Math 351: Homework 5 Due Friday October 19

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Sections 3.2 and 3.3

Section 3.2 Exercises (pg 112-113) work 6,7,8,9,10

6. Let $A \subsetneq \mathbb{R}$. When is $A^\circ = \overline{A}$?

   We know that $A^\circ \subseteq A \subseteq \overline{A}$, so $A^\circ = \overline{A} = A$ in this case. We have previously shown that $A = A^\circ$ if and only if A is open, and that $A = \overline{A}$ if and only if A is closed. So, A must be both open and closed. The only sets for which this can be true are $\mathbb{R}$ and $\varnothing$, and since we have $A \subsetneq \mathbb{R}$, the only case where this is possible is $A = \varnothing$.

7. Is there a set whose interior is empty and whose closure is $\mathbb{R}$?

   The set $\mathbb{Q}$ satisfies this property. Pick any $q \in \mathbb{Q}$ and fix it. For any $\varepsilon > 0$, the open ball $B(q, \varepsilon)$ will contain a nonrational real number due to the order completeness of $\mathbb{R}$. Thus, $\mathbb{Q}$ contains no interior points. In addition, we know that $\mathbb{R}$ consists of all values to which sequences of rational numbers converge, so every real number is the limit of some sequence in $\mathbb{Q}$. Thus every $x \in \mathbb{R}$ is a closure point of $\mathbb{Q}$. So $\mathbb{Q}$ is a set whose interior is empty and whose closure is $\mathbb{R}$.

8. Find the closure and interior of each of the following sets:

   (a) $A = [0, 1) \cup (1, 2)$
   $\overline{A} = [0, 2],\ A^\circ = (0, 1) \cup (1, 2)$

   (b) $\mathbb{N}$
   $\overline{\mathbb{N}} = \mathbb{N},\ \mathbb{N}^\circ = \varnothing$

   (c) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$
   $\overline{A} = A \cup 0,\ A^\circ = \varnothing$

   (d) $A = \mathbb{Q} \cap (0, 1)$
   $\overline{A} = [0, 1],\ A^\circ = \varnothing$

   (e) $A = \mathbb{R} \setminus \mathbb{Q}$
   $\overline{A} = \mathbb{R},\ A^\circ = \varnothing$
   Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$, so A has no interior points.

(f) $A = \{\frac{p}{2^n} : n \in \mathbb{N}, p = 0, \ldots, 2^n\}$
$\overline{A} = \mathbb{R},\ A^\circ = \varnothing$
A is the set of dyadic rationals in $[0, 1]$. In Example 2.2.6, we define a function $d : [0, 1] \to \{0, 1\}^{\mathbb{N}}$ that maps every real number in $[0, 1]$ to a sequence of dyadic intervals of increasing rank. For any $x \in [0, 1]$, if we take the sequence of the left endpoints of these intervals determined by $d(x)$, this sequence will be a subset of our set A and will converge to x.

9. Prove or disprove $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

   We will show that these sets are equal by showing that they are subsets of one another. First, let $x \in \overline{(A \cup B)}$. This means that x is a closure point of $A \cup B$. Thus there exists some sequence $a_n \in (A \cup B)$ that converges to x. There are two possible cases: either a_n is a subset of A or B, or (a_n) contains elements from A and elements from B. If it is the former case, assume without loss of generality that $(a_n) \subseteq A$ (since the following argument is also valid for B). Thus, x is also a closure point of A, so $x \in \overline{A} \subseteq (\overline{A} \cup \overline{B})$. In the case where (a_n) contains elements from both A and B, take either the subsequence $(a_n) \cap A$ or $a_n \cap B$, whichever has an infinite number of elements. We know that if $a_n \to x$, any subsequence of a_n also converges to x. Thus we can find a sequence in either A or B that converges to x, so x is a closure point of either A or B, and it follows that $x \in (\overline{A} \cup \overline{B})$. Since any x in $\overline{A \cup B}$ is in $\overline{A} \cup \overline{B}$, $\overline{A \cup B} \subseteq (\overline{A} \cup \overline{B})$.

   Now take $x \in (\overline{A} \cup \overline{B})$. This means x is either a closure point of A or a closure point of B, so there is some sequence (a_n) of terms in either A or B that converges to x. Each a_n would also be in $A \cup B$, however, which means that x is also a closure point of $A \cup B$, so $x \in \overline{A \cup B}$. Hence $(\overline{A} \cup \overline{B}) \subseteq \overline{A \cup B}$. Since we have both sets as subsets of one another, the sets must be equal.

10. Prove or disprove $\overline{(A \cap B)} = \overline{A} \cap \overline{B}$.

    We disprove this by providing a counterexample. Let $A = (0, 1)$ and $B = (1, 2)$. $\overline{(A \cap B)} = \overline{(0, 1) \cap (1, 2)} = \overline{\varnothing} = \varnothing$. But, $\overline{A} \cap \overline{B} = \overline{(0, 1)} \cap \overline{(1, 2)} = [0, 1] \cap [1, 2] = 1 \neq \varnothing$.

Section 3.3 Exercises (pg 116-117) work 7

7. Find the closure and interior of each of the following sets as subsets of $\mathbb{R}^2$.

   (a) $A = \mathbb{N} \times \mathbb{Q}$
   $\overline{A} = \mathbb{N} \times \mathbb{R},\ A^\circ = \varnothing$

   (b) $A = \mathbb{Z} \times \mathbb{Z}$
   $\overline{A} = A,\ A^\circ = \varnothing$

   (c) $A = \mathbb{Q} \times (0, 1)$
   $\overline{A} = \mathbb{R} \times [0, 1],\ A^\circ = \varnothing$

   (d) $A = \{(x, \sin(\frac{1}{x})) : 0 < x < \pi\}$
   $\overline{A} = A \cup (\{0\} \times [-1, 1]),\ A^\circ = \varnothing$
   For every $y \in [-1, 1]$ there is a sequence in A converging to the element $(0, y)$, namely the set of all (x, y) where x is a solution to the equation $\sin(\frac{1}{x}) - y = 0$, enumerated in decreasing order of x. Thus all $(0, y)$ are closure points. Also, every point in A is a boundary point of A.

Interiors

For $\mathbb{R}^n$, what is the relationship between A° and $(\bar{A})^\circ$? Are they equal? One contained in the other? Prove any claim you make.

Recall that $A \subseteq \overline{A}$. Thus all interior points of A must be elements of $\overline{A}$, and any open balls $B(x, \varepsilon) \subseteq A$ are also subsets of $\overline{A}$. It follows that all interior points of A are also interior points of $\overline{A}$. So, $A^\circ \subseteq (\overline{A})^\circ$. We cannot, however, establish this relationship in the other direction. For instance, take $\mathbb{Q}^n$. $(\mathbb{Q}^n)^\circ = \varnothing$ but $(\overline{\mathbb{Q}^n})^\circ = (\mathbb{R}^n)^\circ = \mathbb{R}^n \neq \varnothing$.

Compactness 1

Suppose K, F are subsets of $\mathbb{R}$, and that K is compact and F is closed. Are the following definitely compact, definitely closed, or possibly neither.

a) $K \cap F$

The intersection of a bounded set with any other set will be bounded, and any intersection of closed sets will be closed, so $K \cap F$ will be closed and bounded, and thus compact by Heine-Borel.

b) $\overline{K^c \cup F^c}$

This set will definitely be closed as it is the closure of some set. However, since K is bounded, K^c will not be bounded, and neither will its union with any other set. The closure of this set will be a superset of an unbounded set, so it will itself be unbounded, and thus not compact.

c) $K \setminus F$

If K and F are disjoint, then $K \setminus F = K$ is compact. However, in the case where $K = [0, 1]$, $F = [\frac{1}{2}, \frac{3}{2}]$, we get $K \setminus F = [0, \frac{1}{2})$, which is not closed and therefore not compact. So $K \setminus F$ can be compact but it need not be.

d) $\overline{K \cap F^c}$

Note that $\overline{K \cap F^c} = \overline{K \setminus F}$. Since $K \setminus F$ is a subset of K, it is definitely bounded. The set in question is therefore the closure of a bounded set, so it is compact.

Compactness 2

Consider the set comprised of closed intervals

$$\ldots \cup [\tfrac{1}{6}, \tfrac{1}{5}] \cup [\tfrac{1}{4}, \tfrac{1}{3}] \cup [\tfrac{1}{2}, 1]$$

That is, let $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for $n \in \mathbb{N}$ and take $B = \bigcup_{n \in \mathbb{N}} I_n$.

a) Show B is bounded but not closed.

Note that, for any n, both endpoints of the interval I_n are positive numbers less than 1. $B \subseteq [0, 1]$, so it is bounded. To show that B is not closed, we show that B^c is not open. Since $0 \notin B$, $0 \in B^c$. Now pick any $\varepsilon > 0$ and consider the open ball $B(0, \varepsilon)$. Recall that $\lim_{n \to \infty} \frac{1}{n} = 0$ means that $\forall \varepsilon > 0\ \exists N$ such that $n > N \implies \frac{1}{n} < \varepsilon$. So, it follows that there are an infinite number of consecutive $\frac{1}{n}$ in $B(0, \varepsilon)$ and thus an infinite number of intervals I_n contained in $B(0, \varepsilon)$. Hence 0 is not an interior point of B^c, so B^c is not open, which means B is not closed.

b) Find a sequence $\{b_n\} \subset B$ that converges to a point not contained in B.

Define $\{b_n\}_{n \in \mathbb{N}}$ such that $b_n = \frac{1}{\frac{(2n)+(2n-1)}{2}} = \frac{1}{2n - \frac{1}{2}}$. Note that for all $n \in \mathbb{N}$,

$b_n \in I_n$, so $b_n \subseteq B$. We find the limit of b_n: $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-\frac{1}{2}} = 0$, which is not an element of B.

c) Find an open cover of B that cannot be reduced to a finite open cover.

Define the interval $I'_n = (\frac{1}{2n} - \frac{1}{100n}, \frac{1}{2n-1} + \frac{1}{100n})$. Note that for all $n \in \mathbb{N}$, I'_n covers I_n and does not cover any I_k for $k \neq n$. Let $G = \bigcup_{n\in\mathbb{N}} I'_n$. Since G is a union of open sets, G is open, so G is an open cover of B. Removing any I'_n from G leaves I_n uncovered, so G cannot be reduced to a finite open cover.

d) Show $\tilde{B} = B \cup \{0\}$ is compact.

We previously showed that B is bounded, so $\tilde{B}$ is bounded since it adds a finite number of elements to B. We will show that $\tilde{B}$ is closed by showing that $\tilde{B}^c$ is open. We can write $\tilde{B}^c = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{n\in\mathbb{N}} \left(\frac{1}{2n+1}, \frac{1}{2n}\right)$. This is a union of open sets and is thus open, so $\tilde{B}$ is closed. By Heine-Borel, $\tilde{B}$ is thus compact.

e) How does the addition of the point $\{0\}$ undo your answers in (b) and (c)? That is, why can't you make similar examples that work for $\tilde{B}$?

In part (b), 0 was the only value outside of B where we could construct a sequence contained in B that converges to it, because it was the only boundary point of B^c. In part (c), we can still construct open covers of $\tilde{B}$. However, due to the argument from part (a), any open set containing 0 will also contain an infinite number of I_n. So any open cover of $\tilde{B}$ can be reduced to the cover that includes whatever open interval was used to cover 0, plus the open intervals used to cover the finitely many intervals I_n that haven't already been covered.

Compactness 3

Prove that, given a bounded open set A and an $\varepsilon > 0$, there exists a finite union of closed intervals $\bigcup C_n \subseteq A$ such that $A \setminus \bigcup C_n$ does not contain any balls of radius ε.

Let a be a lower bound of A and b be an upper bound of A. It follows that $A \subseteq (a-100, b+100)$. So, the set $G = [a-100, b+100] \cap A^c$ is compact, as it is of the form $K \cap F$ from **Compactness 1**. We can cover G with ε

balls, and, since G is compact, reduce this cover to a finite cover $B = \bigcup B_n$. Now consider $B^c \cap [a-100, b+100]$. Since B is a finite collection of open balls of fixed size, this set is a finite union of closed intervals, $\bigcup C_n$. Further, this set is a subset of A, since its complement in $[a-100, b+100]$ covers A's complement in the same interval. Now, notice that if there is a ball of radius epsilon in $A \setminus \bigcup C_n$, it would have to be disjoint from G but would be part of the cover B, which means we can still cover G without that ball and repeat the construction with a modified cover. Hence we can construct $\bigcup C_n$ such that $A \setminus \bigcup C_n$ contains no balls of radius ε.

Compactness 4

Work section 3.4 exercise 3 (page 122).

Let E and F be two disjoint nonempty compact sets in $\mathbb{R}$. We will show that there exists a $\delta > 0$ such that for every interval I, if $|I| < \delta$ then $I \cap E = \varnothing$ or $I \cap F = \varnothing$. Suppose for contradiction that this is false. Then for all $\delta > 0$, there exists an interval I with $|I| < \delta$ and $I \cap E \neq \varnothing$, $I \cap F \neq \varnothing$. We will construct sequences $\{e_n\}$ and $\{f_n\}$ by considering points in I for different values of δ. Pick some δ_1 and find an interval I_1 satisfying the conditions. This interval will contain some $e_1 \in E$ and $f_1 \in F$. Now pick any positive $\delta_2 < \delta_1$ and find a corresponding I_2, containing some e_2 and f_2. Continuing this process, we create sequences $\{e_n\}$ and $\{f_n\}$ where each e_n and f_n is in I_n, and $|I_n| \to 0$. It follows that $0 \leq |e_n - f_n| \leq |I_n|$, so $|e_n - f_n| \to 0$ by the Squeeze Property. Since E and F are compact, they are bounded, so e_n and f_n converge, and in fact converge to the same L. But E and F are also closed, and L is a closure point of both E and F, so L is in E and F, which contradicts our assumption that they are disjoint sets.

The property is not necessarily true if E and F are closed but not compact.

Let $E = \bigcup_{n\in\mathbb{N}} [2n-1, 2n]$ and $F = \bigcup_{n\in\mathbb{N}} [2n + \frac{1}{3n}, 2n + \frac{1}{2}]$. For both sets, their complements are unions of open sets, so E and F are both closed. For any $\delta > 0$, find n such that $\frac{1}{3n} < \delta$. Now take $I = [2n, 2n + \frac{1}{3n}]$, which contains elements from both E and F.

Compactness 5

And now for something completely different. Consider the natural numbers $\mathbb{N}$ with the trivial metric (defined in example 3.1.7 on page 107).

Show that with this metric, the set $\mathbb{N}$ is closed and bounded, but not sequentially compact.

Define $d(x, y)$ to be the trivial metric, where $d(x, y) = 0 \iff x = y$ and $d(x, y) = 1$ otherwise. Since $\mathbb{N}$ is the universal set of the metric space in this case, $\mathbb{N}$ is closed. Note that for any $x, y \in \mathbb{N}$, $d(x, y) \le 1$, so $\mathbb{N}$ is bounded. Now consider the sequence $\{a_n\}$ where $a_n = n$. For any choice of $\{n_k\}$, the elements of $\{a_{n_k}\}$ fail to get closer to one another than ε when $\varepsilon < 1$. Thus there is no subsequence of $\{a_n\}$ which converges, so $\mathbb{N}$ cannot be sequentially compact in this metric space.

Continuity 1

From section 4.1 exercises (pg 143), work 4 and 8. See example 4.1.7.

4. Let $f(x) = \frac{1}{x}$. Show that $f : (-\infty, 0) \to \mathbb{R}$ is continuous on $(-\infty, 0)$.

By the definition, $f(x) = \frac{1}{x}$ is continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|x_1 - x_2| < \delta \implies \left|\frac{1}{x_1} - \frac{1}{x_2}\right| < \varepsilon$. Pick some $\varepsilon > 0$ and fix it. Choosing $\delta < \varepsilon x_1 x_2$, will satisfy the definition. Keeping in mind that $x_1, x_2 < 0$, we manipulate the inequality:

$$\begin{aligned} |x_1 - x_2| < \delta &\implies |x_1 - x_2| < \varepsilon x_1 x_2 \\ &\implies \tfrac{|x_2 - x_1|}{x_1 x_2} < \varepsilon \\ &\implies \left|\tfrac{x_2 - x_1}{x_1 x_2}\right| < \varepsilon \\ &\implies \left|\tfrac{1}{x_1} - \tfrac{1}{x_2}\right| < \varepsilon. \end{aligned}$$

Hence f is continuous on $(-\infty, 0)$.

8. Prove, using the ε, δ definition, that if $f : A \to [0, \infty)$ is continuous, then

$\sqrt{f} : A \to [0, \infty)$ is continuous.

From the definition, we take as given that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$. What we want to show is that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|x_1 - x_2| < \delta \implies \left|\sqrt{f(x_1)} - \sqrt{f(x_2)}\right| < \varepsilon$. Pick any ε and fix it. Now choose some $\varepsilon_1 < \varepsilon\left(\sqrt{f(x_1)} + \sqrt{f(x_2)}\right)$. We know that we can find a δ_1 such that $|x_1 - x_2| < \delta_1 \implies |f(x_1) - f(x_2)| < \varepsilon_1$. So, choose $\delta \leq \delta_1$:

$$
\begin{aligned}
|x_1 - x_2| < \delta &\implies |x_1 - x_2| < \delta_1 \\
&\implies |f(x_1) - f(x_2)| < \varepsilon_1 \\
&\implies |f(x_1) - f(x_2)| < \varepsilon\left(\sqrt{f(x_1)} + \sqrt{f(x_2)}\right) \\
&\implies \tfrac{|f(x_1) - f(x_2)|}{\sqrt{f(x_1)} + \sqrt{f(x_2)}} < \varepsilon \\
&\implies \left|\tfrac{f(x_1) - f(x_2)}{\sqrt{f(x_1)} + \sqrt{f(x_2)}}\right| < \varepsilon \\
&\implies \left|\left(\sqrt{f(x_1)} - \sqrt{f(x_2)}\right)\tfrac{\sqrt{f(x_1)} + \sqrt{f(x_2)}}{\sqrt{f(x_1)} + \sqrt{f(x_2)}}\right| < \varepsilon \\
&\implies \left|\sqrt{f(x_1)} - \sqrt{f(x_2)}\right| < \varepsilon.
\end{aligned}
$$

Continuity 2

From section 4.1 exercises (pg 143), work 5.

5.

a) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that there exists only one $x \in \mathbb{R}$ for which f is continuous at x.

Define $f : \mathbb{R} \to \mathbb{R}$ such that

$$
f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}
$$

First we see that $f(0) = 0$. For values of $x \in \mathbb{R}$, as $x \to 0$, $|f(x) - 0| \to 0$

without fail. However, at any other point the function is discontinuous. In any δ neighborhood of x_1 we can find both rational and irrational numbers. If we choose $\varepsilon < f(x_1)$ it becomes clear that we will always be able to find an x_2 where $|f(x_1) - f(x_2)| > \varepsilon$.

b) Can you extend this so that f is continuous at exactly k points for arbitrary k?

Yes. Say we want f to be continuous at only the points x_n for $1 \leq n \leq k$. Define a function $g(x) = \prod_{n=1}^{k}(x - x_n)$. We can now define f as

$$f(x) = \begin{cases} g(x) & x \in \mathbb{Q} \\ -g(x) & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

c) What about an example that is continuous on $\mathbb{Z}$ but discontinuous on $\mathbb{R} \setminus \mathbb{Z}$?

Define f as

$$f(x) = \begin{cases} \sin(\pi x) & x \in \mathbb{Q} \\ -\sin(\pi x) & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This function will satisfy the property.