Math 351: Homework 8 Due Friday November 16

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Invitation to Real Analysis

Section 6.1

Exercise 7: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Assume that there is some $c \in [a, b]$ where $f(c) > 0$. Now pick any $0 < \varepsilon < f(c)$. By the continuity of f, we can find a δ such that $|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$. So, pick such a δ. Notice that $f(c - \delta) > f(x) - \varepsilon > 0$ and $f(c + \delta) > f(x) - \varepsilon > 0$. Now define the function

$$g(x) = \begin{cases} 0 & a \leq x < c - \delta \\ f(c) - \varepsilon & c - \delta \leq x \leq c + \delta \\ 0 & c + \delta < b \end{cases}$$

We can see that $\int_a^b g = 2\delta(f(c) - \varepsilon) > 0$, and that $f(x) \geq g(x)$ for all $x \in [a, b]$. So, $\int_a^b f \geq \int_a^b g > 0$, a contradiction. Thus we conclude that $f(c) \not> 0$ for any c, so $f(x) = 0$ for all $x \in [a, b]$.

Section 6.2

Exercise 1: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f$$

First define the function

$$g(x) = \int_a^x f(t)dt$$

and note that g is continuous on $[a, b]$ and differentiable on (a, b). So, by the Mean Value Theorem, there exists a $c \in (a, b)$ such that $g'(c) = \frac{g(b)-g(a)}{b-a}$.

Now, by the Fundamental Theorem of Calculus, we have $g'(c) = f(c)$, and we can rewrite the right hand side as well, giving the equation $f(c) = \frac{\int_a^b f - \int_a^a f}{b-a} = \frac{1}{b-a}(\int_a^b f - 0) = \frac{1}{b-a}\int_a^b f$ for some $c \in (a, b)$.

Section 8.2

Exercise 3: Let $f_n : A \to \mathbb{R}$, for $A \subseteq \mathbb{R}$, be uniformly continuous. Show that if (f_n) converges uniformly to the function f then f is uniformly continuous.

We wish to show that f is uniformly continuous. Let $\varepsilon > 0$. We know there exists $N \in \mathbb{N}$ such that $n \geq N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in A$. Since f_n is uniformly continuous, there exists $\delta > 0$ so that $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$ whenever $|x - y| < \delta$ for any choice of $x, y \in A$. Then

$$\begin{aligned}
|f(y) - f(x)| &= |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)| \\
&\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\
&\leq 3 \cdot \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

whenever $|x - y| < \delta$. Therefore f is uniformly continuous on A.

Exercise 5: Let $x \in [0, \infty)$ and define

$$f_n(x) = \frac{nx}{nx + 1}$$

Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0, \infty)$ and prove it. prove that the convergence is uniform on $[1, \infty)$ and is not uniform on $[0, \infty)$. Is it uniform on $(0, \infty)$?

The pointwise limit is $f(x) = 1$. To prove it, fix $x \in [0, \infty)$. Now take any $N > \frac{1}{x\varepsilon}$. Now, if $n > N$, then

$$\begin{aligned}
n > \tfrac{\frac{1}{\varepsilon} - 1}{x} &\implies nx > \tfrac{1}{\varepsilon} - 1 \\
&\implies nx + 1 > \tfrac{1}{\varepsilon} \\
&\implies \tfrac{1}{nx+1} < \varepsilon \\
&\implies \left|\tfrac{nx - nx - 1}{nx+1}\right| < \varepsilon \\
&\implies \left|\tfrac{nx}{nx+1} - 1\right| < \varepsilon
\end{aligned}$$

And thus f_n converges pointwise to $f(x) = 1$.

Exercise 10: Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ for $x \in [0, \pi/2]$ and $n \in \mathbb{N}$. Prove that f_n converges uniformly to 0 on $[0, \pi/2]$, but f_n' does not converge on $[0, \pi/2]$.

Pick any $\varepsilon > 0$ and fix it. Now take $N > \frac{1}{\varepsilon^2}$. Now we can see that

$$\begin{aligned}
\tfrac{1}{\varepsilon^2} &\implies \tfrac{1}{\varepsilon}\sqrt{n} \\
&\implies 1 < \sqrt{n}\varepsilon \\
&\implies |\sin(nx)| < \sqrt{n}\varepsilon \\
&\implies \left|\tfrac{\sin(nx)}{\sqrt{n}}\right| < \varepsilon
\end{aligned}$$

for any choice of $x \in [0, \pi/2]$. So, $f_n \to f(x) = 0$ uniformly.

Now, we can see that $f_n'(x) = \sqrt{n}\cos(nx)$. We will show that f' fails to converge at $x = 0$. Note that, for any n, $f_n'(0) = \sqrt{n\cos(0)} = \sqrt{n}$. As $n \to \infty$, $\sqrt{n} \to \infty$ as well. So the value of $f_n'(0) \to \infty$ as $n \to \infty$, so f_n' fails to converge at $x = 0$.

Applied Topics