Math 351: Homework 3 (Due September 28)

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Problem 1

Let $x_1 = 2$ and define $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ for $n = 2, 3, \ldots$

Prove that $x_n^2 > 2$ for all $n > 1$

We proceed by induction. To start, consider the case where $n = 2$. We see that $x_2^2 = (\frac{1}{2}x_1 + \frac{1}{x_1})^2 = (\frac{1}{2} \cdot 2 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} > 2$.

For the inductive step we assume that $x_n^2 > 2$, and we want to conclude from this that $x_{n+1}^2 > 2$. We proceed by manipulating the first inequality:

$$\begin{aligned}
x_n^2 > 2 \quad &\implies x_n^2 - 2 > 0 \\
&\implies (x_n^2 - 2)^2 > 0 \\
&\implies x_n^4 - 4x_n^2 + 4 > 0 \\
&\implies \tfrac{1}{4}x_n^2 - 1 + \tfrac{1}{x_n^2} > 0 \\
&\implies \tfrac{1}{4}x_n^2 + 1 + \tfrac{1}{x_n^2} > 2 \\
&\implies (\tfrac{1}{2}x_n + \tfrac{1}{x_n})^2 > 2 \\
&\implies x_{n+1}^2 > 2.
\end{aligned}$$

Thus, by mathematical induction, $x_n^2 > 2$ for all $n > 1$.

Prove that $x_{n+1} < x_n$ for all n

We proceed by manipulating the inequality $x_n^2 > 2$. However, we notice that we have only shown this to be true for $n > 1$. We will cover the case where $n = 1$ later. We will also use the fact that $x_n > 0$ for all n, which we can see must be true as every x_n is the sum of positive numbers in $\mathbb{Q}$. So, we can see that

$$\begin{aligned} x_n^2 > 2 &\implies \tfrac{x_n^2}{2} > 1 \\ &\implies \tfrac{x_n}{2} > \tfrac{1}{x_n} \\ &\implies x_n > \tfrac{1}{x_n} + \tfrac{x_n}{2} \\ &\implies x_n > x_{n+1}. \end{aligned}$$

We have shown that $x_{n+1} < x_n$ for $n > 1$. We finish by considering $n = 1$: $x_2 = \frac{3}{2} < 2 = x_1$. Thus, the sequence (x_n) is monotone decreasing.

Prove that $\lim\limits_{n\to\infty} x_n^2 = 2$

We have shown that the sequence (x_n^2) is monotone decreasing and bounded from below, specifically bounded from below by the number 2. This sequence must therefore converge. In other words, there exists an L such that $\lim_{n\to\infty} x_n^2 = L$. We can see also that $\lim_{n\to\infty} x_{n+1}^2 = L$. With the defintion of x_{n+1} we can expand this to get

$$\lim_{n\to\infty}\left(\frac{x_n}{2} + \frac{1}{x_n}\right)^2 = \lim_{n\to\infty}\frac{x_n^2}{4} + 1 + \frac{1}{x_n^2} = \frac{L}{4} + 1 + \frac{1}{L} = L.$$

Solving for L:

$$\begin{aligned} \tfrac{L}{4} + 1 + \tfrac{1}{L} = L &\implies 3L^2 - 4L - 4 = 0 \\ &\implies (L-2)(3L+2) = 0 \\ &\implies L = 2 \text{ or } L = -\tfrac{2}{3}. \end{aligned}$$

Since $x_n > 0$ for all n, we can dismiss the negative solution. Thus, $L = 2$. In other words, $\lim_{n\to\infty} x_n^2 = 2$.

Problem 2

Prove that if $\limsup_{n\to\infty} a_n = L$ then for all $\varepsilon > 0$ there are only finitely many $n \in \mathbb{N}$ for which $a_n > L + \varepsilon$.

We can prove this through contraposition. We will assume that there exists at least one $\varepsilon > 0$ such that there are infinitely many n that satisfy $a_n > L + \varepsilon$. We want to show from this that $\limsup_{n\to\infty} a_n \neq L$. So, find such an ε and fix it. Since there are infinitely many n such that $a_n > L + \varepsilon$, there exists some subsequence (a_{n_k}) where for all k, $a_{n_k} > L+\varepsilon$. This subsequence has been constructed such that it is bounded below by $L+\varepsilon$.

There are two possible cases for the upper bound of this subsequence. Let us first consider the case where (a_{n_k}) is bounded above by some $L' \in \mathbb{R}$. By the Bolzano-Weierstrass Theorem, there must be some subsequence $(a_{n_{k_l}})$ that converges to a value in $[L+\varepsilon, L']$. This value is greater than L. Since $(a_{n_{k_l}})$ is a subsequence of a_n, L cannot be the lim sup of a_n because we have shown that there exists a subsequence that converges to a greater value.

Now consider the case where (a_{n_k}) is not bounded above by any real value. This means that there must be some subsequence $(a_{n_{k_l}})$ which diverges to $+\infty$. Since this subsequence is a subsequence of (a_n), the lim sup of (a_n) would be $+\infty$ in this case. Thus, in both cases, $\limsup_{n\to\infty} a_n \neq L$, which completes our proof.

A similar argument can be used to show that $\limsup_{n\to\infty} a_n = L$ implies that for all $\varepsilon > 0$ there exist only finitely many $n \in \mathbb{N}$ for which $a_n < L - \varepsilon$.

Section 2.1

Problem 18

Let (a_n) be a sequence and let $L \in \mathbb{R}$. We want to prove that if every subsequence (a_{n_k}) has a subsequence $(a_{n_{k_l}})$ that converges to L, then the sequence (a_n) converges to L.

We will prove this using the contrapositive. We hope to show that if (a_n) does not converge to L then there exists some subsequence (a_{n_k}) such that all of its subsequences $(a_{n_{k_l}})$ fail to converge to L. By the negation of the definition of convergence to L, we can assume that there exists an $\varepsilon > 0$ where for all N, there is some $n > N$ such that $|a_n - L| \geq \varepsilon$. Find such an ε and fix it. Set $N = 1$. So, there is some $n_1 > 1$ so that $|a_{n_1} - L| \geq \varepsilon$. Now consider $N = n_1$. We must also be able to find an $n_2 > n_1$ where $|a_{n_2} - L| \geq \varepsilon$. We can continue this argument and find a sequence (n_k) such that, for all k, $|a_{n_k} - L| \geq \varepsilon$. This subsequence of (a_n) has been constructed such that it does not converge to L. More specifically, it has been constructed such that there are no values k where $|a_{n_k} - L| < \varepsilon$. In other words, we cannot find any subsequences of (a_{n_k}) that converge to L, because for such a subsequence $(a_{n_{k_l}})$ to converge, there would have to be infinitely many values of l for which $\left|a_{n_{k_l}} - L\right| < \varepsilon$ for every ε. We conclude that if (a_n) does not converge to L, we can find a subsequence of (a_n) that does not have any subsequences that converge to L. So, by contraposition, if every subsequence of (a_n) has a subsequence that converges to L, then (a_n) converges to L.

Section 2.2

Problem 25

Make sense of the following expression as a limit and find its value:

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Define a sequence (a_n) where $a_1 = 1 + \frac{1}{1} = 2$ and $a_{n+1} = 1 + \frac{1}{1 + a_n}$. The value of the continued fraction above can be interpreted as $\lim_{n\to\infty} a_n$. Assume that (a_n) converges to some value L, so $\lim_{n\to\infty} a_n = L$, and note that $\lim_{n\to\infty} a_{n+1} = L$ as well. We can use the recursive definition to expand this:

$$\begin{aligned}
\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} 1 + \frac{1}{1 + a_n} \\
&= 1 + \frac{1}{1 + L} = L.
\end{aligned}$$

We solve for L:

$$\begin{aligned}
L = 1 + \frac{1}{1 + L} &\implies L(L+1) = (L+1) + 1 \\
&\implies L^2 + L = L + 2 \\
&\implies L^2 - 2 = 0 \\
&\implies L = \sqrt{2} \text{ or } L = -\sqrt{2}.
\end{aligned}$$

Since each term of (a_n) is positive, we can exclude the negative solution for L. So we find that $L = \sqrt{2}$, which is the value of the continued fraction.

Problem 26

Make sense of the following expression as a limit and find its value:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$$

Define a sequence (a_n) where $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. The value of the expression above can be interpreted as $\lim_{n\to\infty} a_n$. Assume that (a_n) converges to some value L, so $\lim_{n\to\infty} a_n = L$, and note that $\lim_{n\to\infty} a_{n+1} = L$ as well. We can use the recursive definition to expand this:

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{2 + a_n} \\ &= \sqrt{2 + L} = L. \end{aligned}$$

We solve for L:

$$\begin{aligned} \sqrt{2 + L} = L \quad &\implies 2 + L = L^2 \\ &\implies L^2 - L - 2 = 0 \\ &\implies (L - 2)(L + 1) = 0 \\ &\implies L = 2 \text{ or } L = -1. \end{aligned}$$

Since each term of (a_n) is positive, we can exclude the negative solution for L. So we find that $L = 2$, which is the value of the expression.

Problem 29

Let $(a_n), (b_n)$ be sequences. Prove both of the following:

$$\limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n$$

$$\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$$

Also show that strict inequalities are possible.

To simplify explanations, define (c_n) where $c_n = a_n + b_n$ for all n. In addition, let $\overline{A} = \limsup a_n$, $\overline{B} = \limsup b_n$, $\underline{A} = \liminf a_n$, and $\underline{B} = \liminf b_n$. We can rewrite the statements as

$$\limsup c_n \leq \overline{A} + \overline{B}$$

$$\liminf c_n \geq \underline{A} + \underline{B}$$

We start by proving the first statement. We want to show that for all $\varepsilon > 0$ there are no subsequences of (c_n) that converge to a value greater than $\overline{A} + \overline{B} + \varepsilon$. Take any ε_1 such that $0 < \varepsilon_1 < \varepsilon$ and let $\varepsilon_2 = \varepsilon - \varepsilon_1$. By Problem 2, there are only finitely many values of n such that $a_n > \overline{A} + \varepsilon_1$ and finitely many values of n such that $b_n > \overline{B} + \varepsilon_2$. Thus, there are only finitely many values n such that $c_n = a_n + b_n > (\overline{A} + \varepsilon_1) + (\overline{B} + \varepsilon_2) = \overline{A} + \overline{B} + \varepsilon$. Since the number of such n values is finite, there are no subsequences of (c_n) where for all k, $c_{n_k} > \overline{A} + \overline{B} + \varepsilon$. Thus $\limsup c_n$ is at most $\overline{A} + \overline{B}$.

A similar argument can be used to prove the second statement by way of showing that for all $\varepsilon > 0$ there are no subsequences of (c_n) that converge to a value less than $\underline{A} + \underline{B} - \varepsilon$. Define $\varepsilon_1, \varepsilon_2$ the same way as above. By Problem 2, there are only finitely many values of n such that $a_n < \underline{A} - \varepsilon_1$ and finitely many values of n such that $b_n < \underline{B} - \varepsilon_2$. Thus, the number of values n where $c_n = a_n + b_n < (\underline{A} - \varepsilon_1) + (\underline{B} - \varepsilon_2) = \underline{A} + \underline{B} - \varepsilon$ is finite. So there are no subsequences (c_{n_k}) where for all k, $c_{n_k} < \underline{A} + \underline{B} - \varepsilon$, which means $\liminf c_n$ is at least $\underline{A} + \underline{B}$.

We can find an example of two sequences $(a_n), (b_n)$ for which the above statements are true with strict inequality. Let $a_n = 1 + (-1)^n$ and $b_n = -1 - (-1)^n$ for all $n \in \mathbb{N}$. Note that $a_n = 2$ for even n and $a_n = 0$ for odd n, whereas $b_n = -2$ for even n and $b_n = 0$ for odd n. Further, we can see that $a_n + b_n = 0$ for all values of n. Evaluating $\liminf$ and $\limsup$ for these sequences is straightforward:

$$\begin{aligned}
\limsup a_n &= 2 \\
\limsup b_n &= 0 \\
\liminf a_n &= 0 \\
\liminf b_n &= -2 \\
\liminf = \limsup = \lim(a_n + b_n) &= 0
\end{aligned}$$

We can see that $\limsup(a_n + b_n) = 0 < 2 = \limsup a_n + \limsup b_n$ and that $\liminf(a_n + b_n) = 0 > -2 = \liminf a_n + \liminf b_n$, so these sequences satisfy the properties above with strict inequality.

Problem 31

Let $(a_n), (b_n)$ be sequences and suppose $\lim_{n\to\infty} b_n = b$. We will prove the following statements:

$$\limsup(a_n + b_n) = \limsup a_n + b$$

$$\limsup(a_n \cdot b_n) = \limsup a_n \cdot b, \text{ provided } a_n \geq 0, b_n \geq 0.$$

We start with the first statement. Given that b_n converges to b, we know that any subsequence (b_{n_k}) also converges to b by Lemma 2.1.27. So the limit of any subsequence $\lim_{k\to\infty}(a_{n_k} + b_{n_k}) = \lim_{k\to\infty} a_{n_k} + b$. We can see that the largest value to which any subsequence of $(a_n + b_n)$ converges is equal to the largest value to which any subsequence of (a_n) converges, plus b. In other words, $\limsup(a_n + b_n) = \limsup a_n + b$.

Now we consider the second limit, which follows a similar argument. We still know that any subsequence (b_{n_k}) converges to b, so $\lim_{n\to\infty}(a_{n_k} \cdot b_{n_k}) = \lim_{n\to\infty} a_{n_k} \cdot b$. Keeping in mind that $a_n, b_n \geq 0$, we can conclude that the largest value to which the product of the sequences converge is equal to the largest value to which a_n converges, times b. In other words, we have shown that $\limsup(a_n \cdot b_n) = \limsup a_n \cdot b$.