1. let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find $A + A^2 + \cdots + A^{2002}$

sol)

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\vdots$

$\therefore Ans = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \times 1001$

$= \boxed{\begin{bmatrix} 1001 & 1001 \\ 1001 & 1001 \end{bmatrix}}$

2.

Problem)

Let

$$A = \begin{bmatrix} \cos \frac{\pi}{2014} & -\sin \frac{\pi}{2014} \\ \sin \frac{\pi}{2014} & \cos \frac{\pi}{2014} \end{bmatrix}$$

Find b_{11} where

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = A + A^2 + \cdots + A^{2014}$$

sol)

let

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$B^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$

note, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\therefore B^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$$

in fer,

maybe $B^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$?

mathematical induction,

if $B^{n-1} = \begin{bmatrix} \cos (n-1)\theta & -\sin (n-1)\theta \\ \sin (n-1)\theta & \cos (n-1)\theta \end{bmatrix}$,

prove $B^n = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

$$B^{n-1} \times B = B^n = \begin{bmatrix} \cos (n-1)\theta & -\sin (n-1)\theta \\ \sin (n-1)\theta & \cos (n-1)\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

10nn

$$= \begin{bmatrix} \cos (n-1)\theta \cos\theta - \sin (n-1)\theta \sin\theta & -\cos (n-1)\theta \sin\theta - \sin (n-1)\theta \cos\theta \\ \sin (n-1)\theta \cos\theta + \cos(n-1)\theta \sin\theta & -\sin (n-1)\theta \sin\theta + \cos (n-1)\theta \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}$$

$$\therefore b_{11} = \cos\left(\frac{\pi}{2014}\right) + \cos\left(\frac{2\pi}{2014}\right) + \cdots + \cos\left(\frac{2014\pi}{2014}\right)$$

since $\cos(\theta) + \cos(\pi - \theta) = 0$,

$$b_{11} = \cos\left(\frac{2014\pi}{2014}\right) = -1$$

problem 3)

For a complex number $z = x + iy$ $(x, y \in \mathbb{R})$, let $M(z) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$

(1) prove that $M(z + z') = M(z) + M(z')$,

$$M(zz') = M(z)\, M(z')$$

→ $M(z+z')$, if we let $z = a + bi$, $z' = a' + b'i$ $(a, b, a', b' \in \mathbb{R})$

$$M(z + z') = M((a+a') + (b+b')i)$$

$$= \begin{bmatrix} a+a' & b+b' \\ -b-b' & a+a' \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix}$$

→ $M(zz') = M(aa' - bb' + (ab' + a'b)i)$

$$= \begin{bmatrix} aa' - bb' & ab' + a'b \\ -ab' - a'b & aa' - bb' \end{bmatrix}$$

$$M(z) \times M(z')$$

$$= \begin{bmatrix} aa' - bb' & ab' + a'b \\ -ab' - a'b & aa' - bb' \end{bmatrix}$$

(2) Show that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

sol)

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = M(\cos\theta + i\sin\theta)$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^n = M((\cos\theta + i\sin\theta)^n)$$

de moivre's theorm,

$= M(\cos n\theta + i \sin n\theta)$

$$= \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

(3) Find 3 real 2x2 matrices A s.t. $A^3 = I$ and a 2x2 matrix B s.t. $B^2 - B - I = 0$

$\to$ $I = M(1)$

assume $A \in \{M(x) \mid x \in \mathbb{C}\}$,

and $A = M(x)$

Then, $x^3 = 1$.

solving for x,

$x = 1$, $\dfrac{-1 \pm \sqrt{3}\, i}{2}$

$\Rightarrow$ 3 results.

$\to$ $M(x^2) - M(x) + M(1) = M(0)$

$x^2 - x + 1 = 0$ solution found.

problem 4)

(1) $\exists A$ s.t. $A \neq 0 \wedge A^2 = 0$?

$\Rightarrow$ Yes. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(2) $\exists A$ s.t. $A \neq 0 \wedge A^2 = A$?

$\Rightarrow$ Yes.