1. Stochastic Processes, Discrete-time Markov Chain

- Stochastic process: collection of r.v. indexed by parameter t that belongs to set T
  - t is usually time, $X(t)$ is state of process at time t
  - $T \to$ index set
  - State space $S \to$ all possible values of $X(t), \forall t \in T$
  - T finite, stochastic process is discrete-time
    - T $\infty \to$ continuous-time process

1.2 Discrete-time Markov Chains

- MC w/ $\{X_n, n=0,1,2 \dots n\}$ w/ S finite s.t. $P(X_{n+1}=j \mid X_0=i_0, X_1=i_1, \dots, X_n=i) = P(X_{n+1}=j \mid X_n=i) = P_{ij}$
  - state at time j only depends on state $j-1$
    - Markovian Property (only depends on last state)
- $P_{ij} \to$ 1-step transition probability $\to$ constant for given states i and j
  - $P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} \to$ simple 1-step transition matrix
  - $P = \begin{bmatrix} 0.1 & 0.9 \\ 0 & 1 \end{bmatrix} \to$ rows must add up to 1 (they have to go somewhere)

    from 1
  - 0.1 to stay at 1, 0.9 to go to 2, once at 2, stuck at 2
  - Conditional Prob. $P(X_3=1 \mid X_2=1, X_1=1) = P(X_3=1 \mid X_2=1) = 0.1$
  - Joint Prob. $P(X_0=1, X_1=1, X_2=2) = P(X_2=2 \mid X_1=1, X_0=1) P(X_1=1 \mid X_0=1) P(X_0=1) = P(X_2=2 \mid X_1=1) P(X_1=1 \mid X_0=1) P(X_0=1)$

    $= P_{12} \cdot P_{11} \cdot P(X_0=1) = 0.9 \cdot 0.1 \cdot 1 = 0.09$

1.3 Chapman-Kolmogorov Equations

- n-step transition probability P_{ij}^n is $P(i \to j)$ in n steps $= P_{ij}^n = P(X_{m+n}=j \mid X_m=i)$
  - denote P_{ij}^n w/ $P^{(n)}$. if $n=1$, $P^{(1)}=P$
- Proof: $P^{(n)} = P^n$ based on Chapman-Kolmogorov Equations for $n,m \in \mathbb{N}$, $P^{(n+m)} = P^{(n)} \cdot P^{(m)}$

  Start: WTS $P^{(n+m)} = P^{(n)} \cdot P^{(m)}$ or $P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m$

  $$P_{ij}^{n+m} = P(X_{n+m}=j \mid X_0=i) = \underbrace{\frac{P(X_{n+m}=j, X_0=i)}{P(X_0=i)}} = \sum_{k=0}^{\infty} \underbrace{\frac{P(X_{n+m}=j, X_n=k, X_0=i)}{P(X_0=i)}}_{\text{summing from } k \to n}$$

  $$= \sum P(X_{n+m}=j \mid X_n=k, X_0=i) P(X_n=k \mid X_0=i) \text{ by condit. prob.}$$

  $$= \sum_{k=0}^{\infty} P_{kj}^m P_{ik}^n$$

  following: by induction: $P^{(2)} = P^{(1+1)} = P^{(1)} P^{(1)} = P \cdot P = P^2$, $P^{(3)} = P^{(2)} P = P^3 \dots$

  assuming holds for $n-1$, for n: $P^{(n)} = P^{(n-1)} \cdot P = P^{n-1} \cdot P = P^n$
- Ex: 1.2 $\to$ $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \to P^{(3)} = P^3 = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}^3$

  Can now find something like $P(X_0=1, X_1=2, X_4=1) = P(X_0=1) \cdot P_{12} \cdot P_{21}^3$
- unconditional prob $\to$ $p_j^n = P(X_n=j) = \sum_{i=0}^{\infty} P_{ij}^n p_i^0 \to (p_1^n, p_2^n, \dots) = (p_1^0, p_2^0, \dots) P^{(n)}$
- for init. $[\tfrac{1}{2}, \tfrac{1}{2}] \cdot \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} = (P(X=1), P(X=2), P(X=3)) \mid P_{X_0} = (\tfrac{1}{2}, \tfrac{1}{2})$

1.4 Classification of States

- {chain ever enters state j | it starts in state i} = $\bigcup_{n=0}^{\infty} \{X_n = j \mid X_0 = i\}$
- j accessible from i if i will ever enter j w/ $P > 0$ then $i \to j$
- $i \to j$ j accessible to i $i \leftrightarrow j$ they communicate
  - ↳ class property between i and j
- if all terms fall into 1 class, then the chain is irreducible
- recurrent → can be returned to
- transient → can only be visited limited times
- absorbing → can't leave once it gets there
- reflecting → can't come back once it leaves
- period = # of steps $i \to i$ has to be a multiple of
- to prove aperiodic → show gcm = 1 for steps like $i \xrightarrow{2} i$ ✓ $i \xrightarrow{3} i$

1.5 Stationary distributions → steady state

- satisfies $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ basically take the limit as $n \to \infty$ of the transition matrix and make it converge
- $\pi = (\pi_1, \pi_2, \dots) \to$ limiting probabilities then $\pi = \pi \cdot P$ can't affect them
- can find $1 = \pi_1 + \pi_2 + \dots \pi_n$ for system of equations to solve for limiting distributions
- MC w/ unique steady state → ergodic

$(x_1, x_2) = (x_1, x_2) \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$, $x_1 + x_2 = 1$ → $x_1 = 0.5x_1 + 0.5x_2$, $x_2 = 0.5x_1 + 0.5x_2$ → $0.5x_1 = 0.5x_2 \to x_1 = x_2$, $x_2 = x_2$, $x_1 + x_2 = 1 \to 2x_2 = 1$ $x_2 = 0.5$

$(x_1, x_2) = (0.5, 0.5)$ ergodic → spends half the time in state 1 and half the time in state 2

- Take an example w/ 2 soln. to system of equations $\underset{\pi_1}{(0.25, 0.75)}$ and $\underset{\pi_2}{(0.75, 0.25)}$, then non-ergodic and $\alpha\pi_1 + (1-\alpha)\pi_2$ is a soln.
  - these become invariant measures instead.

Exercises:

1.1 a) $P(X_3 = 2 \mid X_0 = 1, X_1 = 2, X_2 = 3) \overset{\text{by Markov Prop.}}{=} P(X_3 = 2 \mid X_2 = 3) = P_{32} = 0.1$

b) $P(X_4 = 3 \mid X_0 = 2, X_3 = 1) = P(X_4 = 3 \mid X_3 = 1) = P_{13} = 0.3$

c) $P(X_0 = 1, X_1 = 2, X_2 = 3, X_3 = 1) = P_{12} P_{23} P_{31} \cdot 1 = 0.4 \cdot 0.5 \cdot 0.8 = 0.16$

d) $P(X_0 = 1, X_1 = 2, X_3 = 3, X_5 = 1) = 1 \cdot P_{12} \cdot P_{23}^2 \cdot P_{31}^2 = 1 \cdot 0.4 \cdot \frac{13}{50} \cdot \frac{17}{50} = 0.03536$

5. a) Obvious → $P(X_2 = G \mid X_0 = R, X_1 = B) \neq P(X_2 = G \mid X_1 = B)$ as w/o. replacement it has memory

b) even if there is no updating, it is still technically an MC w/ prob $(X_n = G)$ is the same for any n
Bernoulli ✱

6. Binomial ✱

8. Important Note here and in the chapter

if $P(X_3 = n_3 \mid X_2 = n_2, X_1 = n_1) = 0$, DNE

but something like $P(X_3 = n_3 \mid X_2 = n_2) \neq 0$, Does exist

then the Markovian (memoryless) property is broken and therefore

can't exist as an MC