

## Math 351: Homework 2 (Due September 21)

Name: Jack Ellert-Beck

### Section 0.5

#### Problem 7

We will show by induction that for all integers  $n \geq 1, n < 2^n$ .

For the base case, let  $n = 1$ .  $2^n = 2^1 = 2 > 1$ , so the property holds.

Now we will show that if  $n < 2^n$  then  $n + 1 < 2^{n+1}$ . We can add one to both sides of the first inequality:  $n + 1 < 2^n + 1 < 2^n + 2^n < 2^n \cdot 2 = 2^{n+1}$ . This completes the inductive step. Thus, by mathematical induction, the property is true for all  $n \geq 1$ .

#### Problem 8

Let  $\{F_n\}$  be the Fibonacci sequence. We will prove that for all  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^n F_i = F_{n+2} - 1.$$

We proceed by induction. Let  $n = 1$  and recall that  $F_1 = 1$  and  $F_3 = 2$ . We see that  $F_1 = 1 = 2 - 1 = F_3 - 1$ .

For the inductive step we assume  $\sum_{i=0}^n F_i = F_{n+2} - 1$ , and we want to show that  $\sum_{i=0}^{n+1} F_i = F_{n+3} - 1$ . In addition, recall that the definition of  $\{F_n\}$  includes the statement  $F_n = F_{n-1} + F_{n+2}$ . Consider the following equalities:

$$\begin{aligned} \sum_{i=0}^{n+1} F_i &= \sum_{i=0}^n F_i + F_{n+1} \\ &= F_{n+2} - 1 + F_{n+1} \\ &= (F_{n+2} + F_{n+1}) - 1 \\ &= F_{n+3} - 1. \end{aligned}$$

By mathematical induction, this property must hold for all  $n \in \mathbb{N}$ .

### Problem 9

We will show that for all integers  $n \geq 1$ ,

$$\sum_{i=1}^n \frac{1}{2^i} = 1 - \frac{1}{2^n}.$$

We proceed by induction. Consider the base case where  $n = 1$ . We get  $\frac{1}{2^1} = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \frac{1}{2^1}$ , so the base case is true.

For the inductive step we assume  $\sum_{i=1}^n \frac{1}{2^i} = 1 - \frac{1}{2^n}$  and we want to show that  $\sum_{i=1}^{n+1} \frac{1}{2^i} = 1 - \frac{1}{2^{n+1}}$ . Note that the following equalities hold:

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{2^i} &= \sum_{i=1}^n \frac{1}{2^i} + \frac{1}{2^{n+1}} \\ &= 1 - \frac{1}{2^n} + \frac{1}{2^{n+1}} \\ &= 1 - \frac{2^{n+1} - 2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{2 \cdot 2^n - 2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{2^n}{2^n \cdot 2^{n+1}} \\ &= 1 - \frac{1}{2^{n+1}}. \end{aligned}$$

So, by mathematical induction, this is true for all  $n \geq 1$ .

### Problem 11

We will show that for all integers  $n \geq 1$ ,

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

We can show this by induction. We let  $n = 1$  for the base case, and we see that  $1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6}$ .

For the inductive step we assume  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$  and we want to show that  $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$ . Consider the following equalities:

$$\begin{aligned}
\sum_{i=1}^{n+1} i^2 &= \sum_{i=1}^n i^2 + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)(n(2n+1) + 6(n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

By mathematical induction, this property is true for all  $n \geq 1$ .

### Section 1.3

#### Problem 7

Recall that for sets  $A_1$  and  $A_2$ , the Cartesian product is defined as  $A \times A_1 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$ . Define the Cartesian product of  $k$  sets  $A_1 \times A_2 \times \dots \times A_k = \{(a_1, a_2, \dots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_k \in A_k\}$ .

Using this definition we can prove that if the sets  $A_1, \dots, A_k$  are countable, then their Cartesian product is also countable. We proceed by induction. In the case where  $k = 2$ , we have the Cartesian product  $A_1 \times A_2$ , which is countable by Proposition 1.3.8. For the inductive step, assume that  $A_1, \dots, A_{k+1}$  are countable and that  $A_1 \times \dots \times A_k$  is countable. The Cartesian product  $A_1 \times \dots \times A_{k+1}$  can be written as  $(A_1 \times \dots \times A_k) \times A_{k+1}$ . The set  $A_1 \times \dots \times A_k$  is assumed to be countable, as is  $A_{k+1}$ , so their Cartesian product  $(A_1 \times \dots \times A_k) \times A_{k+1}$  is countable by Proposition 1.3.8. So, by induction, the Cartesian product of  $k$  countable sets is itself countable.

#### Argue that the set of finite sequences of 0's and 1's is countable

Consider the set of finite sequences of 0's and 1's, which we will call  $S$ . We can interpret elements of  $S$  as binary representations of natural numbers. This is actually a surjection from  $\mathbb{N}_0$  onto  $S$ , as any sequence in  $S$  can be interpreted as a natural number. Since we have mapped  $\mathbb{N}_0$ , which is countable, onto  $S$ ,  $S$  must also be countable.

### Argue that the set of infinite sequences of 0's and 1's is not countable

Call the set of infinite sequences of 0's and 1's  $S'$ . Cantor's diagonalization argument can be applied to  $S'$ . Assume for contradiction that  $S'$  is countable and that as a result there exists some ordered list of each element of  $S'$ . We can create a new sequence  $a$  whose first term is different from the first term of the first sequence in  $S'$ , whose second term is different from the second term of the second sequence in  $S'$ , and so on. The sequence  $a$  differs from every sequence in  $S'$  in at least one position, so it cannot be on the list even though it is a member of  $S'$ . We have encountered a contradiction, and thus  $S'$  is not countable.

### Problem 13

We will show that the set of algebraic numbers is countable by showing first that the set of polynomials,  $P$ , is countable. Note that any algebraic number  $a$  is by definition the root of some polynomial  $p \in P$ , or, in other words, there exists an injection from the algebraic numbers into  $P$  (the relation  $f$  where  $f(a) = p$  only if  $a$  is a root of  $p$ ), which means that the cardinality of the algebraic numbers is less than or equal to that of  $P$ . So, if we show that  $P$  is countable, we can conclude that the algebraic numbers are countable.

Let  $P_n$  be the set of polynomials of degree  $n$ : specifically  $P_n = \{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \mid a_n, \dots, a_0 \in \mathbb{Z}\}$ . In addition, consider the Cartesian product  $\mathbb{Z}^n$  to be the Cartesian product of  $\mathbb{Z}$   $n$  times, as defined in Problem 7. We can define the relation  $f : \mathbb{Z}^n \rightarrow P_n$  such that, if  $b = (b_0, \dots, b_n) \in \mathbb{Z}^n, p \in P_n$ , then  $f(b) = p$  only if  $p$ 's coefficient  $a_k = b_k$  for all  $0 \leq k \leq n$ . Every  $p \in P_n$  has coefficients from  $\mathbb{Z}$ , so every  $p = f(b)$  for some  $b \in \mathbb{Z}^n$ . Thus,  $f$  is surjective. Since  $\mathbb{Z}$  is countable,  $\mathbb{Z}^n$  is countable by Problem 7, and since  $f$  is a surjection onto  $P_n$ , each  $P_n$  is also countable.

Now, notice that every polynomial  $p \in P$  is in  $P_n$  for some  $n$ . This means that  $P = \bigcup_{n=1}^{\infty} P_n$ . So, by Theorem 1.3.14,  $P$  is itself countable since it is a countable union of countable sets. By the argument above, we can conclude that the algebraic numbers are countable.

## Section 2.1

### Problem 1

We will prove that  $\lim_{n \rightarrow \infty} |x_n| = 0 \iff \lim_{n \rightarrow \infty} x_n = 0$ . By the definition,  $\lim_{n \rightarrow \infty} |x_n| = 0 \iff \forall \varepsilon > 0, \exists N$  where  $n > N \implies ||x_n|| < \varepsilon$ . However,  $||x_n|| = |x_n|$  since  $|x_n| \geq 0$ . Thus  $\exists N$  such that  $|x_n| < \varepsilon$  for any positive  $\varepsilon$ , which is the definition of  $\lim_{n \rightarrow \infty} x_n = 0$ .

### Problem 2

We will use the definition to show that  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ . To satisfy the definition, then for any  $\varepsilon > 0$  we need to find an  $N$  such that  $n > N \implies \left| \frac{n}{n+1} - 1 \right| < \varepsilon$ . Choose  $N = \frac{1}{\varepsilon} - 1$ , so that

$$\begin{aligned} \frac{1}{\varepsilon} - 1 &< n \\ \frac{1}{\varepsilon} &< n + 1 \\ \frac{1}{n+1} &< \varepsilon \\ \left| \frac{-1}{n+1} \right| &< \varepsilon \\ \left| \frac{n-(n+1)}{n+1} \right| &< \varepsilon \\ \left| \frac{n}{n+1} - 1 \right| &< \varepsilon. \end{aligned}$$

Since we can find an  $N$  for any  $\varepsilon > 0$  where this inequality holds true, by definition  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ .

### Problem 4

Assume  $(x_n)$  is a sequence with  $x_n \geq 0$  that converges to  $L$ . We will use the definition to show that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{L}$ .

By the definition, we know that  $\forall \varepsilon_1 > 0, \exists N_1$  such that  $n > N_1 \implies |x_n - L| < \varepsilon_1$ , and we want to show that  $\forall \varepsilon > 0, \exists N$  where  $n > N \implies |\sqrt{x_n} - \sqrt{L}| < \varepsilon$ .

First, consider only the case where  $L \neq 0$ . Take any  $\varepsilon > 0$  and fix it. Now

set  $\varepsilon_1 = \varepsilon\sqrt{L}$ , which we can do since  $x_n \geq 0 \implies L \geq 0$ . By the definition stated above, we know that there exists some  $N_1$  where  $|x_n - L| < \varepsilon_1$  for all  $n > N_1$ . So, when  $n > N_1$ , (and keeping in mind that  $L \neq 0$ ) we can write  $\varepsilon\sqrt{L} > |x_n - L| \iff \varepsilon > \frac{|x_n - L|}{\sqrt{L}} \geq \frac{|x_n - L|}{\sqrt{x_n} + \sqrt{L}} = \left| \frac{(\sqrt{x_n} - \sqrt{L})(\sqrt{x_n} + \sqrt{L})}{\sqrt{x_n} + \sqrt{L}} \right| = \left| \sqrt{x_n} - \sqrt{L} \right|$ . We showed that there exists an  $N$  for any  $\varepsilon$ , namely,  $N = N_1$ .

Now we deal with the case where  $L = 0$ . This simplifies the problem, now we need to find an  $N$  such that when  $n > N$ ,  $|\sqrt{x_n}| < \varepsilon$ . Select some  $\varepsilon > 0$  and fix it, and then set  $\varepsilon_1 = \varepsilon^2$ . We know that there exists some  $N_1$  where for all  $n > N_1$ ,  $|x_n| < \varepsilon_1 = \varepsilon^2$ . We take the square root of both sides (and note that since the left side is always positive, we can discard the negative solution on the right) and see that  $|\sqrt{x_n}| < \varepsilon$  when  $n > N_1$ .

Thus we have satisfied the definition in all cases, so  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{L}$ .

## Problem 6

WORK ON THIS ONE

## Problem 11

Let  $a > 1$ . We will show that  $\lim_{n \rightarrow \infty} a^n = \infty$ . To satisfy the definition of diverging to  $\infty$ , we will show  $\forall B > 0, \exists N$  such that  $n > N \implies a^n > B$ . If we choose  $N = \log_a(B)$ , we can see that  $n > \log_a(B) \implies a^n > a^{\log_a(B)} = B$ .

Now consider if  $0 < a < 1$ . We will show that in this case,  $\lim_{n \rightarrow \infty} a^n = 0$  by showing that  $\forall \varepsilon > 0, \exists N$  such that  $|a^n| < \varepsilon$ . We can choose  $N = \log_a(\varepsilon)$ . So, when  $n > N$ ,  $n > \log_a(\varepsilon)$ . Now we will raise  $a$  to the power of both sides, but we have to switch the inequality sign since  $a^n$  is monotone decreasing. So,  $a^n = |a^n| < a^{\log_a(\varepsilon)} = \varepsilon$ , and we conclude that the sequence converges to 0.

## Problem 13

Let  $(a_n)$  and  $(b_n)$  be sequences where  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = \infty$ . We will show that if  $L < 0$  then  $\lim_{n \rightarrow \infty} a_n b_n = -\infty$ .

We want to show that  $\forall B < 0, \exists N$  such that  $n > N \implies a_n b_n < B$ . By the definitions of convergence to  $L$  and convergence to  $\infty$ , we know that  $\forall \varepsilon > 0, \exists N_1$  such that  $n > N_1 \implies |a_n - L| < \varepsilon$  and that  $\forall M > 0, \exists N_2$  such that  $n > N_2 \implies b_n > M$ . In particular, if we select  $\varepsilon < L$ , we can find an  $N_1$  that ensures  $a_n$  is negative for all  $n > N_1$ . In addition, if we select  $M = \frac{-B}{L}$ , we can find an  $N_2$  to ensure  $b_n > \left|\frac{B}{L}\right|$  for all  $n > N_2$ . Now we take  $N = \max(N_1, N_2)$  which will ensure both conditions when  $n > N$ . Now notice that when  $n > N$ ,  $a_n b_n < L \cdot \frac{B}{L} = B$ . So we have shown that  $\lim_{n \rightarrow \infty} a_n b_n = -\infty$ .

### **Problem 15**

### **Problem 16**