Math 351: Homework 8 Due Friday November 16

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Invitation to Real Analysis

Section 6.1

Exercise 7: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Assume that there is some $c \in [a, b]$ where $f(c) > 0$. Now pick any $0 < \varepsilon < f(c)$. By the continuity of f, we can find a δ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$. So, pick such a δ. Notice that $f(c - \delta) > f(x) - \varepsilon > 0$ and $f(c + \delta) > f(x) - \varepsilon > 0$. Now define the function

$$g(x) = \begin{cases} 0 & a \leq x < c - \delta \\ f(c) - \varepsilon & c - \delta \leq x \leq c + \delta \\ 0 & c + \delta < b \end{cases}$$

We can see that $\int_a^b g = 2\delta(f(c) - \varepsilon) > 0$, and that $f(x) \geq g(x)$ for all $x \in [a, b]$. So, $\int_a^b f \geq \int_a^b g > 0$, a contradiction. Thus we conclude that $f(c) \not> 0$ for any c, so $f(x) = 0$ for all $x \in [a, b]$.

Section 6.2

Exercise 1: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f$$

First define the function

$$g(x) = \int_a^x f(t)dt$$

and note that g is continuous on $[a, b]$ and differentiable on (a, b). So, by the Mean Value Theorem, there exists a $c \in (a, b)$ such that $g'(c) = \frac{g(b)-g(a)}{b-a}$.

Now, by the Fundamental Theorem of Calculus, we have $g'(c) = f(c)$, and we can rewrite the right hand side as well, giving the equation $f(c) = \frac{\int_a^b f - \int_a^a f}{b-a} = \frac{1}{b-a}(\int_a^b f - 0) = \frac{1}{b-a}\int_a^b f$ for some $c \in (a,b)$.

Section 8.2

Exercise 3: Let $f_n : A \to \mathbb{R}$, for $A \subseteq \mathbb{R}$, be uniformly continuous. Show that if (f_n) converges uniformly to the function f then f is uniformly continuous.

We wish to show that f is uniformly continuous. Let $\varepsilon > 0$. We know there exists $N \in \mathbb{N}$ such that $n \geq N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in A$. Since f_n is uniformly continuous, there exists $\delta > 0$ so that $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$ for any choice of $x, y \in A$. Then

$$\begin{aligned}
|f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\
&\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\
&\leq 3 \cdot \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

whenever $|x - y| < \delta$. Therefore f is uniformly continuous on A.

Exercise 5: Let $x \in [0, \infty)$ and define

$$f_n(x) = \frac{nx}{nx+1}$$

Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0,\infty)$ and prove it. prove that the convergence is uniform on $[1,\infty)$ and is not uniform on $[0,\infty)$. Is it uniform on $(0,\infty)$?

The pointwise limit is $f(x) = 1$. To prove it, fix $x \in [0,\infty)$. Now take any $N > \frac{1}{x\varepsilon}$. Now, if $n > N$, then

$$\begin{aligned}
n > \frac{\frac{1}{\varepsilon} - 1}{x} &\implies nx > \tfrac{1}{\varepsilon} - 1 \\
&\implies nx + 1 > \tfrac{1}{\varepsilon} \\
&\implies \tfrac{1}{nx+1} < \varepsilon \\
&\implies \left|\tfrac{nx - nx - 1}{nx+1}\right| < \varepsilon \\
&\implies \left|\tfrac{nx}{nx+1} - 1\right| < \varepsilon
\end{aligned}$$

And thus f_n converges pointwise to $f(x) = 1$.

Exercise 10: Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ for $x \in [0, \pi/2]$ and $n \in \mathbb{N}$. Prove that f_n converges uniformly to 0 on $[0, \pi/2]$, but f_n' does not converge on $[0, \pi/2]$.

Pick any $\varepsilon > 0$ and fix it. Now take $N > \frac{1}{\varepsilon^2}$. Now we can see that

$$\begin{aligned} \tfrac{1}{\varepsilon^2} &\implies \tfrac{1}{\varepsilon}\sqrt{n} \\ &\implies 1 < \sqrt{n}\varepsilon \\ &\implies |\sin(nx)| < \sqrt{n}\varepsilon \\ &\implies \left|\tfrac{\sin(nx)}{\sqrt{n}}\right| < \varepsilon \end{aligned}$$

for any choice of $x \in [0, \pi/2]$. So, $f_n \to f(x) = 0$ uniformly.

Now, we can see that $f_n'(x) = \sqrt{n}\cos(nx)$. We will show that f' fails to converge at $x = 0$. Note that, for any n, $f_n'(0) = \sqrt{n\cos(0)} = \sqrt{n}$. As $n \to \infty$, $\sqrt{n} \to \infty$ as well. So the value of $f_n'(0) \to \infty$ as $n \to \infty$, so f_n' fails to converge at $x = 0$.

Applied Topics

Metric and Norms

We will prove that if a metric is translation invariant and scales, then it induces a norm by $\|\phi\| = d(\phi, 0)$. First, we verify the triangle inequality, using the triangle inequality for the metric d: $\|\phi + \mu\| = d(\phi + \mu, 0) \leq d(\phi + \mu, \mu) + d(\mu, 0) = d(\phi, 0) + d(\mu, 0) = \|\phi\| + \|\mu\|$. Next, we verify that the induced norm scales: $r\|\phi\| = rd(\phi, 0) = d(r\phi, r \cdot 0) = \|r\phi\|$. Finally, we see that $\phi \equiv 0 \iff d(\phi, 0) = 0 \iff \|\phi\| = 0$.

L^2 norm

Let $\mathcal{S} = \{\phi \in C^0_{[0,\pi]}, \phi(0) = \phi(\pi) = 0\}$ and define

$$\|f\| = \sqrt{\int_0^\pi (f(x))^2 dx}$$

We will show that this is a norm. Notice that $(f(x))^2$ is always positive, so $\|f\|$ is the square root of a non-negative number, so $\|f\| \geq 0$ for all $f \in \mathcal{S}$. In addition, by Exercise 7 above, $\|f\| = 0 \implies \int_0^\pi (f(x))^2 dx = 0 \implies f(x)^2 = f(x) \equiv 0$. Going the other direction, we can see that if $f(x) \equiv 0$, then $\|f\| = 0$. Now consider scalar multiplication. For any $r \geq 0$, $\|rf\| = \sqrt{\int_0^\pi (rf(x))^2 dx} = \sqrt{r^2 \int_0^\pi (f(x))^2 dx} = r\sqrt{\int_0^\pi (f(x))^2 dx} = r\|f\|$. Now, we prove the triangle inequality. In order to do this, we will show that the operation $f \cdot g = \int_0^\pi f(x)g(x)dx$ satisfies the properties of a dot product. We can see that $\|f\|^2 = \int_0^\pi f(x)f(x)dx = f \cdot f$. We also have $f \cdot g = \int_0^\pi f(x)g(x)dx = \int_0^\pi g(x)f(x)dx = g \cdot f$ and $f \cdot (g+h) = \int_0^\pi f(x)(g(x)+h(x))dx = \int_0^\pi f(x)g(x) + f(x)h(x)dx = \int_0^\pi f(x)g(x)dx + \int_0^\pi f(x)h(x)dx = f \cdot g + f \cdot h$. So, this operation has the same properties as the dot product. We now want to show that $\|f+g\| \leq \|f\| + \|g\|$. From here we follow the argument Silva uses to show a similar result in Example 3.1.5. Squaring the left hand side, we get $\|f+g\|^2 = (f+g)\cdot(f+g) = \|f\|^2 + 2f \cdot g + \|g\|^2$. On the right hand side, we have $(\|f\| + \|g\|)^2 = \|f\|^2 + 2\|f\|\|g\| + \|g\|^2$. The proof of the triangle inequality follows from showing that $f \cdot g \leq \|a\|\|b\|$. Silva proves this case of the Cauchy-Scwartz Inequality in the example, and we can conclude that the triangle inequality holds, and thus this is a norm.

Continuity

Let $\mathcal{S} = \{\phi \in C^0_{[0,\pi]}, \phi(0) = \phi(\pi) = 0\}$ and define $\|\phi\| = \int_0^\pi |\phi(x)|dx$. Consider the functional $G : \mathcal{S} \to \mathbb{R}$ as $G(\phi) = \int_0^\pi \phi(x)dx$.

Under the metric induced by this norm, G is uniformly continuous. Pick any $\varepsilon > 0$ and fix it. Now choose any positive $\delta < \varepsilon$. Now, for any choice of $\phi, \mu \in \mathcal{S}$ such that $d(\phi, \mu) < \delta$,

$$\begin{aligned}
\varepsilon > \delta &> \int_0^\pi |\phi(x) - \mu(x)|dx \\
&\geq |\int_0^\pi \phi(x) - \mu(x)dx| \\
&= |\int_0^\pi \phi(x)dx - \int_0^\pi \mu(x)dx| \\
&= |G(\phi) - G(\mu)|.
\end{aligned}$$

This satisfies the definition for uniform continuity.

Another Norm

Let $\mathcal{S} = \{\phi \in C^1_{[0,\pi]}, \phi(0) = \phi(\pi) = 0\}$ and define

$$\|\phi\| = \max_{x \in [0,\pi]} |\phi'(x)|$$

We will show that $\mathcal{S}$ is complete under this norm. Let (ϕ_n) be a Cauchy sequence in $\mathcal{S}$ converging to ϕ. Notice that if $\|\phi\| < \varepsilon$, the derivative of ϕ is bounded. By the Mean Value Theorem, the maximum value of $\frac{\phi(a)-\phi(b)}{a-b}$ is ε too. The least upper bound we can give for $|\phi|$ is thus $\varepsilon\pi$. By Corollary 8.2.16, f is differentiable, so f is in our set, and we conclude that the space is complete.

More Continuity

Consider $\mathcal{S} = \{\phi : \phi \in C^0_{[0,\pi]}, \phi(0) = 0\}$ with the norm $\|\phi\| = \int_0^\pi |\phi(x)|dx$. Consider the operator $G : \mathcal{S} \to \mathcal{S}$ as $G(\phi)(x) = \int_0^x \phi(t)dt$.

We can see that $\phi \in \mathcal{S} \implies G(\phi) \in \mathcal{S}$ because every $G(\phi)(x)$ is differentiable, and thus continuous. In addition, for all $\phi \in \mathcal{S}$, $G(\phi)(0) = \int_0^0 \phi(x)dx = 0$. So, G is an operator.

(I think G is continuous but not uniformly contiuous? Don't know how to prove.)

This operator is linear. If we have two functions $\phi, \mu \in \mathcal{S}$, then $G(\phi + \mu) = \int_0^x \phi(t) + \mu(t)dt = \int_0^x \phi(t)dt + \int_0^x \mu(t)dt = G(\phi) + G(\mu)$. Further, for a non-negative scalar r, $G(r\phi) = \int_0^x r\phi(t)dt = r\int_0^x \phi(x)dx = rG(\phi)$. It also has a fixed point: if $f(x) = 0$ for all x, then $G(f)(x) = 0$ for all x as well, and thus $G(f) = f$.

Completeness

Show that $C^0_{[0,\pi]}$ is complete under the sup metric.

We show this by proving that every Cauchy sequence converges to a point in the space. Recall that the sup metric compares functions f, g by the maximum value of $f - g$ over the interval. So, if we have a Cauchy sequence (ϕ_n), we know that $\forall \varepsilon > 0\ \exists N > 0$ such that $n, m > N \implies \max_{x \in [0,\pi]} |\phi_n(x) - \phi_m(x)| < \varepsilon$. At any individual x, $|\phi_n(x) - \phi_m(x)| < \max_{x \in [0,\pi]} |\phi_n(x) - \phi_m(x)| < \varepsilon$, so for that choice of x, $\phi_n(x)$ is a Cauchy sequence in $\mathbb{R}$, which means it converges to some value $\phi(x)$. So $\phi_n \to \phi$ pointwise, and we will now prove that this ϕ is in the set.

Pick an $\varepsilon > 0$ and fix it. Since ϕ_n is Cauchy, we can find N such that $n, m > N \implies |\phi_n(x) - \phi_m(x)| < \varepsilon/3$ for all $x \in [0, \pi]$. Now fix $n = N + 1$. Because ϕ_n is continuous on a closed interval, it is uniformly continuous on that interval. So, for the same ε, there exists a $\delta > 0$ such that, for any $x, y \in [0, \pi]$, $|x - y| < \delta \implies |\phi_n(x) - \phi_n(y)| < \varepsilon$. Because of the pointwise convergence from earlier, we can find for any x an M_x such that $m > M_x \implies |\phi_m(x) - \phi(x)| < \varepsilon/3$, and similarly we can find for any y an M_y so that $m > M_y \implies |\phi_m(y) - \phi(y)| < \varepsilon/3$. If we choose $m > \max(M_x, M_y)$ and keep $|x-y| < \delta$ we see that $|\phi_m(x)-\phi_m(y)|+|\phi_m(x)-\phi(x)|+|\phi_m(y)-\phi(y)| = |\phi(x) - \phi_m(x) + \phi_m(x) + \phi_m(y) - \phi_m(y) - \phi(y)| = |\phi(x) - \phi(y)| < 3 \cdot \varepsilon/3 = \varepsilon$. This satisfies the definition of uniform continuity, so ϕ is continuous on $[0, \pi]$, which means it is a point in our set. Thus all Cauchy sequences converge in this space, so the space is complete.