Math 350: Homework 0 (LaTeXsample)

Name: *Sample*

Problem 1:

Show that $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ using truth tables:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Problem 2:

Show that $(A \cap B)^c = A^c \cup B^c$:

Suppose $x \in (A \cap B)^c$. Then $x \notin A \cap B$, meaning that x is not in both A and B: either $x \notin A$ or $x \notin B$. In other words, $x \in A^c$ or $x \in B^c$, making $x \in A^c \cup B^c$.

Conversely, if $x \in A^c \cup B^c$ then $x \in A^c$ or $x \in B^c$. So $x \notin A$ or $x \notin B$, so it is not possible for x to be in both A and B. That is $x \notin A \cap B$, making $x \in (A \cap B)^c$.

Problem 3:

Consider a function $f : X \to Y$.

a) Show that $A \subset B \Rightarrow f(A) \subset f(B)$:

For any $y \in f(A)$ there is an $x \in A$ with $y = f(x)$. Since $A \subset B$, we have $x \subset B$, so $y \in f(B)$.

b) Show that $U \subset V \Rightarrow f^{-1}(U) \subset f^{-1}(V)$:

For any $x \in f^{-1}(U)$ there is $y \in U$ with $y = f(x)$. Since $y \in U$ and $U \subset V$, we have $y \in V$. Thus $x = f^{-1}(y) \in f^{-1}(V)$.

Problem 4: Solve exercise 1 from section 1.1 (page 3)

(from C.S.) Let $(x, y) \in (A \times B) \cap (C \times D)$. Then (x, y) is in $A \times B$ and in $C \times D$. So x is in A and in C and y is in B and in D. Thus $x \in A \cap C$ and $y \in B \cap D$. So $x \in (A \cap C) \times (B \cap D)$. Therefore $(A \times B) \cap (C \times D) \subset (A \cap C) \times (B \cap D)$.

For the other inclusion, let $(x, y) \in (A \cap C) \times (B \cap D)$. Then $x \in A \cap C$ and $y \in B \cap D$. So $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Thus $(A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D)$.

Problem 5: Prove the Binomial formula

Show that for all numbers a and b,

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$ is the binomial coefficient.

We proceed by induction. The base step is clear as $\binom{1}{0} = 1 = \binom{1}{1}$.

For the inductive step assume that $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$. Then

$$\begin{aligned}
(a + b)^{n+1} &= (a + b)(a + b)^n = (a + b) \cdot \textstyle\sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \\
&= \textstyle\sum_{i=0}^{n} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=0}^{n} \binom{n}{i} a^i b^{n+1-i} \\
&= a^{n+1} + \textstyle\sum_{i=0}^{n-1} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=1}^{n} \binom{n}{i} a^i b^{n+1-i} + b^{n+1} \\
&= a^{n+1} + \textstyle\sum_{i=0}^{n-1} \binom{n}{i} a^{i+1} b^{n-i} + \sum_{i=0}^{n-1} \binom{n}{i+1} a^{i+1} b^{n-i} + b^{n+1} \\
&= a^{n+1} + \textstyle\sum_{i=0}^{n-1} \left[\binom{n}{i} + \binom{n}{i+1} \right] a^{i+1} b^{n-i} + b^{n+1} \\
&= a^{n+1} + \textstyle\sum_{i=0}^{n-1} \binom{n+1}{i+1} a^{i+1} b^{n-i} + b^{n+1} \\
&= a^{n+1} + \textstyle\sum_{i=1}^{n} \binom{n+1}{i} a^i b^{n+1-i} + b^{n+1} \\
&= \textstyle\sum_{i=0}^{n+1} \binom{n+1}{i} a^i b^{n+1-i},
\end{aligned}$$

which completes the inductive step.

We used the identity

$$\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$$

which we now verify by direct calculation from the definition (we omit a few steps).

$$\begin{aligned}
\tbinom{n}{i} + \tbinom{n}{i+1} &= \tfrac{n!}{i!(n-i)!} + \tfrac{n!}{(i+1)!(n-(i+1))!} \\
&= \tfrac{n!(i+1)+n!(n-i))}{(i+1)!(n-i)!} \\
&= \tfrac{n!(i+1+n-i)}{(i+1)!(n-i)!} \\
&= \tfrac{(n+1)!}{(i+1)!(n+1-(i+1))!} \\
&= \tbinom{n+1}{i+1}.
\end{aligned}$$