Math 351: Homework 5 Due Friday October 19

Sections 3.2 and 3.3

Section 3.2 Exercises (pg 112-113) work 6,7,8,9,10

6. Let $A \subsetneq \mathbb{R}$. When is $A^\circ = \overline{A}$?

   We know that $A^\circ \subseteq A \subseteq \overline{A}$, so $A^\circ = \overline{A} = A$ in this case. We have previously shown that $A = A^\circ$ if and only if A is open, and that $A = \overline{A}$ if and only if A is closed. So, A must be both open and closed. The only sets for which this can be true are $\mathbb{R}$ and $\varnothing$, and since we have $A \subsetneq \mathbb{R}$, the only case where this is possible is $A = \varnothing$.

7. Is there a set whose interior is empty and whose closure is $\mathbb{R}$?

   The set $\mathbb{Q}$ satisfies this property. Pick any $q \in \mathbb{Q}$ and fix it. For any $\varepsilon > 0$, the open ball $B(q, \varepsilon)$ will contain a nonrational real number due to the order completeness of $\mathbb{R}$. Thus, $\mathbb{Q}$ contains no interior points. In addition, we know that $\mathbb{R}$ consists of all values to which sequences of rational numbers converge, so every real number is the limit of some sequence in $\mathbb{Q}$. Thus every $x \in \mathbb{R}$ is a closure point of $\mathbb{Q}$. So $\mathbb{Q}$ is a set whose interior is empty and whose closure is $\mathbb{R}$.

8. Find the closure and interior of each of the following sets:

   (a) $A = [0, 1) \cup (1, 2)$
   $\overline{A} = [0, 2],\ A^\circ = (0, 1) \cup (1, 2)$

   (b) $\mathbb{N}$
   $\overline{\mathbb{N}} = \mathbb{N},\ \mathbb{N}^\circ = \varnothing$

   (c) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$
   $\overline{A} = A \cup 0,\ A^\circ = \varnothing$

   (d) $A = \mathbb{Q} \cap (0, 1)$
   $\overline{A} = [0, 1],\ A^\circ = \varnothing$

   (e) $A = \mathbb{R} \setminus \mathbb{Q}$
   $\overline{A} = \mathbb{R},\ A^\circ = \varnothing$
   Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$, so A has no interior points.

(f) $A = \{\frac{p}{2^n} : n \in \mathbb{N}, p = 0, \dots, 2^n\}$
$\overline{A} = \mathbb{R},\ A^\circ = \varnothing$
A is the set of dyadic rationals in $[0,1]$. In Example 2.2.6, we define a function $d : [0,1] \to \{0,1\}^{\mathbb{N}}$ that maps every real number in $[0,1]$ to a sequence of dyadic intervals of increasing rank. For any $x \in [0,1]$, if we take the sequence of the left endpoints of these intervals determined by $d(x)$, this sequence will be a subset of our set A and will converge to x.

9. Prove or disprove $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

   We will show that these sets are equal by showing that they are subsets of one another. First, let $x \in \overline{(A \cup B)}$. This means that x is a closure point of $A \cup B$. Thus there exists some sequence $a_n \in (A \cup B)$ that converges to x. There are two possible cases: either a_n is a subset of A or B, or (a_n) contains elements from A and elements from B. If it is the former case, assume without loss of generality that $(a_n) \subseteq A$ (since the following argument is also valid for B). Thus, x is also a closure point of A, so $x \in \overline{A} \subseteq (\overline{A} \cup \overline{B})$. In the case where (a_n) contains elements from both A and B, take either the subsequence $(a_n) \cap A$ or $a_n \cap B$, whichever has an infinite number of elements. We know that if $a_n \to x$, any subsequence of a_n also converges to x. Thus we can find a sequence in either A or B that converges to x, so x is a closure point of either A or B, and it follows that $x \in (\overline{A} \cup \overline{B})$. Since any x in $\overline{A \cup B}$ is in $\overline{A} \cup \overline{B}$, $\overline{A \cup B} \subseteq (\overline{A} \cup \overline{B})$.

   Now take $x \in (\overline{A} \cup \overline{B})$. This means x is either a closure point of A or a closure point of B, so there is some sequence (a_n) of terms in either A or B that converges to x. Each a_n would also be in $A \cup B$, however, which means that x is also a closure point of $A \cup B$, so $x \in \overline{A \cup B}$. Hence $(\overline{A} \cup \overline{B}) \subseteq \overline{A \cup B}$. Since we have both sets as subsets of one another, the sets must be equal.

10. Prove or disprove $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$.

    We disprove this by providing a counterexample. Let $A = (0,1)$ and $B = (1,2)$. $\overline{(A \cap B)} = \overline{(0,1) \cap (1,2)} = \overline{\varnothing} = \varnothing$. But, $\overline{A} \cap \overline{B} = \overline{(0,1)} \cap \overline{(1,2)} = [0,1] \cap [1,2] = 1 \neq \varnothing$.

Section 3.3 Exercises (pg 116-117) work 7

Interiors

For $\mathbb{R}^n$, what is the relationship between A° and $(\bar{A})^\circ$? Are they equal? One contained in the other? Prove any claim you make.

Compactness 1

Suppose K, F are subsets of $\mathbb{R}$, and that K is compact and F is closed. Are the following definitely compact, definitely closed, or possibly neither.

a) $K \cap F$

b) $\overline{K^c \cup F^c}$

c) $K \backslash F$

d) $\overline{K \cap F^c}$

Compactness 2

Consider the set comprised of closed intervals

$$\ldots \cup [\tfrac{1}{6}, \tfrac{1}{5}] \cup [\tfrac{1}{4}, \tfrac{1}{3}] \cup [\tfrac{1}{2}, 1]$$

That is, let $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for $n \in \mathbb{N}$ and take $B = \bigcup_{n \in \mathbb{N}} I_n$.

a) Show B is bounded but not closed.

b) Find a sequence $\{b_n\} \subset B$ that converges to a point not contained in B.

c) Find an open cover of B that cannot be reduced to a finite open cover.

d) Show $\tilde{B} = B \cup \{0\}$ is compact.

e) How does the addition of the point $\{0\}$ undo your answers in (b) and (c)? That is, why can't you make similar examples that work for $\tilde{B}$?

Compactness 3

Prove that any open cover of any closed set can be reduced to a countable open cover.

(*HINT: Let B be a closed set, and define $B_n = B \cap [n-1, n]$ for $n \in \mathbb{Z}$. Then B_n is compact.*)

Use this to work section 3.4 exercise 2 (page 122).

Compactness 4

Work section 3.4 exercise 3 (page 122).

(*HINT: Suppose not. Then $\forall \delta > 0$, there is an interval I of length δ with $I \cap E \neq \emptyset$ and $I \cap E \neq \emptyset$. Use this to construct convergent e_n in E and f_n in F, with $|e_n - f_n| \to 0$.)*

Compactness 5

And now for something completely different. Consider the natural numbers $\mathbb{N}$ with the trivial metric (defined in example 3.1.7 on page 107).

Show that with this metric, the set $\mathbb{N}$ is closed and bounded, but not sequentially compact.