112 學年第二學期 電機資訊組 工程數學 第 5 次作業

1. Find the eigenvalues and eigenfunctions of the given boundary value problem. Assume that all eigenvalues are real.

(a) $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) = 0$.

(b) $y'' + \lambda y = 0$, $y'(0) = 0$, $y(1) = 0$.

2. Consider the problem

$$y'' + \lambda y = 0, \quad 2y(0) + y'(0) = 0, \quad y(1) = 0.$$

(a) Find the equation satisfied by the positive eigenvalues. Show that there is an infinite sequence of such eigenvalues and that $\lambda_n \approx (2n+1)^2\pi^2/4$ for large n.

(b) Find the equation satisfied by the negative eigenvalues. Show that there is exactly one negative eigenvalue.

3. Consider the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

Show that if $\phi_m(x)$ and $\phi_n(x)$ are eigenfunctions corresponding to the eigenvalues λ_m and λ_n, respectively, with $\lambda_m \neq \lambda_n$, then

$$\int_0^L \phi_m(x)\phi_n(x)\,dx = 0.$$

Hint: Note that

$$\phi_m'' + \lambda_m \phi_m = 0, \qquad \phi_n'' + \lambda_n \phi_n = 0.$$

Multiply the first of these equation by ϕ_n, the second by ϕ_m, and integrate from 0 to L using integration by parts.

4. Determine whether the given function is periodic. If so, find its fundamental period.

(a) $\cos 2\pi x$. (b) $\sinh 2x$. (c) $\tan \pi x$. (d) $f(x) = \begin{cases} 0 & 2n-1 \le x \le 2n \\ 1, & 2n < x \le 2n+1 \end{cases}, \quad n = 0, \pm1, \pm2, \cdots$

5. Find the Fourier series representation for the following periodic function.

$$f(x) = \begin{cases} x, & -\pi \le x < 0 \\ 0, & 0 \le x < \pi \end{cases}, \quad f(x+2\pi) = f(x).$$

6. Given the periodic function

$$f(x) = 3 + \sqrt{3}\cos 2x + \sin 2x + \sin 3x - \frac{1}{2}\cos\left(5x + \frac{\pi}{3}\right),$$

find the Fourier series representation of $f(x)$.

7. Consider the periodic function $f(x) = x + \pi, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x)$.

(a) Find the Fourier series representation of $f(x)$.

(b) Use the result of (a) to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

8. Find the solution of the initial value problem

$$y'' + \omega^2 y = f(t), \quad y(0) = 1, \quad y'(0) = 0,$$

where $\omega \neq (2n-1)\pi$, $n = 1, 2, \cdots$, $f(t)$ is periodic with period 2 and

$$f(t) = \begin{cases} 1 - t, & 0 \leq t < 1 \\ -1 + t, & 1 \leq t < 2 \end{cases}$$

9. Assuming that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

show that

$$\frac{1}{L} \int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This relation between a function f and its Fourier coefficients is known as Parseval's equation.

10. Find the required Fourier series representation for the following function.

(a) $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x < 2 \end{cases}$, sine series, period 4.

(b) $f(x) = L - x, \quad 0 \leq x \leq L, \quad$ cosine series, period $2L$,

1. (a) $y'' + \lambda y = 0$, $y'(1) = 0$

Case 1 : $\lambda = \mu^2 > 0$, $\mu > 0$,

characterstic equation of the ODE

$$s^2 + \mu^2 = 0 \Rightarrow s_{1,2} = \pm \mu i$$

generation solution of the ODE

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \Rightarrow y'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$$

using boundary condition $y(0) = 0$

$$y(0) = c_1 = 0$$

using boundary condition $y'(1) = 0$

$$y'(1) = \mu c_2 \cos \mu = 0 \Rightarrow \mu_n = (2n-1)\pi/2 , \quad n = 1, 2, \ldots$$

eigenvalues and eigenfunctions

$$\lambda_n = \mu_n^2 = \left(\frac{(2n-1)\pi}{2}\right)^2 , \quad \phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2}\right) , \quad n = 1, 2, \ldots$$

Case 2 : $\lambda = 0$,

general solution of the ODE

$$y'' = 0 \Rightarrow y(x) = c_1 x + c_2 \Rightarrow y'(x) = c_1$$

using the boundary condition $y(0) = 0 \Rightarrow y(0) = c_2 = 0$

using the boundary condition $y'(1) = 0 \Rightarrow y'(1) = c_1 = 0$

$$y(x) = 0, \quad -\infty < x < \infty \Rightarrow \text{trivial solution}$$

Case 3 : $\lambda = -\mu^2 > 0$, $\mu > 0$

charastic equation of the ODE

$$s^2 - \mu^2 = 0 \Rightarrow s_{1,2} = \pm u$$

general solution of the ODE

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$$

$$\Rightarrow y'(x) = \mu c_1 \sinh \mu x + \mu c_2 \cosh \mu x$$

using the boundary condition $y(0) = 0$

$$y(0) = c_1 = 0$$

using the boundary condition $y'(1) = 0$

$$y'(1) = \mu c_2 \cosh \mu = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow y(x) = 0, \quad -\infty < x < \infty \Rightarrow \text{trivial solution}$$

1.(b)

Case 1: $\lambda = \mu^2 > 0$, $\mu > 0$

characterstic equation of the ODE

$$s^2 + \mu^2 = 0 \Rightarrow s_{1,2} = \pm \mu i$$

general solution of the ODE

$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x \Rightarrow y'(x) = -\mu c_1 \sin \mu x + \mu c_2 \cos \mu x$$

using boundary condition $y'(0) = 0 \Rightarrow y'(0) = \mu c_2 = 0$, $c_2 = 0$

using boundary condition $y(1) = 0 \Rightarrow y(1) = c_1 \cos \mu = 0 \Rightarrow \mu_n = (2n-1)\pi/2$, $n = 1, 2, \ldots$

eigenvalue and eigenfunction

$$\lambda_n = \mu_n^2 = \left(\frac{(2n-1)\pi}{2}\right)^2, \quad \phi_n(x) = \cos\left(\frac{(2n-1)\pi x}{2}\right), \quad n = 1, 2, \ldots$$

Case 2: $\lambda = 0 \Rightarrow y'' = 0$

general solution of the ODE

$$y(x) = c_1 x + c_2 \Rightarrow y'(x) = c_1$$

using boundary condition $y'(0) = 0 \Rightarrow y'(0) = c_1 = 0$

using boundary condition $y(1) = 0 \Rightarrow y(1) = c_2 = 0$

$\Rightarrow y(x) = 0, \ -\infty < x < \infty \Rightarrow$ trivial solution

Case 3: $\lambda = -\mu^2 > 0, \ \mu > 0$

characterstic equation of the ODE

$$s^2 - \mu^2 = 0 \Rightarrow s_{1,2} = \pm u$$

general solution of the ODE

$$y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$$

$$\Rightarrow y'(x) = \mu c_1 \sinh \mu x + \mu c_2 \cosh \mu x$$

using the boundary condition $y'(0) = 0$

$$\Rightarrow y'(0) = \mu c_2 = 0 \Rightarrow c_2 = 0$$

using the boundary condition $y(1) = 0$

$$\Rightarrow y(1) = c_1 \cosh \mu = 0 \Rightarrow c_1 = 0$$

$\Rightarrow y(x) = 0, \ -\infty < x < \infty \Rightarrow$ trivial solution

2. consider positive eigenvalues, $\lambda = \mu^2 > 0$

characteristic equation of the ODE

$$s^2 + \mu^2 = 0 \quad \Rightarrow \quad s_{1,2} = \pm \mu i$$

general solution of the ODE

$$y(x) = c_1 \cos\mu x + c_2 \sin\mu x \quad \Rightarrow \quad y'(x) = -\mu c_1 \sin\mu x + \mu c_2 \cos\mu x$$

using the boundary condition $2y(0) + y'(0) = 0 \quad \Rightarrow \quad 2y(0) + y'(0) = 2c_1 + \mu c_2 = 0$

using the boundary condition $y(1) = 0 \quad \Rightarrow \quad y(1) = c_1 \cos\mu + c_2 \sin\mu = 0$

solving the above two equation, we have

$$c_2\left(\sin\mu - \frac{\mu}{2}\cos\mu\right) = 0 \quad \Rightarrow \quad \sin\mu - \frac{\mu}{2}\cos\mu = 0 \quad \Rightarrow \quad \tan\mu = \frac{\mu}{2}$$

from the graph $\begin{cases} f(\mu) = \tan\mu \\ g(\mu) = \mu/2 \end{cases}, \mu > 0$

$\Rightarrow$ n becomes large, $\tan\mu = \mu/2 \quad \mu_n \approx \frac{(2n+1)\pi}{2} \quad \Rightarrow \quad \lambda_n = \mu_n^2 \approx \frac{(2n+1)^2\pi^2}{4}$

(b) consider negative eigenvalue $\Rightarrow \lambda = -\mu^2 < 0, \quad \mu > 0$

characterstic equation of the ODE

$$s^2 - \mu^2 = 0 \Rightarrow s_{1,2} = \pm\mu$$

general solution of the ODE

$$y(x) = C_1 \cosh(\mu x) + C_2 \sinh(\mu x) \Rightarrow y'(x) = \mu C_1 \sinh(\mu x) + \mu C_2 \cosh(\mu x)$$

using boundary condition $2y(0) + y'(0) = 0 \Rightarrow 2y(0) + y'(0) = 2C_1 + \mu C_2 = 0$

using boundary condition $y(1) = 0 \Rightarrow y(1) = C_1 \cosh\mu + C_2 \sinh\mu = 0$

solving above two equations, we have

$$C_2\left(\sinh\mu - \frac{\mu}{2}\cosh\right) = 0 \Rightarrow \sin\mu - \frac{\mu}{2}\cosh = 0 \Rightarrow \tanh = \frac{\mu}{2}$$

from the graph $\begin{cases} f(\mu) = \tanh\mu \\ g(\mu) = \mu/2 \end{cases}, \quad u > 0$

$$\mu_1 \approx 1.915 \Rightarrow \lambda_1 = -\mu_1^2 \approx -3.667$$

3. $\phi_n \phi_m'' + \lambda_m \phi_m \phi_n = 0$, $\phi_m \phi_n'' + \lambda_n \phi_n \phi_m = 0$

$$\Rightarrow (\lambda_m - \lambda_n)\phi_m(x)\phi_n(x) = \phi_m(x)\phi_n''(x) - \phi_n(x)\phi_m''(x)$$

integral equation above from 0 to L

$$(\lambda_m - \lambda_n)\int_0^L \phi_m(x)\phi_n(x)\,dx = \int_0^L [\phi_m(x)\phi_n''(x) - \phi_n(x)\phi_m''(x)]\,dx$$

$$\begin{aligned}
\int_0^L \phi_m(x)\phi_n''(x)\,dx &= \phi_m(x)\phi_n'(x)\Big|_0^L - \int_0^L \phi_m'(x)\phi_n'(x)\,dx \\
&= \phi_m(L)\phi_n'(L) - \phi_m(0)\phi_n'(0) - \int_0^L \phi_m'(x)\phi_n'(x)\,dx \\
&= \phi_m(L)\cdot 0 - 0\cdot\phi_n'(0) - \int_0^L \phi_m'(x)\phi_n'(x)\,dx = -\int_0^L \phi_m'(x)\phi_n'(x)\,dx
\end{aligned}$$

$$\begin{aligned}
\int_0^L \phi_n(x)\phi_m''(x)\,dx &= \phi_n(x)\phi_m'(x)\Big|_0^L - \int_0^L \phi_n'(x)\phi_m'(x)\,dx \\
&= \phi_n(L)\phi_m'(L) - \phi(0)\phi_m'(0) - \int_0^L \phi_n'(x)\phi_m'(x)\,dx \\
&= \phi_n(L)\cdot 0 - 0\cdot\phi_m'(0) - \int_0^L \phi_n'(x)\phi_m'(x)\,dx = -\int_0^L \phi_n'(x)\phi_m'(x)\,dx
\end{aligned}$$

we have

$$(\lambda_m - \lambda_n) \int_0^L \phi_m(x)\phi_n(x)\,dx = \int_0^L [\phi_m(x)\phi_n''(x) - \phi_n(x)\phi_m''(x)]\,dx = 0$$

since $\lambda_m \neq \lambda_n$

$$\int_0^L \phi_m(x)\phi_n(x)\,dx = \frac{0}{\lambda_m - \lambda_n} = 0$$

4. (a) $\cos 2\pi x$ is periodic function with fundamental period $T = 1$

(b) $\sinh 2x$ is not a periodic function

(c) $\tan \pi x$ is a periodic function with functional periodic $T = 1$

(d) $f(x) = \begin{cases} 0 & 2n-1 < x \le 2n \\ 1, & 2n < x \le 2n+1 \end{cases}$, $n = 0, \pm 1, \pm 2, \cdots$ is a periodic function with fundamental periodic $T = 2$

5. $2L = 2X \Rightarrow L = \pi$

Fourier series of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{\pi} + b_m \sin\frac{m\pi x}{\pi}\right) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = -\frac{\pi}{2}$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos mx\,dx = \frac{1-\cos m\pi}{\pi m^2} = \begin{cases} 0, & m \text{ is even} \\ \frac{2}{\pi m^2}, & m \text{ is odd} \end{cases}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} x\sin mx\,dx = \frac{-\cos m\pi}{m} = \frac{(-1)^{m+1}}{m}, \quad m = 1, 2, \dots$$

Fourier series of $f(x)$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi(2n-1)^2}\cos((2n-1)x) + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

6.

rewrite $f(x)$

$$f(x) = 3 + \sqrt{3}\cos 2x + \sin 2x + \sin 3x - \frac{1}{4}\cos 5x + \frac{\sqrt{3}}{4}\sin 5x$$

fourier series expansion of $f(x)$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m\cos(mx) + b_m\sin(mx)\right]$$

where

$$\begin{cases} a_0 = 6,\ a_2 = \sqrt{3},\ a_5 = \frac{-1}{4},\ b_2 = 1,\ b_3 = 1,\ b_5 = \frac{\sqrt{3}}{4} \\ a_m = b_m = 0, \quad \text{otherwise} \end{cases}$$

1. (a) since $L = \pi$

fourier series coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\,dx = 2\pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = 0 \quad , \quad m = 1, 2, \ldots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{2}{m}(-1)^{m+1}$$

therefore, fourier series expansion of $f(x)$

$$f(x) = \pi + \sum_{m=1}^{\infty}\frac{2}{m}(-1)^{m+1}\sin(mx) \quad \#$$

(b) from the result of (a)

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \pi = \pi + \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

$$\Rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi/2}{2} = \frac{\pi}{4} \quad \#$$

8. fourier series representation of $f(x)$

since $2L = T = 2$

$$L = 1 \quad , \quad a_0 = \int_0^1 (1-t)\,dt + \int_1^2 (-1+t)\,dt - 1$$

$$a_m = \int_0^1 (1-t)\cos m\pi t\,dt + \int_1^2 (-1+t)\cos m\pi t\,dt = \frac{2(1-\cos m\pi)}{m^2\pi^2} = \begin{cases} \frac{4}{m^2\pi^2} , & m \text{ is odd} \\ 0 , & m \text{ is even} \end{cases}$$

$$b_m = \int_0^1 (1-t)\sin m\pi t\,dt + \int_1^2 (-1+t)\sin m\pi t\,dt = 0$$

therefore

$$f(t) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi t$$

by undetermined coefficients, particular solution $\phi(t)$

$$\phi(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2n-1)\pi t$$

then

$$\phi'(t) = -\sum_{n=1}^{\infty} (2n-1)\pi C_n \sin(2n-1)\pi t \ , \quad \phi''(t) = -\sum_{n=1}^{\infty} (2n-1)^2\pi^2 C_n \cos(2n-1)\pi t$$

Substituting $\phi(t)$ and $\phi''(t)$ into ODE and comparing the coefficient on both sides

$$C_0 = \frac{1}{2w^2}, \quad C_n = \frac{4}{\pi^2(2n-1)^2[w^2-(2n-1)^2\pi^2]}$$

$$y(t) = A_1 \cos wt + A_2 \sin wt + \frac{1}{2w^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[w^2-(2n-1)^2\pi^2]} \cos(2n-1)\pi t$$

$$y(0) = 1$$

$$A_1 + \frac{1}{2w^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[w^2-(2n-1)^2\pi^2]} = 1$$

$$\Rightarrow A_1 = 1 - \frac{1}{2w^2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[w^2-(2n-1)^2\pi^2]}$$

$$y'(t) = -A_1 w \sin wt + A_2 w \cos wt - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)[w^2-(2n-1)^2\pi^2]} \sin(2n-1)\pi t$$

$$y'(0) = 0 \quad \Rightarrow \quad A_2 = 0$$

$$y(t) = \left[1 - \frac{1}{2w^2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[w^2-(2n-1)^2\pi^2]}\right] \cos wt + \frac{1}{2w^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2[w^2-(2n-1)^2\pi^2]} \cos(2n-1)\pi t$$

$$= \cos wt + \frac{1-\cos wt}{2w^2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi t - \cos wt}{(2n-1)^2[w^2-(2n-1)^2\pi^2]}$$

9.

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2 = \frac{1}{L}\int_{-L}^{L} f(x)\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]dx$$

$$= a_0\frac{1}{L}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left[a_n\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx + b_n\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\right]$$

$$= \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \#$$

10.(a)

$$g(x) = \begin{cases} -1, & -2 < x \le -1 \\ x, & -1 < x \le 1 \\ 1, & 1 < x < 2 \end{cases} \Rightarrow \quad g(x) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{2}$$

$$b_m = \frac{2}{2}\int_0^2 f(x)\sin\frac{m\pi x}{2}dx = \frac{2}{2}\int_0^1 x\sin\frac{m\pi x}{2}dx + \frac{2}{2}\int_1^2 \sin\frac{m\pi x}{2}dx$$

$$= \frac{4\sin\frac{m\pi}{2} - 2m\pi\cos m\pi}{m^2\pi^2}, \quad m = 1, 2, \ldots$$

$$f(x) = \sum_{m=1}^{\infty}\left(\frac{4\sin\frac{m\pi}{2} - 2m\pi\cos m\pi}{m^2\pi^2}\right)\sin\frac{m\pi x}{2}, \quad 0 \le x < 2 \quad \#$$

10. (b)

$$g(x) = \begin{cases} x+L, & -L \le x \le 0 \\ L-x, & 0 < x \le L \end{cases} \Rightarrow \quad g(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L g(x)\,dx = \frac{2}{L}\int_0^L (L-x)\,dx = L$$

$$a_m = \frac{2}{L}\int_0^L (L-x)\cos\frac{m\pi x}{L}\,dx = \frac{2L(1-\cos m\pi)}{m^2\pi^2} = \begin{cases} 0, & m \text{ is even} \\ \frac{4L}{m^2\pi^2}, & m \text{ is odd} \end{cases}$$

$$f(x) = \frac{L}{2} + \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}, \quad 0 \le x \le L$$

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