Angel and Shreiner: Interactive Computer Graphics, Sixth Edition

Chapter 10 Solutions

12.1 $(m+1)^3$

10.2 The function $y = f(x)$ describes the same curve in any plane of constant z, or equivalently, a curve in the y, z plane that is extruded in z to form a surface. Likewise the function $z = g(x)$ describes a surface extruded in y from a curve in the z, x plane. The intersection of the two surfaces leaves a curve. Note, that many simple surfaces cannot be described in this manner because we cannot describe many two–dimensional curves by explicit forms.

10.3 As u varies over (a, b), $v = \frac{u-a}{b-a}$ varies over $(0, 1)$. Substituting into the polynomial $p(u) = \sum_{k=0}^{n} c_k u^k$, we have $q(v) = \sum_{i=0}^{v} d_i v^i = \sum_{k=0}^{n} c_k((b-a)v + a)^k$. We can expand the products on the right and match powers of v to obtain $\{d_i\}$.

10.4 The interpolating matrix for interpolating points x_0, x_1, x_2 and x_3 is

$$\mathbf{A} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}.$$

The determinant of this matrix must be of the form

$$|\mathbf{A}| = c(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

because if any pair of the interpolating points are the same the matrix will have two identical rows and a zero determinant. By this same reasoning, if the interpolating points are all distinct, then the determinant cannot be zero and the matrix must have an inverse.

10.5 Consider the Bernstein polynomial

$$b_{kd}(u) = \begin{pmatrix} d \\ k \end{pmatrix} u^k (1-u)^{d-k}.$$

For $k = 0$ or $k = d$, the maximum value of 1 is at one end of the interval (0,1) and the minimum is at the other because all the zeros are at 1 or 0.

For other values of k, the polynomial is 0 at both ends of the interval and we can differentiate to find that the maximum is at $u = k/d$. Substituting into the polynomial, the maximum value is $\frac{d!}{d^d}\frac{k^k}{k!}\frac{(d-k)^{d-k}}{(d-k)!}$ which is always between 0 and 1.

10.6 For the cubic B–spline, the blending functions are

$$\mathbf{b}(u) = \frac{1}{6}\left[\begin{array}{c} (1-u)^3 \\ 4 - 6u^2 + 3u^3 \\ 1 + 3u + 3u^2 - 3u^3 \\ u^3 \end{array}\right],$$

and the first two derivatives are

$$\mathbf{b}'(u) = \frac{1}{2}\left[\begin{array}{c} -(1-u)^2 \\ -4u + 3u^2 \\ 1 + 2u - 3u^2 \\ u^2 \end{array}\right],$$

$$\mathbf{b}''(u) = \left[\begin{array}{c} 1-u \\ -4 + 6u \\ 1 + 3u + 3u^2 - 3u^3 \\ u^3 \end{array}\right].$$

Consider the B–spline $p(u)$ determined by the control points P_{i-2}, P_{i-1}, P_i, and P_{i+1} and next B–spline determined by P_{i-1}, P_i, P_{i+1}, and P_{i+2}. Thus,

$$p(u) = \left[\begin{array}{cccc} P_{i-2} & P_{i-1} & P_i & P_{i+1} \end{array}\right]\mathbf{b}(u)$$

$$q(u) = \left[\begin{array}{cccc} P_{i-1} & P_i & P_{i+1} & P_{i+2} \end{array}\right]\mathbf{b}(u)$$

The values of $p(u)$ and its first two derivatives at $u = 1$ have to be compared with the values of $q(u)$ and its first two derivatives ar $u = 0$. Using the above equations, we find

$$p(1) = q(0) = \frac{1}{6}(P_{i-1} + 4P_i + P_{i+1}),$$

$$p'(1) = q'(0) = \frac{1}{2}(P_{i+1} - P_{i-1}),$$

$$p''(1) = q''(0) = P_{i-1} + 2P_1 + 2P_{i+1}.$$

These results are sufficient to verify C^2 continuity because all the derivatives of polynomials are continuous except possibly at the join points.

10.7 Proceeding as in the text, we have the interpolating control point array $\mathbf{q}$ and can form the interpolating polynomial

$$p(u) = \mathbf{u}^T \mathbf{M}_I \mathbf{q},$$

where $\mathbf{M}_I$ is the interpolating geometry matrix. This polynomial can also be written as a Bezier polynomial

$$p(u) = \mathbf{u}^T \mathbf{M}_B \mathbf{p},$$

for properly chosen control points $\mathbf{p}$. If these representations are to yield the same polynomial, we must have

$$\mathbf{p} = \mathbf{M}_B^{-1} \mathbf{M}_I \mathbf{q}.$$

Thus, we find the correct control points to convert the interpolating polynomial to an equivalent Bezier polynomial and then use our ability to render Bezier polynomials efficiently.

10.8 Any quadric can be written as

$$q(x, y, z) = ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + 2gx + 2hy + 2iz + j = 0,$$

where $a, b, c, d, e, f, g, h, i$ and j are constants. Let $\mathbf{p}^T = \begin{bmatrix} x & y & z & 1 \end{bmatrix}$. Then, we can rewrite the equation as

$$q(\mathbf{p}) = \mathbf{p}^T \mathbf{Q} \mathbf{p} = 0,$$

where

$$\mathbf{Q} = \begin{bmatrix} a & d & e & g \\ d & b & f & h \\ e & f & c & i \\ g & h & i & j \end{bmatrix}.$$

Note that we can also use $\mathbf{p}^T = \begin{bmatrix} x & y & z & w \end{bmatrix}$ where w can be any constant.

10.9 If we have two patches that share an edge and subdivide only the patch on one side of this edge, we can create a crack. The middle shared endpoint on the subdivided patch does not have to lie on original edge. We can either create an extra triangle from the original endpoints and this middle point to fill the crack or we can triangulate the unsubdivided patch to meet the new subdivided edge, i.e. we replace the unsubdivided patch by a set of triangles that use three of the edges of the patch and the subdivided edge.

10.11 Although the curves are continuous, when we have only G^1 continuity, there is a discontinuity in the velocity at which we trace the curve. In an animation we might see changes in velocity as objects move along paths described with only G^1 continuity.

10.13 One simple test is to use the *twist* (page 590). Suppose that the four corners of the patch are given by $\mathbf{p}_{00}$, $\mathbf{p}_{01}$, $\mathbf{p}_{10}$, and $\mathbf{p}_{11}$. These points form a quadrilateral that will be flat if $\mathbf{p}_{00} - \mathbf{p}_{01} + \mathbf{p}_{10} - \mathbf{p}_{11}$ is zero. A simple test of flatness is to measure the magnitude of this term.

10.14 For the given knot sequence, the B–Spline becomes a quadratic Bezier curve in the interval (0,1) and the three blending functions are u^2, $2u(1-u)$, and $(1-u)^2$. We can next form the rational B–spline with the weights $1, w, 1$ and the control points P_0, P_1, and P_2

$$\mathbf{p}(u) = \frac{u^2 P_0 + 2wu(1-u) + P_2(1-u)^2}{u^2 + 2wu(1-u) + (1-u)^2}.$$

10.15 For $r = 0$ we get the line between P_0 and P_2. For $r = \frac{1}{2}$ we get the parabola $u^2 P_0 + 2u(1-u)P_1 + (1-u)^2 P_2$ which passes through P_0 and P_2. For $r > \frac{1}{2}$, we obtain hyperbolas, and for $r < \frac{1}{2}$, we obtain ellipses. Thus, we can use NURBSs to obtain both parametric polynomial curves and surfaces, and to obtain quadric surfaces.

10.16 The four blending functions for the Hermite curve are $(u-1)^2$, $(2u+1)$, $u^2(-2u+3$, $u(1-u)^2$, and $u^2(u-1)$. The zeros of these functions are $(1, 1, -1/2)$, $(0, 0, -3/2)$, $(0, 1, 1)$, and $(0, 1, 1)$. All are either at the edges of the interval (0,1) or outside of it. Each has only a single stationary point in the interval (0,1). Consequently, these blending functions must be smooth inside this interval

10.17 We can write the Hermite surface as

$$\mathbf{p}(u, v) = \mathbf{u}^T \mathbf{M}_H \mathbf{Q} \mathbf{M}_H^T \mathbf{v} = \mathbf{u}^T \mathbf{A} \mathbf{v},$$

where $\mathbf{Q}$ contains the control point data and $\mathbf{M}_H$ is the Hermite geometry matrix. If evaluate $\mathbf{p}$, $\frac{\partial \mathbf{p}}{\partial u}$, $\frac{\partial \mathbf{p}}{\partial v}$, and $\frac{\partial^2 \mathbf{p}}{\partial u \partial v}$ at the corners we find that the 16 values in the matrix $\mathbf{A}$ are the 4 values at the 4 corners of the patch, the first partial derivatives $\frac{\partial \mathbf{p}}{\partial v}$ and $\frac{\partial \mathbf{p}}{\partial u}$ at the corners and the first mixed partial derivative $\frac{\partial^2 \mathbf{p}}{\partial u \partial v}$ at the corners

10.18 Because $1024 = 2^{10} < 1280 < 2048 = 2^{11}$, after at most 11 subdivisions, we are at the resolution of less than a pixel.

10.19 This process creates a quadric curve which interpolates P_0 and P_2 and lies in the triangle defined by P_0, P_1, and P_2

10.20 This process generates the cubic Bezier curve.

10.21 Nothing unusual happens other than the slope at $u = 0$ must be zero as long as the control points are still separated in parameter space.

10.22 This problem is essentially the same as Exercise 10.9.

10.24 The columns of the matrix $\mathbf{M_R}$ contain the coefficients of the blending polynomials which are

$$
\begin{aligned}
p_0(u) &= -u^3 + 2u^2 - u, \\
p_1(u) &= 2u^3 - 5u^2 + 2, \\
p_2(u) &= -3u^3 + 4u^2 - u, \\
&\quad u^3 - u^2.
\end{aligned}
$$

Note that the third and fourth polynomials can be obtained from the first and second by substituting $1 - u$ for u. We zeros of the fourth polynomial are 0, 0, and 1 so the zeros of the first are 0, 1, and 1. We can obtain the zeros of the third by factoring out u which gives a zero at 0 and solving the resulting quadratic equation to find the zeros at $\frac{-3 \pm \sqrt{7}}{2}$. The zeros of the second polynomial are thus 1 and $\frac{1 \pm \sqrt{7}}{2}$.

10.25 The required matrix is

$$
\mathbf{M_C^{-1} M_B} = \mathbf{\frac{1}{2}} \begin{bmatrix} 6 & 12 & -6 & 1 \\ -1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & 3 & 1 \end{bmatrix}.
$$