# Math 351: Homework 7 Due Friday November 2

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## Problem 1

Suppose I is a bounded closed interval and  $f: I \to \mathbb{R}$  is continuous on I. We will prove that f(I) is bounded. Since I is a bounded closed interval on R, it is a compact subset of R. By the Extreme Value Theorem, f has a maximum and a minimum on I. This means that f(I) has a maximum and minimum value, and thus it is bounded.

Suppose I is a bounded open interval and  $f: I \to \mathbb{R}$  is continuous on I. We will show a counterexample to disprove that f(I) is bounded. Let I be (0,1) and define  $f(x) = \frac{1}{x}$ . I is a bounded open interval and we have previously shown f is continuous on I. However,  $f(x) \to \infty$  as  $x \to 0$ , so  $f(I) = (1, \infty)$  is not bounded.

Suppose I is a bounded open interval and  $f: I \to \mathbb{R}$  is uniformly continuous on I. We will prove that f(I) is bounded. It will suffice to show that f(I) must be bounded from above, since to show that f(I) is bounded from below is the same as showing that -f(I) is bounded from above. So, assume for contradiction that f(I) is not bounded from above. This means that  $\forall M \in \mathbb{R} \ \exists e \in f(I)$  such that e > M. Note that for any such e there is some  $x \in I$  where f(x) = e. Now, by the definition of uniform continuity,  $\forall \varepsilon > 0 \; \exists \delta > 0 \; \text{such that} \; x, y \in I \; \text{and} \; |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$ Pick any  $\varepsilon$  and fix it. Now find a corresponding  $\delta$ . Now, let a, b be the endpoints of I such that I=(a,b). Note that at least one of  $f((a,\frac{a+b}{2}])$  or  $f([\frac{a+b}{2},b))$  is not bounded. (If both subintervals were bounded, then  $\sup I$ would be a real number and be equal to the larger of the two supremums of the subintervals.) If the image of the former interval under f is unbounded, let  $a_1 = a$  and  $b_1 = \frac{a+b}{2}$ . If not, then let  $a_1 = \frac{a+b}{2}$  and  $b_1 = b$ . Now let  $I_1 = (a_1, b_1)$ . Again, we can divide  $I_1$  in half and choose whichever half has an unbounded image under f to be  $I_2$ . Continuing this pattern, note that  $(a_n - b_n) = \frac{a-b}{2^n} \to 0$  as  $n \to \infty$ , and that for all  $I_n$ ,  $f(I_n)$  is unbounded. Thus there is some  $I_n$  where  $(a_n - b_n) < \delta$ . Now pick  $y = a_n$  and let  $M = f(y) + \varepsilon$ . By the fact that  $I_n$  is unbounded, there is some  $x \in I_n$ such that  $f(x) > f(y) + \varepsilon$ . But, x is in  $I_n$ , so we have found a case where  $|x-y| < \delta$  and  $|f(x)-f(y)| > \varepsilon$ , a contradiction. So f(I) must be bounded

from above, and following the same argument with -f(I) shows that f(I) is also bounded from below, so f(I) is bounded.

Suppose A is a countable union of bounded open intervals and  $f: A \to \mathbb{R}$  is uniformly continuous on A. We will provide a counterexample to show that f(A) need not be bounded. Let  $A = \ldots \cup (-2, -1) \cup (0, 1) \cup (2, 3) \cup \ldots$  and f(x) = x. We can show that f is uniformly continuous on  $\mathbb{R}$ . Pick any  $\varepsilon > 0$ . If we choose  $\delta = \frac{\varepsilon}{2}$ , we can see that  $|x - y| < \delta \Longrightarrow |f(x) - f(y)| = |x - y| < \frac{\varepsilon}{2} < \varepsilon$ . Since f is uniformly continuous on  $\mathbb{R}$ , it is uniformly continuous on  $A \subseteq \mathbb{R}$ . However, we can see that f(I) is not bounded. If we choose any M > 0, we know that f(M + 1) = M + 1 > M. Note that at least one of M + 1, M + 2, M + 2.1, and M + 3.1 will be in A. Thus we can always find an  $x \in A$  where f(x) > M. So in this case, f is uniformly continuous on a countable union of bounded open intervals and f(I) is not bounded.

## Problem 2

Suppose  $a_n$  and  $b_n$  are Cauchy sequences.

Prove or disprove:  $|a_n - b_n|$  is Cauchy. Since both sequences are Cauchy, we know that  $a_n$  converges to some  $A \in \mathbb{R}$  and  $b_n$  converges to a value  $B \in \mathbb{R}$ . So,  $\forall \varepsilon > 0 \ \exists N_1 > 0$  such that  $n > N_1 \implies |a_n - A| < \varepsilon$  and  $n > N_2 \implies |b_n - B| < \varepsilon$ . Pick an  $\varepsilon$  and fix it. Find any  $N_1, N_2$  corresponding to  $\frac{\varepsilon - (A - B)}{2}$  and set  $N > \max\{N_1, N_2\}$ . Now we have  $|a_n - b_n| = |a_n - A - b_n + B + A - B| = |(a_n - A) + (B - b_n) + (A - B)| < \varepsilon - (A - B) + (A - B) = \varepsilon$ . Thus  $|a_n - b_n|$  converges and is therefore Cauchy.

Prove or disprove:  $(-1)^n a_n$  is Cauchy. Let  $a_n = 1$  This clearly converges and thus is Cauchy. However,  $(-1)^n a_n$  does not converge. To see why, pick  $\varepsilon = \frac{1}{2}$ . For any choice of N,  $|a_{N+2} - a_{N+1}| = 2 > \varepsilon$ . Thus  $(-1)^n a_n$  is not Cauchy.

Prove or disprove: if  $a_n \neq 0$  for all n, then  $\frac{1}{a_n}$  is Cauchy. Let  $a_n = \frac{1}{2^n}$ . This sequence converges to 0, and for all n,  $a_n > 0$ . So  $a_n$  is Cauchy. However,  $\frac{1}{a_n} = 2^n \to \infty$ . So,  $\frac{1}{a_n}$  does not converge, and this counterexample disproves the proposition.

Prove or disprove: if  $a_n > 0.001$  for all n, then  $\frac{1}{a_n}$  is Cauchy. If  $a_n > 0.001$  for all n and  $a_n$  is Cauchy, then  $a_n$  converges to some L > 0.001. Hence  $\frac{1}{a_n} < 1000$  and, by the algebraic properties of limits,  $\frac{1}{a_n}$  converges to  $\frac{1}{L}$ , and

we conclude that  $\frac{1}{a_n}$  is Cauchy.

## Problem 3

A function  $f: \mathbb{R} \to \mathbb{R}$  is monotone if  $x \leq y$  makes  $f(x) \leq f(y)$ . Show that a monotone function can have at most a countable number of points of discontinuity.

First, consider the function  $f(x) = \lfloor x \rfloor$  as an example of a monotone function with a countably infinite number of points of discontinuity. So, clearly a monotone function can have an infinite number of discontinuities.

In order to show this for  $\mathbb{R}$ , we will first show that f must have at most a countable number of discontinuities on the interval [a,b]. At a point of discontinuity  $c \in [a,b]$  we have that  $\exists \varepsilon > 0$  such that  $\forall \delta > 0$ ,  $|x-c| < \delta$  and  $|f(x) - f(c)| > \varepsilon$ . Note that every time we have a discontinuity, the value of f can only jump up, because f is monotone and also defined on all of [a,b]. Now, pick a discontinuity point c and find a corresponding  $\varepsilon$  as per the definition. There must be a finite number of jumps of size  $\varepsilon$  on [a,b] because the set f([a,b]) is bounded. We can find at most a finite set of discontinuities bigger than  $\varepsilon$  for any value of  $\varepsilon$ . So, the set of all discontinuous points on [a,b] is a union of finite sets, so it is at most countable. We can cover  $\mathbb R$  with a countable union of bounded closed sets, each containing a countable number of discontinuities, so the total set of all discontinuities is a countable union of countable sets, which itself must be countable.

#### Section 5.1

4. Prove that f is continuous on [-1,1], but f is not differentiable at x=0 where  $f:[-1,1]\to\mathbb{R}$  is defined by:

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

To show that f is continuous on [-1,1], we can show that for any sequence  $(a_n)$  in [-1,1],  $a_n \to x \implies f(a_n) \to f(x)$ . Pick any sequence  $(a_n)$  in the domain and call its limit L. Then  $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n \sin(1/a_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} \sin(1/a_n)) = L \cdot \lim_{n\to\infty} \sin(1/a_n)$ . Since  $\sin(x)$  is continuous and 1/x is continuous, their composition is

continuous, which means that  $\lim_{n\to\infty} \sin(1/a_n) = \sin\left(\frac{1}{\lim_{n\to\infty} a_n}\right) = \sin(1/L)$ . So,  $\lim_{n\to\infty} f(a_n) = L \cdot (1/L) = f(L)$  and we conclude that f(x) is continuous.

However, we can show that f is not differentiable at 0 by showing that f is not locally linear near 0. We want to try to write  $f(x) = 0 + M \cdot x + o(x)$  for all  $x \in B(0, \varepsilon)$  and  $\varepsilon > 0$ . We know that f(x) = x is continuous, so we can write  $f(x) = x \sin(1/x) = (M_1 \cdot x + o(x))(\sin(1/x)) = M_1 \cdot x \sin(1/x) + o(x) \sin(1/x) = M_1 \cdot x \sin(1/x) + o(x)$ . However  $\sin(1/x)$  is not continuous and thus not differentiable at 0, so it is not locally linear near 0, so we cannot simplify this expression to the desired form. Thus, f is not differentiable at 0.

5. Prove that g is differentiable on (-1,1) but g' is not continuous at x=0 where g is defined by:

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0\\ 0 & x = 0 \end{cases}$$

Ran out of time to figure this one out. I'm confused because, what does it mean for g' to not be continuous at a point if it is defined at that point? I would assume it means that the left-sided limit and the right-sided limit of g' are different. But doesn't that mean g is not differentiable at that point? I had trouble figuring out what I can do to prove anything here.

6. Give an example of a continuous invertible function on (0,1) that is not differentiable.

Let  $f:(0,1)\to\mathbb{R}$  be defined by

$$f(x) = \begin{cases} x & x \le \frac{1}{2} \\ 2x - \frac{1}{2} & x > \frac{1}{2} \end{cases}$$

Note that at  $x = \frac{1}{2}$  the left-sided derivative is 1, but the right-sided derivative is 2, so f is not differentiable. But the inverse of f is well-defined:

$$f(x) = \begin{cases} x & x \le \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & x > \frac{1}{2} \end{cases}$$

7. Prove that a continuous invertible function defined on a closed or open interval is strictly monotone.

Let f be a continuous invertible function and assume for contradiction that f is not strictly monotone. This means that there exist some a < b < c in the domain such that  $f(a) \le f(b)$  and  $f(c) \le f(b)$  or where  $f(a) \ge f(b)$  and  $f(c) \ge f(b)$ . We can assume the former case is true without losing generality. In addition, we consider the case where  $f(a) \le f(c)$ . By the Intermediate Value Theorem, there exists some  $a' \in [a, b]$  such that f(a') = f(c) since  $f(c) \in [f(a), f(b)]$ . But, if f(a') = f(c) then f is not one-to-one, so f is not invertible, a contradiction. So continuous invertible functions must be strictly monotone.

8. Let  $n \in \mathbb{N}$ . Construct a function that is n times differentiable on an interval but fails to be n+1 times differentiable at a point in the interval.

Define  $f:[0,\infty)\to\mathbb{R}$  as  $f(x)=x^{\frac{1}{2}+n}$ . We can differentiate f n times on  $[0,\infty)$ , and  $f^N(x)=\left(\Pi_{i=1}^n(\frac{1}{2}+i)\right)x^{\frac{1}{2}}$ . However,  $x^{\frac{1}{2}}$  is not differentiable at x=0.

#### Section 5.2

1. Let  $f:[a,b]\to\mathbb{R}$  be a function. Suppose that f is continuous on [a,b] and differentiable on (a,b). Show that if  $f'(x)\neq 0$  for all  $x\in (a,b)$ , then f is one-to-one.

We prove this by proving the contrapositive. Assume f is not one-to-one, which means that  $\exists x < y \in (a,b)$  such that f(x) = f(y). By Rolle's Theorem, there must be a  $c \in (x,y)$  such that f'(c) = 0. So we have shown the contrapositive, and thus the original statement is true.

2. Let  $f, g : [a, b] \to \mathbb{R}$  be functions that are continuous on [a, b] and differentiable on (a, b). Show that if f'(x) = g'(x) for all  $x \in (a, b)$  then there exists  $c \in \mathbb{R}$  such that f(x) = g(x) + c for all  $x \in [a, b]$ .

Since f'(a) = g'(a) then f and g are locally linear near a with the same slope M. This means that f(x) = f(a) + M(x - a) + o(x - a) and g(x) = g(a) + M(x - a) + o(x - a). Now subtract g from f: f(x) - g(x) = f(a) + M(x - a) + o(x - a) - g(a) - M(x - a) - o(x - a) = (f(a) - g(a)) + o(x - a) - o(x - a) Now, as  $x \to a$ ,  $f(x) - g(x) = (f(a) - g(a)) \implies f(x) = g(x) + (f(a) - g(a))$ . Note that f(a) - g(a) is a constant value, so we have proven the statement true.

- 3. Let  $f, g : [a, b] \to \mathbb{R}$  be functions that are continuous on [a, b] and differentiable on (a, b) such that f(a) = g(a). Show that if  $f'(x) \le g'(x)$  for all  $x \in (a, b)$  then  $f(x) \le g(x)$  for all  $x \in (a, b)$ .
  - We will prove this by the contrapositive. Assume that f(a) = g(a) and that  $\exists c \in [a,b]$  such that f(c) > g(c). Now define h(x) = g(x) f(x) and note that h(a) = 0 and h(c) = d for some value of d < 0. By the Mean Value Theorem, there is some  $c' \in (a,c)$  such that  $h'(c') = \frac{d-0}{c-a} < 0$ . Note that h'(c') = g'(c') f'(c'), and we have  $g'(c') f'(c') < 0 \implies g'(c') < f'(c')$ . We have shown that if f(x) > g(x) for any x then there must be some value where f'(x) > g'(x). The contrapositive has been shown to be true, so the original proposition is true.
- 8. Let  $f:(a,b)\to\mathbb{R}$  be a differentiable function. Prove that if  $f'(x)\geq 0$  for all  $x\in(a,b)$  then f is an increasing function on (a,b). Furthermore, if f'(x)>0 for all  $x\in(a,b)$ , then f is a strictly increasing function on (a,b).
  - We will prove this by contraposition. Assume that  $\exists x < y \in (a, b)$  such that f(x) > f(y), and we will show that there is some c where f'(c) < 0. By the Mean Value Theorem, we can see that  $\exists c \in (x, y)$  such that  $f'(c) = \frac{f(y) f(x)}{y x}$ . Since y > x and f(y) < f(x), f'(c) is negative. So we have shown the contrapositive to be true, and we conclude that the original statement is true.
- 9. Prove that the function  $g:(-1,1)\to\mathbb{R}$  given by  $g(x)=\sqrt[3]{x}$  is not differentiable at 0.
  - The derivative of  $g(x) = \sqrt[3]{(x)} = x^{1/3}$  on  $(-1,0) \cup (0,1)$  is  $g'(x) = \frac{1}{3}x^{-2/3}$ . This function approaches infinity as  $x \to 0$  from either side, so g' is not defined at 0.