111-2 Signal and Systems HW #1 Solution

CT1.2.3 The *time-scaled* version of a waveform $x(t)$, is defined as $x(at)$, where a is a nonzero constant. Consider the waveform

$$x(t) = \begin{cases} 0; & t < 0 \\ t; & 0 \le t < 1 \\ 0; & 1 \le t \end{cases} \qquad (20)$$

$\frac{y}{x}$

Plot:

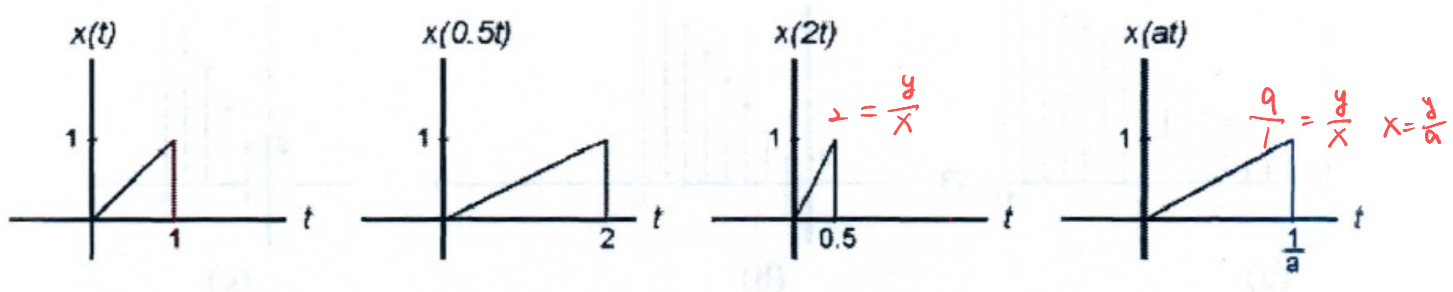
a) $x(t)$ versus t. $\quad x(t) = 1t$

b) $x(at)$ versus t, where $a = 0.5$. $\quad x(t) = 0.5t$

c) $x(at)$ versus t, where $a = 2$. $\quad x(t) = 2t$

d) $x(at)$ versus t, where a is shown as an arbitrary parameter on the plot. Assume that $a > 0$. $\quad x(at) = at$

Ans:



(a) (b) (c) (d)

For each of the following systems, the system output and input are related as below.
Determine whether the system is i) linear, ii) time-invariant or not.

Please justify your answers.

a) S_1: $e^{j5t} \overset{S_1}{\rightarrow} t \cdot e^{j5t}$

b) S_2: $e^{j5t} \overset{S_2}{\rightarrow} e^{j5(t-1)}$

$x_1(t) = e^{j5t} \rightarrow \square \rightarrow t e^{j5t} = y_1(t) = t x_1(t)$

$x_1(t) \rightarrow \square \rightarrow y_1(t) = t x_1(t)$

$x_2(t) \rightarrow \square \rightarrow y_2(t) = t x_2(t)$

$x(t) = \alpha x_1(t) + \beta x_2(t) \rightarrow \square \rightarrow y(t) = \alpha y_1(t) + \beta y_2(t)$

$= t[\alpha x_1(t) + \beta x_2(t)]$

$x_1(t) \rightarrow \square \rightarrow y_1(t)$

Ans:

a) $x(t) = e^{j5t} \rightarrow \square \rightarrow y(t) = t \cdot e^{j5t}$

$= t \cdot x(t)$

so, $x_1(t) \rightarrow \square \rightarrow y_1(t) = t \cdot x_1(t)$

$x_2(t) \rightarrow \square \rightarrow y_2(t) = t \cdot x_2(t)$

$a \cdot x_1(t) + b x_2(t) \rightarrow \square \rightarrow y(t)$

$y(t) = t \cdot [a x_1(t) + b x_2(t)]$

$= a \cdot y_1(t) + b \cdot y_2(t)$

$\therefore$ linear.

Now, $x_1(t) \rightarrow \boxed{\quad} \rightarrow y_1(t) = t \cdot x_1(t)$

$x(t) = x_1(t-2) \rightarrow \boxed{\quad} \rightarrow y(t) = t \cdot x(t)$

$= t \cdot x_1(t-2)$

$\neq y_1(t-2)$

$\therefore$ not time-invariant #

1b) $x(t) = e^{j5t} \rightarrow \boxed{\quad} \rightarrow y(t) = e^{j5(t-1)}$

$= x(t-1)$

So $x_1(t) \rightarrow \boxed{\quad} \rightarrow y_1(t) = x_1(t-1)$

$x_2(t) \rightarrow \boxed{\quad} \rightarrow y_2(t) = x_2(t-1)$

$a x_1(t) + b x_2(t) \rightarrow \boxed{\quad} \rightarrow y(t)$

$= a x_1(t-1)$
$+ b x_2(t-1)$

$= a \cdot y_1(t) + b y_2(t)$

$\therefore$ linear.

Now, $x_1(t) \rightarrow \boxed{\quad} \rightarrow y_1(t) = x_1(t-1)$

$x(t) = x_1(t-2) \rightarrow \boxed{\quad} \rightarrow y(t) = x(t-1)$

$= x_1(t-1-2)$

$= y_1(t-2)$

$\therefore$ time-invariant #

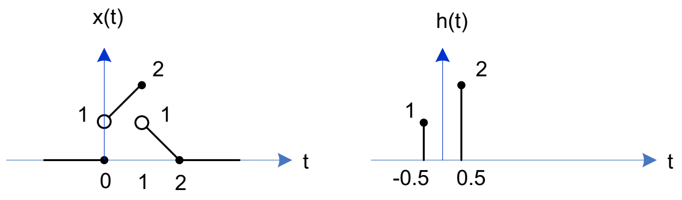
3. Consider a LTI system with input $x(t)$, output $y(t)$ and impulse response $h(t)$. It is known that $y(t) = x(t) * h(t)$, where ' * ' is the convolution operator.

Assume $x(t) = \begin{cases} t+1 & 0 \le t \le 1 \\ 2-t & 1 < t \le 2 \\ 0 & elsewhere \end{cases}$ and $h(t) = \delta(t+0.5) + 2\delta(t-0.5)$

a) Plot x(t) and h(t)
b) Is this system BIBO stable? Is this system causal? Justify your answer.
c) Find the range of t where $y(t) \neq 0$. using sliding tape method
d) Find expression for y(t)
e) Plot y(t)

Ans:

a)



b) This system is BIBO stable.

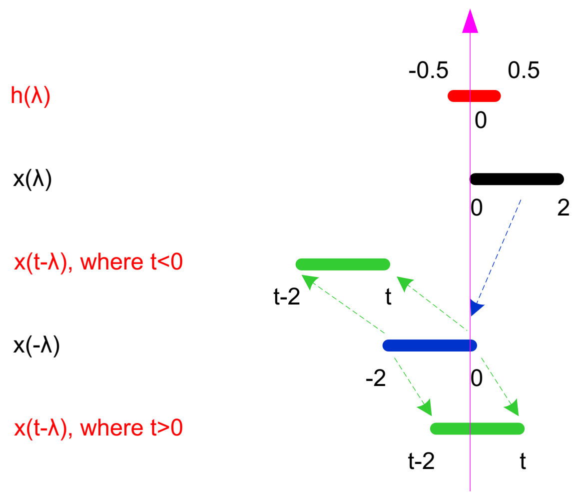
This is because $\int_{t=-\infty}^{\infty}|h(t)|dt = \int_{t=-\infty}^{\infty}\delta(t+0.5) + 2\delta(t-0.5)dt = 3 < \infty$

This system is not causal.
This is because $h(t) \neq 0$ when $t = -0.5 < 0$

c) For CTLTI system, $y(t) = x(t) * h(t) = \int_{\lambda=-\infty}^{\infty} h(\lambda)\cdot x(t-\lambda)d\lambda$

To find the range of t where $y(t) \neq 0$, we can plot range of $x(\lambda)$ and $h(t-\lambda)$ using tape method first.

The following plot shows how the range of $x(\lambda)$ and $h(t-\lambda)$ can be derived.



From the red and blue tape, we found as long as $-0.5 \leq t \leq 2.5$
$=>$ $h(\lambda)\cdot x(t-\lambda) \neq 0$.

d) Now we want to calculate y(t) from convolution integral of h(t)*x(t).
One can always calculate $y(t)$ directly by performing convolution integral using

$$y(t) = x(t) * h(t) = \int_{\lambda=-\infty}^{\infty} h(\lambda)\cdot x(t-\lambda)d\lambda$$

However, there may exist other easier ways to calculate the above.

Noticing that $h(t) = \delta(t+0.5) + 2\delta(t-0.5)$.

Also from homework, we learn that $x(t) * \delta(t \pm \tau) = x(t \pm \tau)$

We can rewrite the system output $y(t) = x(t) * \{\delta(t+0.5) + 2\delta(t-0.5)\}$

This results in $y(t) = x(t+0.5) + 2x(x-0.5)$.

Where $x(t+0.5)$ *is* $x(t)$ shift to the left by 0.5, and $x(t-0.5)$ *is* $x(t)$ shift to the right by 0.5

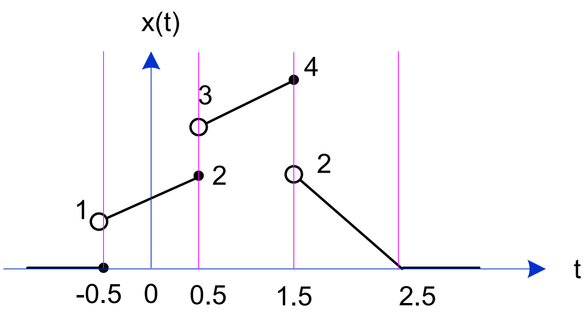
We can now write equation for $x(t+0.5)$ *and* $2x(x-0.5)$

$$x(t+0.5) = \begin{cases} (t+0.5)+1 & -0.5 < t \leq 0.5 \\ 2-(t+0.5) & 0.5 < t \leq 1.5 \\ 0 & elsewhere \end{cases}$$

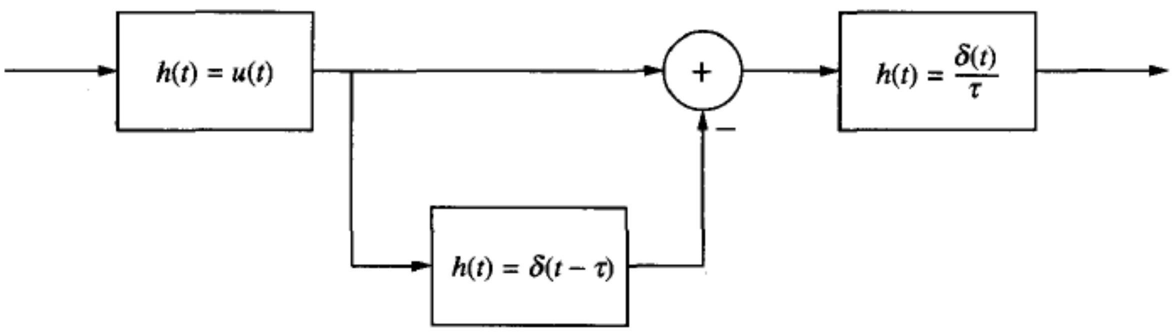
$$2 \cdot x(t-0.5) = \begin{cases} 2 \cdot \{(t-0.5)+1\} & 0.5 < t \leq 1.5 \\ 2 \cdot \{2-(t-0.5)\} & 1.5 < t \leq 2.5 \\ 0 & elsewhere \end{cases}$$

$$y(t) = \begin{cases} t+1.5 & -0.5 < t \leq 0.5 \\ t+2.5 & 0.5 < t \leq 1.5 \\ 5-2t & 1.5 < t \leq 2.5 \\ 0 & else \end{cases}$$

e)



4. Find the impulse response of the system below



Ans:

$$h(t) = \frac{1}{\tau}[u(t) - u(t-\tau)]$$

2CT.9.3 Use the graphical method of convolution to show that

$$\sqcap\left(\frac{t}{\tau}\right) * \sqcap\left(\frac{t}{\tau}\right) = \tau \wedge \left(\frac{t}{\tau}\right)$$

Ans:

$$\sqcap\left(\frac{t}{\tau}\right) * \sqcap\left(\frac{t}{\tau}\right) = \int_{-\infty}^{\infty} \sqcap\left(\frac{t-\lambda}{\tau}\right) \sqcap\left(\frac{\lambda}{\tau}\right) d\lambda$$

The stated result follows by plotting $\sqcap\left(\frac{t-\lambda}{\tau}\right)$, $\sqcap\left(\frac{\lambda}{\tau}\right)$ and $\sqcap\left(\frac{t-\lambda}{\tau}\right)\sqcap\left(\frac{\lambda}{\tau}\right)$ versus λ for different values of t. For example, plot these functions for $0 \le t < \tau$ to show that

$$\int_{-\infty}^{\infty} \sqcap\left(\frac{t-\lambda}{\tau}\right) \sqcap\left(\frac{\lambda}{\tau}\right) d\lambda = \tau - t$$

when $0 \le t < \tau$. Plot the functions for $\tau < t$ to show that

$$\int_{-\infty}^{\infty} \sqcap\left(\frac{t-\lambda}{\tau}\right) \sqcap\left(\frac{\lambda}{\tau}\right) d\lambda = 0$$

when $\tau < t$ and so on.

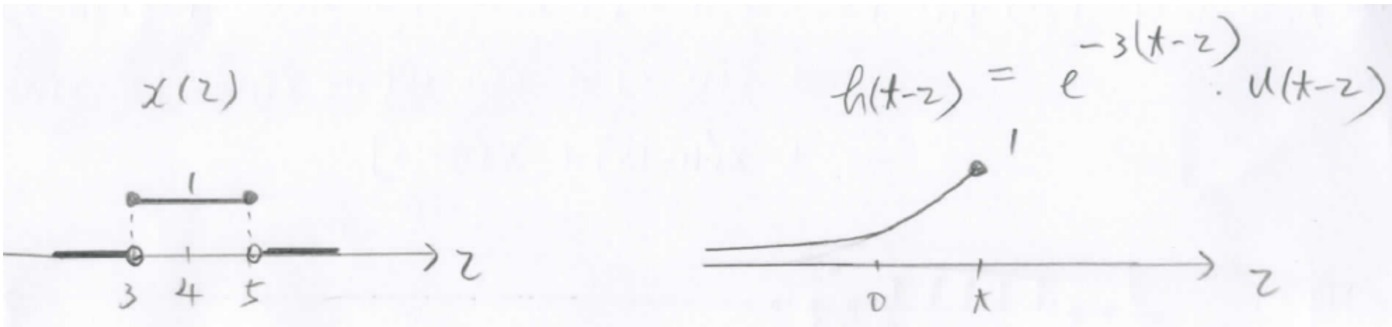
6. Let $x(t) = \text{rect}(\frac{t-4}{2})$ and $h(t) = e^{-3t} \cdot u(t)$

   a) Plot x(t) and h(t)
   b) Compute $y_1(t) = x(t) \otimes h(t)$ where $\otimes$ is convolution operation
   c) Compute and plot $y_2(t) = (\frac{dx(t)}{dt}) \otimes h(t)$
   d) How is $y_2(t)$ related to $y_1(t)$?

Ans:

a)

b)



when $t<3$ $\quad y(t) = x(t) \circledast h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = 0$

when $3 \le t < 5$ $\quad y(t) = \int_3^t 1 \cdot e^{-3(t-\tau)} d\tau = \int_3^t e^{-3t} \cdot e^{3\tau} d\tau$

$$= e^{-3t} \cdot \frac{1}{3} e^{3\tau} \Big|_3^t = \frac{1}{3} \cdot e^{-3t} \cdot \left(e^{3t} - e^{9}\right) = \frac{1}{3} - \frac{1}{3} e^{-3(t-3)}$$

when $t \ge 5$ $\quad y(t) = \int_3^5 e^{-3(t-\tau)} d\tau = e^{-3t} \frac{1}{3} e^{3\tau} \Big|_3^5 = \frac{1}{3} \cdot e^{-3t} \left(e^{15} - e^{9}\right)$

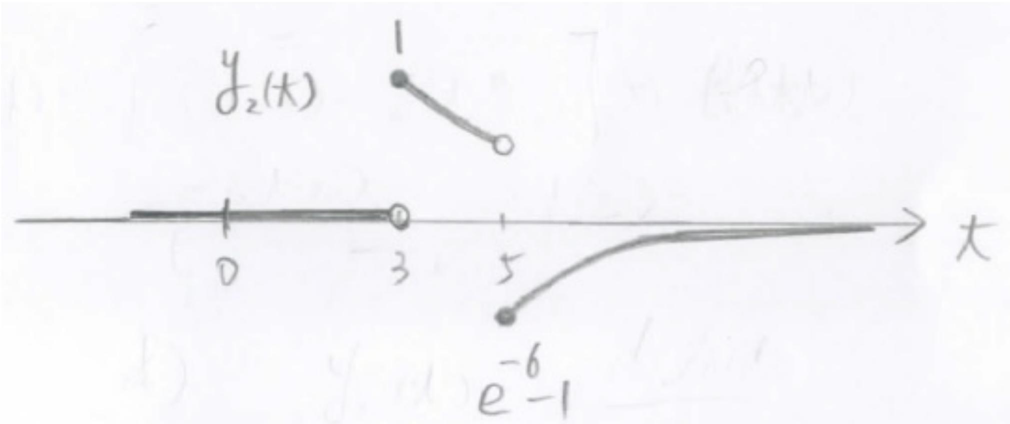
c)

$$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5), \quad y_2(t) = \left[\delta(t-3) - \delta(t-5)\right] \circledast h(t)$$

$$= e^{-3(t-3)} \cdot u(t-3) - e^{-3(t-5)} u(t-5)$$

d)

$$y_2(t) = \frac{d}{dt}\left[x(t) \circledast h(t)\right] = \frac{dy_1(t)}{dt}$$



7. For each of the following systems, the output y(t) and input x(t) are related as y(t) =T(x(t)). Determine whether the system is i) linear, ii) time-invariant or not. Please justify your answers.

c) $y(t) = T(x(t)) = \int_{-\infty}^{2t} x(\tau) d\tau$

d) $y(t) = T(x(t)) = e^{-x(t)}$

Ans:

a)

$$y_1(t) = \mathrm{T}\big(x_1(\mathrm{t})\big) = \int_{-\infty}^{2t} x_1(\tau)d\tau$$

$$y_2(t) = \mathrm{T}\big(x_2(\mathrm{t})\big) = \int_{-\infty}^{2t} x_2(\tau)d\tau$$

$$y(t) = \mathrm{T}\big(ax_1(\mathrm{t}) + bx_2(\mathrm{t})\big) = \int_{-\infty}^{2t} ax_1(\tau) + bx_2(\tau)d\tau = ay_1(\mathrm{t}) + by_2(\mathrm{t})$$

System is linear

$$y_1(t) = \mathrm{T}\big(x_1(\mathrm{t})\big) = \int_{-\infty}^{2t} x_1(\tau)d\tau$$

$$y(t) = \mathrm{T}\big(x_1(\mathrm{t}-\lambda)\big) = \int_{-\infty}^{2t} x_1(\tau-\lambda)d\tau \neq y_1(t-\lambda)$$

System is not time-invariant

b) $y_1(t) = \mathrm{T}\big(x_1(\mathrm{t})\big) = e^{-x_1(t)}$

$$y_2(t) = \mathrm{T}\big(x_2(\mathrm{t})\big) = e^{-x_2(t)}$$

$$y(t) = \mathrm{T}\big(ax_1(\mathrm{t}) + bx_2(\mathrm{t})\big) = e^{-ax_1(t)-bx_2(t)} \neq ay_1(\mathrm{t}) + by_2(\mathrm{t})$$

System is not linear

$$y_1(t) = \mathrm{T}\big(x_1(\mathrm{t})\big) = e^{-x_1(t)}$$

$$y(t) = \mathrm{T}\big(x_1(\mathrm{t}-\tau)\big) = e^{-x_1(t-\tau)} = y_1(t-\tau)$$

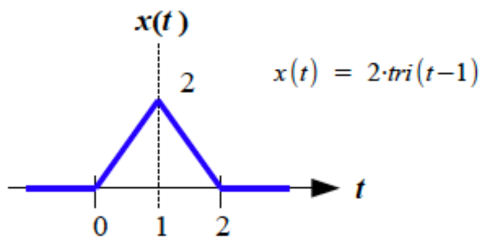
System is time-invariant

8. Consider a LTI system with input x(t) and output y(t).
It is known that $\mathrm{y(t)} = \mathrm{x(t)} * h(t)$, where ‘ * ‘ is the convolution operator.
Assume $\mathrm{y(t)} = \int_{-\infty}^{t} [x(\tau) - x(\tau-4)]d\tau$, and $\mathrm{x(t)} = 2 \cdot \Lambda(\mathrm{t}-1)$

a) Plot x(t)
b) Find h(t) and plot h(t)
c) Find expression for y(t)
d) Plot y(t)
e) Is this system BIBO stable? Is this system causal? Justify your answer.

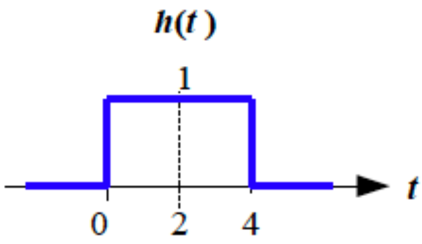
Ans:

(a)



(b)(e)

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} [\ x(\tau) - x(\tau-4)\ ]d\tau \\ &= \int_{-\infty}^{t} x(\tau)d\tau - \int_{-\infty}^{t} x(\tau-4)d\tau \qquad (\ s=\tau-4\ ,\ s=-\infty\sim\ t-4\ ) \\ &= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau - \int_{-\infty}^{t-4} x(s)ds \\ &= x(t)\otimes u(t) - \int_{-\infty}^{+\infty} x(s)u(t-s-4)ds \\ &= x(t)\otimes u(t) - x(t)\otimes u(t-4) \\ &= x(t)\ \otimes\ [u(t)-u(t-4)] \\ &= x(t)\ \otimes\ rect\left(\frac{t-2}{4}\right) \end{aligned}$$

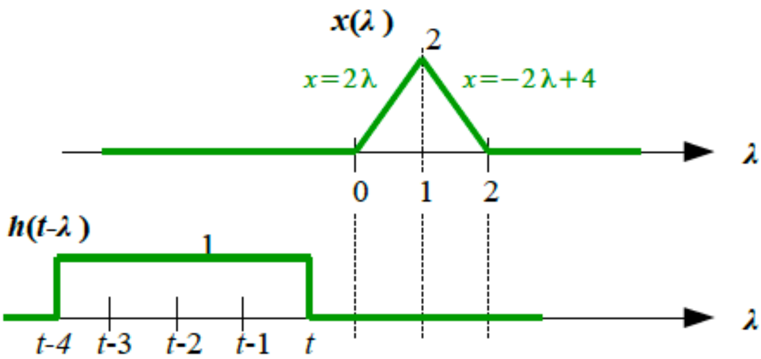


$$\Rightarrow \quad h(t) = rect\left(\frac{t-2}{4}\right) \qquad \#$$

$\Rightarrow$ Because $h(t<0)=0$, the system is causal.

$\Rightarrow$ Because $\int_{-\infty}^{\infty} |\ h(t)\ | dt = 4 < \infty$, the system is BIBO stable.

(c)(d)



① $t<0$

$$\Rightarrow \quad y(t) = 0$$

② $0\le t<1$

$$\Rightarrow \quad y(t) = \int_{0}^{t}(2\lambda)\cdot(1)d\lambda = t^2$$

③ $1\le t<2$

$$\Rightarrow \quad y(t) = \int_{0}^{1}(2\lambda)\cdot(1)d\lambda + \int_{1}^{t}(-2\lambda+4)\cdot(1)d\lambda = -t^2+4t-2$$

④ $t\ge 2$ & $(t-4)<0$ i.e., $2\le t<4$

$$\Rightarrow \quad y(t) = \int_{0}^{1} 2\cdot tri(\lambda-1)d\lambda = 2$$

⑤ $0\le(t-4)<1$ i.e., $4\le t<5$

$$\Rightarrow \quad y(t) = \int_{t-4}^{1}(2\lambda)\cdot(1)d\lambda + \int_{1}^{2}(-2\lambda+4)\cdot(1)d\lambda = -t^2+8t-14$$

⑥ $1\le(t-4)<2$ i.e., $5\le t<6$

$$\Rightarrow \quad y(t) = \int_{t-4}^{2}(-2\lambda+4)\cdot(1)d\lambda = t^2-12t+36$$

$$\Rightarrow \quad y(t) = 0$$

$$y(t) = \begin{cases} t^2 & , \quad 0 \le t < 1 \\ -t^2+4t-2 & , \quad 1 \le t < 2 \\ 2 & , \quad 2 \le t < 4 \\ -t^2+8t-14 & , \quad 4 \le t < 5 \\ t^2-12t+36 & , \quad 5 \le t < 6 \\ 0 & , \quad otherwise \end{cases}$$