Chapter 3: Poisson Processes

3.1 Poisson Defn. and Facts

stochastic $\{N(t), t \geq 0\}$ → counting process if $N(t)$ gives # by time t

independent if $N(5) - N(0)$ is independent of $N(10) - N(5)$ events between those times

stationary increments if $N(t) - N(0) = N(t+s) - N(s)$ so only depends on t and not s

counting process $\{N(t), t \geq 0\}$ → Poisson Process w/ rate λ if:

1) no events at $t=0$, i.e. $N(0)=0$
2) independent increments, i.e. $N(5) - N(0) \perp\!\!\!\perp N(10) - N(5)$
3) stationary increments, i.e. $N(s+t) - N(s) = N(t) - N(0)$
4) $P(N(t)=n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$, $n = 0,1,2\ldots$ $\quad E(N(t)) = Var(N(t)) = \lambda t$

$\lambda \perp\!\!\!\perp t$ therefore constant and can be referred to as homogenous Poisson Process

interarrival time: time between 2 consecutive occurences of events $(n-1)^{st}$ - n^{th} event → $T_n : \{T_1, T_2 .. T_n\}$ w/ $T_1 = E_1$

waiting time: $S_n = T_1 + T_2 \ldots + T_n$ → time at event n

Prop 3.1: Interarrival times T_n, $n=1,2,\ldots$ are independent exp r.v. w/ density $f_{T_n}(t) = \lambda e^{-\lambda t}$, $t \geq 0$

Proof: note $P(T_1 > t) = P(N(t)=0) = e^{-\lambda t}$ so $T_1 \sim Exp(\lambda)$

so $P(T_2 > t) = \int_0^\infty P(T_2 > t \mid T_1 = s)\, \lambda e^{-\lambda s} ds = \int_0^\infty P(N(t+s) - N(s) = 0 \mid N(s) = 1)\, \lambda e^{-\lambda s} ds =$

$\int_0^\infty P(N(t)=0)\, \lambda e^{-\lambda s} = e^{-\lambda t} \int_0^\infty \overset{\text{PDF}}{\lambda e^{-\lambda s} ds} = e^{-\lambda t}$ so $T_2 \sim Exp(\lambda)$

therefore by using same logic for T_n or induction will show $P(T_{n+1} > t) = e^{-\lambda t}$ → $\prod T_1 \to T_n \to f_{T_n}(t) = \lambda e^{-\lambda t}$

Remark 3.1: $\exp(x)$ → memoryless and therefore $pois(\lambda)$ or T_n should inherit that

thought: ($\lambda = 2$ per hour then $E(e) = 1/2$ an hour)

Prop 3.2: $S_n \sim Gamma(n, \lambda)$ so $E(S_n) = \frac{n}{\lambda}$, $Var(S_n) = \frac{n}{\lambda^2}$ density $f_{S_n}(s) = \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s}$, $s \geq 0$

Proof: $S_n = T_1 + \ldots + T_n$ $\quad \prod \exp(\lambda) = gamma(n, \lambda)$

Remark 3.2: Poisson Process → continuous-time Markov Chain

Remark 3.3: Each event can only happen by itself like can't have 2 at once

Example 3.1: Poisson: ppl entering bank Not Poisson: bus time arrivals (schedule)

Example 3.2: Tour bus w/ 50 ppl come every 15 minutes ind. and exp.

a) $E(S_5) = 5/4 = 1.25$ hours $\quad Var(S_5) = 5/4^2 = 5/16$ hours

b) $E(T$ between $11 \to 1{:}30) = E(2.5 \text{ hrs}) = 10$

Prop 3.3: can break $Poiss(\lambda)$ into $N(t) = \overset{p}{N_1(t)} + \overset{1-p}{N_2(t)}$ $\quad N_1 \to Poiss(\lambda p)$ $\quad N_2 \to Poiss(\lambda(1-p))$

but N_1 and N_2 are independent, with both being thinned from the superposition $N(t)$

Application 3.2: Sports Analytics of Goal Opening

assume points scored by team A ~ pois(λ_A) → $\{N_A(t), t \geq 0\}$

and team B ~ pois(λ_B) → $\{N_B(t), t \geq 0\}$

have to assume independence

a) superposition $N(t) = N_A(t) + N_B(t)$ w/ rate $\lambda_A + \lambda_B$ for either team to score

then $1/\lambda_A$ → Team A scores, $1/\lambda_B$ → Team B scores, $1/(\lambda_A+\lambda_B)$ either team score

assume team A every 10 min and team B every 12 minute $E(\text{either team score}) = \frac{1}{1/10 + 1/12} = 5.45$

b) Probability of first team scoring: $E(T_A) = 1/\lambda_A \quad E(T_B) = 1/\lambda_B$

$P(\text{team A scores before team B}) = P(T_A < T_B) = \int_0^\infty P(T_B > t) f_A(t)\,dt = \int_0^\infty e^{-\lambda_B t} \lambda_A e^{-\lambda_A t}\,dt = \frac{\lambda_A}{\lambda_A + \lambda_B} = \frac{1/10}{1/10 + 1/12} = .545$

$P(\text{B before A}) = P(T_B < T_A) = \frac{\lambda_B}{\lambda_A + \lambda_B} = 1 - .545 = .455$

c) tie, team A wins, team B wins

T → end of game

$P(\text{game ties}) = \sum_{n=0}^{\infty} P(N_A(T) = n, N_B(T) = n) = e^{-(\lambda_A+\lambda_B)T} \sum_{n=0}^{\infty} \frac{(\lambda_A \lambda_B T^2)^n}{(n!)^2}$

$P(\text{team A wins}) = P(N_A(T) > N_B(T)) = \sum\sum P(N_A(T) = n+k, N_B(T) = n) = e^{-(\lambda_A+\lambda_B)T} \sum\left[\frac{(\lambda_A\lambda_B T^2)^n}{n!} \cdot \sum \frac{(\lambda_A T)^k}{(n+k)!}\right]$

substitute for team B or 1 − ties − A = B

if $T = 60$, $P(\text{ties}) = e^{-(1/10 + 1/12)(60)} \sum \frac{((1/10)(1/12)(60)^2)^n}{(n!)^2} = e^{-11} \sum \frac{30^n}{(n!)^2} = 0.1166$

$P(A) = e^{-11} \sum\left[\frac{30^n}{n!} \cdot \sum \frac{6^k}{(n+k)!}\right] = 0.5590$

$P(B) = 1 - \text{ties} - A = 0.3244$

Application 3.3: Poisson pedestrian vs. traffic flow

Exercise 3.1: $\{N(t), t \geq 0\}$ Poiss(λ), joint pdf $P(N(s) = m, N(t) = n)$, $t \geq s \geq 0$, $n \geq m \geq 0$

$P(N(s) = m, N(t) = n) = P(N(t) - N(s) = n-m, N(s) = m) = P(N(t) - N(s) = n-m)\,P(N(s) = m) =$

$P(N(t-s) = n-m)\,P(N(s) = m) = \frac{(\lambda(t-s))^{n-m}}{(n-m)!} e^{-\lambda(t-s)} \cdot \frac{(\lambda s)^m}{m!} e^{-\lambda s} = \frac{(t-s)^{n-m} s^m}{(n-m)!\,m!} \lambda^n e^{-\lambda t}$

Exercise 3.2: $\text{Cov}(N(s), N(t)) = E[N(s)N(t)] - E[N(s)]\,E[N(t)]$ (λs, λt) (2)

= through terrible properties → $E[N(t-s)]E[N(s)] + Var[N(s)] + [E(N(s))]^2$

$\lambda(t-s)\lambda s + \lambda s + (\lambda s)^2$ (1)

$= (1) - (2) = \lambda s$

Exercise 3.4: 5 min per call, .15 for buying

a) 2 hours → 5 min → 24 · .15 = 3.6

b) within 60 → 15 calls → 5 successes $P(N(1) = 15, N_s(1) = 5)$ → $P(N_s = 5, N_{NS} = 10) = \frac{(1.8)^5}{5!} e^{-1.8} \frac{(10.2)^{10}}{10!} e^{-10.2} = .003$

c) $P_4 = 10$, $P_1 = 3$ → $P_3 = 7$ → $\frac{(1.8)(3))^7}{7!} e^{-1.8(3)} = 0.119$