Graph Connector

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1 Introduction

When I took AP Calculus in high school, one of the projects we were assigned was a roller coaster project. In this project, we connected graphs in such a way that if two graphs intersected at a point, their first derivatives also intersected at that same point, then their 1st derivative also had an intersection at the same point. This LaTeX document and the subsequent code detail a simplified version of the algorithm I used to create functions that connect any two points. The derivative at each endpoint is also dependent on choice. I chose this model because having both endpoints with a derivative of zero leads to some interesting simplifications. This project is on display as it demonstrates my ability to tackle novel questions and use coding with abstract mathematics to efficiently produce functions that complete the task.

2 Background

We will need two concepts: using a matrix to solve a system of equations, the definition of a derivative, and what it means for the derivative to equal zero.

Examine the graph x^2 and look at the origin point (0,0) which is a point on the graph x^2, shown in the 1st figure. By the diagram provided, we can make an observation that as our x value get closer

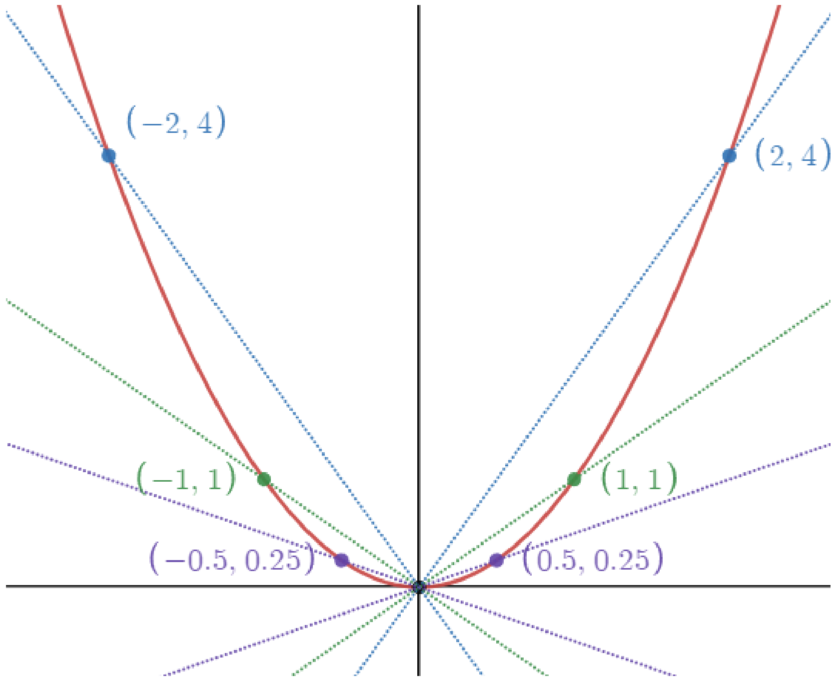

Figure 1: As the points get closer to zero, the linear line slope formed by the points and the origin gets closer to zero, weather we use points to the left or right of zero. Tables below shows this relationship.

Color	Point	Slope
● (blue)	(-2,4)	-2
● (green)	(-1,1)	-1
● (purple)	(-0.5, 0.25)	$\frac{-1}{2}$

Color	Point	Slope
● (blue)	(2,4)	2
● (green)	(1,1)	1
● (purple)	(0.5, 0.25)	$\frac{1}{2}$

to zero, the slope of the line connecting the origin to the given x point gets closer to zero. An intuitive way to think of the derivative of a function at a given point is the slope of the line, which the 2 points for that line is the given point and a point very close to the given point. When the derivative is zero, the line will be in the form of y=c, where c is a constant.

The previous figure showed the derivative of the point (0,0) on the function x^2, we will now show the derivative of (2,4) on that same function. Figure 2 is dedicated to this example.

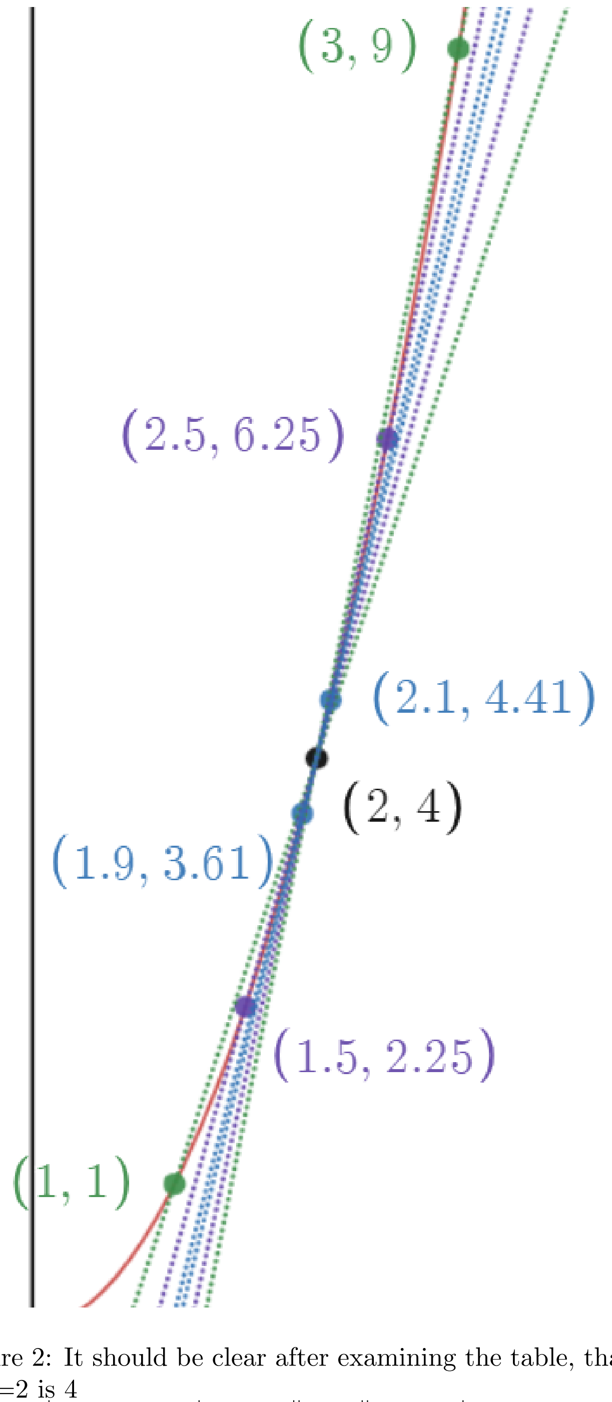

Figure 2: It should be clear after examining the table, that the derivative of x^2 at x=2 is 4

Color	Point	Slope
green	(1,1)	3
purple	(1.5,2.25)	3.5
blue	(1.9, 3.61)	3.9

Color	Point	Slope
green	(3,9)	5
purple	(2.5,6.25)	4.5
blue	(2.1, 4.41)	4.1

Although there are many kinds of function we can take the derivative of such as the trig functions (sin,cos, etc) and the exponential function e^x we are only interested in looking at the derivative of polynomial functions, thus we will need something called the power rule.

The power rule is just this simple identity

$$\frac{d}{dx}[ax^n] = anx^{n-1}$$

Note that if n=0 (ie, no x is present), then the derivative is also 0. This is a very easy identity and these examples will show that.

Example 1: $\frac{d}{dx}[3x^6 + 4x^2 + x - 4]$

$$\frac{d}{dx}[3x^6+4x^2+x^1-4] = 3*6*x^{6-1}+4*2*x^{2-1}+1*x^{1-1} = 18x^5+8x+1$$

Example 2: $\frac{d}{dx}[2x^{3/2} - 4x^{1.1}]$

$$\frac{d}{dx}[2x^{\frac{3}{2}} - 4x^{1.1}] = 2 * \frac{3}{2} * x^{\frac{3}{1}-1} - 4 * 1.1 * x^{1.1-1} = 3x^{\frac{1}{2}} - 4.4x^{0.1}$$

Although we can use the power rule for non-integer of n examples, we will only need to use the power rule on 3rd degree polynomials which only contain integer values of n.

Examine the 3rd figure, and notice that is is not defied on the domain $x < 0$. We can not find a traditional derivative for the function x^2 at 0 because there are no points to the left of zero, but we can define a right hand derivative for this, with the value of this right hand derivative being 0. Although we are only taking one-sided derivatives, we can still use power rule in this case.

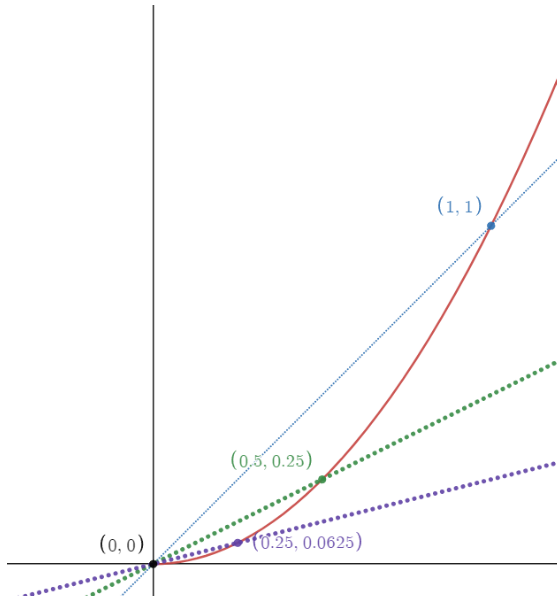

Figure 3: This is an example of a Right hand derivative, as we only used points from the right to define this value.

The other topic we need to cover are matrix's, and there applications to solving equations, It is not important to understand the work, all you need to know is the following

Let x,y be variables we are trying to solve for, and let A,B,C,D,E,F be fixed constant values.Then...

$$C = Ax + By$$
$$F = Dx + Ey$$

$$\rightarrow \begin{bmatrix} C \\ F \end{bmatrix} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} A & B \\ D & E \end{bmatrix}^{-1} * \begin{bmatrix} C \\ F \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Input this into a calculator to get us the solution to our problem. We now follow up with an Example.

Let 5 = 3x + 2y and let 12 = 4x - y, To solve this equation for x,y we put the following matrix equation into a calculator,

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}^{-1} * \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

3 Calculations

Let (x_1, y_1) and (x_2, y_2) be 2 point on the x,y plane such that $x_1 < x_2$, an example of this is shown in the figure above. We want to connect the points by a curve with that curve having the these properties

- (x_1, y_1) and (x_2, y_2) are the endpoints of that curve
- the left hand derivative and right hand derivative of the points are equal to zero respectively.

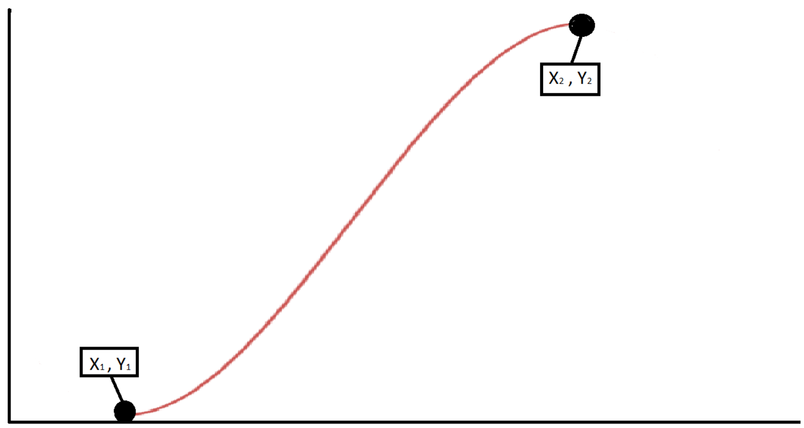

Figure 4: x_1, y_1 and x_2, y_2 are 2 chosen points on the x,y plane. Our goal is to find the equation of the red line.

Assume a function f, did exist with these properties, then we could describe f as such

$$f(x_1) = y_1 \quad f(x_2) = y_2$$

$$f'(x_1) = 0 \quad f'(x_2) = 0$$

Back in high school algebra , if you were given 2 points and used a polynomial of degree one (a linear function) to connect the two points. Because we are given 4 restrictions for our function, a good guess to what kind of equation would help us solve this problem is a cubic function.

Assume $f(x) = a(x - x_1)^3 + b(x - x_1) + c(x - x_1) + d$. With our given restrictions, plus our choice of centering the polynomial around x_1, we can simplify this equation quite a bit.

Since $f(x_1) = y_1$

$$f(x_1) = y_1 = a(x_1 - x_1)^3 + b(x_1 - x_1)^2 + c(x_1 - x_1) + d = d \rightarrow y_1 = d$$

Since $f'(x_1) = 0$

$$f'(x_1) = 0 = 3a(x_1 - x_1)^2 + 2b(x_1 - x_1) + c \rightarrow 0 = c$$

Thus we only need to solve for a and b. Given that $y_1 = d$ and $0 = c$, we can re-write $f(x) = a(x - x_1)^3 + b(x - x_1) + y_1$. We now use the last two descriptions of f using x_2 to find the values of a,b.

Since $f(x_2) = y_2$

$$f(x_2) = y_2 = a(x_2 - x_1)^3 + b(x_2 - x_1)^2 + y_1$$

$$\rightarrow y_2 - y_1 = a(x_2 - x_1)^3 + b(x_2 - x_1)^2$$

Since $f'(x_2) = 0$

$$f'(x_2) = 0 = 3a(x_2 - x_1)^2 + 2b(x_2 - x_1)$$

We now re-write this equation as a matrix and try to solve the equation.

$$\begin{gathered} y_2 - y_1 = a(x_2 - x_1)^3 + b(x_2 - x_1)^2 \\ 0 = 3a(x_2 - x_1)^2 + 2b(x_2 - x_1) \end{gathered}$$

$$\rightarrow \begin{bmatrix} y_1 - y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)^3 & (x_2 - x_1)^2 \\ 3(x_2 - x_1)^2 & 2(x_2 - x_1) \end{bmatrix} * \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} (x_2 - x_1)^3 & (x_2 - x_1)^2 \\ 3(x_2 - x_1)^2 & 2(x_2 - x_1) \end{bmatrix}^{-1} * \begin{bmatrix} y_1 - y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

We let the code take care of the math and we are done.