111學年度第二學期 電機資訊組 工程數學 期中考參考解答

1. (10%) 考慮由向量 $\mathbf{a} = (2, -6, 3)$ 與 $\mathbf{b} = (4, 3, -1)$ 所張開的平面，求該平面的單位法線向量。

<u>*Solution:*</u> The unit vector $\mathbf{n}$ is given by

$$\mathbf{n} = \pm \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}.$$

Since

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & 3 \\ 4 & 3 & -1 \end{vmatrix} = -3\,\mathbf{i} + 14\,\mathbf{j} + 30\,\mathbf{k}, \quad |\mathbf{A} \times \mathbf{B}| = \sqrt{(-3)^2 + 14^2 + 30^2} = \sqrt{1105}.$$

we have

$$\mathbf{n} = \pm \frac{1}{\sqrt{1105}} \left(-3\,\mathbf{i} + 14\,\mathbf{j} + 30\,\mathbf{k} \right).$$

2. (10%) 求曲線 $C: \mathbf{r}(t) = (e^t \cos t,\ e^t \sin t,\ e^t),\ -\infty < t < \infty$ 的曲度(curvature) κ.

<u>*Solution:*</u> The tangent vector of the curve is given by

$$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t) \implies |\mathbf{r}'(t)| = \sqrt{3} e^t.$$

Moreover,

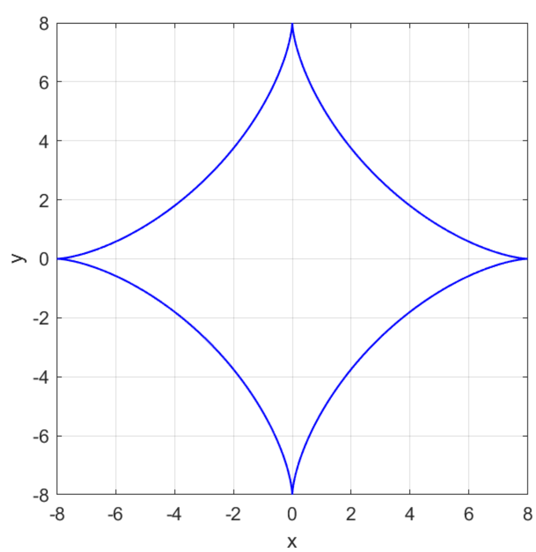
$$\mathbf{r}''(t) = (-2e^t \sin t,\ 2e^t \cos t,\ e^t).$$

Then the curvature κ is given by

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{|(e^{2t} \sin t - e^{2t} \cos t,\ -e^{2t} \cos t - e^{2t} \sin t,\ 2e^{2t})|}{3\sqrt{3} e^{3t}} = \frac{\sqrt{6} e^{2t}}{3\sqrt{3} e^{3t}} = \frac{\sqrt{2}}{3} e^{-t}.$$

3. (10%) 求曲線 $x^{2/3} + y^{2/3} = 4$ 的長度。

<u>*Solution:*</u> The curve is shown in the following figure.



The parametric representation of the curve is given by

$$\mathbf{r}(t) = (8\cos^3 t, 8\sin^3 t), \quad 0 \le t \le 2\pi.$$

Since

$$\mathbf{r}'(t) = (-24\cos^2 t \sin t, 24\sin^2 t \cos t),$$

the length of $\mathbf{r}'(t)$ is given by

$$|\mathbf{r}'(t)| = \sqrt{24^2\cos^4 t \sin^2 t + 24^2 \sin^4 t \cos^2 t} = \sqrt{24^2 \cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)}$$

$$= \sqrt{24^2 \sin^2 t \cos^2 t} = 12|\sin 2t|, \quad 0 \le t \le 2\pi,$$

the length of the curve is

$$\int_0^{2\pi} 12|\sin 2t|\, dt = 4\int_0^{\pi/2} 12 \sin 2t\, dt = 4 \times 12 = 48.$$

4. (10%) 求曲面 $x^2 + y^2 - z = 0$ 在點 $\mathbf{r} = (1,1,2)$ 的切平面單位法線向量 $\mathbf{n}$。

<u>*Solution:*</u> The level surface is given by

$$f(x,y,z) = x^2 + y^2 - z = 0.$$

The unit normal vector $\mathbf{n}$ of the surface S at point $\mathbf{r} = (1,1,2)$ is given by

$$\mathbf{N} = \nabla f(x,y,z)\Big|_{\mathbf{r}=(1,1,2)} = (2x, 2y, -1)\Big|_{\mathbf{r}=(1,1,2)} = (2,2,-1)$$

$$\Longrightarrow \quad \mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

Of course, $-\mathbf{n}$ is also a solution.

5. (10%) 求純量函數 $f(x,y,z) = 2x + y + z^2$ 在點 $\mathbf{r} = (0,0,0)$ 沿著向量 $\mathbf{a} = (1,1,1)$ 的方向導數。

<u>*Solution:*</u> We first normalize the directional vector as

$$\mathbf{b} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{3}}(1,1,1).$$

Then the directional derivative of $f(x,y,z)$ at $\mathbf{r} = (0,0,0)$ along the direction $\mathbf{b}$ is given by

$$D_{\mathbf{b}} f(\mathbf{r}) = \nabla f(\mathbf{r}) \cdot \mathbf{b} = (2,1,2z)\Big|_{\mathbf{r}=(0,0,0)} \cdot \mathbf{b} = (2,1,0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \sqrt{3}.$$

6. (10%) 考慮純量函數 $f(x,y,z)=x^2-y^2-z^2$ 與點 $\mathbf{r}=(1,1,0)$，求

(a) 最大方向導數的方向 $\mathbf{a}$。

<u>*Solution:*</u> The direction of the maximum rate of change is the direction of the gradient vector given by

$$\nabla f(\mathbf{r})=(2x,-2y,-2z)\Big|_{\mathbf{r}=(1,1,0)}=(2,-2,0).$$

(b) 考慮 $f(x,y,z)=x^2-y^2-z^2=0$ 所表示的等值曲面，求通過點 $\mathbf{r}=(1,1,0)$ 的法線參數式。

<u>*Solution:*</u> The normal vector $\mathbf{N}$ at $\mathbf{r}=(1,1,0)$ is given by

$$\nabla f(\mathbf{r})=(2x,-2y,-2z)\Big|_{\mathbf{r}=(1,1,0)}=(2,-2,0).$$

The parametric representation of the normal line is

$$\frac{x-1}{2}=\frac{y-1}{-2}=\frac{z}{0}=t \quad\Longrightarrow\quad x=2t+1,\ y=-2t+1,\ z=0,\ t\in\mathbb{R}.$$

7. (10%) 考慮向量函數 $\mathbf{F}(x,y,z)=xz\,\mathbf{i}+(x-y)^2\,\mathbf{j}+2x^2yz\,\mathbf{k}$，求 (a) $\nabla\cdot\mathbf{F}$ (b) $\nabla\times\mathbf{F}$.

<u>*Solution:*</u> (a) The divergence of $\mathbf{F}$ is given by

$$\nabla\cdot\mathbf{F}=z-2(x-y)+2x^2y.$$

(b) The curl of $\mathbf{F}$ is given by

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xz & (x-y)^2 & 2x^2yz\end{vmatrix}=(2x^2z,x-4xyz,2(x-y)).$$

8. (10%) 求線積分 $\int_C\mathbf{F}(\mathbf{r})\cdot d\mathbf{r}$，其中積分路徑 C 的方程式爲 $x^2+y^2=9$ (逆時鐘方向)，向量函數爲 $\mathbf{F}=(2x-y+z)\,\mathbf{i}+(x+y-z^2)\,\mathbf{j}+(3x-2y+4z)\,\mathbf{k}$。

<u>*Solution:*</u> The parametric representation of the circle C is

$$\mathbf{r}(t)=(3\cos t,3\sin t,0),\quad 0\le t\le 2\pi \quad\Longrightarrow\quad \mathbf{r}'(t)=(-3\sin t,3\cos t,0).$$

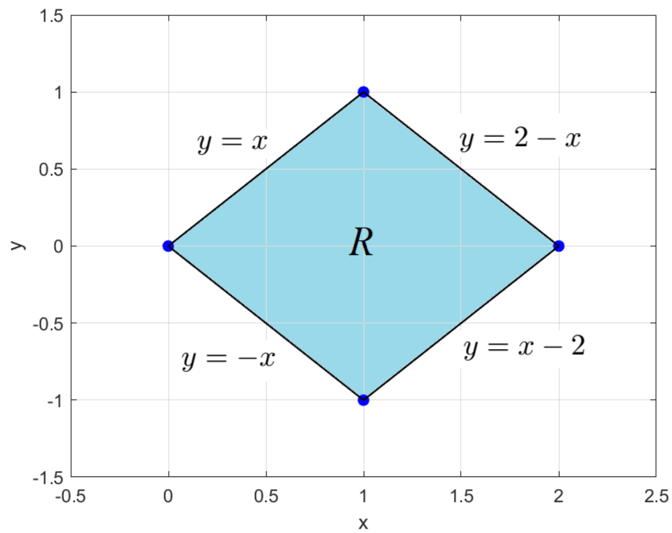
Then the line integral is

$$\int_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(6\cos t-3\sin t,3\cos t+3\sin t,9\cos t-6\sin t)\cdot(-3\sin t,3\cos t,0)\,dt$$

$$=\int_0^{2\pi}(9-9\sin t\cos t)\,dt=\int_0^{2\pi}(9-4.5\sin 2t)\,dt=9\cdot 2\pi=18\pi.$$

9. (10%) 求雙重積分 $\iint_R (x^2+y^2)\,dxdy$ 的值，其中 R 為一四邊形區域，該區域的四個頂點分別為 $(0,0)$，$(1,1)$，$(1,-1)$，$(2,0)$。

Solution: The region R is shown in the following figure.



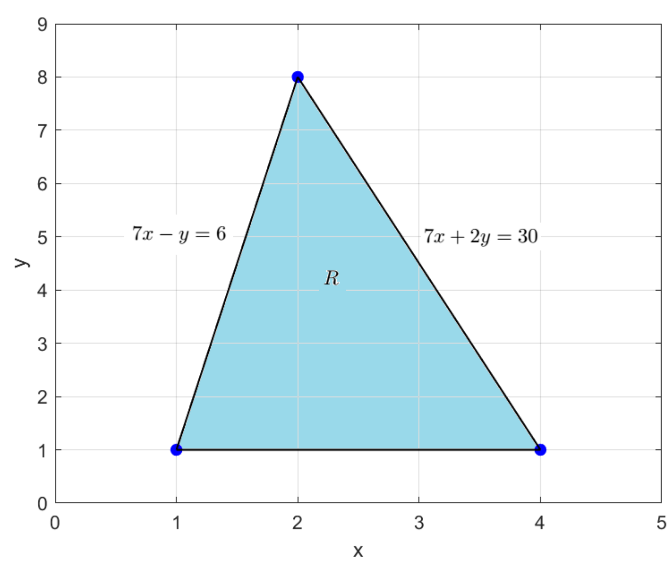
The double integral is given by

$$\int_0^1 \int_{-x}^{x} (x^2+y^2)\,dydx + \int_1^2 \int_{x-2}^{2-x} (x^2+y^2)\,dydx = \int_0^1 \left(x^2y+\frac{y^3}{3}\right)\Big|_{-x}^{x} dx + \int_1^2 \left(x^2y+\frac{y^3}{3}\right)\Big|_{x-2}^{2-x} dx$$

$$= \int_0^1 \frac{8}{3}x^3\,dx + \int_1^2 \left(-\frac{8}{3}x^3+8x^2-8x+\frac{16}{3}\right) dx = \frac{2}{3}x^4\Big|_0^1 + \left(-\frac{2}{3}x^4+\frac{8}{3}x^3-4x^2+\frac{16}{3}x\right)\Big|_1^2 = \frac{8}{3}.$$

10. (10%) 考慮向量函數 $\mathbf{F}(x,y)=x^2\,\mathbf{i}-2xy\,\mathbf{j}$，積分路徑 C 為三角形區域的邊界，該三角形區域的三個頂點座標分別為 $(1,1)$，$(4,1)$，與 $(2,8)$。求線積分 $\oint_C \mathbf{F}\cdot d\mathbf{r}$ 的值。

Solution: The region R is shown in the following figure.



Since $M(x,y)=x^2$ and $N(x,y)=-2xy$, by Green's theorem, the line integral is equal to the following double integral

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dA = \iint_R -2y\, dA = \int_1^8 \int_{\frac{y+6}{7}}^{\frac{30-2y}{7}} -2y\, dx dy$$

$$= \int_1^8 -2xy\Big|_{\frac{y+6}{7}}^{\frac{30-2y}{7}} dy = -\frac{2}{7}\int_1^8 (24y-3y^2)\, dy = -70.$$