112學年第二學期 電機資訊組 工程數學 第4次作業

1. Find a parametric representation for the following surface.

(a) The plane through the origin that contains the vectors $\mathbf{i}-\mathbf{j}$ and $\mathbf{j}-\mathbf{k}$.

(b) The part of the hyperboloid $4x^2-4y^2-z^2=4$ that lies in front of the yz-plane (i.e., $x\geq 0$).

(c) The part of the sphere $x^2+y^2+z^2=16$ that lies between the planes $z=-2$ and $z=2$.

2. Find an equation of the tangent plane to the given parametric surface at the specified point P.

(a) $x=u+v$, $y=3u^2$, $z=u-v$; point: $P=(2,3,0)$.

(b) $\mathbf{r}(u,v)=\sin u\,\mathbf{i}+\cos u\sin v\,\mathbf{j}+\sin v\,\mathbf{k}$; point $P=\mathbf{r}\left(\frac{\pi}{6},\frac{\pi}{6}\right)$.

3. Find the area of the surface.

(a) The part of the plane $3x+2y+z=6$ that lies in the first octant.

(b) The part of the sphere $x^2+y^2+z^2=b^2$ that lies inside the cylinder $x^2+y^2=a^2$, where $0<a<b$.

4. Compute the surface integral $\iint_S x^2\,dA$, where S is the unit sphere $x^2+y^2+z^2=1$.

5. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x,y,z)=y\mathbf{i}+x\mathbf{j}+z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z=1-x^2-y^2$ and the plane $z=0$.

6. Use Gauss's law to find the charge enclosed by the cube with vertices $(\pm 1,\,\pm 1,\,\pm 1)$, i.e.,

$$Q=\epsilon_0\oiint_S \mathbf{E}\cdot d\mathbf{S},$$

if the electric field is

$$\mathbf{E}(x,y,z)=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}.$$

7. Evaluate the line integral

$$\oint_C (y+\sin x)\,dx+(z^2+\cos y)\,dy+x^3\,dz,$$

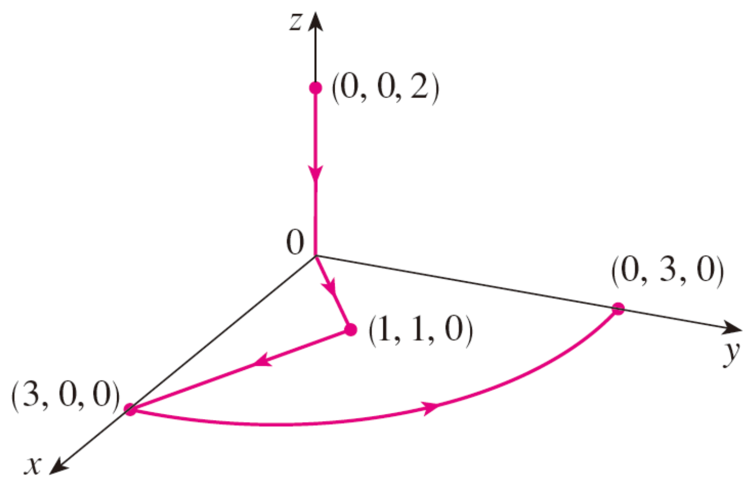
where

$$C:\mathbf{r}(t)=(\sin t,\,\cos t,\,\sin 2t),\quad 0\leq t\leq 2\pi.$$

8. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\mathbf{k},$$

evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(0, 3, 0)$ shown in the figure.



9. Find the surface integral $\oiint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k},$$

and S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

a, b, and c are all positive numbers.

10. If $\mathbf{a}$ is a constant vector, $\mathbf{r}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary C, show that

$$\iint_s 2\mathbf{a} \cdot d\mathbf{S} = \oint_C (a \times \mathbf{r}) \cdot d\mathbf{r}.$$

1、 (a)

$$N = (i - j) \times (j - k) = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = i + j + k$$

equation of the plane

$$(x-0, y-0, z-0) \cdot N = 0 \Rightarrow x + y + z = 0$$

parametric representation (not unique) of the plane

$$S : r(u,v) = (u, v, -u, -v), \quad -\infty < u, v < \infty \quad \#$$

(b)

$$4x^2 = 4 + 4y^2 + z^2 \Rightarrow x^2 = 1 + y^2 + \frac{z^2}{4}$$

parametric representation of the plane

$$S : r(u,v) = \left(\sqrt{1 + u^2 + \frac{v^2}{4}}, u, v \right), \quad -\infty < u, v < \infty \quad \#$$

1 (c)

parametric representation of the sphere

$$x = 4\sin\phi\cos\theta,\quad y = 4\sin\phi\cos\theta,\quad z = 4\cos\phi,\quad 0 \le \theta \le 2\pi,\quad 0 \le \phi \le \pi$$

let $-2 \le z \le 2$

$$-2 \le 4\cos\phi \le 2 \Longrightarrow -\frac{1}{2} \le \cos\phi \le \frac{1}{2} \Longrightarrow \frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$$ #

2 (a)

parametric representation of the point P

$$u+v=2\ ,\quad u-v=0 \quad\Rightarrow\quad u=1\ ,\quad v=1$$

parametric representation of the surface S

$$S:\ r(u,v)=(u+v,\ 3u^2,\ u-v)$$

$$\Rightarrow\ r_u(u,v)=(1,\ 6u,\ 1)\ ,\quad r_v(u,v)=(1,0,-1)$$

normal vector of the tangent plane

$$N(u,v)=r_u\times r_v=\begin{vmatrix} i & j & k \\ 1 & 6u & 1 \\ 1 & 0 & -1 \end{vmatrix}=(-6u,\ 2,\ -6u)$$

at point P

$$N(1,1)=(-6,2,-6)$$

equation of the tangent plane

$$(x-2,\ y-3,\ z-0)\cdot(-6,2,-6)=0 \Rightarrow -6x+12+2y-6-6z=0 \Rightarrow 3x-y+3z=3$$ #

2 (b)

tangent vectors $r_u(u,v)$ and $r_v(u,v)$

$$r_u(u,v) = (\cos u, -\sin u \sin v, 0) \quad , \quad r_u(u,v) = (0, \cos u \cos v, \cos v)$$

and the normal vector of the tangent plane

$$N(u,v) = r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos u & -\sin u \sin v & 0 \\ 0 & \cos u \cos v & \cos v \end{vmatrix}$$

$$= (-\sin u \sin v \cos v, -\cos u \cos v, \cos^2 u \cos v)$$

at point P

$$N\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{8}, -\frac{3}{4}, \frac{3\sqrt{3}}{8}\right)$$

equation of the tangent plane

$$\left(x-\frac{1}{2}, y-\frac{\sqrt{3}}{4}, z-\frac{1}{2}\right) \cdot \left(\frac{-\sqrt{3}}{8}, -\frac{3}{4}, \frac{3\sqrt{3}}{8}\right) = 0$$

$$\Rightarrow -\frac{\sqrt{3}}{8}\left(x-\frac{1}{2}\right) - \frac{3}{4}\left(y-\frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z-\frac{1}{2}\right) = 0 \Rightarrow \frac{\sqrt{3}}{8}x + \frac{3}{4}y - \frac{3\sqrt{3}}{8}z = \frac{\sqrt{3}}{16} \quad \#$$

3 (a)

parametric of the surface S

$$S : r(u,v) = (u, v, 6-3u-2v), \quad 0 \le u \le 2, \quad 0 \le v \le 3, \quad 3u + 2v \le 6$$

$$r_u(u,v) = (1, 0, -3), \quad r_u(u,v) = (0, 1, -2)$$

$$\Rightarrow N = r_u(u,v) \times r_v(u,v) = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{vmatrix} = 3i + 2j + 1k$$

$$\Rightarrow |N| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

area of the surface is given by

$$A = \int_0^2 \int_0^{(6-3u)/2} \sqrt{14}\, dv\, du = \sqrt{14} \int_0^2 \left(3 - \frac{3}{2}u\right) du = 3\sqrt{14} \ \#$$

3(b)

upper part of the surface

$$S : r(u,v) = (b\cos v\cos u,\ b\cos v\sin u,\ b\sin v),\ \frac{\pi}{2} - \sin^{-1}\left(\frac{a}{b}\right) \le v \le \frac{\pi}{2}$$

$$0 \le u \le 2\pi$$

normal vector and its magnitude

$$r_u(u,v) = (-b\cos v\sin u,\ b\cos v\cos u,\ 0),$$

$$r_v(u,v) = (-b\sin v\cos u,\ -b\sin v\sin u,\ b\cos v),$$

$$r_u(u,v) \times r_v(u,v) = b^2\cos v \Longrightarrow |r_u(u,v) \times r_v(u,v)| = b^2\cos v,$$

by symmetric, area of the surface

$$A = 2\int_{\frac{\pi}{2}-\sin^{-1}(\frac{a}{b})}^{\frac{\pi}{2}} \int_0^{2\pi} b^2 \cos v\,du\,dv = 4\pi b^2 \int_{\frac{\pi}{2}-\sin^{-1}(\frac{a}{b})}^{\frac{\pi}{2}} \cos v\,dv$$

$$= 4\pi b^2 \sin v \Big|_{\frac{\pi}{2}-\sin^{-1}(\frac{a}{b})}^{\frac{\pi}{2}} = 4\pi b^2\left(1 - \frac{\sqrt{b^2-a^2}}{b}\right) = 4\pi b\left(b - \sqrt{b^2-a^2}\right) \#$$

4

parametric representation of the surface

$$S : r(u,v) = (\cos v \cos u, \cos v \sin u, \sin v), \quad 0 \le u \le 2\pi, \ -\frac{\pi}{2} \le v \le \frac{\pi}{2}$$

$$|r_u(u,v) \times r_v(u,v)| = \cos v$$

scalar surface intergal

$$\iint_S x^2 \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos^2 v \cos^2 u \cdot \cos v \, du \, dv = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 v \int_0^{2\pi} \frac{1 + \cos 2\pi}{2} \, du \, dv$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 v) \cos v \, dv = \pi \left(\sin v - \frac{1}{3}\sin^3 v\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4\pi}{3}$$

$$\iint_S x^2 \, dA = \iint_S y^2 \, dA = \iint_S z^2 \, dA$$

$$\iint_S (x^2 + y^2 + z^2) \, dA = \iint_S 1 \, dA = \text{Area of } S = 4\pi \Longrightarrow \iint_S x^2 \, dA = \frac{4\pi}{3}$$

#

5 surface S

$$S_1 : r(u,v) = (u\cos v,\ u\sin v,\ 1-u^2),\quad 0 \le u \le 1,\quad 0 \le v \le 2\pi$$

$$S_2 : \hat{r}(u,v) = (u\cos v,\ u\sin v,\ 0),\quad 0 \le u \le 1,\quad 0 \le v \le 2\pi$$

normal vectors $N_1(u,v)$ and $N_2(u,v)$

$$N_1(u,v) = r_u(u,v) \times r_v(u,v) = (\cos v, \sin v, -2u) \times (-u\sin v, u\cos v, 0)$$

$$= (2u^2\cos v, 2u^2\sin v, u)$$

$$N_2(u,v) = \hat{r}_u(u,v) \times \hat{r}_v(u,v) = (\cos v, \sin v, 0) \times (-u\sin v, u\cos v, 0) = (0,0,u)$$

surface integral

$$\iint_S F\, dS = \iint_{S_1} F \cdot dS = \iint_{S_2} F \cdot dS = \int_0^1\int_0^{2\pi} (u\sin v, u\cos v, 1-u^2)\cdot(2u^2\cos v, 2u^2\sin v, u)\, dv\, du$$

$$- \int_0^1\int_0^{2\pi} (u\sin u, u\cos v, 0)\cdot(0,0,u)\, dv\, du$$

$$= \int_0^1\int_0^{2\pi} [4u^3\sin v\cos v + u(1-u^2)]\, dv\, du = \int_0^1 2\pi u(1-u^2)\, du = 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}$$ #

6

by symmetry, surface $z=1$

$S_1 : r(u,v) = (u, v, 1)$, $-1 \leq u \leq 1$, $-1 \leq u \leq 1$

normal vector of surface $S_1 \rightarrow N_1 = (0,0,1)$

surface integral

$$\epsilon_0 \iint_S E \cdot ds = \epsilon_0 \int_{-1}^{1} \int_{-1}^{1} (u,v,1) \cdot (0,0,1)\, du dv = \epsilon_0 \int_{-1}^{1} \int_{-1}^{1} 1\, du dv = 4\epsilon_0$$

$$Q = \epsilon_0 \oiint_S E \cdot ds = 6 \times 4\epsilon_0 = 24\epsilon_0$$ #

vector feild F

7.

$$F(x,y,z) = (y+\sin x,\ x^2+\cos y,\ x^3),$$

$$\nabla \times F = (-2z, -3x^2, -1) \neq 0$$

by Stoke's thm.

1. surface S

$$S:\ r(u,v) = (u, v, 2uv),\quad 0 \le u^2+v^2 \le 1$$

2. normal vector

$$r_u(u,v) = (1,0,2v),\quad r_v(u,v) = (0,1,2u) \Rightarrow r_u(u,v) \times r_v(u,v)$$

$$\Rightarrow N(u,v) = (2v, 2u, -1) \qquad\qquad = (-2v, -2u, 1)$$

$$\Rightarrow \oint_C (y+\sin x)dx + (z^2+\cos y)dy + x^3 dz = \iint_S \nabla \times F \cdot dS$$

$$= \iint_{u^2+v^2\le 1} (-4uv, -3u^2, -1)\cdot(2v, 2u, -1)\, du\, dv = \iint_{u^2+v^2\le 1} (-8uv^2-6u^3+1)\, du\, dv$$

7. cont.

variable transformation, $u = r\cos\theta$, $v = r\sin\theta$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$

$$\oint_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

$$= \int_0^1 \int_0^{2\pi} (-8r^3\cos\theta\sin^2\theta - 6r^3\sin^3\theta + 1)\,r\,dr$$

$$= \int_0^1 2\pi r\,dr = \pi r^2 \Big|_0^1 = \pi$$

8.

$\because \nabla \times F = 0 \quad \therefore$ scalar potential function $f(x, y, z)$

$\nabla f = F \longrightarrow f(x,y,z) = x^3yz - 3xy + z^2$

$\int_C F \cdot dr = f(0,?,0) - f(0,0,2) = 0 - 4 = -4$

9.

by divergence thm.

$$\oiint_S F \cdot dS = \iiint_D \nabla \cdot F \, dV = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} (1+x)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1-\frac{x}{a}-\frac{y}{b}\right)(1+x)\, dy dx = \int_0^a c(1+x) \int_0^{b(1-\frac{x}{a})} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy dx$$

$$= \int_0^a c(1+x) \cdot \frac{b}{2}\left(1-\frac{x}{a}\right)^2 dx = \frac{a(a+4)bc}{24}$$

#

10

let $a = (a_1, a_2, a_3)$

$$a \times r = (a_2 z - a_3 y,\ a_3 x - a_1 z,\ a_1 y - a_2 x)$$

$$\nabla \times (a \times r) = (2a_1, 2a_2, 2a_3) = 2a$$

by Stoke's thm.

$$\iint_S \nabla \times F \cdot dS = \oint_C F \cdot dr \Longrightarrow \iint_S 2a \cdot dS = \oint_C (a \times r) \cdot dr$$ #