Math 351: Homework 1 (Due September 14)

Name: Jack Ellert-Beck

Section 0.2

Problem 1

Prove the following equivalences:

a) $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

We can write a truth table that lists the truth values of $\neg(P \vee Q)$ and $(\neg P \wedge \neg Q)$ for all truth values of P and Q:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$(\neg P \wedge \neg Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Since the two statements have the same truth value in all cases, they are equivalent.

b) $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

We can write a truth value to test every case:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P \vee \neg Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Since both statements have the same truth value for any value of P or Q, they are equivalent.

Problem 2

Prove that $P \implies Q \equiv (\neg P) \vee Q$. Deduce that the negation of $P \implies Q$ is $P \wedge (\neg Q)$.

We write a truth table to show that $P \implies Q$ has the same truth value as $(\neg P) \vee Q$ in all cases:

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \implies Q$
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

Since $P \implies Q$ is equivalent to $(\neg P) \vee Q$, their negations will be equivalent. So we negate $(\neg P) \vee Q$ and manipulate using equivalences from Problem 1 to find the negation:

$$\begin{aligned} \neg(P \implies Q) &= \neg(\neg P \vee Q) \\ &= \neg(\neg P \vee \neg(\neg Q)) \\ &= \neg(\neg(P \wedge (\neg Q))) \\ &= P \wedge (\neg Q). \end{aligned}$$

We also used the fact that $\neg(\neg P) \equiv P$, which we now show with a truth table:

P	$\neg P$	$\neg(\neg P)$
T	F	**T**
F	T	**F**

Problem 3

Find the negation of the following statements.

a) $\neg(P \wedge \neg Q) \vee R$

$$\begin{aligned} \neg(\neg(P \wedge \neg Q) \vee R) &= (P \wedge \neg Q) \wedge \neg R \\ &= P \wedge \neg Q \wedge \neg R. \end{aligned}$$

b) $P \implies (Q \vee R)$

$$\begin{aligned} \neg(P \implies (Q \vee R)) &= P \wedge \neg(Q \vee R) \\ &= P \wedge \neg Q \wedge \neg R. \end{aligned}$$

c) $\neg(P \vee Q) \implies (R \vee S)$

$$\begin{aligned} \neg(\neg(P \vee Q) \implies (R \vee S)) &= \neg((P \vee Q) \vee (R \vee S)) \\ &= \neg(P \vee Q) \wedge \neg(R \vee S) \\ &= \neg P \wedge \neg Q \wedge \neg R \wedge \neg S. \end{aligned}$$

Problem 5

Prove that an implication is equivalent to its contrapositive.

The contrapositive of $P \implies Q$ is the statement $\neg Q \implies \neg P$. We can manipulate the statement $\neg P \vee Q$, noting that it is equivalent to $P \implies Q$:

$$\begin{aligned} \neg P \vee Q &= Q \vee \neg P \\ &= \neg(\neg Q) \vee \neg P \\ &= \neg Q \implies \neg P. \end{aligned}$$

Problem 6

An example of an implication whose converse is not true:

If polygon $ABCD$ is a square, then its interior angles are all 90 degrees.

An example of an implcation whose converse is true:

$$(x = 4) \implies (x + 2 = 6)$$

Problem 7

Negate the following statements:

a) For every ice cream flavor there is a pie that goes with that flavor

There exists an ice cream flavor for which there are no pies that go with it.

b) For every race car there is a driver who can drive that car.

There exists at least one race car with no driver who can drive it.

c) There exists a race car that every driver can drive.

For each race car there exists at least one driver who cannot drive it.

d) There exists a driver that can drive every race car.

For all race car drivers there exists at least one car they cannot drive.

Section 0.3

Problem 1

Let $A, B \subseteq X$. Which of the statements is equivalent to $A \cup B \neq \varnothing$?

Answer: (b) $A \neq \varnothing \vee B \neq \varnothing$. For the union of A and B to be nonempty it is sufficient for at least one of A or B to be nonempty.

Problem 2

Let $A, B \subseteq X$. Which of the statements is equivalent to $A \cap B = \varnothing$?

Answer: (b) $A = \varnothing \vee B = \varnothing$. If any one of A or B is empty, then their intersection will be empty.

Problem 3

Let A and B be sets. Prove that $A = (A \cap B) \cup (A \setminus B)$.

We consider the definitions of $A \cap B$ and $A \setminus B$. The set $A \cap B$ is the set of all elements of A that are also in B. The set $A \setminus B$ is the set of all elements of A that are not in B. The union of these two sets, $(A \cap B) \cup (A \setminus B)$, is the set of all elements of A that are also in B or that are not also in B. Since everything is either in B or not in B, this set contains all elements of A. Thus, $A = (A \cap B) \cup (A \setminus B)$.

Problem 4

Let A and B be sets. Prove that $A \setminus (A \setminus B) = B$.

The premise is actually false. Let $A = \{1, 2\}$ and $B = \{2, 3\}$. We can see that $A \setminus B = \{1\}$ and that $A \setminus \{1\} = \{2\} \neq \{2, 3\}$.

If instead the condition had been added that $B \subseteq A$, it would have been a good problem.

Problem 5

Let $A, B \subseteq X$. Prove that $A \subseteq B \iff B^c \subseteq A^c$.

We first prove that $A \subseteq B \implies B^c \subseteq A^c$. $A \subseteq B$ means that all elements of A are also in B. Also, by definition, $x \in B^c \implies x \notin B$. So, $\forall x \notin B^c$, x cannot be in A, which means that $x \in A^c$. Thus, $B^c \subseteq A^c$.

We now prove the other direction, that $B^c \subseteq A^c \implies A \subseteq B$. Note that any $x \in B^c$ is also in A^c. In other words, x cannot be in B^c and also be in A. So, any element of A is also in B, which means $A \subseteq B$.

Problem 17

Let A and B be two sets such that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Let S be any element of $\mathcal{P}(A)$. By the definition of $\mathcal{P}(A)$, $S \subseteq A$. Since $A \subseteq B$, every element of S is in B as well, so $S \subseteq B$. Hence, $S \in \mathcal{P}(B)$. So, every $S \in \mathcal{P}(A)$ is also in $\mathcal{P}(B)$, which means $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Section 0.4

Problem 10

Let $f : X \to Y$ be a function.

a) Prove that $f(f^{-1}(B)) \subseteq B$ for all $B \subseteq Y$.

Take any $y \in f(f^{-1}(B))$. This means that $y = f(x)$ for some $x \in f^{-1}(B)$. By the definition of $f^{-1}(B)$, $f(x) \in B$, so $y \in B$. Thus, $f(f^{-1}(B)) \subseteq B$.

b) Prove that $f(f^{-1}(B)) = B$ when f is surjective.

We have already shown that $f(f^{-1}(B)) \subseteq B$ in general, so it is also true if f is surjective. We will show that if f is surjective, then $B \subseteq f(f^{-1}(B))$, and thus that $f(f^{-1}(B)) = B$. For f to be surjective, it is necessary that for all $y \in Y$ there exists some $x \in X$ such that $f(x) = y$. In particular, for all $y \in B$ there exists an $x \in f^{-1}(B)$ where $f(x) = y$. Hence, $y \in f(f^{-1}(B))$ for any $y \in B$. So, when f is surjective, $B \subseteq f(f^{-1}(B))$, and thus $B = f(f^{-1}(B))$.

Problem 17

Let $f : X \to Y$ be a function. Prove that $f(A \cup B) = f(A) \cup f(B)$ for all $A, B \in X$.

Problem 18

Problem 19