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634. Find Derangements

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In combinatorial mathematics, a derangement is a permutation of the elements of a set, such that no element appears in its original position.

There's originally an array consisting of `n` integers from 1 to `n` in ascending order, you need to find the number of derangement it can generate.

Also, since the answer may be very large, you should return the output mod $10^9 + 7$.

Example 1:

```
Input: 3
Output: 2
Explanation: The original array is [1,2,3]. The two derangements are [2,3,1] and [3,1,
```

Note:

`n` is in the range of [1, 10^6].

Solution

Approach 1: Brute Force

The simplest solution is to consider every possible permutation of the given numbers from 1 to n and count the number of permutations which are dereangements of the original arrangement.

Complexity Analysis

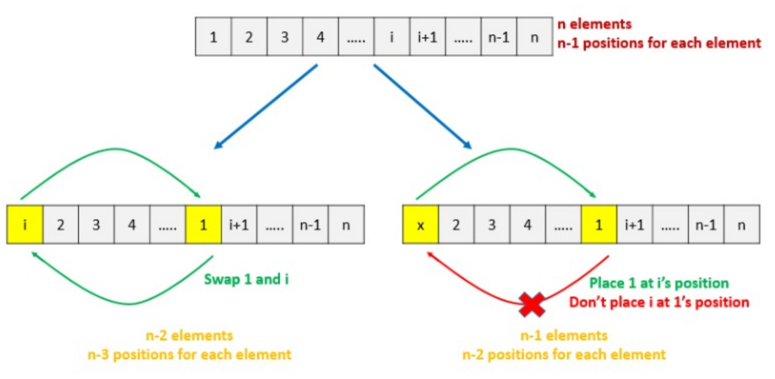
- Time complexity : $O((n+1)!)$. $n!$ permutations are possible for n numbers. For each permutation, we need to traverse over the whole arrangement to check if it is a derangement or not, which takes $O(n)$ time.
- Space complexity : $O(n)$. Each permutation would require n space to be stored.

Approach 2: Recursion

Algorithm

In order to find the number of derangements for n numbers, firstly we can consider the the original array to be `[1,2,3,...,n]`. Now, in order to generate the derangements of this array, assume that firstly, we move the number 1 from its original position and place at the place of the number i. But, now, this i^{th} position can be chosen in $n-1$ ways. Now, for placing the number i we have got two options:

1. We place i at the place of 1: By doing this, the problem of finding the derangements reduces to finding the derangements of the remaining $n-2$ numbers, since we've got $n-2$ numbers and $n-2$ places, such that every number can't be placed at exactly one position.
2. We don't place i at the place of 1: By doing this, the problem of finding the derangements reduces to finding the derangements for the $n-1$ elements(except 1). This is because, now we've got $n-1$ elements and these $n-1$ elements can't be placed at exactly one location(with i not being placed at the first position).



Based, on the above discussion, if $d(n)$ represents the number of derangements for n elements, it can be obtained as:

$d(n) = (n-1)\cdot[d(n-1)+d(n-2)]$

This is a recursive equation and can thus, be solved easily by making use of a recursive function.

But, if we go with the above method, a lot of duplicate function calls will be made, with the same parameters being passed. This is because the same state can be reached through various paths in the recursive tree. In order to avoid these duplicate calls, we can store the result of a function call, once its made, into a memoization array. Thus, whenever the same function call is made again, we can directly return the result from this memoization array. This helps to prune the search space to a great extent.

Java | Copy

```java
public class Solution {
    public int findDerangement(int n) {
        Integer[] memo = new Integer[n + 1];
        return find(n, memo);
    }
    public int find(int n, Integer[] memo) {
        if (n == 0)
            return 1;
        if (n == 1)
            return 0;
        if (memo[n] != null)
            return memo[n];
        memo[n] = (int)(((n - 1L) * (find(n - 1, memo) + find(n - 2, memo))) % 1000000007);
        return memo[n];
    }
}
```

Complexity Analysis

- Time complexity : $O(n)$. $memo$ array of length n is filled once only.
- Space complexity : $O(n)$. $memo$ array of length n is used.

Approach 3: Dynamic Programming

Algorithm

As we've discussed above, the recursive formula for finding the derangements for n elements is given by:

$d(n) = (n-1)\cdot[d(n-1)+d(n-2)]$

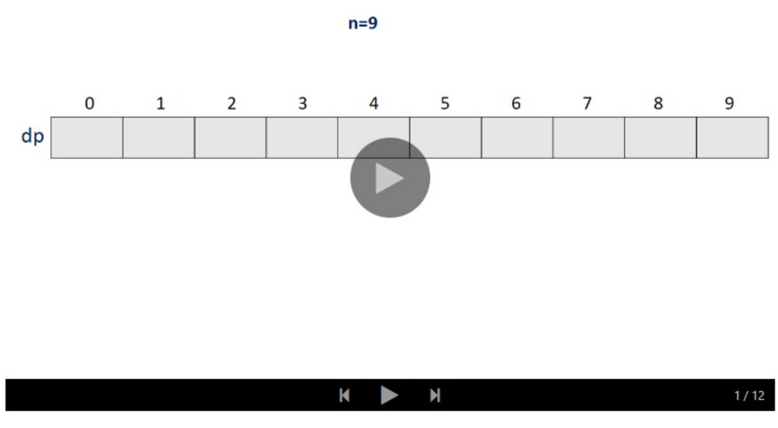
From this expression, we can see that the result for derangements for i numbers depends only on the result of the derangments of numbers lesser than i. Thus, we can solve the given problem by making use of Dynamic Programming.

The equation for Dynamic Programming remains identical to the recursive equation.

$dp[i] = (i-1)\cdot(dp[i-1]+dp[i-2])$

Here, $dp[i]$ is used to store the number of derangements for i elements. We start filling the dp array from $i=0$ and move towards the larger values of i. At the end, the value of $dp[n]$ gives the required result.

The following animation illustrates the dp filling process.



Java | Copy

```java
public class Solution {
    public int findDerangement(int n) {
        if (n == 0)
            return 1;
        if (n == 1)
            return 0;
        int[] dp = new int[n + 1];
        dp[0] = 1;
        dp[1] = 0;
        for (int i = 2; i <= n; i++)
            dp[i] = (int)(((i - 1L) * (dp[i - 1] + dp[i - 2])) % 1000000007);
        return dp[n];
    }
}
```

Complexity Analysis

- Time complexity : $O(n)$. Single loop upto n is required to fill the dp array of size n.
- Space complexity : $O(n)$. dp array of size n is used.

Approach 4: Constant Space Dynamic Programming

Algorithm

In the last approach, we can easily observe that the result for $dp[i]$ depends only on the previous two elements, $dp[i-1]$ and $dp[i-2]$. Thus, instead of maintaining the entire 1-D array, we can just keep a track of the last two values required to calculate the value of the current element. By making use of this observation, we can save the space required by the dp array in the last approach.

Java | Copy

```java
public class Solution {
    public int findDerangement(int n) {
        if (n == 0)
            return 1;
        if (n == 1)
            return 0;
        int first = 1, second = 0;
        for (int i = 2; i <= n; i++) {
            int temp = second;
            second = (int)(((i - 1L) * (first + second)) % 1000000007);
            first = temp;
        }
        return second;
    }
}
```

Complexity Analysis

- Time complexity : $O(n)$. Single loop upto n is required to find the required result.
- Space complexity : $O(1)$. Constant extra space is used.

Approach 5: Formula

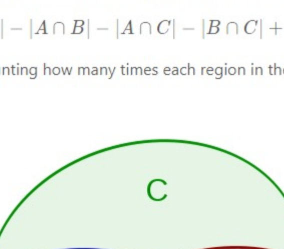
Algorithm

Before discussing this approach, we need to look at some preliminaries.

In combinatorics (combinatorial mathematics), the *inclusion–exclusion principle* is a counting technique which generalizes the familiar method of obtaining the number of elements in the union of two finite sets; symbolically expressed as

$|A \cup B| = |A| + |B| - |A \cap B|$

where A and B are two finite sets and $|S|$ indicates the cardinality of a set S (which may be considered as the number of elements of the set, if the set is finite).

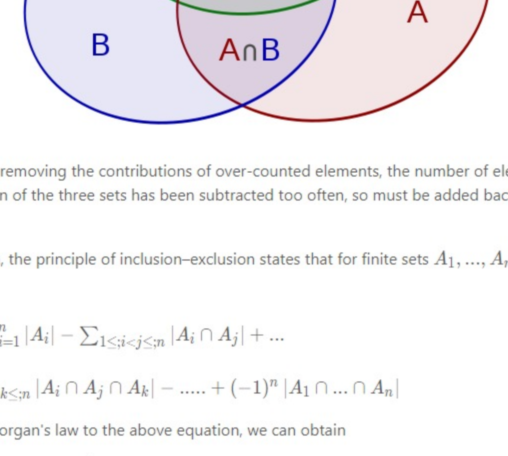


The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

The principle is more clearly seen in the case of three sets, which for the sets A, B and C is given by

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

This formula can be verified by counting how many times each region in the Venn diagram figure shown below.



In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.

In its general form, the principle of inclusion-exclusion states that for finite sets $A_1, \ldots, A_n$, one has the identity

$\left|\bigcup_{i=1}^n A_i\right| = \sum_{i=1}^n |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \ldots$
$+ \ldots + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap \ldots \cap A_n|$

By applying De-Morgan's law to the above equation, we can obtain

$\left|\bigcap_{i=1}^n \bar{A_i}\right| = \left|S - \bigcup_{i=1}^n A_i\right| = |S| - \sum_{i=1}^n |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \ldots +$
$(-1)^n |A_1 \cap \ldots \cap A_n|$

Here, S represents the universal set containing all of the A_i and $\bar{A_i}$ denotes the complement of A_i in S.

Now, let A_i denote the set of permutations which leave A_i in its natural position. Thus, the number of permutations in which the i^{th} element remains at its natural position is $(n-1)!$. Thus, the component $\sum_{i=1}^n |A_i|$ above becomes $\binom{n}{1}(n-1)!$. Here, $\binom{n}{1}$ represents the number of ways of choosing 1 element out of n elements.

Making use of this notation, the required number of derangements can be denoted by $\left|\bigcap_{i=1}^n \bar{A_i}\right|$ term.

This is the same term which has been expanded in the last equation. Putting appropriate values of the elements, we can expand the above equation as:

$\left|\bigcap_{i=1}^n \bar{A_i}\right| = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \ldots + (-1)^p\binom{n}{p}(n-p)! + \ldots +$
$(-1)^n\binom{n}{n}(n-n)!$
$= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \ldots + (-1)^n\frac{n!}{n!}$

We can make use of this formula to obtain the required number of derangements.

Java | Copy

```java
public class Solution {
    public int findDerangement(int n) {
        long mul = 1, sum = 0, M = 1000000007;
        for (int i = n; i >= 0; i--) {
            sum = (sum + M + mul * (i % 2 == 0 ? 1 : -1)) % M;
            mul = (mul * i) % M;
        }
        return (int) sum;
    }
}
```

Complexity Analysis

- Time complexity : $O(n)$. Single loop upto n is used.
- Space complexity : $O(1)$. Constant space is used.

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In approach 4

the second equation dp[i]=(n−1)∗(dp[i−1]+dp[i−2]) need to revised to dp[i]=({\color{red} i} −1)∗(dp[i−1]+dp[i−2]) ?

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