1. let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find $A + A^2 + \cdots + A^{2002}$

sol)

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vdots$

$\therefore Ans = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \times 1001$

$= \boxed{\begin{bmatrix} 1001 & 1001 \\ 1001 & 1001 \end{bmatrix}}$

2.

Problem)

Let

$$A = \begin{bmatrix} \cos\frac{\pi}{2014} & -\sin\frac{\pi}{2014} \\ \sin\frac{\pi}{2014} & \cos\frac{\pi}{2014} \end{bmatrix}$$

Find b_{11} where

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = A + A^2 + \cdots + A^{2014}$$

sol)

let

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$B^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

note, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore B^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

in fer,

maybe $B^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$?

mathematical induction,

if $B^{n-1} = \begin{bmatrix} \cos(n-1)\theta & -\sin(n-1)\theta \\ \sin(n-1)\theta & \cos(n-1)\theta \end{bmatrix}$,

prove $B^n = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$$B^{n-1} \times B = B^n = \begin{bmatrix} \cos(n-1)\theta & -\sin(n-1)\theta \\ \sin(n-1)\theta & \cos(n-1)\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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$$= \begin{bmatrix} \cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta & -\cos(n-1)\theta\sin\theta - \sin(n-1)\theta\cos\theta \\ \sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta & -\sin(n-1)\theta\sin\theta + \cos(n-1)\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

$$\therefore b_{11} = \cos\left(\frac{\pi}{2014}\right) + \cos\left(\frac{2\pi}{2014}\right) + \cdots + \cos\left(\frac{2014\pi}{2014}\right)$$

since $\cos(\theta) + \cos(\pi - \theta) = 0$,

$$b_{11} = \cos\left(\frac{2014\pi}{2014}\right) = -1$$

problem 3)

For a complex number $z = x + iy$

$(x, y \in \mathbb{R})$, let $M(z) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$

(1) prove that $M(z+z') = M(z) + M(z')$,

$M(zz') = M(z)M(z')$

$\rightarrow M(z+z')$, if we let $z = a+bi$, $z' = a'+b'i$ $(a, b, a', b' \in \mathbb{R})$

$$M(z+z') = M\left((a+a') + (b+b')i\right)$$

$$= \begin{bmatrix} a+a' & b+b' \\ -b-b' & a+a' \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix}$$

$$\rightarrow M(zz') = M\left(aa' - bb' + (ab' + a'b)i\right)$$

$$= \begin{bmatrix} aa' - bb' & ab' + a'b \\ -ab' - a'b & aa' - bb' \end{bmatrix}$$

$$M(z) \times M(z')$$

$$= \begin{bmatrix} aa' - bb' & ab' + a'b \\ -ab' - a'b & aa' - bb' \end{bmatrix}$$

(2) Show that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

sol)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = M(\cos\theta + i\sin\theta)$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = M\left((\cos\theta + i\sin\theta)^n\right)$$

de moivre's theorm,

$$= M(\cos n\theta + i \sin n\theta)$$

$$= \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

(3) Find 3 real 2x2 matrices A s.t $A^3 = I$ and a 2x2 matrix B s.t $B^2 - B - I = 0$

$\rightarrow I = M(1)$

assume $A \in \{M(x) \mid x \in \mathbb{C}\}$,

and $A = M(x)$

Then, $x^3 = 1$.

solving for x,

$x = 1, \quad \frac{-1 \pm \sqrt{3}\, i}{2}$

$\Rightarrow$ 3 results.

$\rightarrow M(x^2) - M(x) + M(1) = M(0)$

$x^2 - x + 1 = 0$ solution found.

Problem 4)

(1) $\exists A$ s.t. $A \neq 0 \wedge A^2 = 0$?

$\Rightarrow$ Yes. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(2) $\exists A$ s.t. $A \neq 0 \wedge A^2 = A$?

$\Rightarrow$ Yes. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Problem 5)

$X + Y = I \qquad X, Y \in M_{2\times2}$

$XY = 0$

find X, Y

sol)

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad Y = \begin{pmatrix} 1-x_{11} & -x_{12} \\ -x_{21} & 1-x_{22} \end{pmatrix}$$

$$XY = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$x_{11} - x_{11}^2 - x_{12} \cdot x_{21} = 0 \quad \cdots \quad (1)$$

$$-x_{11} \cdot x_{12} + x_{12}(1 - x_{22}) = 0$$

$$x_{12}(1 - x_{11} - x_{22}) = 0 \quad \cdots \quad (2)$$

$$x_{21}(1 - x_{11}) - x_{21} \cdot x\, x_{22} = 0$$

$$x_{21}(1 - x_{11} - x_{22}) = 0$$

$$-x_{21} \cdot x_{12} + x_{22} - x_{22}^2 = 0$$

$$x_{11} + x_{22} = 1$$

problem 6)

let $[P,Q] = PQ - QP$ when $P, Q \in M_{2\times 2}$

1. show $[A,A] = 0$

$$[A,A] = AA - AA = A^2 - A^2 = 0$$

2. show $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$

(given) $= \Sigma [AB - BA, C]$

$$= \Sigma \left((AB - BA) \times C - C \times (AB - BA) \right)$$

$$= \Sigma (ABC - BAC - CAB + CBA)$$

$$= 0$$

3. show $[A,B] = I \Rightarrow [A, B^k] = k B^{k-1}$

given) $AB - BA = I$

$$[A, B^k] = AB^k - B^k A$$
$$= A B \times B^{k-1} - B B^{k-1} A$$

$$[A, B^2] = AB^2 - B^2 A$$
$$= ABB - BBA$$

since $AB - BA = I$,

$AB = (I + BA)$

$$\therefore ABB - B BA = (I + BA) \times B - BBA$$

$$= B + BAB - BBA$$

$$BBA = B(AB - I) = BAB - B$$

$$\therefore 2B.$$

6. Is there really two 2×2 matrices s.t. $[A, B] = I$?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\times b_{11} + a_{12}\times b_{21} & a_{11}\times b_{12} + a_{12}\times b_{22} \\ a_{21}\times b_{11} + a_{22}\times b_{21} & a_{21}\times b_{12} + a_{22}\times b_{22} \end{bmatrix}$$

$$\therefore [A, B] = \begin{bmatrix} a_{12} b_{21} - a_{21} b_{12} & a_{11}\times b_{12} + a_{12}\times b_{22} \; - \\ a_{21}\times b_{11} + a_{22}\times b_{21} & \end{bmatrix}$$

$$- b_{21}\times a_{11} + b_{22}\times a_{21}$$