112學年第二學期 電機資訊組 工程數學 第2次作業

1. Given the scalar function $f(x, y, z) = x^2yz + 4xz^2$ and $\mathbf{r} = (1, -2, -1)$, find the directional derivative of $f(x, y, z)$ at $\mathbf{r}$ along the direction $\mathbf{a} = (2, -1, -2)$.

2. Consider the surface S characterized by $2xz^2 - 3xy - 4x = 7$, find the equation of the tangent plane at $\mathbf{r} = (1, -1, 2)$.

3. Consider two surfaces S_1 and S_2 characterized by $S_1 : \ x^2 + y^2 + z^2 = 9$ and $S_2 : \ z = x^2 + y^2 - 3$, respectively. Find the angle between the two surfaces S_1 and S_2 at the point $\mathbf{r} = (2, -1, 2)$.

4. Consider the vector function

$$\mathbf{F}(x, y, z) = (y^2 - 2xyz^3)\,\mathbf{i} + (3 + 2xy - x^2z^3)\,\mathbf{j} + (6z^3 - 3x^2yz^2)\,\mathbf{k},$$

find a scalar function $f(x, y, z)$ such that $\nabla f = \mathbf{F}$.

5. Given the scalar function $f(x, y, z) = x^2yz^3$, what is the direction at point $\mathbf{r} = (2, 1, -1)$ having the maximum directional derivative? Find the maximum value of the directional derivative.

6. For the vector function $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (-x^2 + y)\,\mathbf{j} + z\,\mathbf{k}$, determine $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

7. Let $\mathbf{r} = (x, y, z)$ and $r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, determine the value of n such that $\nabla \cdot (r^n\mathbf{r}) = 0$.

8. Let $\mathbf{r} = (x, y, z)$, $r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$, and $f(r)$ is a scalar function of r. Determine $\nabla \cdot (f(r)\,\mathbf{r})$ and $\nabla \times (f(r)\,\mathbf{r})$.

9. Given $\mathbf{G}(x, y, z) = (xy - 1)\,\mathbf{i} - xz\,\mathbf{j} + (2 - yz)\,\mathbf{k}$, find the vector function $\mathbf{F}(x, y, z)$ such that $\nabla \times \mathbf{F} = \mathbf{G}$. Is the vector function $\mathbf{F}(x, y, z)$ unique?

10. Show that $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies the Laplace equation

$$\nabla^2 f = \frac{\partial^2 f(x, y, z)}{\partial x^2} + \frac{\partial^2 f(x, y, z)}{\partial y^2} + \frac{\partial^2 f(x, y, z)}{\partial z^2} = 0.$$

1.

$$\nabla f(r) = (2xyz + 4z^2,\ x^2 z,\ x^2 y + 8xz)\Big|_{(x,y,z)=(1,-2,-1)} = (8, -1, -10)$$

$$b = \frac{a}{|a|} = \frac{(2,-1,-2)}{\sqrt{2^2+(-1)^2+(-2)^2}} = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

$$D_b f = \nabla f(r) \cdot b = \frac{16+1+20}{3} = \frac{37}{3} \quad \#$$

2.

Let $f(x,y,z) = 2xz^2 - 3xy - 4x$

$$n = \nabla f(x) = (2z^2 - 3y - 4,\ -3x,\ 4xz)\Big|_{r=(1,-1,2)} = (7, -3, 8)$$

equation of tangent plane

$$(7,-3,8) \cdot (x-1, y+1, z-2) = 0 \Rightarrow 7(x-1) - 3(y+1) + 8(z-2) = 0$$

3.

Let $f_1(x,y,z) = x^2+y^2+z^2$ normal vectors n_1 and n_2 of the surface S_1 and S_2

$f_2(x,y,z) = x^2+y^2-z$ at $r=(2,-1,2)$ are given by

$$n_1 = \nabla f_1(r) = (2x, 2y, 2z)\Big|_{r=(2,-1,2)} = (4,-2,4)$$

$$n_2 = \nabla f_2(r) = (2x, 2y, -1)\Big|_{r=(2,-1,2)} = (4,-2,-1)$$

angle between S_1 and S_2 at $r=(2,-1,2)$

$$\theta = \cos^{-1}\frac{n_1\cdot n_2}{|n_1||n_2|} = \cos^{-1}\frac{16}{\sqrt{36\cdot 21}} = \cos^{-1}\frac{8}{3\sqrt{21}} = 54.4147^\circ$$

4. $F(x, y, z) = (y^2 - 2xyz^3)\,i + (3 + 2xy - x^2z^3)\,j + (6z^3 - 3x^2yz^2)\,k$

$$\frac{\partial f}{\partial x} = y^2 - 2xyz^3 \longrightarrow f(x,y,z) = xy^2 - x^2yz^3 + g(y,z)$$

$$\frac{\partial f}{\partial y} = 3 + 2xy - x^2z^3 \longrightarrow f(x,y,z) = 3y + xy^2 - x^2yz^3 + h(x,z)$$

$$\frac{\partial f}{\partial z} = 6z^3 - 3x^2yz^2 \longrightarrow f(x,y,z) = \frac{3}{2}z^4 - x^2yz^3 + k(x,y)$$

$$\Rightarrow \quad f(x,y,z) = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + c$$

5. The direction of $f(x,y,z)$ at $r=(2,1,-1)$ having the maximum directional directive is given by

$$a = \nabla f(r) = (2xyz^3,\ x^2z^3,\ 3x^2yz^2)\Big|_{r=(2,1,-1)} = (-4, -4, 12)$$

maximum directional derivative

$$|a| = |\nabla f(x)| = \sqrt{(-4)^2 + (-4)^2 + 12^2} = \sqrt{176} = 4\sqrt{11}$$

6.

divergence $\nabla \cdot F$

$$\nabla \cdot F = 1 + 1 + 1 = 3$$

curl $\nabla \times F$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & -x^2+y & z \end{vmatrix} = -2(x+y)\,k$$

7.

Let $r = (x, y, z)$ $\quad r = |r| = (x^2+y^2+z^2)^{1/2}$, determine the value of n such that $\nabla \cdot (r^n r) = 0$

$$r^n r = x(x^2+y^2+z^2)^{n/2}\, i + y(x^2+y^2+z^2)^{n/2}\, j + z(x^2+y^2+z^2)^{n/2}\, k$$

$$\nabla \cdot (r^n r) = (3+n)\, r^n = 0 \quad \rightarrow \quad n = -3$$

8.

$$\nabla \cdot (f(r)\, r) = \frac{\partial f(x)}{\partial x} x + f(x) + \frac{\partial f(r)}{\partial y} y + f(r) + \frac{\partial f(r)}{\partial z} z + f(r)$$

$$= f'(r) \cdot \frac{x^2}{r} + f(r) + f'(r) \cdot \frac{y^2}{r} + f(r) + f'(r) \cdot \frac{z^2}{r} + f(r)$$

$$= r f'(r) + 3 f(r)$$

$$\nabla \times (f(r)\, r) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$$

$$= \left(z \frac{\partial f(r)}{\partial y} - y \frac{\partial f(r)}{\partial z} \right) i - \left(z \frac{\partial f(r)}{\partial x} - x \frac{\partial f(r)}{\partial z} \right) j + \left(y \frac{\partial f(x)}{\partial x} - x \frac{\partial f(x)}{\partial y} \right) k = 0$$

9. Let $F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1(x,y,z) & f_2(x,y,z) & f_3(x,y,z) \end{vmatrix} = (xy-1)i - xy\,j + (z-yz)k$$

$$\Rightarrow \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} = xy-1, \quad \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} = -xz, \quad \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = z - yz$$

Assume $f_3(x,y,z)$ is independent of y

$$-\frac{\partial f_2}{\partial z} = xy-1 \Rightarrow f_2(x,y,z) = -xyz + z + C_2$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = -yz - 1 \Rightarrow f_2(x,y,z) = -xyz + z + C_2$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = -yz - \frac{\partial f_1}{\partial y} = z - yz \Rightarrow f_1(x,y,z) = -zy + C_1$$

$$\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} = -\frac{\partial f_2}{\partial x} = -xz \Rightarrow f_3(x,y,z) = \frac{1}{2}x^2 z + C_3$$

function independent by y

$$F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)) = \left(-zy, -xyz+z, \frac{x^2 z}{2}\right)$$

10.

$$\frac{\partial f(x,y,z)}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

$$\Rightarrow \frac{\partial^2 f(x,y,z)}{\partial x^2} + \frac{\partial^2 f(x,y,z)}{\partial y^2} + \frac{\partial^2 f(x,y,z)}{\partial z^2}$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z)(x^2+y^2+z^2)^{-5/2} = 0$$