Math 351: Homework 1 (Due September 14)

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Section 0.2

Problem 1

Prove the following equivalences:

a) $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

We can write a truth table that lists the truth values of $\neg(P \vee Q)$ and $(\neg P \wedge \neg Q)$ for all truth values of P and Q:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P \wedge \neg Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Since the two statements have the same truth value in all cases, they are equivalent.

b) $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

We can write a truth value to test every case:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P \vee \neg Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Since both statements have the same truth value for any value of P or Q, they are equivalent.

Problem 2

Prove that $P \implies Q \equiv (\neg P) \vee Q$. Deduce that the negation of $P \implies Q$ is $P \wedge (\neg Q)$.

We write a truth table to show that $P \implies Q$ has the same truth value as $(\neg P) \vee Q$ in all cases:

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \implies Q$
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

Since $P \implies Q$ is equivalent to $(\neg P) \vee Q$, their negations will be equivalent. So we negate $(\neg P) \vee Q$ and manipulate using equivalences from Problem 1 to find the negation:

$$\begin{aligned}
\neg(P \implies Q) \quad &= \neg(\neg P \vee Q) \\
&= \neg(\neg P \vee \neg(\neg Q)) \\
&= \neg(\neg(P \wedge (\neg Q))) \\
&= P \wedge (\neg Q).
\end{aligned}$$

We also used the fact that $\neg(\neg P) \equiv P$, which we now show with a truth table:

P	$\neg P$	$\neg(\neg P)$
T	F	**T**
F	T	**F**

Problem 3

Find the negation of the following statements.

a) $\neg(P \wedge \neg Q) \vee R$

$$\begin{aligned}
\neg(\neg(P \wedge \neg Q) \vee R) \quad &= (P \wedge \neg Q) \wedge \neg R \\
&= P \wedge \neg Q \wedge \neg R.
\end{aligned}$$

b) $P \implies (Q \vee R)$

$$\begin{aligned} \neg(P \implies (Q \vee R)) &= P \wedge \neg(Q \vee R) \\ &= P \wedge \neg Q \wedge \neg R. \end{aligned}$$

c) $\neg(P \vee Q) \implies (R \vee S)$

$$\begin{aligned} \neg(\neg(P \vee Q) \implies (R \vee S)) &= \neg((P \vee Q) \vee (R \vee S)) \\ &= \neg(P \vee Q) \wedge \neg(R \vee S) \\ &= \neg P \wedge \neg Q \wedge \neg R \wedge \neg S. \end{aligned}$$

Problem 5

Prove that an implication is equivalent to its contrapositive.

The contrapositive of $P \implies Q$ is the statement $\neg Q \implies \neg P$. We can manipulate the statement $\neg P \vee Q$, noting that it is equivalent to $P \implies Q$:

$$\begin{aligned} \neg P \vee Q &= Q \vee \neg P \\ &= \neg(\neg Q) \vee \neg P \\ &= \neg Q \implies \neg P. \end{aligned}$$

Problem 6

An example of an implication whose converse is not true:

If polygon $ABCD$ is a square, then its interior angles are all 90 degrees.

An example of an implcation whose converse is true:

$$(x = 4) \implies (x + 2 = 6)$$

Problem 7

Negate the following statements:

a) For every ice cream flavor there is a pie that goes with that flavor

There exists an ice cream flavor for which there are no pies that go with it.

b) For every race car there is a driver who can drive that car.

There exists at least one race car with no driver who can drive it.

c) There exists a race car that every driver can drive.

For each race car there exists at least one driver who cannot drive it.

d) There exists a driver that can drive every race car.

For all race car drivers there exists at least one car they cannot drive.

Section 0.3

Problem 1

Let $A, B \subseteq X$. Which of the statements is equivalent to $A \cup B \neq \varnothing$?

Answer: (b) $A \neq \varnothing \vee B \neq \varnothing$. For the union of A and B to be nonempty it is sufficient for at least one of A or B to be nonempty.

Problem 2

Let $A, B \subseteq X$. Which of the statements is equivalent to $A \cap B = \varnothing$?

Answer: (b) $A = \varnothing \vee B = \varnothing$. If any one of A or B is empty, then their intersection will be empty.

Problem 3

Let A and B be sets. Prove that $A = (A \cap B) \cup (A \setminus B)$.

We consider the definitions of $A \cap B$ and $A \setminus B$. The set $A \cap B$ is the set of all elements of A that are also in B. The set $A \setminus B$ is the set of all elements of A that are not in B. The union of these two sets, $(A \cap B) \cup (A \setminus B)$, is the set of all elements of A that are also in B or that are not also in B. Since everything is either in B or not in B, this set contains all elements of A. Thus, $A = (A \cap B) \cup (A \setminus B)$.

Problem 4

Let A and B be sets. Prove that $A \setminus (A \setminus B) = B$.

The premise is actually false. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. We can see that $A \setminus B = \{1\}$ and that $A \setminus \{1\} = \{2\} \neq \{2, 3\}$.

If instead the condition had been added that $B \subseteq A$, it would have been a good problem.

Problem 5

Let $A, B \subseteq X$. Prove that $A \subseteq B \iff B^c \subseteq A^c$.

We first prove that $A \subseteq B \implies B^c \subseteq A^c$. $A \subseteq B$ means that all elements of A are also in B. Also, by definition, $x \in B^c \implies x \notin B$. So, $\forall x \notin B^c$, x cannot be in A, which means that $x \in A^c$. Thus, $B^c \subseteq A^c$.

We now prove the other direction, that $B^c \subseteq A^c \implies A \subseteq B$. Note that any $x \in B^c$ is also in A^c. In other words, x cannot be in B^c and also be in A. So, any element of A is also in B, which means $A \subseteq B$.

Problem 17

Let A and B be two sets such that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let S be any element of $\mathcal{P}(A)$. By the definition of $\mathcal{P}(A)$, $S \subseteq A$. Since $A \subseteq B$, every element of S is in B as well, so $S \subseteq B$. Hence, $S \in \mathcal{P}(B)$. So, every $S \in \mathcal{P}(A)$ is also in $\mathcal{P}(B)$, which means $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Section 0.4

Problem 10

Let $f : X \to Y$ be a function.

a) Prove that $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq Y$.

Take any $y \in f(f^{-1}(B))$. This means that $y = f(x)$ for some $x \in f^{-1}(B)$. By the definition of $f^{-1}(B)$, $f(x) \in B$, so $y \in B$. Thus, $f(f^{-1}(B)) \subseteq B$.

b) Prove that $f(f^{-1}(B)) = B$ when f is surjective.

We have already shown that $f(f^{-1}(B)) \subseteq B$ in general, so it is also true if f is surjective. We will show that if f is surjective, then $B \subseteq f(f^{-1}(B))$, and thus that $f(f^{-1}(B)) = B$. For f to be surjective, it is necessary that for all $y \in Y$ there exists some $x \in X$ such that $f(x) = y$. In particular, for all $y \in B$ there exists an $x \in f^{-1}(B)$ where $f(x) = y$. Hence, $y \in f(f^{-1}(B))$ for any $y \in B$. So, when f is surjective, $B \subseteq f(f^{-1}(B))$, and thus $B = f(f^{-1}(B))$.

Problem 17

Let $f : X \to Y$ be a function. Prove that $f(A \cup B) = f(A) \cup f(B)$ for all $A, B \in X$.

Let $y \in Y$ and $x \in X$ such that $f(x) = y$ and $y \in f(A \cup B)$. $y \in f(A \cup B)$ if and only if $x \in A \cup B$. Equivalently we can say $x \in A$ or $x \in B$, which is true if and only if $y \in f(A)$ or $y \in f(B)$. By definition, this means $y \in f(A) \cup f(B)$. Since $y \in f(A \cup B) \iff y \in f(A) \cup f(B)$, $f(A \cup B) = f(A) \cup f(B)$ for all $A, B \in X$.

Problem 18

Let $f : X \to Y$ be a function.

a) Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$ for all sets $A, B \subseteq X$ but that equality does not hold in general.

Let $y \in f(A \cap B)$. There must exist at least one $x \in A \cap B$ such that $f(x) = y$. By definition $x \in A$ and $x \in B$, so $y \in f(A)$ and $y \in f(B)$. Since this is true for any $y \in f(A \cap B)$, it must be that $f(A \cap B) \subseteq f(A) \cap f(B)$.

We can show a case where $f(A \cap B) \subsetneq f(A) \cap f(B)$.

Take $X = \{1, 2, 3\}, Y = \{8, 9\}$ and $A, B \subseteq X$ where $A = \{1, 2\}$ and $B = \{1, 3\}$. Define $f : X \to Y$ such that $f(1) = 9, f(2) = 8, f(3) = 8$. Now we find that $(A \cap B) = \{1\}, f(A \cap B) = \{9\}, f(A) = \{8, 9\}, f(B) = \{8, 9\}$, and $(f(A) \cap f(B)) = \{8, 9\}$. Note that $\{9\} \subsetneq \{8, 9\}$ in this case, but it still satisfies the property we proved above.

b) Prove that if f is injective, $f(A \cap B) = f(A) \cap f(B)$.

We already showed that for general f, $f(A \cap B) \subseteq f(A) \cap f(B)$. So we only need to prove that $f(A) \cap f(B) \subseteq f(A \cap B)$ in order to conclude that $f(A \cap B) \subseteq f(A) \cap f(B)$. Let $y \in f(A) \cap f(B)$, which means that $y \in f(A)$ and $y \in f(B)$. Since f is injective, there exists a unique x such that $f(x) = y$. This means that x must be in both A and B. Thus, $x \in A \cap B$, so $y \in f(A \cap B)$. Since all $y \in f(A) \cap f(B)$ is also in $f(A \cap B)$, $f(A) \cap f(B) \subseteq f(A \cap B)$ and thus $f(A \cap B) = f(A) \cap f(B)$.

Problem 19

Let $f : X \to Y$ be a function and let $A, B \subseteq Y$.

Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$:

For all $x \in X$,

$$\begin{aligned} x \in f^{-1}(A \cup B) &\iff f(x) \in A \cup B \\ &\iff f(x) \in A \vee f(x) \in B \\ &\iff x \in f^{-1}(A) \vee x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cup f^{-1}(B). \end{aligned}$$

Since $x \in f^{-1}(A \cup B) \iff x \in f^{-1}(A) \cup f^{-1}(B)$, we conclude that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$:

For all $x \in X$,

$$\begin{aligned} x \in f^{-1}(A \cap B) &\iff f(x) \in A \cap B \\ &\iff f(x) \in A \wedge f(x) \in B \\ &\iff x \in f^{-1}(A) \wedge x \in f^{-1}(B) \\ &\iff x \in f^{-1}(A) \cap f^{-1}(B). \end{aligned}$$

Since $x \in f^{-1}(A \cap B) \iff x \in f^{-1}(A) \cap f^{-1}(B)$, we conclude that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Show that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$:

For all $x \in X$,

$$\begin{aligned}
x \in f^{-1}(A \setminus B) \quad &\iff f(x) \in A \setminus B \\
&\iff f(x) \in A \wedge f(x) \notin B \\
&\iff x \in f^{-1}(A) \wedge x \notin f^{-1}(B) \\
&\iff x \in f^{-1}(A) \setminus f^{-1}(B).
\end{aligned}$$

Since $x \in f^{-1}(A \setminus B) \iff x \in f^{-1}(A) \setminus f^{-1}(B)$, we conclude that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.