Math 351: Homework 2 (Due September 21)

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Section 0.5

Problem 7

We will show by induction that for all integers $n \geq 1, n < 2^n$.

For the base case, let $n = 1$. $2^n = 2^1 = 2 > 1$, so the property holds.

Now we will show that if $n < 2^n$ then $n + 1 < 2^{n+1}$. We can add one to both sides of the first inequality: $n + 1 < 2^n + 1 < 2^n + 2^n < 2^n \cdot 2 = 2^{n+1}$. This completes the inductive step. Thus, by mathematical induction, the property is true for all $n \geq 1$.

Problem 8

Let $\{F_n\}$ be the Fibonacci sequence. We will prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1.$$

We proceed by induction. Let $n = 1$ and recall that $F_1 = 1$ and $F_3 = 2$. We see that $F_1 = 1 = 2 - 1 = F_3 - 1$.

For the inductive step we assume $\sum_{i=0}^{n} F_i = F_{n+2} - 1$, and we want to show that $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$. In addition, recall that the definition of $\{F_n\}$ includes the statement $F_n = F_{n-1} + F_{n+2}$. Consider the following equalities:

$$\begin{aligned}
\textstyle\sum_{i=0}^{n+1} F_i &= \textstyle\sum_{i=0}^{n} F_i + F_{n+1} \\
&= F_{n+2} - 1 + F_{n+1} \\
&= (F_{n+2} + F_{n+1}) - 1 \\
&= F_{n+3} - 1.
\end{aligned}$$

By mathematical induction, this property must hold for all $n \in \mathbb{N}$.

Problem 9

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}.$$

We proceed by induction. Consider the base case where $n = 1$. We get $\frac{1}{2^1} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$, so the base case is true.

For the inductive step we assume $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$ and we want to show that $\sum_{i=1}^{n+1} \frac{1}{2^i} = 1 - \frac{1}{2^{n+1}}$. Note that the following equalities hold:

$$\begin{aligned}
\textstyle\sum_{i=1}^{n+1} \frac{1}{2^i} &= \textstyle\sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^{n+1}} \\
&= 1 - \tfrac{1}{2^n} + \tfrac{1}{2^{n+1}} \\
&= 1 - \tfrac{2^{n+1} - 2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{2 \cdot 2^n - 2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{1}{2^{n+1}}.
\end{aligned}$$

So, by mathematical induction, this is true for all $n \geq 1$.

Problem 11

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can show this by induction. We let $n = 1$ for the base case, and we see that $1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6}$.

For the inductive step we assume $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and we want to show that $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$. Consider the following equalities:

$$\begin{aligned}\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\ &= \tfrac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \tfrac{n(n+1)(2n+1)+6(n+1)^2}{6} \\ &= \tfrac{(n+1)(n(2n+1)+6(n+1))}{6} \\ &= \tfrac{(n+1)(2n^2+7n+6)}{6} \\ &= \tfrac{(n+1)(n+2)(2n+3)}{6}.\end{aligned}$$

By mathematical induction, this property is true for all $n \geq 1$.

Section 1.3

Problem 7

Recall that for sets A_1 and A_2, the Cartesian product is defined as $A \times A_1 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$. Define the Cartesian product of k sets $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k\}$.

Using this definition we can prove that if the sets $A_1, \ldots, A_k$ are countable, then their Cartesian product is also countable. We proceed by induction. In the case where $k = 2$, we have the Cartesian product $A_1 \times A_2$, which is countable by Proposition 1.3.8. For the inductive step, assume that $A_1, \ldots, A_{k+1}$ are countable and that $A_1 \times \ldots \times A_k$ is countable. The Cartesian product $A_1 \times \ldots \times A_{k+1}$ can be written as $(A_1 \times \ldots \times A_k) \times A_{k+1}$. The set $A_1 \times \ldots \times A_k$ is assumed to be countable, as is A_{k+1}, so their Cartesian product $(A_1 \times \ldots \times A_k) \times A_{k+1}$ is countable by Proposition 1.3.8. So, by induction, the Cartesian product of k countable sets is itself countable.

Argue that the set of finite sequences of 0's and 1's is countable

Consider the set of finite sequences of 0's and 1's, which we will call S. We can interpret elements of S as binary representations of natural numbers. This is actually a surjection from $\mathbb{N}_0$ onto S, as any sequence in S can be interpreted as a natural number. Since we have mapped $\mathbb{N}_0$, which is countable, onto S, S must also be countable.

Argue that the set of infinite sequences of 0's and 1's is not countable

Call the set of infinite sequences of 0's and 1's S'. Cantor's diagonalization argument can be applied to S'. Assume for contradiction that S' is countable and that as a result there exists some ordered list of each element of S'. We can create a new sequence a whose first term is different from the first term of the first sequence in S', whose second term is different from the second term of the second sequence in S', and so on. The sequence a differs from every sequence in S' in at least one position, so it cannot be on the list even though it is a member of S'. We have encountered a contradicton, and thus S' is not countable.

Problem 13

We will show that the set of algebraic numbers is countable by showing first that the set of polynomials, P, is countable. Note that any algebraic number a is by definition the root of some polynomial $p \in P$, or, in other words, there exists an injection from the algebraic numbers into P (the relation f where $f(a) = p$ only if a is a root of p), which means that the cardinality of the algebraic numbers is less than or equal to that of P. So, if we show that P is countable, we can conclude that the algebraic numbers are countable.

Let P_n be the set of polynomials of degree k: specifically $P_n = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \mid a_n, \ldots, a_0 \in \mathbb{Z}\}$. In addition, consider the Cartesian product $\mathbb{Z}^n$ to be the Cartesian product of $\mathbb{Z}$ n times, as defined in Problem 7. We can define the relation $f : \mathbb{Z}^n \to P_n$ such that, if $b = (b_0, \ldots, b_n) \in \mathbb{Z}^n, p \in P_n$, then $f(b) = p$ only if p's coefficient $a_k = b_k$ for all $0 \leq k \leq n$. Every $p \in P_n$ has coefficients from $\mathbb{Z}$, so every $p = f(b)$ for some $b \in \mathbb{Z}^n$. Thus, f is surjective. Since $\mathbb{Z}$ is countable, $\mathbb{Z}^n$ is countable by Problem 7, and since f is a surjection onto P_n, each P_n is also countable.

Now, notice that every polynomial $p \in P$ is in P_n for some n. This means that $P = \bigcup_{n=1}^{\infty} P_n$. So, by Theorem 1.3.14, P is itself countable since it is a countable union of countable sets. By the argument above, we can conclude that the algebraic numbers are countable.

Section 2.1

Problem 1

We will prove that $\lim_{n\to\infty} |x_n| = 0 \iff \lim_{n\to\infty} x_n = 0$. By the definition, $\lim_{n\to\infty} |x_n| = 0 \iff \forall\varepsilon > 0, \exists N$ where $n > N \implies ||x_n|| < \varepsilon$. However, $||x_n|| = |x_n|$ since $|x_n| \geq 0$. Thus $\exists N$ such that $|x_n| < \varepsilon$ for any positive ε, which is the definition of $\lim_{n\to\infty} x_n = 0$.

Problem 2

We will use the definition to show that $\lim_{n\to\infty} \frac{n}{n+1} = 1$. To satisfy the definition, then for any $\varepsilon > 0$ we need to find an N such that $n > N \implies \left|\frac{n}{n+1} - 1\right| < \varepsilon$. Choose $N = \frac{1}{\varepsilon} - 1$, so that

$$\begin{aligned}
\tfrac{1}{\varepsilon} - 1 &< n \\
\tfrac{1}{\varepsilon} &< n + 1 \\
\tfrac{1}{n+1} &< \varepsilon \\
\left|\tfrac{-1}{n+1}\right| &< \varepsilon \\
\left|\tfrac{n-(n+1)}{n+1}\right| &< \varepsilon \\
\left|\tfrac{n}{n+1} - 1\right| &< \varepsilon.
\end{aligned}$$

Since we can find an N for any $\varepsilon > 0$ where this inequality holds true, by definition $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Problem 4

Assume (x_n) is a sequence with $x_n \geq 0$ that converges to L. We will use the definition to show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}$.

By the definition, we know that $\forall\varepsilon_1 > 0, \exists N_1$ such that $n > N_1 \implies |x_n - L| < \varepsilon_1$, and we want to show that $\forall\varepsilon > 0, \exists N$ where $n > N \implies \left|\sqrt{x_n} - L\right| < \varepsilon$.

First, consider only the case where $L \neq 0$. Take any $\varepsilon > 0$ and fix it. Now set $\varepsilon_1 = \varepsilon\sqrt{L}$, which we can do since $x_n \geq 0 \implies L \geq 0$. By the definition stated above, we know that there exists some N_1 where $|x_n - L| < \varepsilon_1$ for all $n > N_1$. So, when $n > N_1$, (and keeping in mind that $L \neq 0$) we can write $\varepsilon\sqrt{L} > |x_n - L| \iff \varepsilon > \frac{|x_n-L|}{\sqrt{L}} \geq \frac{|x_n-L|}{\sqrt{x_n}+\sqrt{L}} = \left|\frac{(\sqrt{x_n}-\sqrt{L})(\sqrt{x_n}+\sqrt{L})}{\sqrt{x_n}+\sqrt{L}}\right| = \left|\sqrt{x_n} - \sqrt{L}\right|$. We showed that there exists an N for any ε, namely, $N = N_1$.

Now we deal with the case where $L = 0$. This simplifies the problem, now we need to find an N such that when $n > N$, $\left|\sqrt{x_n}\right| < \varepsilon$. Select some $\varepsilon > 0$ and fix it, and then set $\varepsilon_1 = \varepsilon^2$. We know that there exists some N_1 where for all $n > N_1$, $|x_n| < \varepsilon_1 = \varepsilon^2$. We take the square root of both sides (and note that since the left side is always positive, we can discard the negative solution on the right) and see that $\left|\sqrt{x_n}\right| < \varepsilon$ when $n > N_1$.

Thus we have satisfied the definition in all cases, so $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{L}$.

Problem 6

For the following problems, we will use results from other problems, base cases we proved in class (such as $\lim_{n\to\infty} \frac{1}{n^p} = 0$ when $p > 0$), and Proposition 2.1.13 to break down complicated limits into solveable problems.

a) $\lim_{n\to\infty} (-1)^n \frac{5}{n}$

From Problem 1, we know that if the sequence of the absolute values converges to 0, then this sequence will too. So, we evaluate $\lim_{n\to\infty} \left|(-1)^n \frac{5}{n}\right| = \lim_{n\to\infty} \frac{5}{n} = \lim_{n\to\infty} 5 \cdot \lim_{n\to\infty} \frac{1}{n} = 5 \cdot 0 = 0$. Since this converges to 0, so does the original sequence.

b) $\lim_{n\to\infty} \frac{5n^2-3n+2}{n^2-n}$

Divide the top and the bottom by n^2 and evaluate:

$\lim_{n\to\infty} \frac{5-3\frac{1}{n}+2\frac{1}{n^2}}{1-\frac{1}{n}} = (\lim_{n\to\infty} 5 - 3\lim_{n\to\infty} \frac{1}{n} + 2\lim_{n\to\infty} \frac{1}{n^2}) \cdot \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} = (5 - 0 + 0) \cdot \frac{1}{1-\lim_{n\to\infty} \frac{1}{n}} = 5$. The sequence converges to 5.

c) $\lim_{n\to\infty} \frac{n^2+5n+2}{5n^3+n}$

Divide the top and the bottom by n^3 and evaluate:

$\lim_{n\to\infty} \frac{\frac{1}{n}+5\frac{1}{n^2}+2\frac{1}{n^3}}{5+\frac{1}{n^2}} = (\lim_{n\to\infty} \frac{1}{n}+5\lim_{n\to\infty} \frac{1}{n^2}+2\lim_{n\to\infty} \frac{1}{n^3})\cdot\lim_{n\to\infty} \frac{1}{5+\frac{1}{n^2}} = (0+5\cdot 0+2\cdot 0)\cdot \lim_{n\to\infty} \frac{1}{5+\frac{1}{n^2}} = 0\cdot \frac{1}{5+\lim_{n\to\infty} \frac{1}{n^2}} = 0\cdot \frac{1}{5} = 0$. The sequence converges to 0.

d) $\lim_{n\to\infty}(-1)^n n$

This sequence diverges. It is the product of an oscillating value with constant magnitude and a value that grows without bound.

e) $\lim_{n\to\infty} \frac{1}{(-1)^n n+2n}$

We can use Proposition 2.1.13 and part (d) to evaluate:

$\frac{1}{\lim_{n\to\infty}(-1)^n n+\lim_{n\to\infty} 2n} = \frac{1}{0+2\lim_{n\to\infty} n} = \frac{\lim_{n\to\infty} \frac{1}{n}}{2} = \frac{0}{2} = 0$. The sequence converges to 0.

f) $\lim_{n\to\infty} \sqrt[3]{n+1} - \sqrt[3]{n}$

We rewrite the expression to utilize the $\frac{1}{n^p}$ property:

$\lim_{n\to\infty}(n+1)^{\frac{1}{3}} - n^{\frac{1}{3}} = \lim_{n\to\infty}(n+1)^{\frac{1}{3}} - \lim_{n\to\infty} n^{\frac{1}{3}} = 0-0=0$. The sequence converges to 0.

Problem 11

Let $a > 1$. We will show that $\lim_{n\to\infty} a^n = \infty$. To satisfy the definition of diverging to ∞, we will show $\forall B > 0, \exists N$ such that $n > N \implies a^n > B$. If we choose $N = \log_a(B)$, we can see that $n > \log_a(B) \implies a^n > a^{log_a(B)} = B$.

Now consider if $0 < a < 1$. We will show that in this case, $\lim_{n\to\infty} a^n = 0$ by showing that $\forall \varepsilon > 0, \exists N$ such that $|a^n| < \varepsilon$. We can choose $N = log_a(\varepsilon)$. So, when $n > N$, $n > log_a(\varepsilon)$. Now we will raise a to the power of both sides, but we have to switch the inequality sign since a^n is monotone decreasing. So, $a^n = |a^n| < a^{log_a(\varepsilon)} = \varepsilon$, and we conclude that the sequence converges to 0.

Problem 13

Let (a_n) and (b_n) be sequences where $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = \infty$. We will show that if $L < 0$ then $\lim_{n\to\infty} a_n b_n = -\infty$.

We want to show that $\forall B < 0, \exists N$ such that $n > N \implies a_n b_n < B$. By the definitions of convergence to L and convergence to ∞, we know that $\forall \varepsilon > 0, \exists N_1$ such that $n < N_1 \implies |a_n - L| < \varepsilon$ and that $\forall M > 0, \exists N_2$ such that $n > N_2 \implies b_n > M$. In particular, if we select $\varepsilon < L$, we can find an N_1 that ensures a_n is negative for all $n > N_1$. In addition, if we select $M = \frac{-B}{L}$, we can find an N_2 to ensure $b_n > \left|\frac{B}{L}\right|$ for all $n > N_2$. Now we take $N = \max(N_1, N_2)$ which will ensure both conditions when $n > N$. Now notice that when $n > N$, $a_n b_n < L \cdot \frac{B}{L} = B$. So we have shown that $\lim_{n\to\infty} a_n b_n = -\infty$.

Problem 15

Suppose (a_n) is a sequence with $a_n > 0$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$. We want to show that if $L < 1$, then $\lim_{n\to\infty} a_n = 0$. By separability, there must exist some $\varepsilon > 0$ so that $L+\varepsilon < 1$. Further, by the definition of convergence to L, there exists some $N \in \mathbb{N}$ such that $n \geq N \implies \frac{a_{n+1}}{a_n} < L + \varepsilon$. Specifically, $\frac{a_{N+1}}{a_N} < L + \varepsilon$. We can rearrange this as $a_{N+1} < (L + \varepsilon) \cdot a_N$. Every ratio between successive terms of a_n past $n = N$ will be smaller than $L+\varepsilon$, so it is valid to say that, in general, $a_{N+k} < (L+\varepsilon) \cdot a_{N+k-1}$. If we keep expanding a_{N+k-1} using the same principle, then we can write $a_{N+k} < (L + \varepsilon)^k \cdot a_N$. Recall that $a_n > 0$, and note that the expression on the right side converges to 0 as $k \to \infty$ since $0 < (L + \varepsilon) < 1$, as shown in Problem 11. So, by the Squeeze theorem, we can see that as $k \to \infty$, a_{N+k} converges to 0 as it is always greater than 0 and is less than another sequence that also converges to 0.

Problem 16

In the case where $L > 1$, $a_n \to \infty$ because even if the ratio between successive terms converges to 1.01, the sequence will grow without bound. If $L = 1$, the sequence a_n will converge to some value. The ratio between successive terms approaches 1, so the sequence will tend to a single value as $n \to \infty$.