# Math 351: Homework 7 Due Friday November 2

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## Problem 1

Suppose $I$ is a bounded closed interval and $f : I \to \mathbb{R}$ is continuous on $I$. We will prove that $f(I)$ is bounded. Since $I$ is a bounded closed interval on $R$, it is a compact subset of $R$. By the Extreme Value Theorem, $f$ has a maximum and a minimum on $I$. This means that $f(I)$ has a maximum and minimum value, and thus it is bounded.

Suppose $I$ is a bounded open interval and $f : I \to \mathbb{R}$ is continuous on $I$. We will show a counterexample to disprove that $f(I)$ is bounded. Let $I$ be $(0, 1)$ and define $f(x) = \frac{1}{x}$. $I$ is a bounded open interval and we have previously shown $f$ is continuous on $I$. However, $f(x) \to \infty$ as $x \to 0$, so $f(I) = (1, \infty)$ is not bounded.

Suppose $I$ is a bounded open interval and $f : I \to \mathbb{R}$ is uniformly continuous on $I$. We will prove that $f(I)$ is bounded. It will suffice to show that $f(I)$ must be bounded from above, since to show that $f(I)$ is bounded from below is the same as showing that $-f(I)$ is bounded from above. So, assume for contradiction that $f(I)$ is not bounded from above. This means that $\forall M \in \mathbb{R}\ \exists e \in f(I)$ such that $e > M$. Note that for any such $e$ there is some $x \in I$ where $f(x) = e$. Now, by the definition of uniform continuity, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $x, y \in I$ and $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Pick any $\varepsilon$ and fix it. Now find a corresponding $\delta$. Now, let $a, b$ be the endpoints of $I$ such that $I = (a, b)$. Note that at least one of $f((a, \frac{a+b}{2}])$ or $f([\frac{a+b}{2}, b))$ is not bounded. (If both subintervals were bounded, then $\sup I$ would be a real number and be equal to the larger of the two supremums of the subintervals.) If the image of the former interval under $f$ is unbounded, let $a_1 = a$ and $b_1 = \frac{a+b}{2}$. If not, then let $a_1 = \frac{a+b}{2}$ and $b_1 = b$. Now let $I_1 = (a_1, b_1)$. Again, we can divide $I_1$ in half and choose whichever half has an unbounded image under $f$ to be $I_2$. Continuing this pattern, note that $(a_n - b_n) = \frac{a-b}{2^n} \to 0$ as $n \to \infty$, and that for all $I_n$, $f(I_n)$ is unbounded. Thus there is some $I_n$ where $(a_n - b_n) < \delta$. Now pick $y = a_n$ and let $M = f(y) + \varepsilon$. By the fact that $I_n$ is unbounded, there is some $x \in I_n$ such that $f(x) > f(y) + \varepsilon$. But, $x$ is in $I_n$, so we have found a case where $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon$, a contradiction. So $f(I)$ must be bounded from above, and following the same argument with $-f(I)$ shows that $f(I)$ is also bounded from below, so $f(I)$ is bounded.

Suppose $A$ is a countable union of bounded open intervals and $f : A \to \mathbb{R}$ is uniformly continuous on $A$. We will provide a counterexample to show that $f(A)$ need not be bounded. Let $A = \ldots \cup (-2, -1) \cup (0, 1) \cup (2, 3) \cup \ldots$ and $f(x) = x$. We can show that $f$ is uniformly continuous on $\mathbb{R}$. Pick any $\varepsilon > 0$. If we choose $\delta = \frac{\varepsilon}{2}$, we can see that $|x-y| < \delta \implies |f(x)-f(y)| = |x-y| < \frac{\varepsilon}{2} < \varepsilon$. Since $f$ is uniformly continuous on $\mathbb{R}$, it is uniformly continuous on $A \subseteq \mathbb{R}$. However, we can see that $f(I)$ is not bounded. If we choose any $M > 0$, we know that $f(M + 1) = M + 1 > M$. Note that at least one of $M + 1, M + 2, M + 2.1$, and $M + 3.1$ will be in $A$. Thus we can always find an $x \in A$ where $f(x) > M$. So in this case, $f$ is uniformly continuous on a countable union of bounded open intervals and $f(I)$ is not bounded.

## Problem 2

Suppose $a_n$ and $b_n$ are Cauchy sequences.

Prove or disprove: $|a_n - b_n|$ is Cauchy. Since both sequences are Cauchy, we know that $a_n$ converges to some $A \in \mathbb{R}$ and $b_n$ converges to a value $B \in \mathbb{R}$. So, $\forall \varepsilon > 0\ \exists N_1 > 0$ such that $n > N_1 \implies |a_n - A| < \varepsilon$ and $n > N_2 \implies |b_n - B| < \varepsilon$. Pick an $\varepsilon$ and fix it. Find any $N_1, N_2$ corresponding to $\frac{\varepsilon-(A-B)}{2}$ and set $N > \max\{N_1, N_2\}$. Now we have $|a_n - b_n| = |a_n - A - b_n + B + A - B| = |(a_n - A) + (B - b_n) + (A - B)| < \varepsilon - (A - B) + (A - B) = \varepsilon$. Thus $|a_n - b_n|$ converges and is therefore Cauchy.

Prove or disprove: $(-1)^n a_n$ is Cauchy. Let $a_n = 1$ This clearly converges and thus is Cauchy. However, $(-1)^n a_n$ does not converge. To see why, pick $\varepsilon = \frac{1}{2}$. For any choice of $N$, $|a_{N+2} - a_{N+1}| = 2 > \varepsilon$. Thus $(-1)^n a_n$ is not Cauchy.

Prove or disprove: if $a_n \neq 0$ for all $n$, then $\frac{1}{a_n}$ is Cauchy. Let $a_n = \frac{1}{2^n}$. This sequence converges to 0, and for all $n$, $a_n > 0$. So $a_n$ is Cauchy. However, $\frac{1}{a_n} = 2^n \to \infty$. So, $\frac{1}{a_n}$ does not converge, and this counterexample disproves the proposition.

Prove or disprove: if $a_n > 0.001$ for all $n$, then $\frac{1}{a_n}$ is Cauchy. If $a_n > 0.001$ for all $n$ and $a_n$ is Cauchy, then $a_n$ converges to some $L > 0.001$. Hence $\frac{1}{a_n} < 1000$ and, by the algebraic properties of limits, $\frac{1}{a_n}$ converges to $\frac{1}{L}$, and we conclude that $\frac{1}{a_n}$ is Cauchy.

## Problem 3

A function $f : \mathbb{R} \to \mathbb{R}$ is monotone if $x \leq y$ makes $f(x) \leq f(y)$. Show that a monotone function can have at most a countable number of points of discontinuity.

First, consider the function $f(x) = \lfloor x \rfloor$ as an example of a monotone function with a countably infinite number of points of discontinuity. So, clearly a monotone function can have an infinite number of discontinuities.

In order to show this for $\mathbb{R}$, we will first show that $f$ must have at most a countable number of discontinuities on the interval $[a, b]$. At a point of discontinuity $c \in [a, b]$ we have that $\exists \varepsilon > 0$ such that $\forall \delta > 0$, $|x - c| < \delta$ and $|f(x) - f(c)| > \varepsilon$. Note that every time we have a discontinuity, the value of $f$ can only jump up, because $f$ is monotone and also defined on all of $[a, b]$. Now, pick a discontinuity point $c$ and find a corresponding $\varepsilon$ as per the definition. There must be a finite number of jumps of size $\varepsilon$ on $[a, b]$ because the set $f([a, b])$ is bounded. We can find at most a finite set of discontinuities bigger than $\varepsilon$ for any value of $\varepsilon$. So, the set of all discontinuous points on $[a, b]$ is a union of finite sets, so it is at most countable. We can cover $\mathbb{R}$ with a countable union of bounded closed sets, each containing a countable number of discontinuities, so the total set of all discontinuities is a countable union of countable sets, which itself must be countable.

## Section 5.1

4. Prove that $f$ is continuous on $[-1, 1]$, but $f$ is not differentiable at $x = 0$ where $f : [-1, 1] \to \mathbb{R}$ is defined by:

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   To show that $f$ is continuous on $[-1, 1]$, we can show that for any sequence $(a_n)$ in $[-1, 1]$, $a_n \to x \implies f(a_n) \to f(x)$. Pick any sequence $(a_n)$ in the domain and call its limit $L$. Then $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n \sin(1/a_n) = (\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} \sin(1/a_n)) = L \cdot \lim_{n\to\infty} \sin(1/a_n)$. Since $\sin(x)$ is continuous and $1/x$ is continuous, their composition is continuous, which means that $\lim_{n\to\infty}\sin(1/a_n) = \sin\left(\frac{1}{\lim_{n\to\infty} a_n}\right) = \sin(1/L)$. So, $\lim_{n\to\infty} f(a_n) = L\cdot(1/L) = f(L)$ and we conclude that $f(x)$ is continuous.

However, we can show that $f$ is not differentiable at 0 by showing that $f$ is not locally linear near 0. We want to try to write $f(x) = 0+M\cdot x+o(x)$ for all $x \in B(0,\varepsilon)$ and $\varepsilon > 0$. We know that $f(x) = x$ is continuous, so we can write $f(x) = x\sin(1/x) = (M_1\cdot x + o(x))(\sin(1/x)) = M_1 \cdot x\sin(1/x) + o(x)\sin(1/x) = M_1\cdot x\sin(1/x) + o(x)$. However $\sin(1/x)$ is not continuous and thus not differentiable at 0, so it is not locally linear near 0, so we cannot simplify this expression to the desired form. Thus, $f$ is not differentiable at 0.

5. Prove that $g$ is differentiable on $(-1,1)$ but $g'$ is not continuous at $x=0$ where $g$ is defined by:

$$f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Ran out of time to figure this one out. I'm confused because, what does it mean for $g'$ to not be continuous at a point if it is defined at that point? I would assume it means that the left-sided limit and the right-sided limit of $g'$ are different. But doesn't that mean $g$ is not differentiable at that point? I had trouble figuring out what I can do to prove anything here.

6. Give an example of a continuous invertible function on $(0,1)$ that is not differentiable.

Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & x \leq \frac{1}{2} \\ 2x - \frac{1}{2} & x > \frac{1}{2} \end{cases}$$

Note that at $x = \frac{1}{2}$ the left-sided derivative is 1, but the right-sided derivative is 2, so $f$ is not differentiable. But the inverse of $f$ is well-defined:

$$f(x) = \begin{cases} x & x \leq \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & x > \frac{1}{2} \end{cases}$$

7. Prove that a continuous invertible function defined on a closed or open interval is strictly monotone.

Let $f$ be a continuous invertible function and assume for contradiction that $f$ is not strictly monotone. This means that there exist some $a < b < c$ in the domain such that $f(a) \leq f(b)$ and $f(c) \leq f(b)$ or where $f(a) \geq f(b)$ and $f(c) \geq f(b)$. We can assume the former case is true without losing generality. In addition, we consider the case where $f(a) \leq f(c)$. By the Intermediate Value Theorem, there exists some $a' \in [a, b]$ such that $f(a') = f(c)$ since $f(c) \in [f(a), f(b)]$. But, if $f(a') = f(c)$ then $f$ is not one-to-one, so $f$ is not invertible, a contradiction. So continuous invertible functions must be strictly monotone.

8. Let $n \in \mathbb{N}$. Construct a function that is $n$ times differentiable on an interval but fails to be $n+1$ times differentiable at a point in the interval.

   Define $f : [0, \infty) \to \mathbb{R}$ as $f(x) = x^{\frac{1}{2}+n}$. We can differentiate $f$ $n$ times on $[0, \infty)$, and $f^N(x) = \left(\Pi_{i=1}^{n}(\frac{1}{2}+i)\right) x^{\frac{1}{2}}$. However, $x^{\frac{1}{2}}$ is not differentiable at $x = 0$.

## Section 5.2

1. Let $f : [a, b] \to \mathbb{R}$ be a function. Suppose that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Show that if $f'(x) \neq 0$ for all $x \in (a, b)$, then $f$ is one-to-one.

   We prove this by proving the contrapositive. Assume $f$ is not one-to-one, which means that $\exists x < y \in (a, b)$ such that $f(x) = f(y)$. By Rolle's Theorem, there must be a $c \in (x, y)$ such that $f'(c) = 0$. So we have shown the contrapositive, and thus the original statement is true.

2. Let $f, g : [a, b] \to \mathbb{R}$ be functions that are continuous on $[a, b]$ and differentiable on $(a, b)$. Show that if $f'(x) = g'(x)$ for all $x \in (a, b)$ then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in [a, b]$.

   Since $f'(a) = g'(a)$ then $f$ and $g$ are locally linear near $a$ with the same slope $M$. This means that $f(x) = f(a) + M(x - a) + o(x - a)$ and $g(x) = g(a) + M(x - a) + o(x - a)$. Now subtract $g$ from $f$: $f(x) - g(x) = f(a) + M(x-a) + o(x-a) - g(a) - M(x-a) - o(x-a) = (f(a) - g(a)) + o(x - a) - o(x - a)$ Now, as $x \to a$, $f(x) - g(x) = (f(a) - g(a)) \implies f(x) = g(x) + (f(a) - g(a))$. Note that $f(a) - g(a)$ is a constant value, so we have proven the statement true.

3. Let $f, g : [a, b] \to \mathbb{R}$ be functions that are continuous on $[a, b]$ and differentiable on $(a, b)$ such that $f(a) = g(a)$. Show that if $f'(x) \leq g'(x)$ for all $x \in (a, b)$ then $f(x) \leq g(x)$ for all $x \in (a, b)$.

   We will prove this by the contrapositive. Assume that $f(a) = g(a)$ and that $\exists c \in [a, b]$ such that $f(c) > g(c)$. Now define $h(x) = g(x) - f(x)$ and note that $h(a) = 0$ and $h(c) = d$ for some value of $d < 0$. By the Mean Value Theorem, there is some $c' \in (a, c)$ such that $h'(c') = \frac{d-0}{c-a} < 0$. Note that $h'(c') = g'(c') - f'(c')$, and we have $g'(c') - f'(c') < 0 \implies g'(c') < f'(c')$. We have shown that if $f(x) > g(x)$ for any $x$ then there must be some value where $f'(x) > g'(x)$. The contrapositive has been shown to be true, so the original proposition is true.

8. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Prove that if $f'(x) \geq 0$ for all $x \in (a, b)$ then $f$ is an increasing function on $(a, b)$. Furthermore, if $f'(x) > 0$ for all $x \in (a, b)$, then $f$ is a a strictly increasing function on $(a, b)$.

   We will prove this by contraposition. Assume that $\exists x < y \in (a, b)$ such that $f(x) > f(y)$, and we will show that there is some $c$ where $f'(c) < 0$. By the Mean Value Theorem, we can see that $\exists c \in (x, y)$ such that $f'(c) = \frac{f(y)-f(x)}{y-x}$. Since $y > x$ and $f(y) < f(x)$, $f'(c)$ is negative. So we have shown the contrapositive to be true, and we conclude that the original statement is true.

9. Prove that the function $g : (-1, 1) \to \mathbb{R}$ given by $g(x) = \sqrt[3]{x}$ is not differentiable at 0.