Chapter 2: Random Walk

2.1 Defn. of Random Walk

Basically only p and $1-p$ on X_{n-1} and X_{n+1} so can only go $+1$ or -1 as a SRW (simple random walk)

symmetric if $p = 0.5$

has many variations: infinite, finite, two/three dimension

can start anywhere

TM :
$$\begin{bmatrix} 0 & p & 0 & \cdots & 0 & p-1 \\ p & 0 & p-1 & \cdots & 0 & 0 \\ 0 & p & 0 & p-1 & \cdots & 0 \\ 0 & 0 & p & 0 & p-1 \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 \end{bmatrix}$$

can add delays or absorbing barriers or even on a graph

Ex 2.1 Gamblers with absorbing barrier and $p = 0.45$ to win and $1-p = 0.55$ to lose

2.2 Facts about Random Walk

$\{X_n, n = 0,1,2\}$ an asymmetric 1-d RW w/ prob $p \to +$ $1-p \to -$ Init. at 0, $X_0 = 0$

Prop 2.1 : $E(X_n) = (2p-1)n$ at time n

$Var(X_n) = 4p(1-p)n$ at time n

because $E(X_i) = 1(p) + (-1)(1-p) = 2p-1$ $E(X_n) = n(2p-1)$

$Var(X_i) = (1)^2(p) + (-1)^2(1-p) - (2p-1)^2 = 1-(2p-1)^2 = 4p(1-p)$ $Var(X_n) = 4p(1-p)n$

for symmetric: $p = 1/2$ $E(X_n) = 0$ $Var(X_n) = n$

Prop 2.2 : Prob. of a return

$$P(X_n = x \mid X_0 = x) = \binom{n}{n/2} p^{n/2}(1-p)^{n/2} \text{ if } n \text{ even o/w } 0$$

Prop 2.3 : Recurrence vs Transience of 1-D RW

1-D RW only recurrent if $p = 1/2$ o/w will visit origin limited time

Can use CLT to show asympt. weak $\xrightarrow{D}$ n-d that it strays far from origin

Prop 2.4 : 2d RW recurrent if symmetric. o/w all other 2-D and all 3-D+ are transient

2.4 Applications of Random Walk

Gambler's Ruin: Sit-rep: start fortune $\$i$ move up $\$1$ w/ $P = p$, down $\$1$ w/ $P = 1-p$ until goes to 0 or fortune $\$N$

recurrence relation: $P_j = qP_{j-1} + pP_{j+1}$ → to go back to square $P_0 = 0,\ P_n = 1$

$$pP_j + qP_j = qP_{j-1} + pP_{j+1} \to P_{j+1} - P_j = \frac{q}{p}(P_j - P_{j-1}) \qquad P_2 - P_1 = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{i-1}\right] P_1 \to P_i = \left\{ \frac{1-(q/p)^i}{} \right.$$

so $P \to \$N = \dfrac{1-(q/p)^i}{1-(q/p)^N}$ P of ruin $= 1 - P_{\$N}$

$$E_i(\text{reach boundary}) = \frac{i - NP_i}{q-p}$$

Random Walk on Graph:

Just predicting # of steps to get to a point starting at another