112學年第二學期 電機資訊組 工程數學 第1次作業解答

1. Show that the two lines

$$L_1: \quad x = t-3, \quad y = 1-2t, \quad z = 2t+5$$

$$L_2: \quad x = 4-2t, \quad y = 4t+3, \quad z = 6-4t$$

are parallel, and find an equation for the plane that contains them.

Solution: The direction of the line L_1 is $\mathbf{a} = (1,-2,2)$ and the direction of the line L_2 is $\mathbf{b} = (-2,4,-4)$. Since $\mathbf{b} = -2\mathbf{a}$, we know the two lines L_1 and L_2 are parallel.

From the parametric representation of L_1, we know $\mathbf{x}_1 = (-3,1,5)$ is on the line L_1. Similarly, from the parametric representation of L_2, we know $\mathbf{x}_2 = (4,3,6)$ is on the line L_2. We first compute the vector

$$\mathbf{c} = \mathbf{x}_1 - \mathbf{x}_2 = (-7,-2,-1).$$

The normal vector of the plane that contains L_1 and L_2 should be orthogonal to both $\mathbf{a}$ and $\mathbf{c}$. Therefore,

$$\mathbf{n} = \mathbf{a}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ -7 & -2 & -1 \end{vmatrix} = 6\,\mathbf{i} - 13\,\mathbf{j} - 16\,\mathbf{k}.$$

Finally, the equation for the plane that contains L_1 and L_2 is given by

$$(x+3, y-1, z-5)\cdot(6,-13,-16) = 0 \quad \Longrightarrow \quad 6(x+3) - 13(y-1) - 16(z-5) = 0.$$

2. The median of a triangle is the line segment that joins a vertex of a triangle to the midpoint of the opposite side. The purpose of this problem is to use vectors to show that the medians of a triangle all meet at a point.

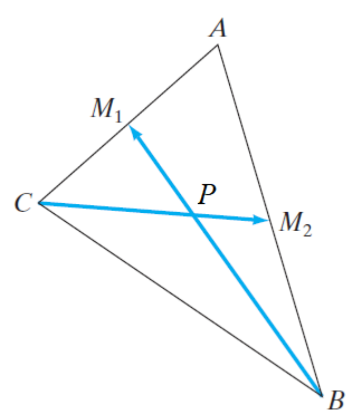

Figure 1

(a) Consider the triangle shown in Figure 1, express the vectors $\overrightarrow{BM_1}$ and $\overrightarrow{CM_2}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution: From Figure 1, we have

$$\overrightarrow{BM_1} + \frac{1}{2}\overrightarrow{CA} = \overrightarrow{BA} \quad \Longrightarrow \quad \overrightarrow{BM_1} = -\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC},$$

$$\overrightarrow{CM_2} + \frac{1}{2}\overrightarrow{BA} = \overrightarrow{CA} \quad \Longrightarrow \quad \overrightarrow{CM_2} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}.$$

(b) Let P be the point of intersection of $\overline{BM_1}$ and $\overline{CM_2}$. Express $\overrightarrow{BP}$ and $\overrightarrow{CP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

> *Solution:* Let $\overrightarrow{BP} = \alpha\overrightarrow{BM_1}$ and $\overrightarrow{CP} = \beta\overrightarrow{CM_2}$, $0 < \alpha < 1$, $0 < \beta < 1$, then
>
> $$\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{BP} + \overrightarrow{PC} = \overrightarrow{BP} - \overrightarrow{CP} = \alpha\overrightarrow{BM_1} - \beta\overrightarrow{CM_2}$$
>
> $$= -\alpha\overrightarrow{AB} + \frac{\alpha}{2}\overrightarrow{AC} - \frac{\beta}{2}\overrightarrow{AB} + \beta\overrightarrow{AC} = \left(-\alpha - \frac{\beta}{2}\right)\overrightarrow{AB} + \left(\frac{\alpha}{2} + \beta\right)\overrightarrow{AC}$$
>
> $$\implies \quad \alpha + \frac{\beta}{2} = 1, \quad \frac{\alpha}{2} + \beta = 1 \quad \implies \quad \alpha = \frac{2}{3}, \quad \beta = \frac{2}{3}.$$
>
> Therefore,
>
> $$\overrightarrow{BP} = \frac{2}{3}\overrightarrow{BM_1} = -\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}, \qquad \overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM_2} = \frac{1}{3}\overrightarrow{AB} - \frac{2}{3}\overrightarrow{AC}.$$

(c) Use the fact that $\overrightarrow{CB} = \overrightarrow{CP} + \overrightarrow{PB} = \overrightarrow{CA} + \overrightarrow{AB}$ to show that P must lie two-thirds of the way from B to M_1 and two-thirds of the way from C to M_2.

> *Solution:* In part (b), we have shown $\alpha = 2/3$ and $\beta = 2/3$. That means P must lie two-thirds of the way from B to M_1 and two-thirds of the way from C to M_2.

(d) Use part (c) to show why all three medians must meet at point P.

> *Solution:* Let M_3 be the midpoint $\overrightarrow{BC}$ and Q be the point of intersection of $\overline{BM_1}$ and $\overline{AM_3}$. By similar reasoning, we know Q must lie two-thirds of the way from B to M_1 and two-thirds of the way from A to M_3. Since P and Q both lie two-thirds of the way from B to M_1, they must be the same point.

3. Consider the scalar function

$$f(x, y, z) = \frac{xy^2 - x^2z}{y^2 + z^2 + 1}$$

Find the values of the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ at point $\mathbf{r} = (-1, 2, 1)$.

> *Solution:* The partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ of the scalar function $f(x, y, z)$ at point $\mathbf{r} = (-1, 2, 1)$ are given by
>
> $$\left.\frac{\partial f}{\partial x}\right|_{\mathbf{r}=(-1,2,1)} = \left.\frac{(y^2 - 2xz)(y^2 + z^2 + 1)}{(y^2 + z^2 + 1)^2}\right|_{\mathbf{r}=(-1,2,1)} = 1,$$
>
> $$\left.\frac{\partial f}{\partial y}\right|_{\mathbf{r}=(-1,2,1)} = \left.\frac{2xy(y^2 + z^2 + 1) - 2y(xy^2 - x^2z)}{(y^2 + z^2 + 1)^2}\right|_{\mathbf{r}=(-1,2,1)} = -\frac{1}{9},$$
>
> $$\left.\frac{\partial f}{\partial z}\right|_{\mathbf{r}=(-1,2,1)} = \left.\frac{-x^2(y^2 + z^2 + 1) - 2z(xy^2 - x^2z)}{(y^2 + z^2 + 1)^2}\right|_{\mathbf{r}=(-1,2,1)} = \frac{1}{9}.$$

4. Let $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ and $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))$ be vector functions in $\mathbb{R}^3$.

(a) Show that

$$\frac{d}{dt}(\mathbf{x}(t)\cdot\mathbf{y}(t)) = \frac{d\mathbf{x}(t)}{dt}\cdot\mathbf{y}(t) + \mathbf{x}(t)\cdot\frac{d\mathbf{y}(t)}{dt}.$$

<u>*Solution:*</u>

$$\frac{d}{dt}(\mathbf{x}(t)\cdot\mathbf{y}(t)) = \frac{d}{dt}\left(x_1(t)y_1(t) + x_2(t)y_2(t) + x_3(t)y_3(t)\right)$$

$$= \frac{dx_1(t)}{dt}y_1(t) + x_1(t)\frac{dy_1(t)}{dt} + \frac{dx_2(t)}{dt}y_2(t) + x_2(t)\frac{dy_2(t)}{dt} + \frac{dx_3(t)}{dt}y_3(t) + x_3(t)\frac{dy_3(t)}{dt}$$

$$= \frac{dx_1(t)}{dt}y_1(t) + \frac{dx_2(t)}{dt}y_2(t) + \frac{dx_3(t)}{dt}y_3(t) + x_1(t)\frac{dy_1(t)}{dt} + x_2(t)\frac{dy_2(t)}{dt} + x_3(t)\frac{dy_3(t)}{dt}$$

$$= \left(\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}, \frac{dx_3(t)}{dt}\right)\cdot(y_1(t), y_2(t), y_3(t)) + (x_1(t), x_2(t), x_3(t))\cdot\left(\frac{dy_1(t)}{dt}, \frac{dy_2(t)}{dt}, \frac{dy_3(t)}{dt}\right)$$

$$= \frac{d\mathbf{x}(t)}{dt}\cdot\mathbf{y}(t) + \mathbf{x}(t)\cdot\frac{d\mathbf{y}(t)}{dt}.$$

(b) Show that

$$\frac{d}{dt}(\mathbf{x}(t)\times\mathbf{y}(t)) = \frac{d\mathbf{x}(t)}{dt}\times\mathbf{y}(t) + \mathbf{x}(t)\times\frac{d\mathbf{y}(t)}{dt}.$$

<u>*Solution:*</u>

$$\frac{d}{dt}(\mathbf{x}(t)\times\mathbf{y}(t)) = \frac{d}{dt}\left(x_2(t)y_3(t) - x_3(t)y_2(t), x_3(t)y_1(t) - x_1(t)y_3(t), x_1(t)y_2(t) - x_2(t)y_1(t)\right)$$

$$= \left[x_2(t)y_3'(t) + x_2'(t)y_3(t) - x_3(t)y_2'(t) - x_3'(t)y_2(t)\right]\mathbf{i} +$$

$$\left[x_3(t)y_1'(t) + x_3'(t)y_1(t) - x_1(t)y_3'(t) - x_1'(t)y_3(t)\right]\mathbf{j} +$$

$$\left[x_1(t)y_2'(t) + x_1'(t)y_2(t) - x_2(t)y_1'(t) - x_2'(t)y_1(t)\right]\mathbf{k} +$$

$$= \left[(x_2'(t)y_3(t) - x_3'(t)y_2(t))\mathbf{i} + (x_3'(t)y_1(t) - x_1'(t)y_3(t))\mathbf{j} + (x_1'(t)y_2(t) - x_2'(t)y_1(t))\mathbf{k}\right] +$$

$$+ \left[(x_2(t)y_3'(t) - x_3(t)y_2'(t))\mathbf{i} + (x_3(t)y_1'(t) - x_1(t)y_3'(t))\mathbf{j} + (x_1(t)y_2'(t) - x_2(t)y_1'(t))\mathbf{k}\right] +$$

$$= \frac{d\mathbf{x}(t)}{dt}\times\mathbf{y}(t) + \mathbf{x}(t)\times\frac{d\mathbf{y}(t)}{dt}.$$

5. Show that if $\mathbf{x}(t)$ has constant length, i.e., $|\mathbf{x}(t)| = c$ for all t, then $\mathbf{x}(t)$ is orthogonal to its derivative $d\mathbf{x}(t)/dt$.

<u>*Solution:*</u> Since $\mathbf{x}(t)\cdot\mathbf{x}(t) = |\mathbf{x}(t)|^2 = c$, we have

$$\frac{d}{dt}(\mathbf{x}(t)\cdot\mathbf{x}(t)) = \frac{d\mathbf{x}(t)}{dt}\cdot\mathbf{x}(t) + \mathbf{x}(t)\cdot\frac{d\mathbf{x}(t)}{dt} = \frac{dc}{dt} = 0 \implies 2\mathbf{x}(t)\cdot\frac{d\mathbf{x}(t)}{dt} = 0 \implies \mathbf{x}(t)\cdot\frac{d\mathbf{x}(t)}{dt} = 0.$$

That means $\mathbf{x}(t)$ is orthogonal to its derivative $d\mathbf{x}(t)/dt$.

6. If $w = f\left(\frac{xy}{x^2+y^2}\right)$ is a differentiable function of $u = \frac{xy}{x^2+y^2}$, show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.$$

<u>*Solution:*</u> By the chain rule, we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial w}{\partial u} \cdot \frac{y(x^2+y^2) - xy \cdot 2x}{(x^2+y^2)^2} = \frac{\partial w}{\partial u} \cdot \frac{y(-x^2+y^2)}{(x^2+y^2)^2},$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{x(x^2+y^2) - xy \cdot 2y}{(x^2+y^2)^2} = \frac{\partial w}{\partial u} \cdot \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

Therefore,

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u} \cdot \frac{xy(-x^2+y^2)}{(x^2+y^2)^2} + \frac{\partial w}{\partial u} \cdot \frac{xy(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

7. The polar coordinate (r, θ) and the Cartesian coordinate (x, y) are related by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Consider a function $f(x, y)$ in the Cartesian coordinate which can be transformed into the polar coordinate as $g(r, \theta) = f(r\cos\theta, r\sin\theta)$, show that

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial^2 g(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g(r,\theta)}{\partial \theta^2}.$$

<u>*Solution:*</u> By the chain rule, we compute the first-order and the second-order partial derivative as

$$\frac{\partial g(r,\theta)}{\partial r} = \frac{\partial f(x,y)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f(x,y)}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f(x,y)}{\partial x}\cos\theta + \frac{\partial f(x,y)}{\partial y}\sin\theta,$$

$$\frac{\partial g(r,\theta)}{\partial \theta} = \frac{\partial f(x,y)}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f(x,y)}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f(x,y)}{\partial x} \cdot -r\sin\theta + \frac{\partial f(x,y)}{\partial y} \cdot r\cos\theta.$$

$$\frac{\partial^2 g(r,\theta)}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial f(x,y)}{\partial x}\cos\theta + \frac{\partial f(x,y)}{\partial y}\sin\theta\right]$$

$$= \frac{\partial^2 f(x,y)}{\partial x^2}\cos^2\theta + \frac{\partial^2 f(x,y)}{\partial x \partial y}2\sin\theta\cos\theta + \frac{\partial^2 f(x,y)}{\partial y^2}\sin^2\theta,$$

$$\frac{\partial^2 g(r,\theta)}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left[\frac{\partial f(x,y)}{\partial x} \cdot -r\sin\theta + \frac{\partial f(x,y)}{\partial y} \cdot r\cos\theta\right]$$

$$= \frac{\partial^2 f(x,y)}{\partial \theta \partial x} \cdot -r\sin\theta + \frac{\partial f(x,y)}{\partial x} \cdot -r\cos\theta + \frac{\partial^2 f(x,y)}{\partial \theta \partial y} \cdot r\cos\theta + \frac{\partial f(x,y)}{\partial y} \cdot -r\sin\theta$$

$$= \frac{\partial^2 f(x,y)}{\partial x^2}r^2\sin^2\theta - \frac{\partial^2 f(x,y)}{\partial x \partial y}2r^2\sin\theta\cos\theta + \frac{\partial^2 f(x,y)}{\partial y^2}r^2\cos^2\theta - \frac{\partial f(x,y)}{\partial x} \cdot r\cos\theta - \frac{\partial f(x,y)}{\partial y} \cdot r\sin\theta.$$

$$\Longrightarrow \quad \frac{\partial^2 g(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g(r,\theta)}{\partial \theta^2} = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial f(x,y)}{\partial y^2}$$

8. Consider the plane curve C parametrized by $\mathbf{r}(t) = (t^2,\ t^3 - t)$ shown in Figure 2.

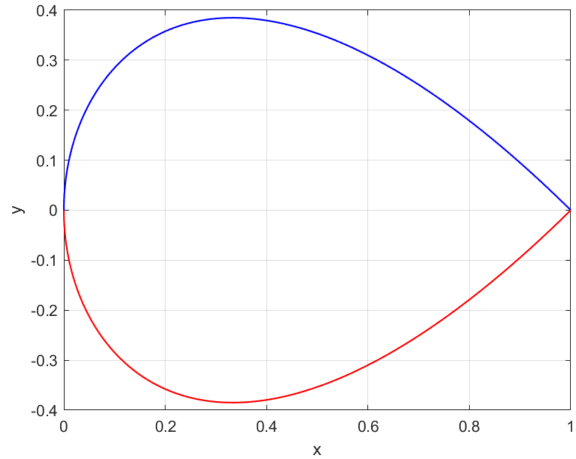

Figure 2

(a) Show that this curve intersects itself, there are numbers t_1 and t_2 ($t_1 \neq t_2$) such that $\mathbf{x}(t_1) = \mathbf{x}(t_2)$.

Solution: Since $\mathbf{x}(t_1) = \mathbf{x}(t_2)$ and $t_1 \neq t_2$, we have

$$t_1^2 = t_2^2 \implies t_1^2 - t_2^2 = 0 \implies (t_1 - t_2)(t_1 + t_2) = 0 \implies t_1 = -t_2.$$

$$t_1^3 - t_1 = t_2^3 - t_2 \implies 2t_1^3 - 2t_1 = 0 \implies 2t_1(t_1^2 - 1) = 0 \implies t_1 = 1 \text{ or } -1.$$

Therefore, $\mathbf{x}(-1) = \mathbf{x}(1) = (1, 0)$ and the curve intersects itself at two different values of t.

(b) At the point where the curve intersects itself, it makes sense to say that the curve has two tangent lines. What is the angle between these tangent lines?

Solution: The tangent vector of the curve C is given by

$$\mathbf{r}'(t) = (2t, 3t^2 - 1) \implies \mathbf{r}'(-1) = (-2, 2),\ \ \mathbf{r}'(1) = (2, 2).$$

The angle between the two vectors $(-2, 2)$ and $(2, 2)$ is

$$\gamma = \cos^{-1} \frac{(-2,2)\cdot(2,2)}{2\sqrt{2}\cdot 2\sqrt{2}} = \cos^{-1} 0 = \frac{\pi}{2}.$$

9. Consider the curve C parametrized by

$$\mathbf{r}(t) = \left(t,\ \frac{1}{3}(1+t)^{3/2},\ \frac{1}{3}(1-t)^{3/2}\right), \quad -1 < t < 1.$$

Find the length of the curve L and the curvature κ.

Solution: The tangent vector $\mathbf{r}'(t)$ is given by

$$\mathbf{r}'(t) = \left(1, \frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}\right) \implies |\mathbf{r}'(t)| = \sqrt{\frac{3}{2}}.$$

The length of the curve C is given by

$$L = \int_{-1}^{1} |\mathbf{r}'(t)|\, dt = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, dt = \sqrt{6}.$$

Since

$$\mathbf{r}''(t) = \left(0, \frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}\right),$$

the curvature κ is given by

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{3\sqrt{2(1-t^2)}}.$$

10. Consider the plane curve parametrized by

$$x(s) = \int_0^s \cos[g(t)]\, dt, \qquad y(s) = \int_0^s \sin[g(t)]\, dt,$$

where $g(t)$ is a differentiable function of t.

(a) Show that the parameter s is the arclength parameter.

Solution: The curve C is parametrized by

$$\mathbf{r}(s) = (x(s), y(s)) \implies \mathbf{r}'(s) = (\cos[g(s)], \sin[g(s)]) \implies |\mathbf{r}'(s)| = 1.$$

The arclength of the curve is given by

$$\int_0^s |\mathbf{r}'(t)|\, dt = \int_0^s 1\, dt = s.$$

That indicates the parameter s is the arclength parameter.

(b) Find the curvature κ.

Solution: Since

$$\mathbf{r}''(s) = \left(-\sin[g(s)]g'(s), \cos[g(s)]g'(s)\right),$$

the curvature κ is given by

$$\kappa = |\mathbf{r}''(s)| = \sqrt{(\sin^2[g(s)])[g'(s)]^2 + (\cos^2[g(s)])[g'(s)]^2} = \sqrt{[g'(s)]^2} = |g'(s)|.$$

(c) Use part (b) to explain how you can create a parametrized plane curve with any specified continuous, non-negative curvature function $\kappa(s)$.

Solution: Given the curvature function $\kappa(s)$, we can find the function $g(s)$ satisfying $g'(s) = \kappa(s)$. Then the curve parametrized by $x(s) = \int_0^s \cos[g(t)]\, dt$ and $y(s) = \int_0^s \sin[g(t)]\, dt$ has the curvature $\kappa(s)$.

(d) Give a set of parametric equations for a curve whose curvature $\kappa(s) = |s|$. (Your answer should involve integrals.)

Solution: From part (b), we have

$$|g'(s)| = |s| \implies g(s) = \frac{s^2}{2} \text{ or } g(s) = -\frac{s^2}{2}.$$

Case 1: If $g(s) = s^2/2$, then

$$x(s) = \int_0^s \cos[g(t)]\,dt = \int_0^s \cos(t^2/2)\,dt,$$

$$y(s) = \int_0^s \sin[g(t)]\,dt = \int_0^s \sin(t^2/2)\,dt.$$

Case 2: If $g(s) = -s^2/2$, then

$$x(s) = \int_0^s \cos[g(t)]\,dt = \int_0^s \cos(-t^2/2)\,dt = \int_0^s \cos(t^2/2)\,dt,$$

$$y(s) = \int_0^s \sin[g(t)]\,dt = \int_0^s \sin(-t^2/2)\,dt = -\int_0^s \sin(t^2/2)\,dt.$$

(e) (Optional) Use a computer program to plot the curve you found in part (d).

Solution: