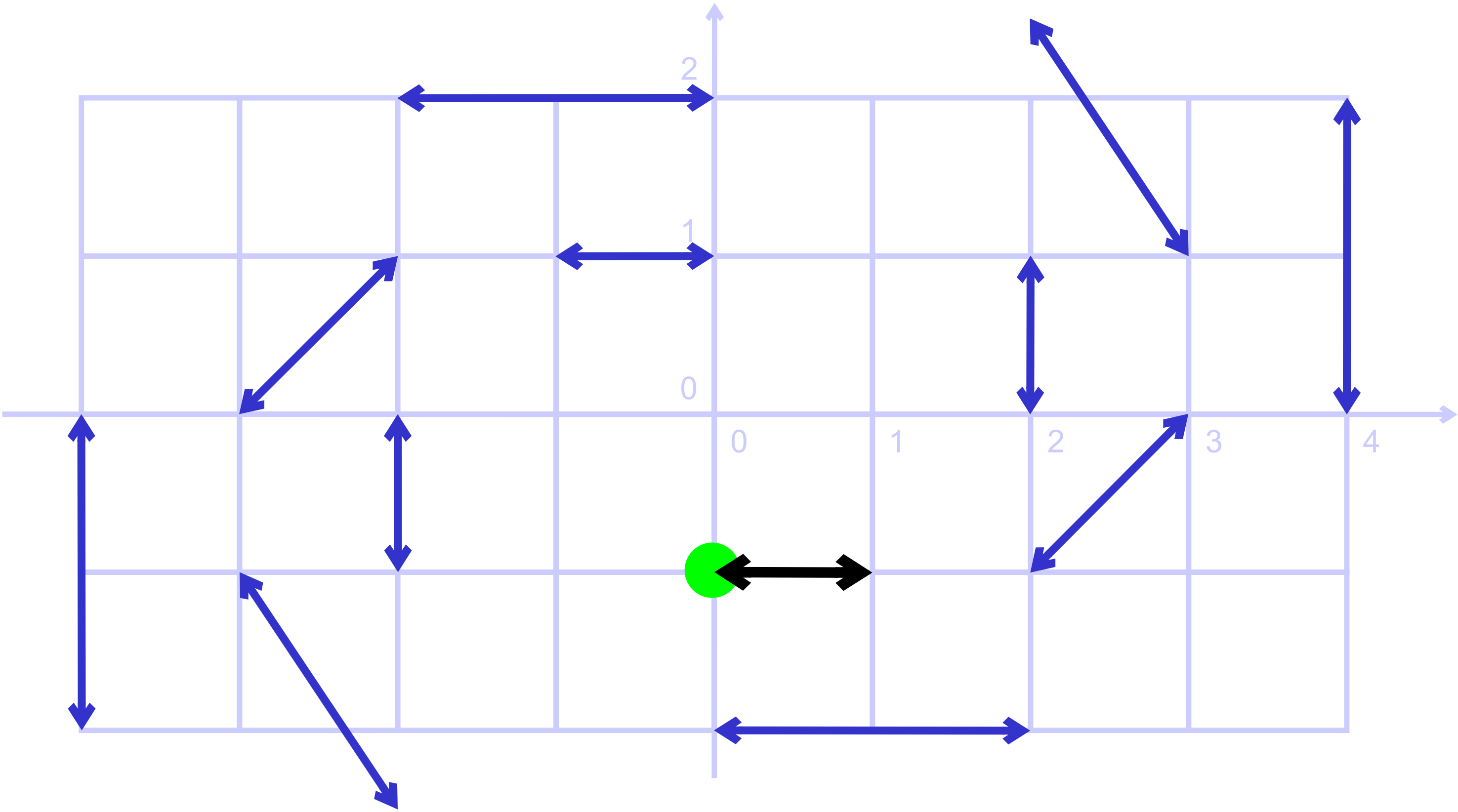
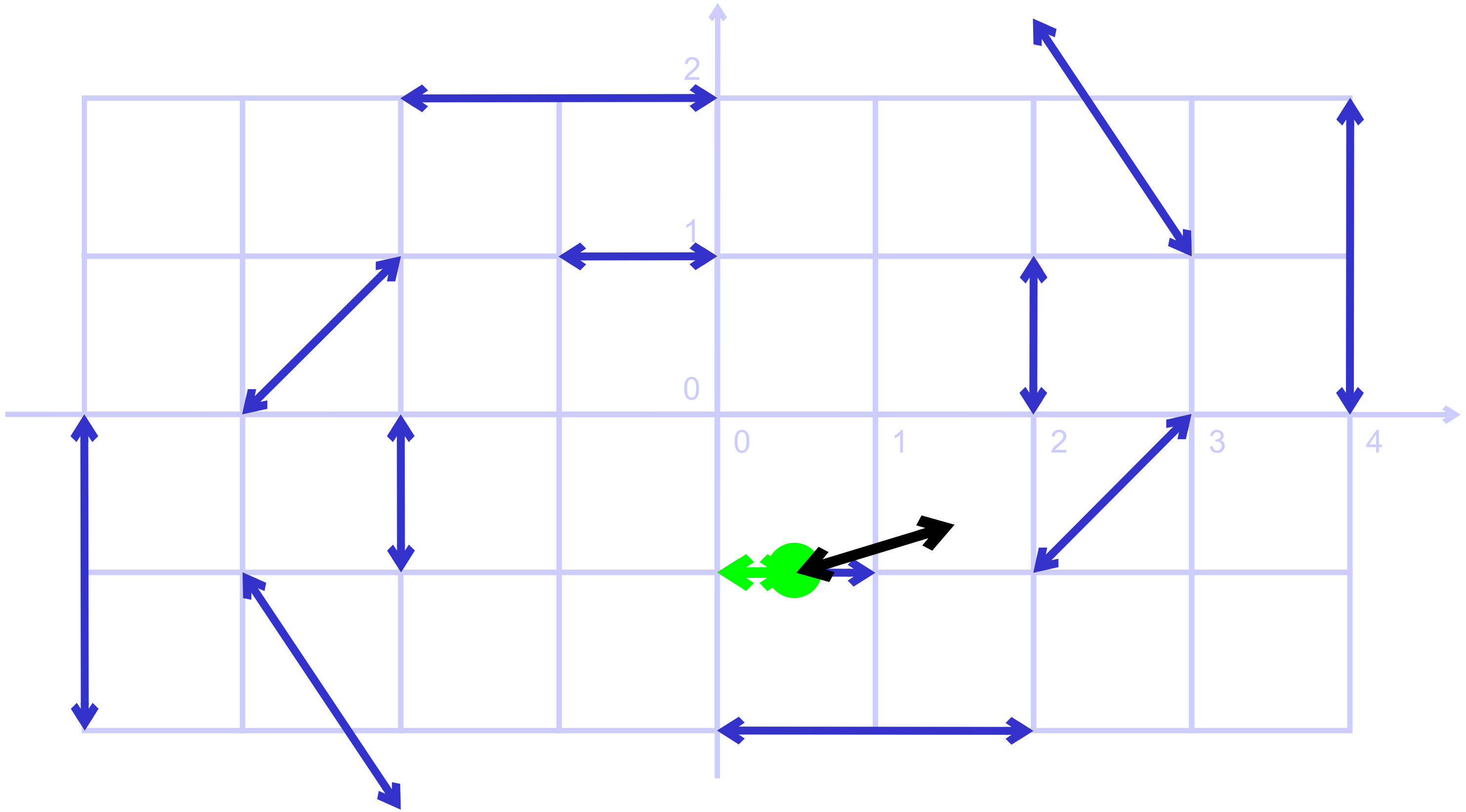
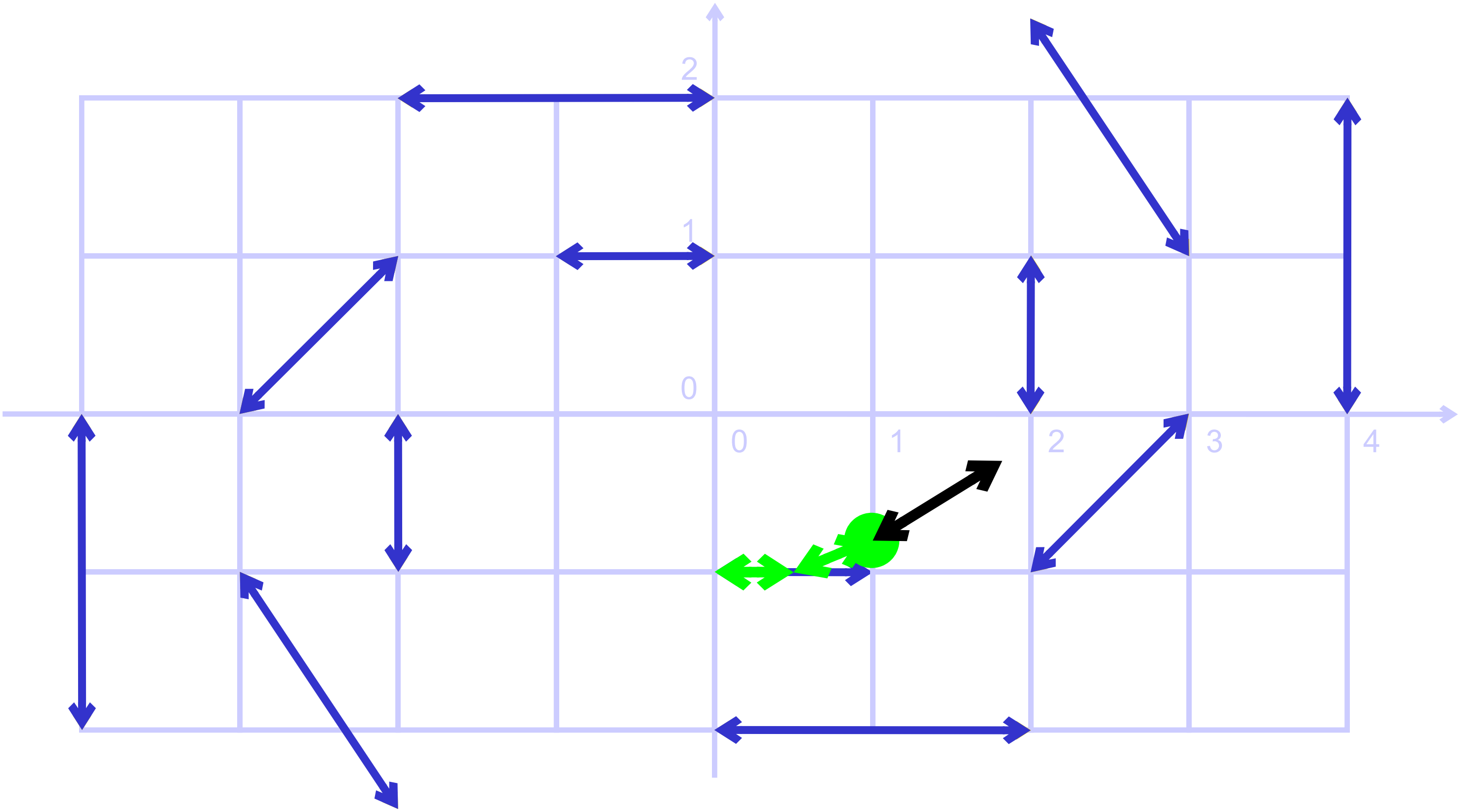


Streamline creation



Streamline creation



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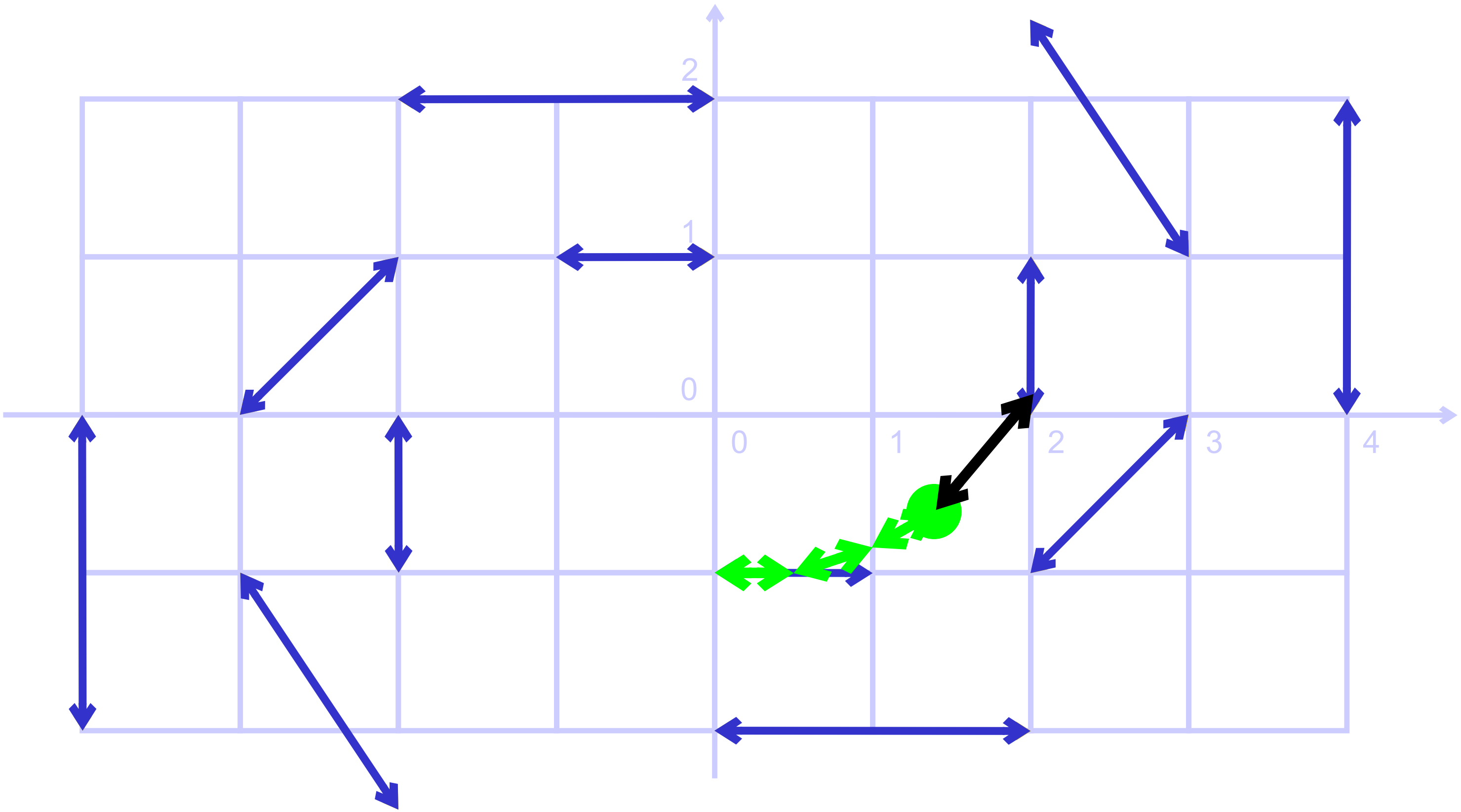

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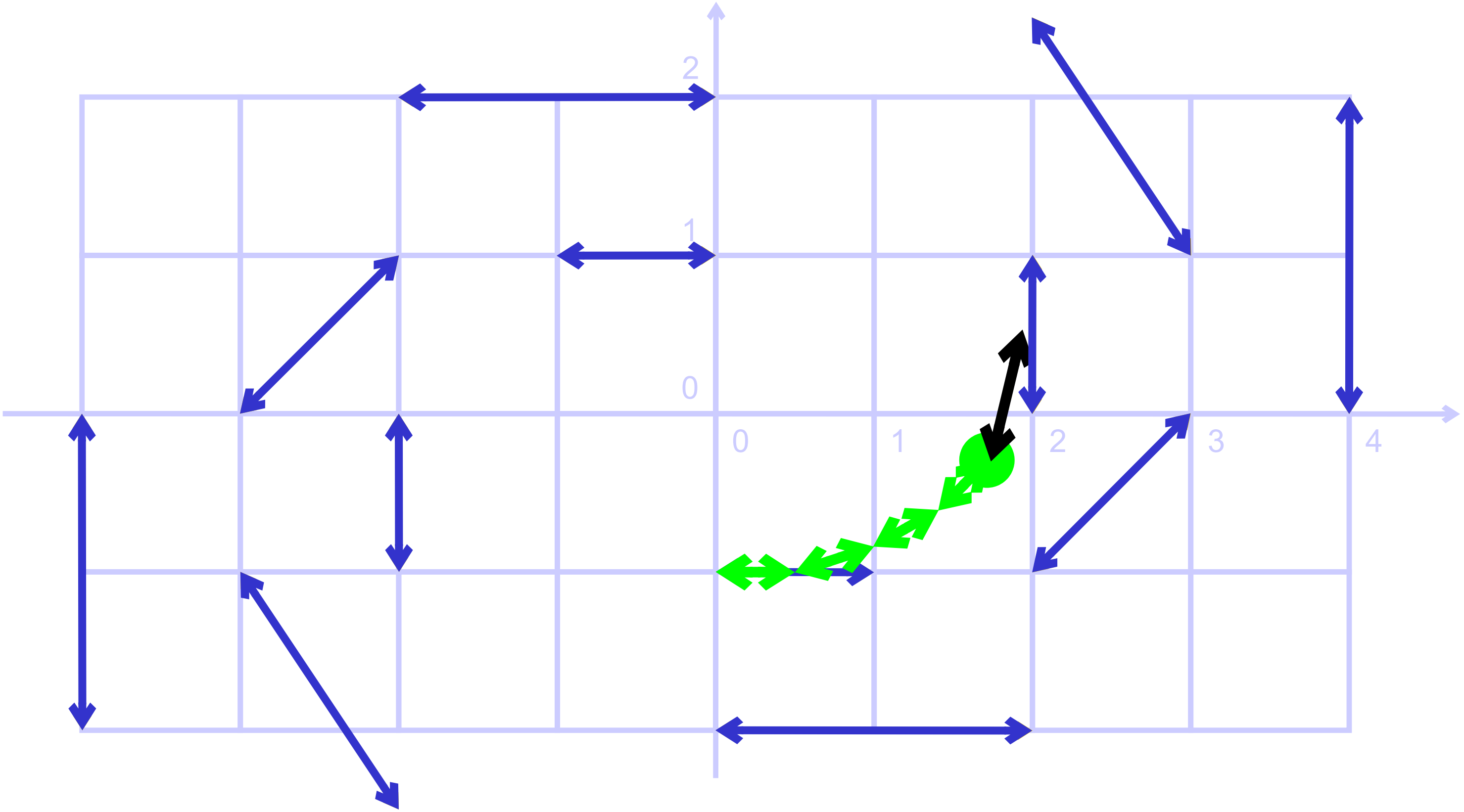

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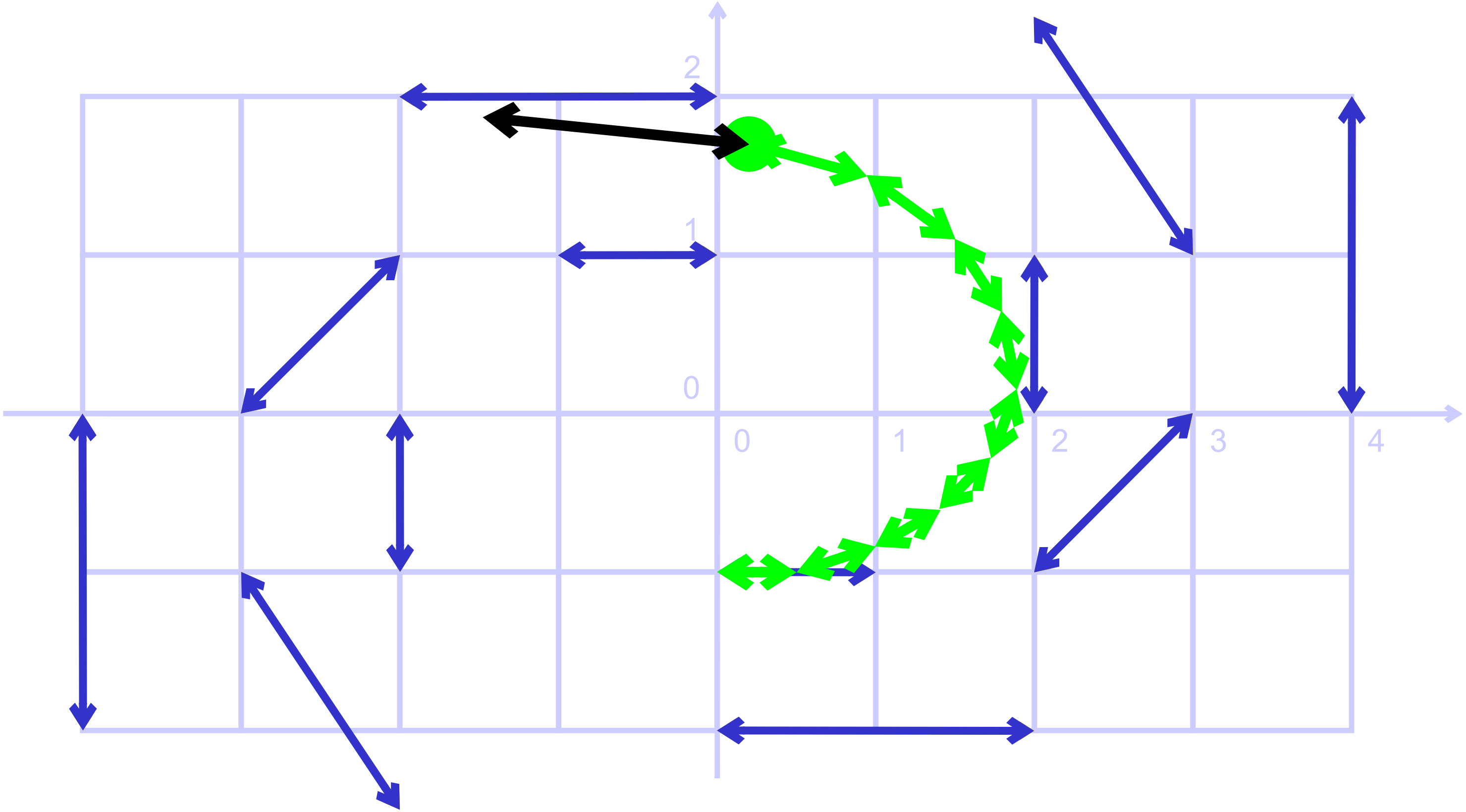
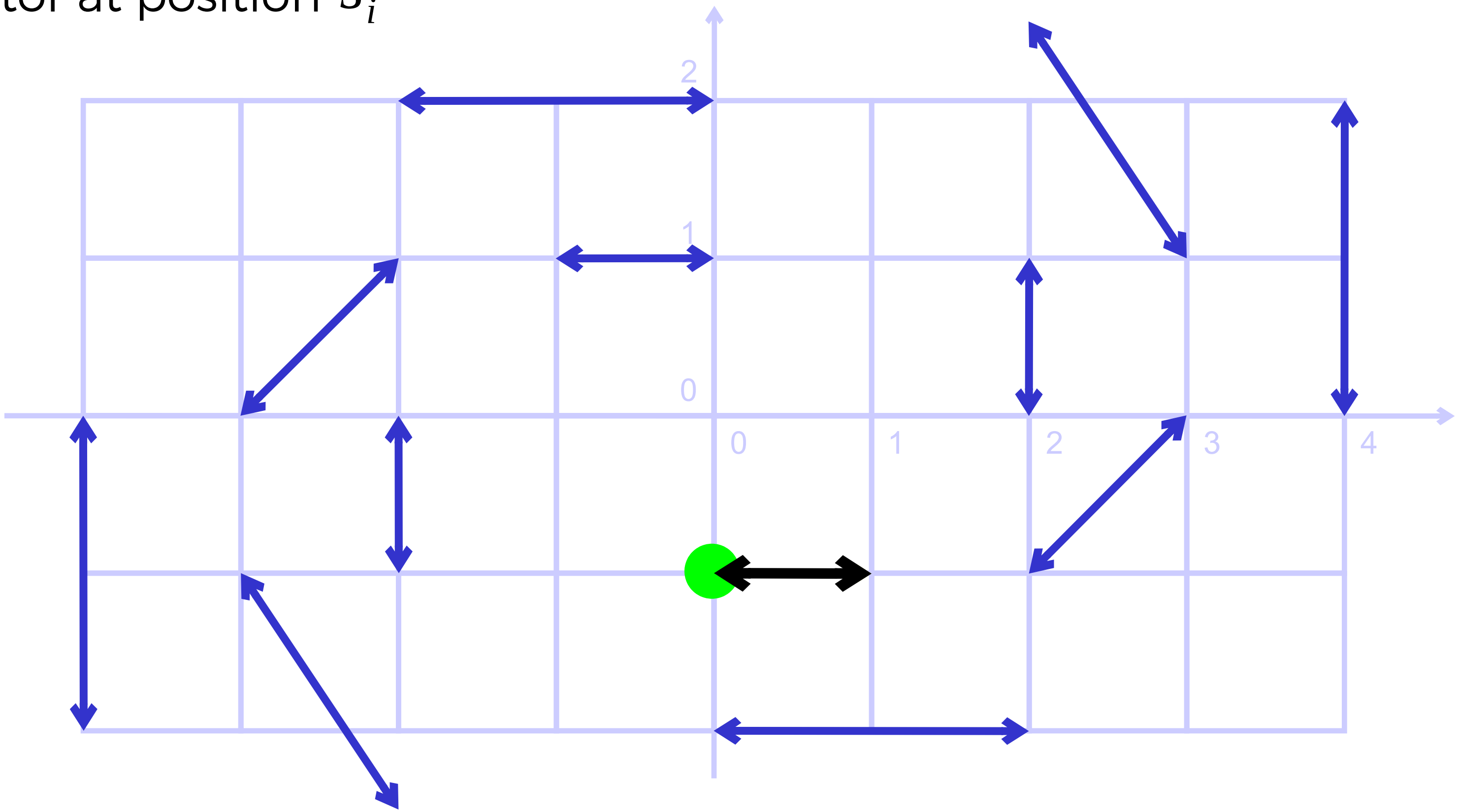


Mathematics

s_i Position at time step i

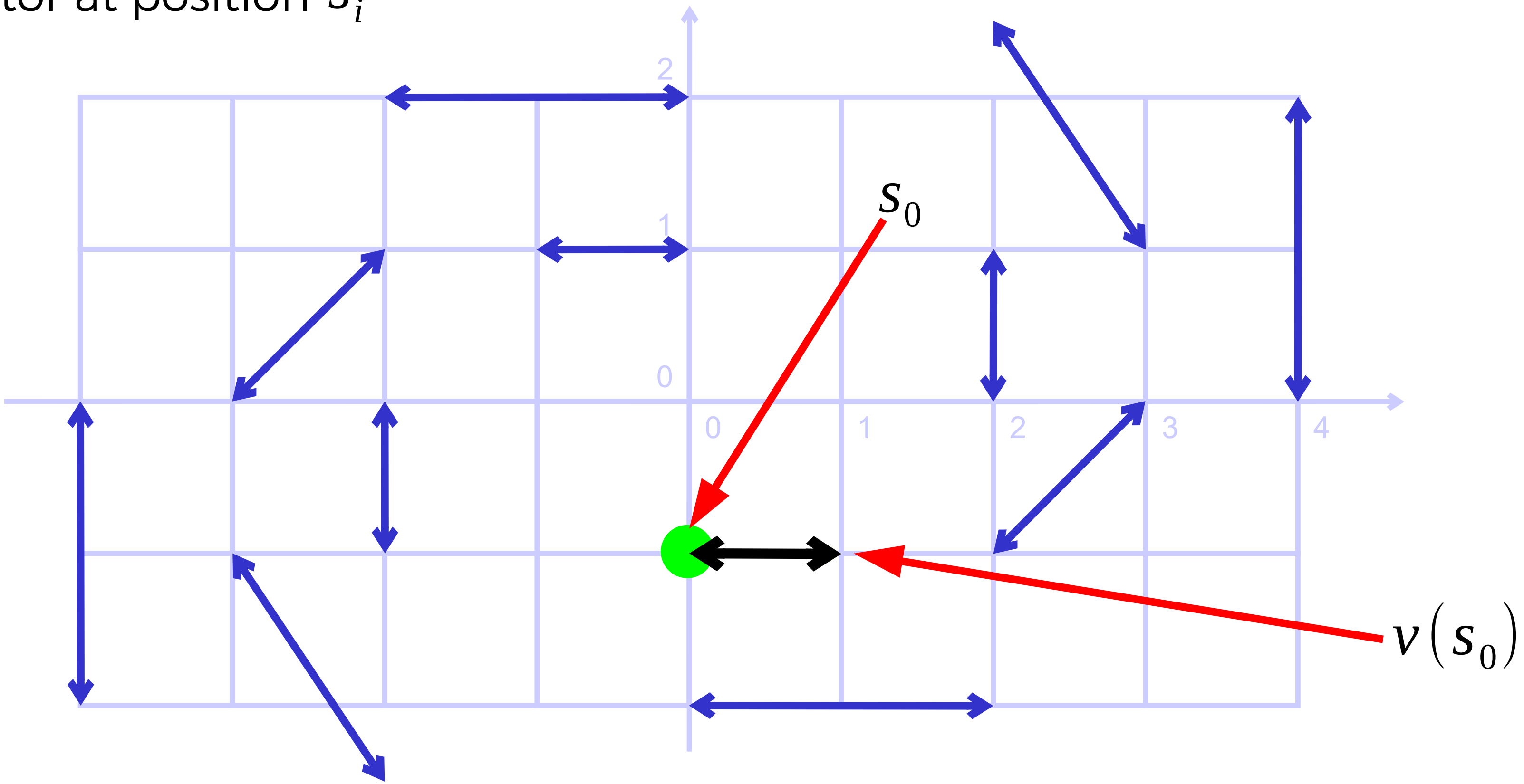
$v(s_i)$ Vector at position s_i



Position at time 0

s_i Position at time step i

$v(s_i)$ Vector at position s_i

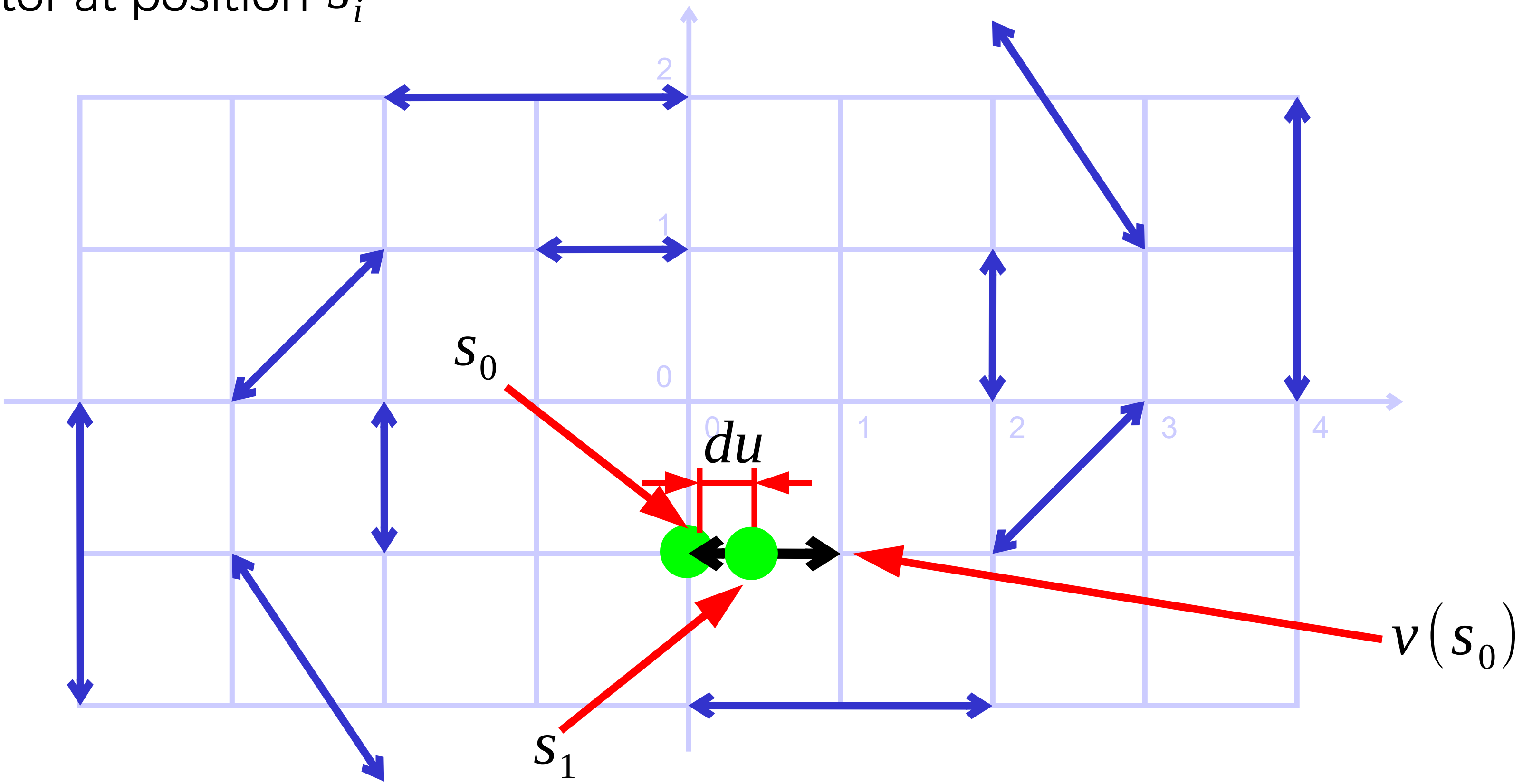


Position at time 1

s_i Position at time step i

$v(s_i)$ Vector at position s_i

$$s_1 = s_0 + v(s_0)\, du$$



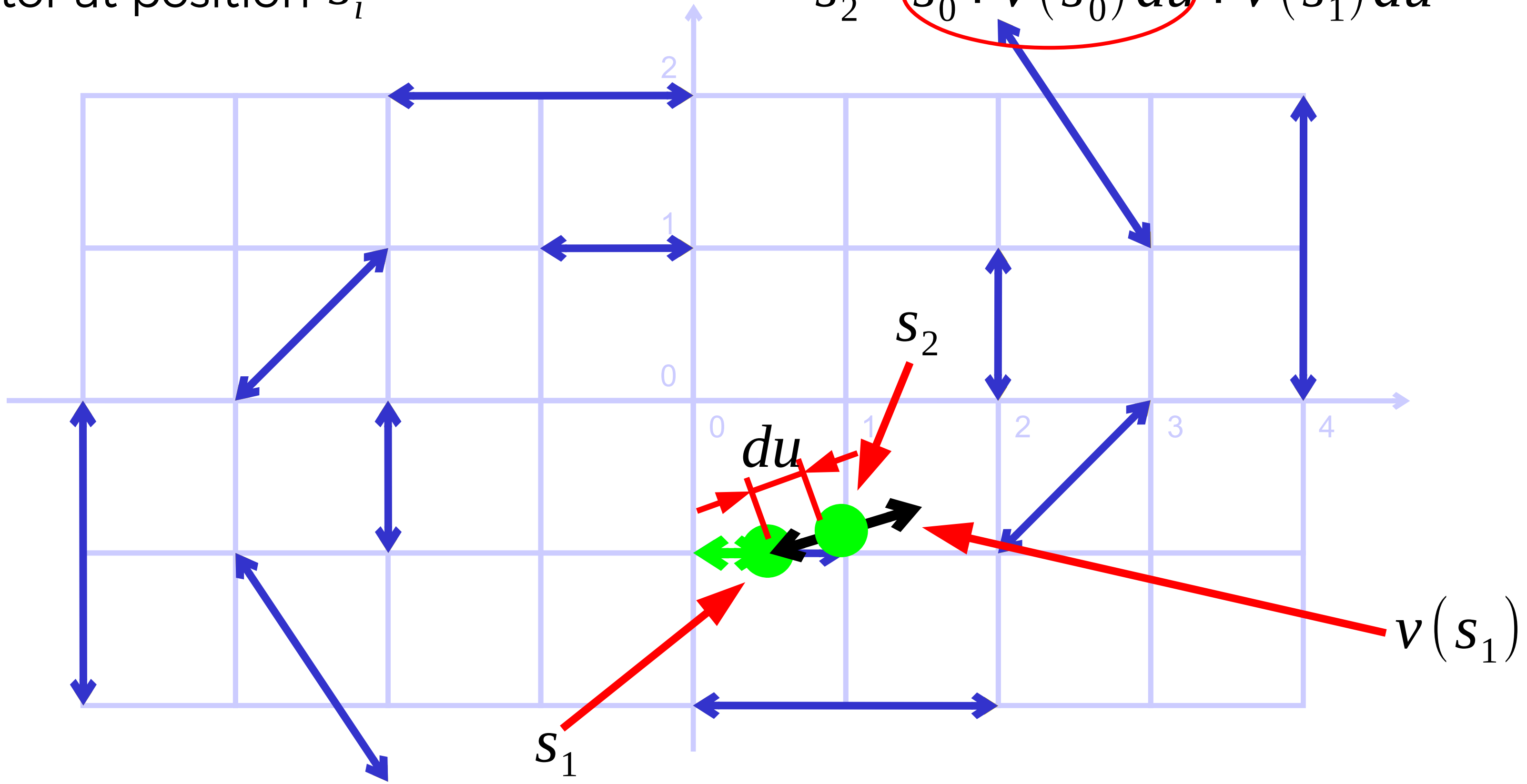
Position at time 2

s_i Position at time step i

$v(s_i)$ Vector at position s_i

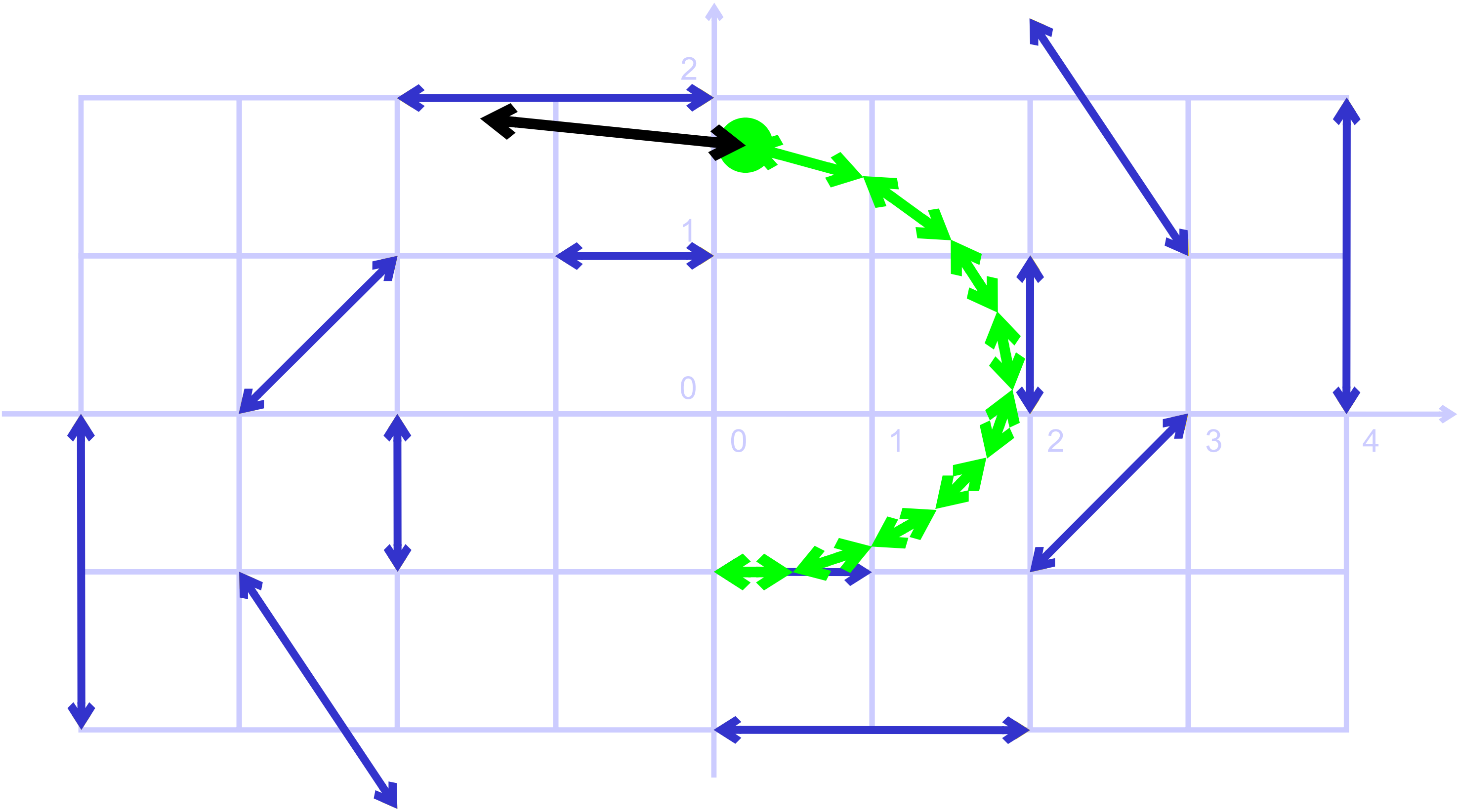
$$s_2 = s_1 + v(s_1)\,du$$

$$s_2 = s_0 + v(s_0)\,du + v(s_1)\,du$$



Position at time *i*

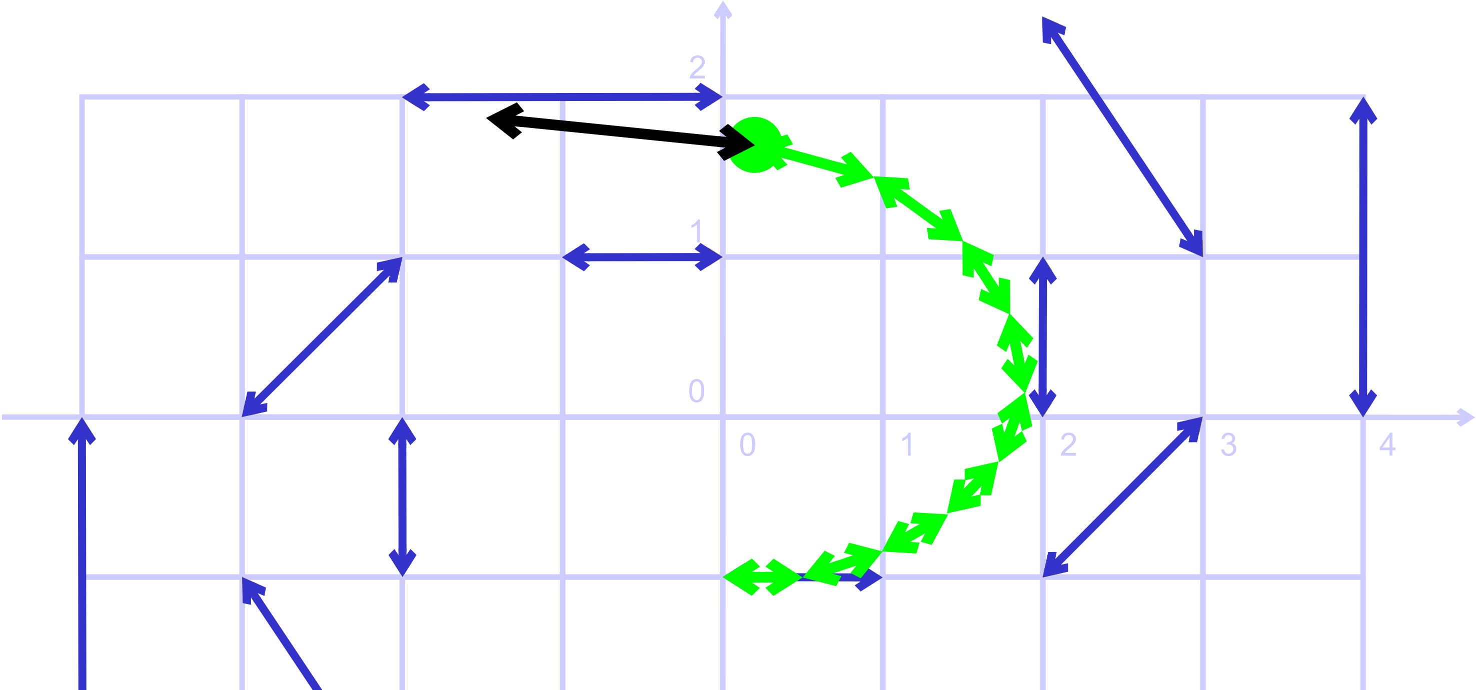
$$s_i = s_0 + v(s_0)\,du + v(s_1)\,du + \cdots + v(s_{i-1})\,du$$



Position at time *i*

$$s_i = s_0 + v(s_0)\,du + v(s_1)\,du + \cdots + v(s_{i-1})\,du$$

$$s_i = s_0 + \sum_{u=0}^{i} v(s_u)\,du$$

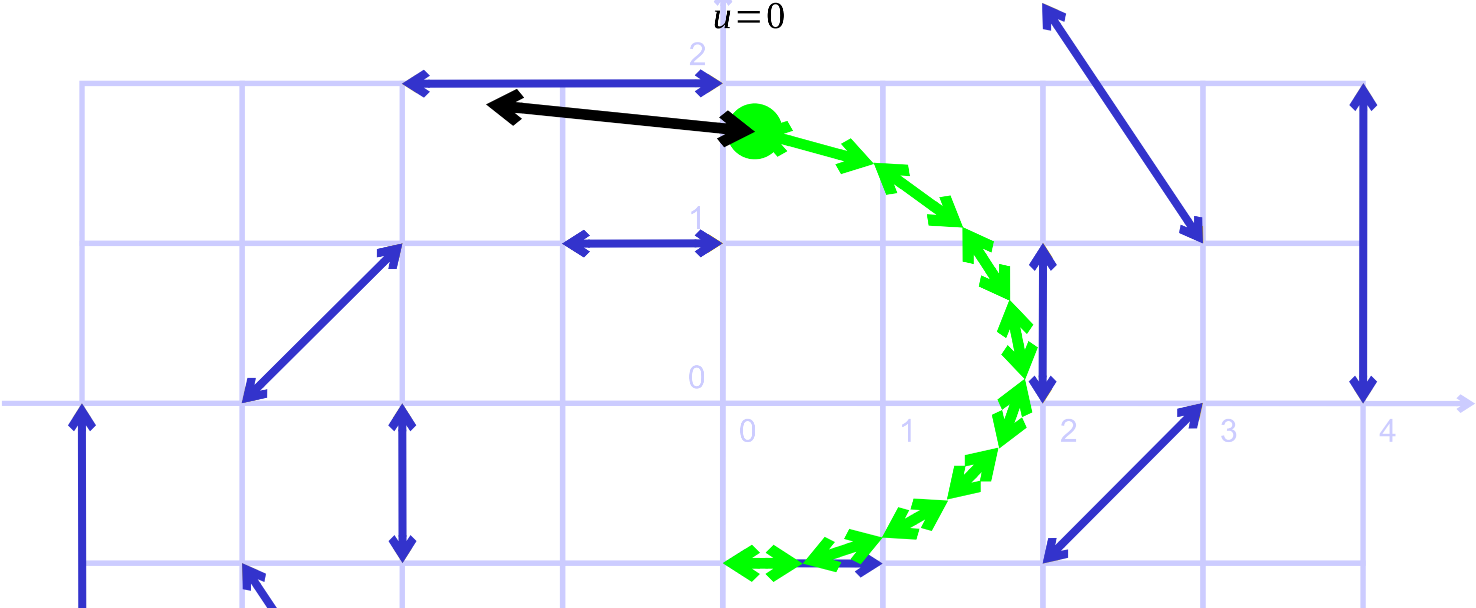


How does this make a line?

$$s_i = s_0 + \sum_{u=0}^{i} v(s_u)\, du$$

A line is just this at every *i* from 0 to T (the end)

$$f(T) = s_0 + \sum_{u=0}^{T} v(s_u)\, du$$



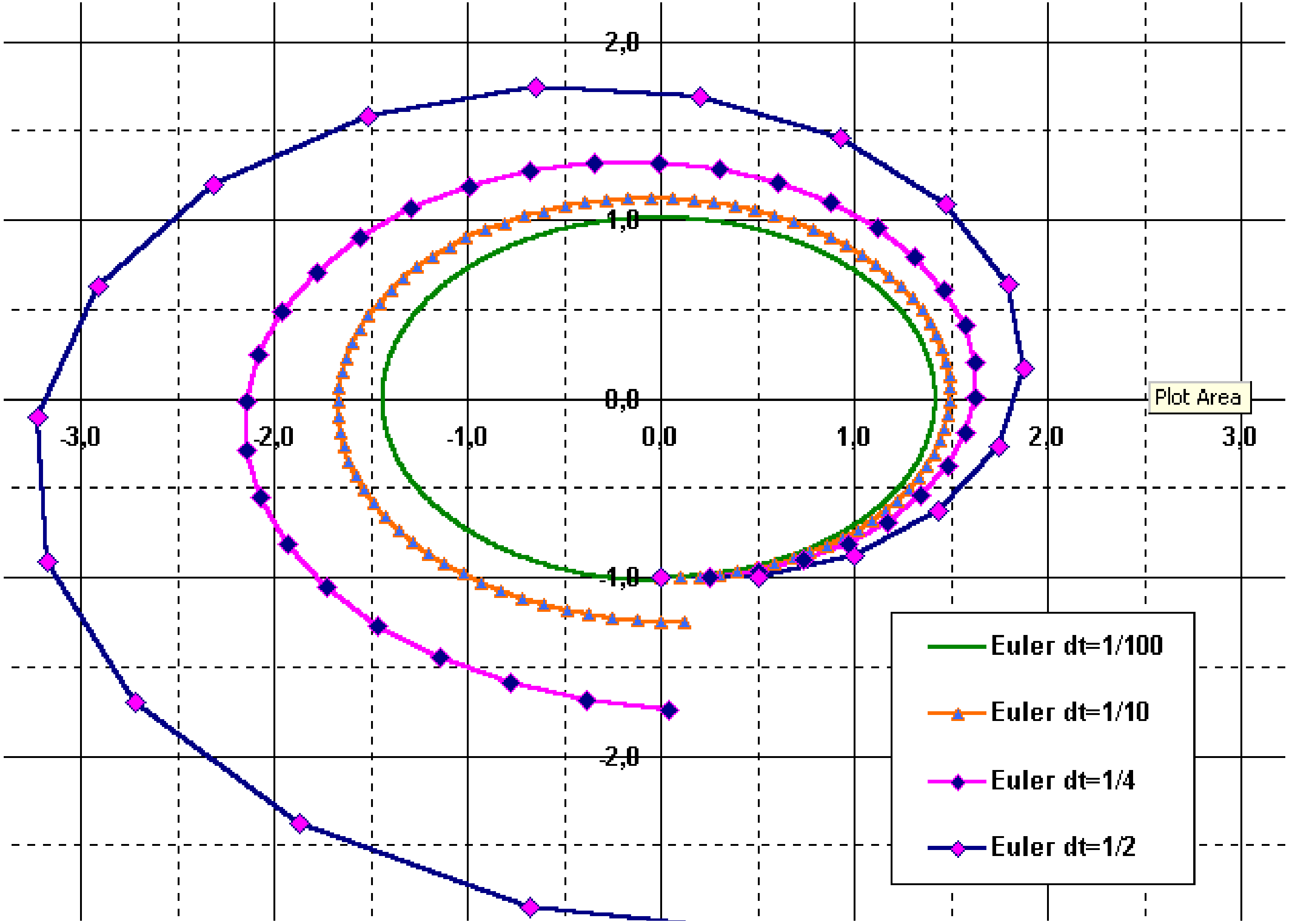
Discrete to continuous

- Computers are discrete
- Continous version is the correct version of the streamline
- Euler integration

Discrete: $$f(T) = s_0 + \sum_{u=0}^{T} v(s_u)\, du$$ Fixed, small, number

Continuous: $$f(T) = s_0 + \int_{u=0}^{T} v(s_u)\, du$$ Infinitesimally small

Issues with Euler



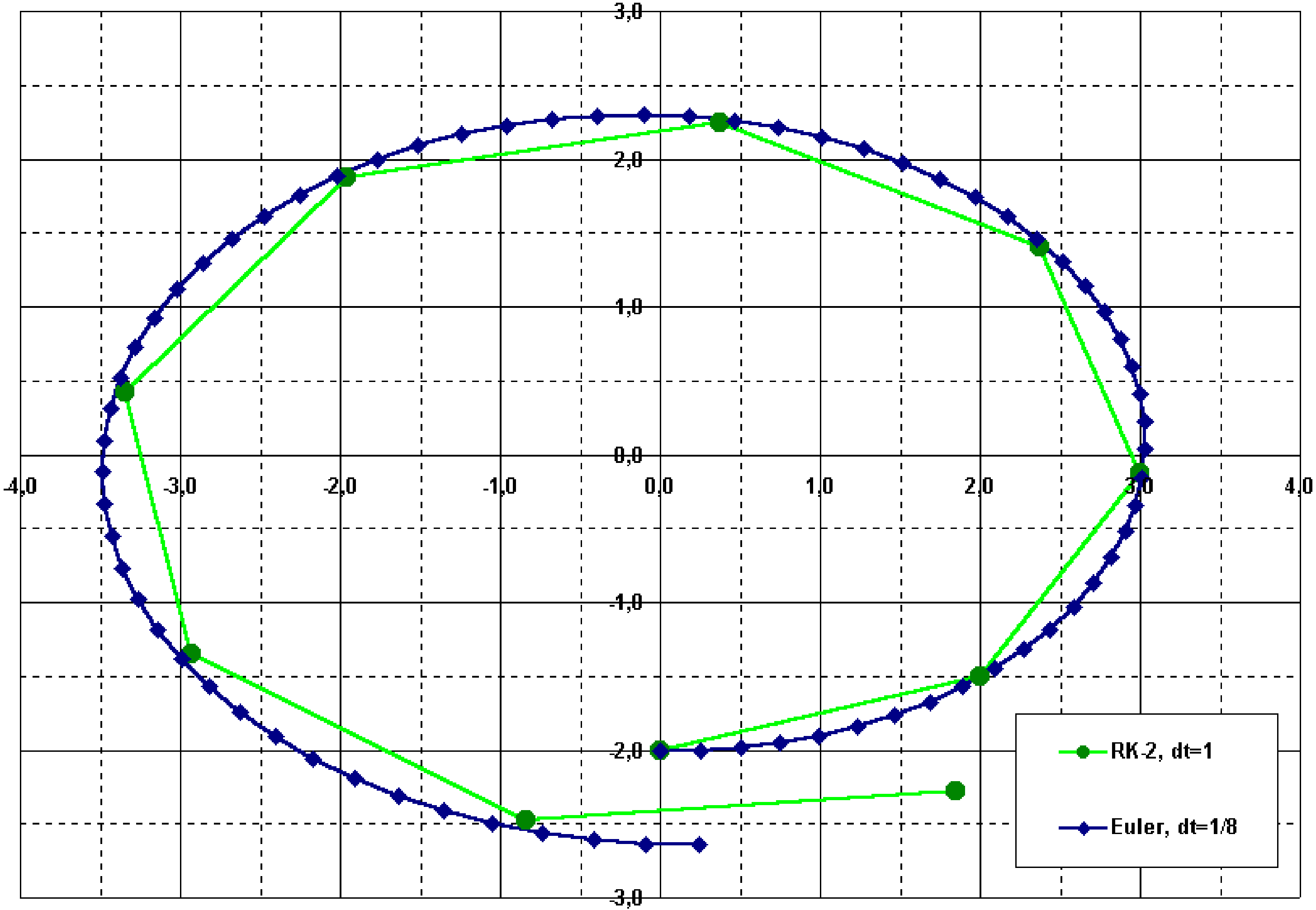
Integration methods

- Analytical
- Euler
- Runge-Kutte

Runge-Kutte to the rescue!

- Actually a family of methods
- Developed around 1900
- Idea is to cut the curve arc shorter
- RK-2 and RK-4 most popular

Runge-Kutte to the rescue!



9 steps (RK-2) vs 72 (Euler)

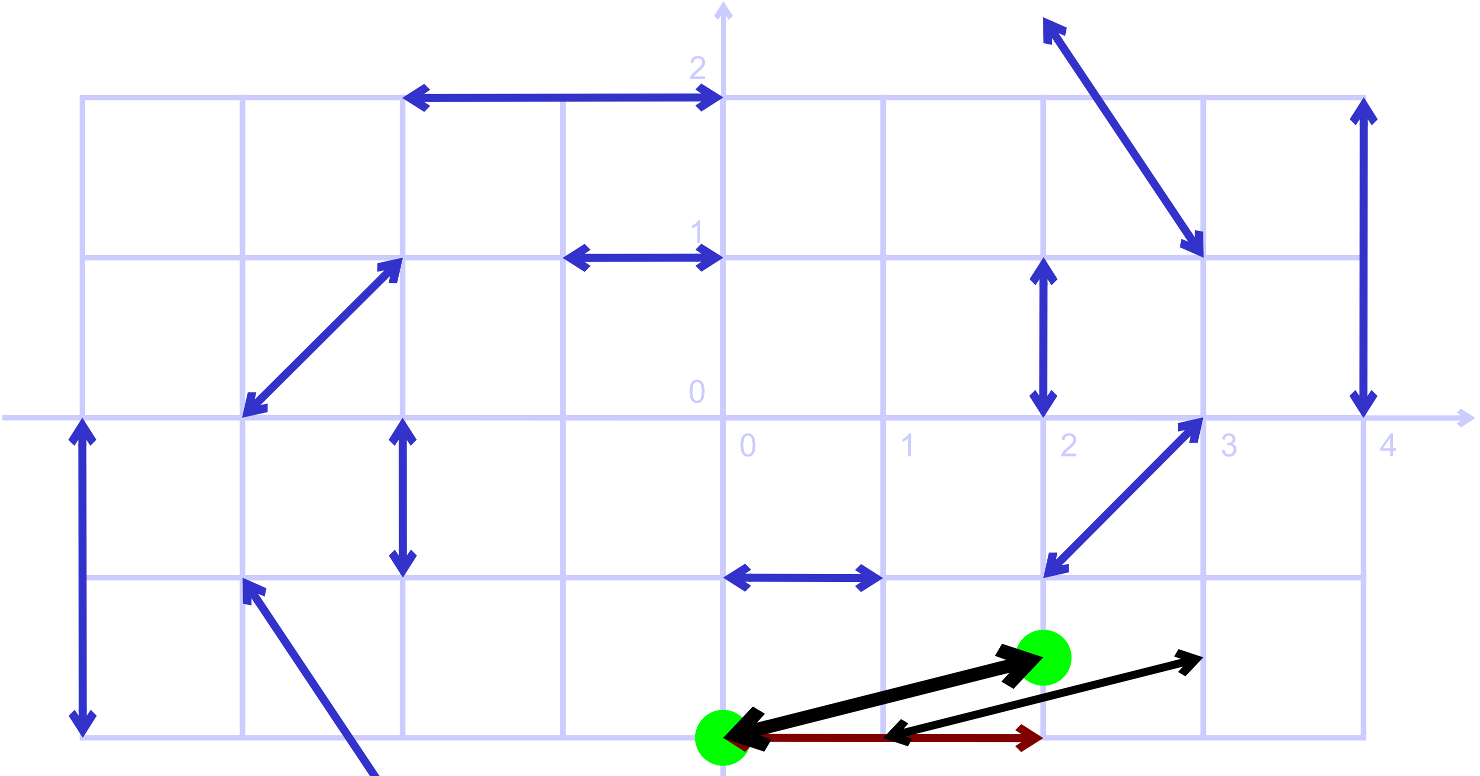
Runge-Kutte 2

$$s_i = s_{i-1} + v\left(s_{i-1} + v\left(s_{i-1}\right) du/2\right) du$$

1) Do half an Euler step
2) Evaluate the flow vector there
3) Use it at the original point
4) Average the 2 vectors

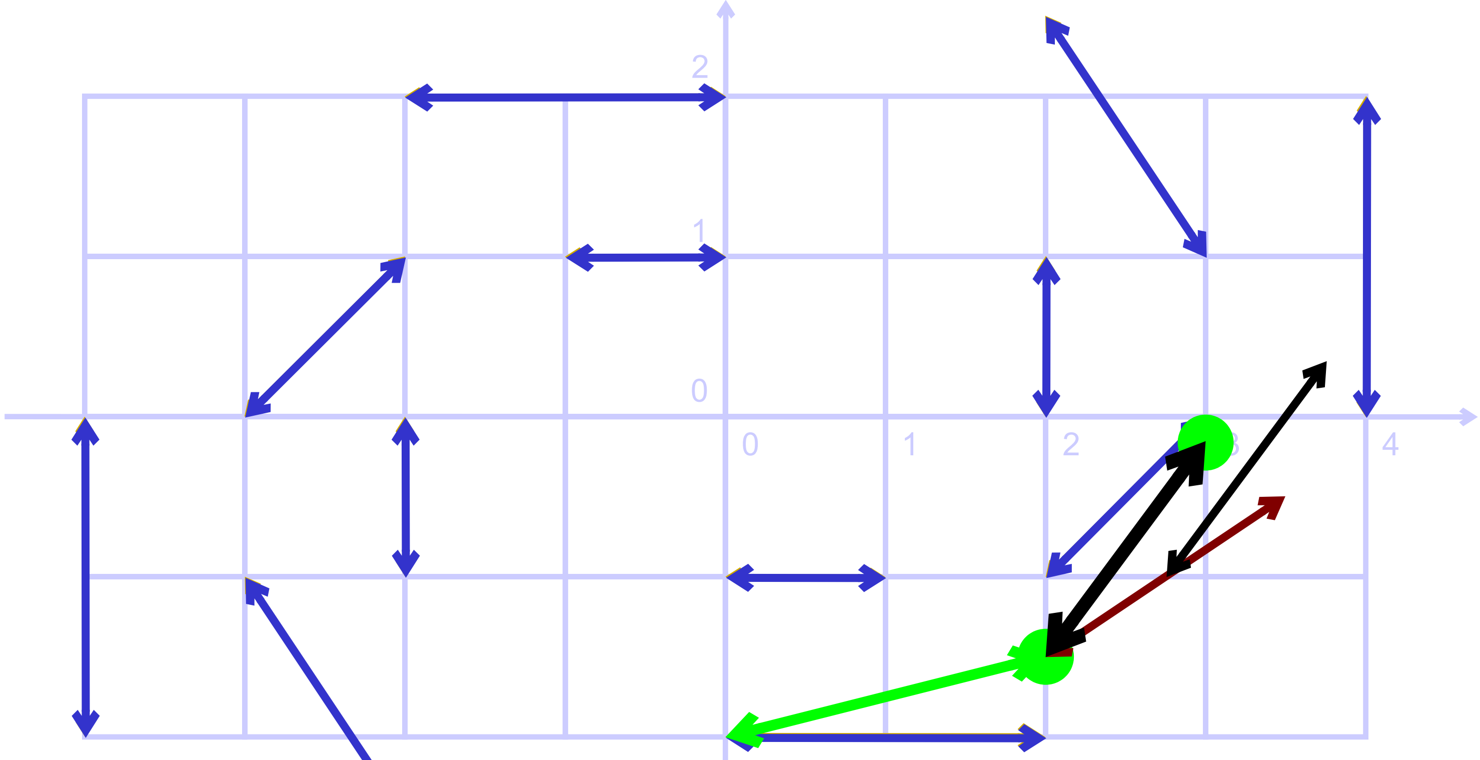
Runge-Kutte 2

Seed point $\mathbf{s}_0 = (0, -2)^T$;
current flow vector $\mathbf{v}(\mathbf{s}_0) = (2, 0)^T$;
preview vector $\mathbf{v}(\mathbf{s}_0 + \mathbf{v}(\mathbf{s}_0) \cdot du/2) = (2, 0.5)^T$;
$du = 1$



Runge-Kutte 2

Seed point $\mathbf{s}_1 = (2, -1.5)^T$;
current flow vector $\mathbf{v}(\mathbf{s}_1) = (1.5, 1)^T$;
preview vector $\mathbf{v}(\mathbf{s}_1 + \mathbf{v}(\mathbf{s}_1) \cdot du/2) = (1, 1.4)^T$;
$du = 1$

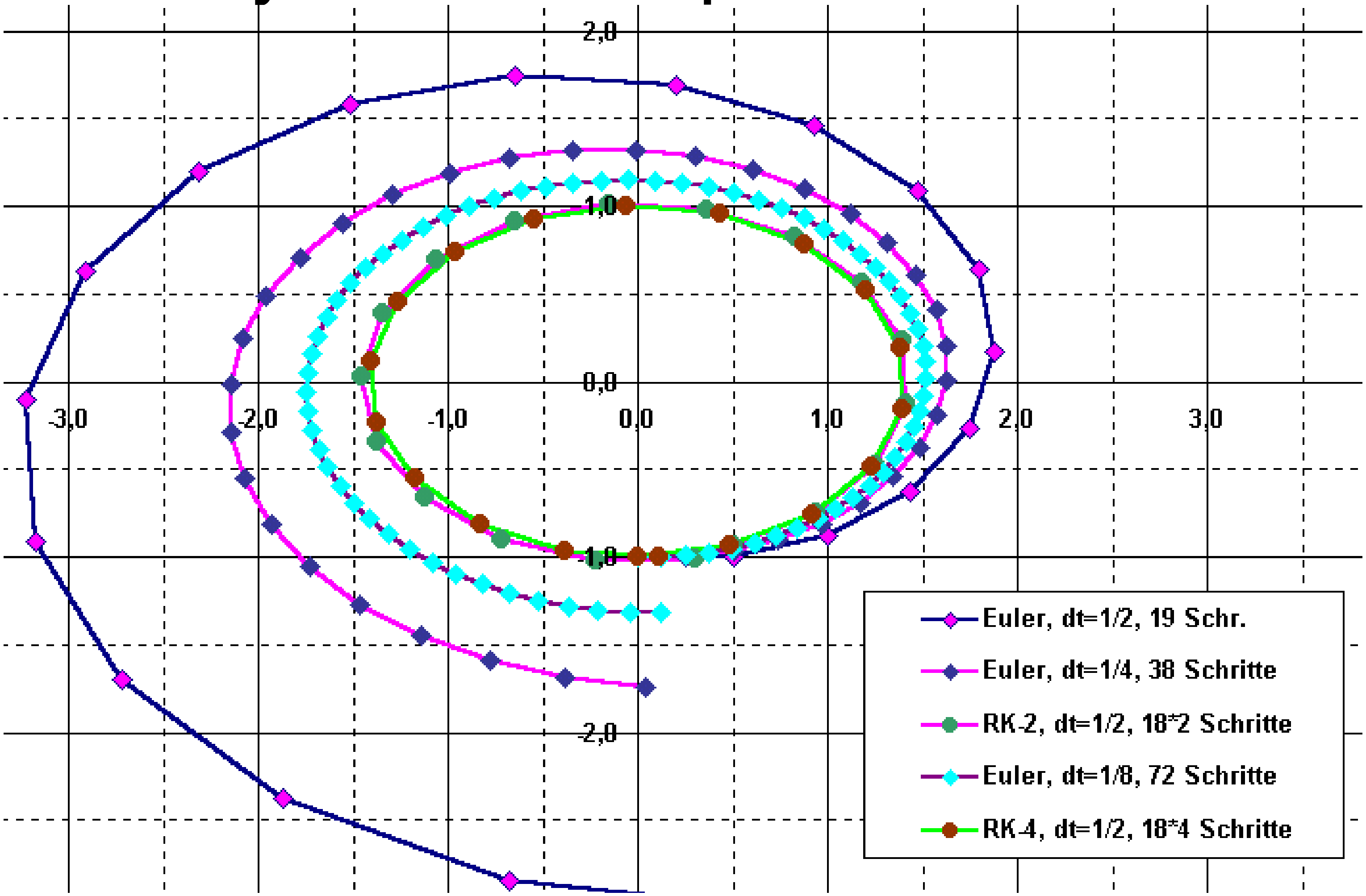


Runge-Kutte 4

Even better: fourth order RK:

- four vectors **a**, **b**, **c**, **d**
- one step is a convex combination:
  $\mathbf{s}_{i+1} = \mathbf{s}_i + (\mathbf{a} + 2 \cdot \mathbf{b} + 2 \cdot \mathbf{c} + \mathbf{d})/6$
- vectors:
  - $\mathbf{a} = \mathrm{d}u \cdot \mathbf{v}(\mathbf{s}_i)$ … original vector
  - $\mathbf{b} = \mathrm{d}u \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{a}/2)$ … RK-2 vector
  - $\mathbf{c} = \mathrm{d}u \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{b}/2)$ … use RK-2 …
  - $\mathbf{d} = \mathrm{d}u \cdot \mathbf{v}(\mathbf{s}_i + \mathbf{c})$ … and again

RK-4 only for complex flows



Unsteady flow visualization

- **Streamline**: a curve that is everywhere tangent to the flow (release 1 massless particle)
- **Pathline**: a curve that is everywhere tangent to an unsteady flow field (release 1 massless particle)
- **Streakline**: a curve traced by the continues release of particles in unsteady flow from the same position in space (release infinitely many massless particles)

$$f(T) = s_0 + \int_{u=0}^{T} v(s_u, t=\tau)\,du$$

$$f(T) = s_0 + \int_{u=0}^{T} v(s_u, t=u)\,du$$

1) Pick an s_0
2) Find all pathlines that pass through that point
3) Get the end points of each of these lines at time T
4) Connect all the points from step 3

Conclusion

Summary

- analytic determination of streamlines usually not possible
- Euler: simple, imprecise, esp. with small dt
- RK: more accurate in higher orders
- Other lines are integrations over time/space

Acknowledgements

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