Math 351: Homework 2 (Due September 21)

Name: Jack Ellert-Beck

Section 0.5

Problem 7

We will show by induction that for all integers $n \geq 1, n < 2^n$.

For the base case, let $n = 1$. $2^n = 2^1 = 2 > 1$, so the property holds.

Now we will show that if $n < 2^n$ then $n + 1 < 2^{n+1}$. We can add one to both sides of the first inequality: $n + 1 < 2^n + 1 < 2^n + 2^n < 2^n \cdot 2 = 2^{n+1}$. This completes the inductive step. Thus, by mathematical induction, the property is true for all $n \geq 1$.

Problem 8

Let $\{F_n\}$ be the Fibonacci sequence. We will prove that for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1.$$

We proceed by induction. Let $n = 1$ and recall that $F_1 = 1$ and $F_3 = 2$. We see that $F_1 = 1 = 2 - 1 = F_3 - 1$.

For the inductive step we assume $\sum_{i=0}^{n} F_i = F_{n+2} - 1$, and we want to show that $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$. In addition, recall that the definition of $\{F_n\}$ includes the statement $F_n = F_{n-1} + F_{n+2}$. Consider the following equalities:

$$\begin{aligned}
\textstyle\sum_{i=0}^{n+1} F_i &= \textstyle\sum_{i=0}^{n} F_i + F_{n+1} \\
&= F_{n+2} - 1 + F_{n+1} \\
&= (F_{n+2} + F_{n+1}) - 1 \\
&= F_{n+3} - 1.
\end{aligned}$$

By mathematical induction, this property must hold for all $n \in \mathbb{N}$.

Problem 9

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}.$$

We proceed by induction. Consider the base case where $n = 1$. We get $\frac{1}{2^1} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$, so the base case is true.

For the inductive step we assume $\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$ and we want to show that $\sum_{i=1}^{n+1} \frac{1}{2^i} = 1 - \frac{1}{2^{n+1}}$. Note that the following equalities hold:

$$\begin{aligned}
\textstyle\sum_{i=1}^{n+1} \frac{1}{2^i} &= \textstyle\sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^{n+1}} \\
&= 1 - \tfrac{1}{2^n} + \tfrac{1}{2^{n+1}} \\
&= 1 - \tfrac{2^{n+1} - 2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{2 \cdot 2^n - 2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{2^n}{2^n \cdot 2^{n+1}} \\
&= 1 - \tfrac{1}{2^{n+1}}.
\end{aligned}$$

So, by mathematical induction, this is true for all $n \geq 1$.

Problem 11

We will show that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can show this by induction. We let $n = 1$ for the base case, and we see that $1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6}$.

For the inductive step we assume $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and we want to show that $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$. Consider the following equalities:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^{n} i^2 + (n+1)^2 \\
&= \tfrac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \tfrac{n(n+1)(2n+1)+6(n+1)^2}{6} \\
&= \tfrac{(n+1)(n(2n+1)+6(n+1))}{6} \\
&= \tfrac{(n+1)(2n^2+7n+6)}{6} \\
&= \tfrac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

By mathematical induction, this property is true for all $n \geq 1$.

Section 1.3

Problem 7

Argue that the set of finite sequences of 0's and 1's is countable

Argue that the set of infinite sequences of 0's and 1's is not countable

Problem 11

Section 2.1

Problem 1

Problem 2

Problem 4

Problem 6