# Math 351: Homework 10 Due Friday December 7

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## Chopping

Use Lemma 3.3.4 to complete the proof of the chopping lemma, Lemma 3.2.1

[Use angle difference formulas. Argue that the above argument for sin works for cos as well. Create a period of $2\pi/k$ with some error that disappears as $k \to \infty$ (that part might not quite be right)]

### 0.1 Jensen's

Exercise 5.1: Prove lemma 5.2.2.

Lemma: If $\rho$ is convex on $I$, and $x_1, x_2 \in I$, and equality holds

$$\rho(wx_1 + (1-w)x_2) \leq w\rho(x_1) + (1-w)\rho(x_2)$$

for some $0 < w < 1$, then either $x_1 = x_2$ or $\rho$ is linear on $[x_1, x_2]$.

[BY CONTRADICTION? CONFUSING]

Exercise 5.2: Prove lemma 5.2.3.

Lemma: If $\rho$ is a convex function on an interval $I$, $x_1, \ldots, x_n \in I$, and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum w_i = 1$, then

$$\rho\left(\sum w_i x_i\right) \leq \sum w_i \rho(x_i)$$

with equality holding iff $\rho$ is linear on an interval containing the $x_i$'s, or all $x_i$'s are the same.

We proceed by induction. The base case is where $n = 2$, which is true by Lemma 5.2.2.

[FINISH THIS ONE BOY]

Exercise 5.3: Prove that convex functions are continuous.

We will show that if a function $\rho : I \to \mathbb{R}$ (where $I = [a, b]$) has the property that, for $x_1, x_2 \in I$ and all $0 \leq w \leq 1$, $\rho(wx_1 + (1-w)x_2) \leq w\rho(x_1) + (1-w)\rho(x_2)$, then it is continuous on $(a, b)$, meaning that, at any $c \in I$, $\forall \varepsilon > 0\ \exists \delta$ such that $|x - a| < \delta \implies |\rho(x) - \rho(c)| < \varepsilon$.

Pick any $\varepsilon > 0$. If $\varepsilon \leq \max(|\rho(x) - \rho(b)|, |\rho(a) - \rho(x)|)$, then any choice of $\delta$ satisfies the definition of continuity. So, we now consider only cases where $\varepsilon < \max(|\rho(x) - \rho(b)|, |\rho(a) - \rho(x)|)$.

## 0.2 Euler-Lagrange

Exercise 4.1: Use Euler-Lagrange to find the extremals for:

a) $\int_1^2 y'^2/x^3 dx$ with $y(1) = 2, y(2) = 17$

$$\begin{aligned}
g(x, y, y') = y'^2/x^3 \quad & \text{and } \tfrac{\partial g}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial g}{\partial y'}\right) = 0 \\
& \implies 0 - \tfrac{d}{dx}\left(\tfrac{2y'}{x^3}\right) = 0 \\
& \implies \tfrac{2y'}{x^3} = C_0 \text{ for some constant } C_0 \\
& \implies y' = \tfrac{C_0}{2}x^3 \\
& \implies y = \int \tfrac{C_0}{2}x^3 dx = C_1x^4 + C_2 \\
\text{so } 1C_1 + C_2 = 2 \text{ and } 16C_1 + C_2 = 17 \quad & \implies C_1 = 1,\ C_2 = 1 \\
& \implies y = x^4 + 1
\end{aligned}$$

b) $\int_0^{\pi/2} y^2 - y'^2 - 2y\sin(x)dx$ with $y(0) = 1, y(\pi/2) = 1$

$$\begin{aligned}
g(x, y, y') = y^2 - y'^2 - 2y\sin(x) \quad & \text{and } \tfrac{\partial g}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial g}{\partial y'}\right) = 0 \\
& \implies (2y - 2\sin(x)) - \tfrac{d}{dx}(-2y') = 0 \\
& \implies y'' = \sin(x) - y \\
& \implies y = C_2\sin(x) + C_1\cos(x) - \tfrac{1}{2}x\cos(x) \\
& \text{(from Mathematica)} \\
\text{so } 0C_2 + 1C_1 - 0 = 1 \text{ and } 1C_2 + 0C_1 - 0 = 1 \quad & \implies C_1 = 1,\ C_2 = 1 \\
& \implies y = \sin(x) + \cos(x) - \tfrac{1}{2}x\cos(x)
\end{aligned}$$

c) $\int_0^\pi y'^2 + 2y\sin(x)dx$ with $y(0) = 0, y(\pi) = 0$

$$\begin{aligned}
g(x, y, y') = y'^2 + 2y\sin(x) \quad & \text{and } \tfrac{\partial g}{\partial y} - \tfrac{d}{dx}\left(\tfrac{\partial g}{\partial y'}\right) = 0 \\
& \implies 2\sin(x) - \tfrac{d}{dx}(2y') = 0 \\
& \implies y'' = \sin(x) \\
& \implies y = -\sin(x) + C_1 x + C_2 \\
& \text{(from Mathematica)} \\
\text{so } 0C_1 + C_2 = 0 \text{ and } \pi C_1 + C_2 = 0 \quad & \implies C_1 = 0,\ C_2 = 0 \\
& \implies y = -\sin(x)
\end{aligned}$$