Math 351: Homework 3 (Due September 28)

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Problem 1

Let $x_1 = 2$ and define $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ for $n = 2, 3, \ldots$

Prove that $x_n^2 > 2$ for all $n > 1$

We proceed by induction. To start, consider the case where $n = 2$. We see that $x_2^2 = (\frac{1}{2}x_1 + \frac{1}{x_1})^2 = (\frac{1}{2} \cdot 2 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4} > 2$.

For the inductive step we assume that $x_n^2 > 2$, and we want to conclude from this that $x_{n+1}^2 > 2$. We proceed by manipulating the first inequality:

$$\begin{aligned}
x_n^2 > 2 &\implies x_n^2 - 2 > 0 \\
&\implies (x_n^2 - 2)^2 > 0 \\
&\implies x_n^4 - 4x_n^2 + 4 > 0 \\
&\implies \tfrac{1}{4}x_n^2 - 1 + \tfrac{1}{x_n^2} > 0 \\
&\implies \tfrac{1}{4}x_n^2 + 1 + \tfrac{1}{x_n^2} > 2 \\
&\implies (\tfrac{1}{2}x_n + \tfrac{1}{x_n})^2 > 2 \\
&\implies x_{n+1}^2 > 2.
\end{aligned}$$

Thus, by mathematical induction, $x_n^2 > 2$ for all $n > 1$.

Prove that $x_{n+1} < x_n$ for all n

We proceed by manipulating the inequality $x_n^2 > 2$. However, we notice that we have only shown this to be true for $n > 1$. We will cover the case where $n = 1$ later. We will also use the fact that $x_n > 0$ for all n, which we can see must be true as every x_n is the sum of positive numbers in $\mathbb{Q}$. So, we can see that

$$\begin{aligned} x_n^2 > 2 \quad &\implies \tfrac{x_n^2}{2} > 1 \\ &\implies \tfrac{x_n}{2} > \tfrac{1}{x_n} \\ &\implies x_n > \tfrac{1}{x_n} + \tfrac{x_n}{2} \\ &\implies x_n > x_{n+1}. \end{aligned}$$

We have shown that $x_{n+1} < x_n$ for $n > 1$. We finish by considering $n = 1$: $x_2 = \frac{3}{2} < 2 = x_1$. Thus, the sequence (x_n) is monotone decreasing.

Prove that $\lim\limits_{n\to\infty} x_n^2 = 2$

We have shown that the sequence (x_n^2) is monotone decreasing and bounded from below, specifically bounded from below by the number 2. This sequence must therefore converge. In other words, there exists an L such that $\lim_{n\to\infty} x_n^2 = L$. We can see also that $\lim_{n\to\infty} x_{n+1}^2 = L$. With the defintion of x_{n+1} we can expand this to get

$$\lim_{n\to\infty} \left(\frac{x_n}{2} + \frac{1}{x_n} \right)^2 = \lim_{n\to\infty} \frac{x_n^2}{4} + 1 + \frac{1}{x_n^2} = \frac{L}{4} + 1 + \frac{1}{L} = L.$$

Solving for L:

$$\begin{aligned} \tfrac{L}{4} + 1 + \tfrac{1}{L} = L \quad &\implies 3L^2 - 4L - 4 = 0 \\ &\implies (L-2)(3L+2) = 0 \\ &\implies L = 2 \text{ or } L = -\tfrac{2}{3}. \end{aligned}$$

Since $x_n > 0$ for all n, we can dismiss the negative solution. Thus, $L = 2$. In other words, $\lim_{n\to\infty} x_n^2 = 2$.

Section 2.1

Problem 16 (Bonus)

Problem 18

Let (a_n) be a sequence and let $L \in \mathbb{R}$. We want to prove that if every subsequence (a_{n_k}) has a subsequence $(a_{n_{k_l}})$ that converges to L, then the sequence (a_n) converges to L.

We will prove this using the contrapositive. We hope to show that if (a_n) does not converge to L then there exists some subsequence (a_{n_k}) such that all of its subsequences $(a_{n_{k_l}})$ fail to converge to L. By the negation of the definition of convergence to L, we can assume that there exists an $\varepsilon > 0$ where for all N, there is some $n > N$ such that $|a_n - L| \geq \varepsilon$. Find such an ε and fix it. Set $N = 1$. So, there is some $n_1 > 1$ so that $|a_{n_1} - L| \geq \varepsilon$. Now consider $N = n_1$. We must also be able to find an $n_2 > n_1$ where $|a_{n_2} - L| \geq \varepsilon$. We can continue this argument and find a sequence (n_k) such that, for all k, $|a_{n_k} - L| \geq \varepsilon$. This subsequence of (a_n) has been constructed such that it does not converge to L. More specifically, it has been constructed such that there are no values k where $|a_{n_k} - L| < \varepsilon$. In other words, we cannot find any subsequences of (a_{n_k}) that converge to L, because for such a subsequence $(a_{n_{k_l}})$ to converge, there would have to be infinitely many values of l for which $\left|a_{n_{k_l}} - L\right| < \varepsilon$ for every ε. We conclude that if (a_n) does not converge to L, we can find a subsequence of (a_n) that does not have any subsequences that converge to L. So, by contraposition, if every subsequence of (a_n) has a subsequence that converges to L, then (a_n) converges to L.

Section 2.2

Problem 25

Make sense of the following expression as a limit and find its value:

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

Define a sequence (a_n) where $a_1 = 1 + \frac{1}{1} = 2$ and $a_{n+1} = 1 + \frac{1}{1 + a_n}$. The value of the continued fraction above can be interpreted as $\lim_{n\to\infty} a_n$. Assume that (a_n) converges to some value L, so $\lim_{n\to\infty} a_n = L$, and note that $\lim_{n\to\infty} a_{n+1} = L$ as well. We can use the recursive definition to expand this:

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} 1 + \frac{1}{1+a_n} \\ &= 1 + \frac{1}{1+L} = L.\end{aligned}$$

We solve for L:

$$\begin{aligned}L = 1 + \frac{1}{1+L} &\implies L(L+1) = (L+1) + 1 \\ &\implies L^2 + L = L + 2 \\ &\implies L^2 - 2 = 0 \\ &\implies L = \sqrt{2} \text{ or } L = -\sqrt{2}.\end{aligned}$$

Since each term of (a_n) is positive, we can exclude the negative solution for L. So we find that $L = \sqrt{2}$, which is the value of the continued fraction.

Problem 26

Make sense of the following expression as a limit and find its value:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$$

Define a sequence (a_n) where $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$. The value of the expression above can be interpreted as $\lim_{n\to\infty} a_n$. Assume that (a_n) converges to some value L, so $\lim_{n\to\infty} a_n = L$, and note that $\lim_{n\to\infty} a_{n+1} = L$ as well. We can use the recursive definition to expand this:

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{2 + a_n} \\ &= \sqrt{2 + L} = L.\end{aligned}$$

We solve for L:

$$
\begin{aligned}
\sqrt{2+L} = L \quad &\implies 2 + L = L^2 \\
&\implies L^2 - L - 2 = 0 \\
&\implies (L-2)(L+1) = 0 \\
&\implies L = 2 \text{ or } L = -1.
\end{aligned}
$$

Since each term of (a_n) is positive, we can exclude the negative solution for L. So we find that $L = 2$, which is the value of the expression.