Math 351: Homework 7 Due Friday November 2

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Problem 1

Suppose I is a bounded closed interval and $f : I \to \mathbb{R}$ is continuous on I. We will prove that $f(I)$ is bounded. Since I is a bounded closed interval on R, it is a compact subset of R. By the Extreme Value Theorem, f has a maximum and a minimum on I. This means that $f(I)$ has a maximum and minimum value, and thus it is bounded.

Suppose I is a bounded open interval and $f : I \to \mathbb{R}$ is continuous on I. We will show a counterexample to disprove that $f(I)$ is bounded. Let I be $(0, 1)$ and define $f(x) = \frac{1}{x}$. I is a bounded open interval and we have previously shown f is continuous on I. However, $f(x) \to \infty$ as $x \to 0$, so $f(I) = (1, \infty)$ is not bounded.

Suppose I is a bounded open interval and $f : I \to \mathbb{R}$ is uniformly continuous on I. We will prove that $f(I)$ is bounded. It will suffice to show that $f(I)$ must be bounded from above, since to show that $f(I)$ is bounded from below is the same as showing that $-f(I)$ is bounded from above. So, assume for contradiction that $f(I)$ is not bounded from above. This means that $\forall M \in \mathbb{R}\ \exists e \in f(I)$ such that $e > M$. Note that for any such e there is some $x \in I$ where $f(x) = e$. Now, by the definition of uniform continuity, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $x, y \in I$ and $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Pick any ε and fix it. Now find a corresponding δ. Now, let a, b be the endpoints of I such that $I = (a, b)$. Note that at least one of $f((a, \frac{a+b}{2}])$ or $f([\frac{a+b}{2}, b))$ is not bounded. (If both subintervals were bounded, then $\sup I$ would be a real number and be equal to the larger of the two supremums of the subintervals.) If the image of the former interval under f is unbounded, let $a_1 = a$ and $b_1 = \frac{a+b}{2}$. If not, then let $a_1 = \frac{a+b}{2}$ and $b_1 = b$. Now let $I_1 = (a_1, b_1)$. Again, we can divide I_1 in half and choose whichever half has an unbounded image under f to be I_2. Continuing this pattern, note that $(a_n - b_n) = \frac{a-b}{2^n} \to 0$ as $n \to \infty$, and that for all I_n, $f(I_n)$ is unbounded. Thus there is some I_n where $(a_n - b_n) < \delta$. Now pick $y = a_n$ and let $M = f(y) + \varepsilon$. By the fact that I_n is unbounded, there is some $x \in I_n$ such that $f(x) > f(y) + \varepsilon$. But, x is in I_n, so we have found a case where $|x - y| < \delta$ and $|f(x) - f(y)| > \varepsilon$, a contradiction. So $f(I)$ must be bounded from above, and following the same argument with $-f(I)$ shows that $f(I)$ is also bounded from below, so $f(I)$ is bounded.

Suppose A is a countable union of bounded open intervals and $f : A \to \mathbb{R}$ is uniformly continuous on A. We will provide a counterexample to show that $f(A)$ need not be bounded. Let $A = \ldots \cup (-2,-1) \cup (0,1) \cup (2,3) \cup \ldots$ and $f(x) = x$. We can show that f is uniformly continuous on $\mathbb{R}$. Pick any $\varepsilon > 0$. If we choose $\delta = \frac{\varepsilon}{2}$, we can see that $|x-y| < \delta \implies |f(x)-f(y)| = |x-y| < \frac{\varepsilon}{2} < \varepsilon$. Since f is uniformly continuous on $\mathbb{R}$, it is uniformly continuous on $A \subseteq \mathbb{R}$. However, we can see that $f(I)$ is not bounded. If we choose any $M > 0$, we know that $f(M+1) = M+1 > M$. Note that at least one of $M+1, M+2, M+2.1$, and $M+3.1$ will be in A. Thus we can always find an $x \in A$ where $f(x) > M$. So in this case, f is uniformly continuous on a countable union of bounded open intervals and $f(I)$ is not bounded.

Problem 2

Suppose a_n and b_n are Cauchy sequences.

Prove or disprove: $|a_n - b_n|$ is Cauchy. Since both sequences are Cauchy, we know that a_n converges to some $A \in \mathbb{R}$ and b_n converges to a value $B \in \mathbb{R}$. So, $\forall \varepsilon > 0\ \exists N_1 > 0$ such that $n > N_1 \implies |a_n - A| < \varepsilon$ and $n > N_2 \implies |b_n - B| < \varepsilon$. Pick an ε and fix it. Find any N_1, N_2 corresponding to $\frac{\varepsilon-(A-B)}{2}$ and set $N > \max\{N_1, N_2\}$. Now we have $|a_n - b_n| = |a_n - A - b_n + B + A - B| = |(a_n - A) + (B - b_n) + (A - B)| < \varepsilon - (A - B) + (A - B) = \varepsilon$. Thus $|a_n - b_n|$ converges and is therefore Cauchy.

Prove or disprove: $(-1)^n a_n$ is Cauchy. Let $a_n = 1$ This clearly converges and thus is Cauchy. However, $(-1)^n a_n$ does not converge. To see why, pick $\varepsilon = \frac{1}{2}$. For any choice of N, $|a_{N+2} - a_{N+1}| = 2 > \varepsilon$. Thus $(-1)^n a_n$ is not Cauchy.

Prove or disprove: if $a_n \neq 0$ for all n, then $\frac{1}{a_n}$ is Cauchy. Let $a_n = \frac{1}{2^n}$. This sequence converges to 0, and for all n, $a_n > 0$. So a_n is Cauchy. However, $\frac{1}{a_n} = 2^n \to \infty$. So, $\frac{1}{a_n}$ does not converge, and this counterexample disproves the proposition.

Prove or disprove: if $a_n > 0.001$ for all n, then $\frac{1}{a_n}$ is Cauchy. If $a_n > 0.001$ for all n and a_n is Cauchy, then a_n converges to some $L > 0.001$. Hence $\frac{1}{a_n} < 1000$ and, by the algebraic properties of limits, $\frac{1}{a_n}$ converges to $\frac{1}{L}$, and we conclude that $\frac{1}{a_n}$ is Cauchy.

Problem 3

A function $f : \mathbb{R} \to \mathbb{R}$ is monotone if $x \leq y$ makes $f(x) \leq f(y)$. Show that a monotone function can have at most a countable number of points of discontinuity.

Section 5.1

4. Prove a thing
5. Prove a thing
6. Give an example
7. Prove a thing
8. Construct a function

Section 5.2

1. Who knows
2. not me
3. yikes
8. 8 really
9. 9 huh