Math 351: Homework 10 Due Friday December 7

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Chopping

Use Lemma 3.3.4 to complete the proof of the chopping lemma, Lemma 3.2.1

[Use angle difference formulas. Argue that the above argument for sin works for cos as well. Create a period of $2\pi/k$ with some error that disappears as $k \to \infty$ (that part might not quite be right)]

0.1 Jensen's

Exercise 5.1: Prove lemma 5.2.2.

Lemma: If ρ is convex on I, and $x_1, x_2 \in I$, and equality holds

$$\rho(wx_1 + (1-w)x_2) \leq w\rho(x_1) + (1-w)\rho(x_2)$$

for some $0 < w < 1$, then either $x_1 = x_2$ or ρ is linear on $[x_1, x_2]$.

[BY CONTRADICTION? CONFUSING]

Exercise 5.2: Prove lemma 5.2.3.

Lemma: If ρ is a convex function on an interval I, $x_1, \ldots, x_n \in I$, and $w_1, \ldots, w_n$ are non-negative real numbers with $\sum w_i = 1$, then

$$\rho\left(\sum w_i x_i\right) \leq \sum w_i \rho(x_i)$$

with equality holding iff ρ is linear on an interval containing the x_i's, or all x_i's are the same.

We proceed by induction. The base case is where $n = 2$, which is true by Lemma 5.2.2.

For the inductive step, assume that

$$\rho\left(\sum_{i=0}^{n} w_i x_i\right) = \sum_{i=0}^{n} w_i \rho(x_i)$$

implies that either all x_i's are the same or ρ is linear for some $n \geq 2$. We will show that

$$\rho\left(\sum_{i=0}^{n+1} w_i x_i\right) = \sum_{i=0}^{n+1} w_i \rho(x_i)$$

implies that either all x_i's are the same or ρ is linear. We rewrite the latter equation:

$$\begin{aligned} & \rho\left(\sum_{i=0}^{n+1} w_i x_i\right) = \rho\left(\left(\sum_{i=0}^{n} w_i x_i\right) + w_{n+1}x_{n+1}\right) \\ \text{by Lemma 5.2.2} \quad & \implies \sum_{i=0}^{n} w_i \rho(x_i) + w_{n+1}\rho(x_{n+1}) \\ & \implies \sum_{i=0}^{n+1} w_i x_i \\ & \implies \text{ and that either } x_i\text{'s are the same or } \rho \text{ is linear.} \end{aligned}$$

[FINISH THIS ONE BOY]

Exercise 5.3: Prove that convex functions are continuous.

We will show that if a function $\rho : I \to \mathbb{R}$ (where $I = (a,b)$) has the property that, for $x_1, x_2 \in I$ and all $0 \leq w \leq 1$, $\rho(wx_1 + (1-w)x_2) \leq w\rho(x_1) + (1-w)\rho(x_2)$, then it is continuous on (a,b), meaning that, at any $c \in I$, $\forall \varepsilon > 0\ \exists \delta$ such that $|x - a| < \delta \implies |\rho(x) - \rho(c)| < \varepsilon$.

First, we will see that $\rho(x)$ is bounded on I. Let $A = \min(\rho(a), \rho(b))$. Note that any $c \in [a,b]$ can be written as $wa + (1-w)b$ when $w = \frac{b-c}{b-a}$. Thus, by convexity, $\rho(c) \leq w\rho(a) + (1-w)\rho(b) \leq wA + (1-w) = A$.

Pick any $\varepsilon > 0$. If $\varepsilon \geq \max(|\rho(c) - \rho(b)|, |\rho(a) - \rho(c)|)$, then any choice of δ satisfies the definition of continuity. So, we now consider only cases where $\varepsilon < A - \rho(c)$. Let $c_1 = \frac{a-c}{2}$. If $|\rho(c_1) - \rho(c)| < \varepsilon$, then pick $\delta < |c_1 - c|$. If not, let $c_2 = \frac{c_1 - c}{c}$. Note that $c < c_2 < c_1$. Keep picking c_n in this way until, for some N, $|\rho(c_N) - \rho(c)| < \varepsilon$. Choosing $0 < \delta < c_N$ will make $|f(c) - f(x)| < \varepsilon$. Thus, ρ is continuous on I.

0.2 Euler-Lagrange

Exercise 4.1: Use Euler-Lagrange to find the extremals for:

a) $\int_1^2 y'^2/x^3 dx$ with $y(1)=2, y(2)=17$

$$\begin{aligned}
g(x,y,y') = y'^2/x^3 \quad & \text{and } \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0 \\
& \implies 0 - \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0 \\
& \implies \frac{2y'}{x^3} = C_0 \text{ for some constant } C_0 \\
& \implies y' = \frac{C_0}{2}x^3 \\
& \implies y = \int \frac{C_0}{2}x^3 dx = C_1x^4 + C_2 \\
\text{so } 1C_1 + C_2 = 2 \text{ and } 16C_1 + C_2 = 17 \quad & \implies C_1 = 1,\ C_2 = 1 \\
& \implies y = x^4 + 1
\end{aligned}$$

b) $\int_0^{\pi/2} y^2 - y'^2 - 2y\sin(x) dx$ with $y(0)=1, y(\pi/2)=1$

$$\begin{aligned}
g(x,y,y') = y^2 - y'^2 - 2y\sin(x) \quad & \text{and } \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0 \\
& \implies (2y - 2\sin(x)) - \frac{d}{dx}(-2y') = 0 \\
& \implies y'' = \sin(x) - y \\
& \implies y = C_2\sin(x) + C_1\cos(x) - \frac{1}{2}x\cos(x) \\
& \text{(from Mathematica)} \\
\text{so } 0C_2 + 1C_1 - 0 = 1 \text{ and } 1C_2 + 0C_1 - 0 = 1 \quad & \implies C_1 = 1,\ C_2 = 1 \\
& \implies y = \sin(x) + \cos(x) - \frac{1}{2}x\cos(x)
\end{aligned}$$

c) $\int_0^{\pi} y'^2 + 2y\sin(x) dx$ with $y(0)=0, y(\pi)=0$

$$\begin{aligned}
g(x,y,y') = y'^2 + 2y\sin(x) \quad & \text{and } \frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right) = 0 \\
& \implies 2\sin(x) - \frac{d}{dx}(2y') = 0 \\
& \implies y'' = \sin(x) \\
& \implies y = -\sin(x) + C_1x + C_2 \\
& \text{(from Mathematica)} \\
\text{so } 0C_1 + C_2 = 0 \text{ and } \pi C_1 + C_2 = 0 \quad & \implies C_1 = 0,\ C_2 = 0 \\
& \implies y = -\sin(x)
\end{aligned}$$