

Math 351: Homework 2 (Due September 21)

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Section 0.5

Problem 7

We will show by induction that for all integers $n \geq 1, n < 2^n$.

For the base case, let $n = 1$. $2^n = 2^1 = 2 > 1$, so the property holds.

Now we will show that if $n < 2^n$ then $n + 1 < 2^{n+1}$. We can add one to both sides of the first inequality: $n + 1 < 2^n + 1 < 2^n + 2^n < 2^n \cdot 2 = 2^{n+1}$. This completes the inductive step. Thus, by mathematical induction, the property is true for all $n \geq 1$.

Problem 8

Let $\{F_n\}$ be the Fibonacci sequence. We will prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^n F_i = F_{n+2} - 1.$$

We proceed by induction. Let $n = 1$ and recall that $F_1 = 1$ and $F_3 = 2$. We see that $F_1 = 1 = 2 - 1 = F_3 - 1$.

For the inductive step we assume $\sum_{i=0}^n F_i = F_{n+2} - 1$, and we want to show that $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$. In addition, recall that the definition of $\{F_n\}$ includes the statement $F_n = F_{n-1} + F_{n+2}$. Consider the following equalities:

$$\begin{aligned} \sum_{i=0}^{n+1} F_i &= \sum_{i=0}^n F_i + F_{n+1} \\ &= F_{n+2} - 1 + F_{n+1} \\ &= (F_{n+2} + F_{n+1}) - 1 \\ &= F_{n+3} - 1. \end{aligned}$$

By mathematical induction, this property must hold for all $n \in \mathbb{N}$.

Problem 9

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^n \frac{1}{2^i} = 1 - \frac{1}{2^n}.$$

We proceed by induction. Consider the base case where $n = 1$. We get $\frac{1}{2^1} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$, so the base case is true.

For the inductive step we assume $\sum_{i=1}^n \frac{1}{2^i} = 1 - \frac{1}{2^n}$ and we want to show that $\sum_{i=1}^{n+1} \frac{1}{2^i} = 1 - \frac{1}{2^{n+1}}$. Note that the following equalities hold:

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{2^i} &= \sum_{i=1}^n \frac{1}{2^i} + \frac{1}{2^{n+1}} \\ &= 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} \\ &= 1 - \frac{2^{n+1} - 2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{2 \cdot 2^n - 2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{1}{2^{n+1}}. \end{aligned}$$

So, by mathematical induction, this is true for all $n \geq 1$.

Problem 11

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can show this by induction. We let $n = 1$ for the base case, and we see that $1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6}$.

For the inductive step we assume $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ and we want to show that $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$. Consider the following equalities:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

By mathematical induction, this property is true for all $n \geq 1$.

Section 1.3

Problem 7

Recall that for sets A_1 and A_2 , the Cartesian product is defined as $A \times A_1 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$. Define the Cartesian product of k sets $A_1 \times A_2 \times \dots \times A_k = \{(a_1, a_2, \dots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_k \in A_k\}$.

Using this definition we can prove that if the sets A_1, \dots, A_k are countable, then their Cartesian product is also countable. We proceed by induction. In the case where $k = 2$, we have the Cartesian product $A_1 \times A_2$, which is countable by Proposition 1.3.8. For the inductive step, assume that A_1, \dots, A_{k+1} are countable and that $A_1 \times \dots \times A_k$ is countable. The Cartesian product $A_1 \times \dots \times A_{k+1}$ can be written as $(A_1 \times \dots \times A_k) \times A_{k+1}$. The set $A_1 \times \dots \times A_k$ is assumed to be countable, as is A_{k+1} , so their Cartesian product $(A_1 \times \dots \times A_k) \times A_{k+1}$ is countable by Proposition 1.3.8. So, by induction, the Cartesian product of k countable sets is itself countable.

Argue that the set of finite sequences of 0's and 1's is countable

Consider the set of finite sequences of 0's and 1's, which we will call S . We can interpret elements of S as binary representations of natural numbers. This is actually a surjection from \mathbb{N}_0 onto S , as any sequence in S can be interpreted as a natural number. Since we have mapped \mathbb{N}_0 , which is countable, onto S , S must also be countable.

Argue that the set of infinite sequences of 0's and 1's is not countable

Call the set of infinite sequences of 0's and 1's S' . Cantor's diagonalization argument can be applied to S' . Assume for contradiction that S' is countable and that as a result there exists some ordered list of each element of S' . We can create a new sequence a whose first term is different from the first term of the first sequence in S' , whose second term is different from the second term of the second sequence in S' , and so on. The sequence a differs from every sequence in S' in at least one position, so it cannot be on the list even though it is a member of S' . We have encountered a contradiction, and thus S' is not countable.

Problem 13

We will show that the set of algebraic numbers is countable by showing first that the set of polynomials, P , is countable. Note that any algebraic number a is by definition the root of some polynomial $p \in P$, or, in other words, there exists an injection from the algebraic numbers into P (the relation f where $f(a) = p$ only if a is a root of p), which means that the cardinality of the algebraic numbers is less than or equal to that of P . So, if we show that P is countable, we can conclude that the algebraic numbers are countable.

Let P_n be the set of polynomials of degree n : specifically $P_n = \{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \mid a_n, \dots, a_0 \in \mathbb{Z}\}$. In addition, consider the Cartesian product \mathbb{Z}^n to be the Cartesian product of \mathbb{Z} n times, as defined in Problem 7. We can define the relation $f : \mathbb{Z}^n \rightarrow P_n$ such that, if $b = (b_0, \dots, b_n) \in \mathbb{Z}^n, p \in P_n$, then $f(b) = p$ only if p 's coefficient $a_k = b_k$ for all $0 \leq k \leq n$. Every $p \in P_n$ has coefficients from \mathbb{Z} , so every $p = f(b)$ for some $b \in \mathbb{Z}^n$. Thus, f is surjective. Since \mathbb{Z} is countable, \mathbb{Z}^n is countable by Problem 7, and since f is a surjection onto P_n , each P_n is also countable.

Now, notice that every polynomial $p \in P$ is in P_n for some n . This means that $P = \bigcup_{n=1}^{\infty} P_n$. So, by Theorem 1.3.14, P is itself countable since it is a countable union of countable sets. By the argument above, we can conclude that the algebraic numbers are countable.

Section 2.1

Problem 1

We will prove that $\lim_{n \rightarrow \infty} |x_n| = 0 \iff \lim_{n \rightarrow \infty} x_n = 0$. By the definition, $\lim_{n \rightarrow \infty} |x_n| = 0 \iff \forall \varepsilon > 0, \exists N$ where $n > N \implies ||x_n|| < \varepsilon$. However, $||x_n|| = |x_n|$ since $|x_n| \geq 0$. Thus $\exists N$ such that $|x_n| < \varepsilon$ for any positive ε , which is the definition of $\lim_{n \rightarrow \infty} x_n = 0$.

Problem 2

We will use the definition to show that $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$. To satisfy the definition, then for any $\varepsilon > 0$ we need to find an N such that $n > N \implies \left| \frac{n}{n+1} - 1 \right| < \varepsilon$. Choose $N = \frac{1}{\varepsilon} - 1$, so that

$$\begin{aligned} \frac{1}{\varepsilon} - 1 &< n \\ \frac{1}{\varepsilon} &< n + 1 \\ \frac{1}{n+1} &< \varepsilon \\ \left| \frac{-1}{n+1} \right| &< \varepsilon \\ \left| \frac{n-(n+1)}{n+1} \right| &< \varepsilon \\ \left| \frac{n}{n+1} - 1 \right| &< \varepsilon. \end{aligned}$$

Since we can find an N for any $\varepsilon > 0$ where this inequality holds true, by definition $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$.

Problem 4

Assume (x_n) is a sequence with $x_n \geq 0$ that converges to L . We will use the definition to show that $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{L}$.

By the definition, we know that $\forall \varepsilon_1 > 0, \exists N_1$ such that $n > N_1 \implies |x_n - L| < \varepsilon_1$, and we want to show that $\forall \varepsilon > 0, \exists N$ where $n > N \implies |\sqrt{x_n} - \sqrt{L}| < \varepsilon$.

First, consider only the case where $L \neq 0$. Take any $\varepsilon > 0$ and fix it. Now

set $\varepsilon_1 = \varepsilon\sqrt{L}$, which we can do since $x_n \geq 0 \implies L \geq 0$. By the definition stated above, we know that there exists some N_1 where $|x_n - L| < \varepsilon_1$ for all $n > N_1$. So, when $n > N_1$, (and keeping in mind that $L \neq 0$) we can write $\varepsilon\sqrt{L} > |x_n - L| \iff \varepsilon > \frac{|x_n - L|}{\sqrt{L}} \geq \frac{|x_n - L|}{\sqrt{x_n} + \sqrt{L}} = \left| \frac{(\sqrt{x_n} - \sqrt{L})(\sqrt{x_n} + \sqrt{L})}{\sqrt{x_n} + \sqrt{L}} \right| = \left| \sqrt{x_n} - \sqrt{L} \right|$. We showed that there exists an N for any ε , namely, $N = N_1$.

Now we deal with the case where $L = 0$. This simplifies the problem, now we need to find an N such that when $n > N$, $|\sqrt{x_n}| < \varepsilon$.

Problem 6

Problem 11

Problem 13

Problem 15

Problem 16