Calculus BC Important Info Sheet

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Part I

Information

1 Functions

1.1 Even and Odd Functions

A function $y = f(x)$ is **even** if $f(-x) = f(x)$ for every x in the function's domain. Every even function is summertric about the y-axis.

$$\text{Example:} \qquad (-x)^2 = x^2$$

A function $y = f(x)$ is **odd** if $f(x) = -f(x)$ for every x in the function's domain. Every odd function is symmertric about the origin.

$$\text{Example:} \qquad (-x)^3 = -(x)^3$$

1.2 Periodicity

A function $f(x)$ is **periodic** with period p $(p > 0)$ if $f(x + p) = f(x)$ for every value of x.

Sinusoids are examples of periodic functions. Specifically, the period of the function $y = A\sin(Bx + C)$ or $y = A\cos(Bx + C)$ is $\frac{2\pi}{|B|}$. The amplitude is $|A|$. The period of $y = \tan(x)$ is π.

1.3 Composition of Functions

Given two functions f and g, the composite function $f(g(x))$ can be written as $(f \circ g)(x)$. Note that $f \circ g$ is not necessarily equal to $g \circ f$.

1.4 Inverse Functions

The inverse of a funtion $f(x)$ is often written as $f^{-1}(x)$, or given a new name such as $g(x)$.

If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f, then f and g are inverse functions of one another.

A function f has an inverse if and only if no horizontal line intersects its graph more than once.

If f is either increasing or decreasing in an interval, then f has an inverse.

For information on the derivatives of inverse functions, see Section 3.4.

2 Limits

2.1 Definition of a Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number. Then $\lim_{x\to c} f(x) = L$ means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.

2.1.1 Symbolic Definitions of Limits

- Definition of a finite limit at a specific point:

$$\lim_{x\to c} = L \Leftrightarrow (\forall \varepsilon > 0, \exists \delta > 0, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

- Definition of an undefined limit at a specific point:

$$\lim_{x\to c} = \infty \Leftrightarrow (\forall M > 0, \exists \delta > 0, 0 < |x - c| < \delta \Rightarrow f(x) > M)$$

$$\lim_{x\to c} = -\infty \Leftrightarrow (\forall M < 0, \exists \delta > 0, 0 < |x - c| < \delta \Rightarrow f(x) < M)$$

- Definition of a finite limit at infinity:

$$\lim_{x\to\infty} = L \Leftrightarrow (\forall \varepsilon > 0, \exists M, x > M \Rightarrow |f(x) - L| < \varepsilon)$$

$$\lim_{x\to-\infty} = L \Leftrightarrow (\forall \varepsilon > 0, \exists M, x < M \Rightarrow |f(x) - L| < \varepsilon)$$

2.2 Continuity

A function $y = f(x)$ is **continuous** at $x = a$ if $f(a)$ exists, $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x) = f(a)$. $y = f(x)$ is continuous on (a, b) if $f(x)$ is continuous for every $x \in (a, b)$.

2.3 Horizontal and Vertical Asymptotes

A line $y = b$ is a **horizontal asymptote** of the graph of $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

A line $x = a$ is a **vertical asymptote** of the graph of $y = f(x)$ if either $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

2.4 Evaluating Limits

2.4.1 Limits of Rational Functions as $x \to \pm\infty$

- $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = 0$ if the degree of $f(x)$ is less than the degree of $g(x)$.

$$\text{Example:} \qquad \lim_{x\to\infty} \frac{x^2 - 2x}{x^3 + 3} = 0$$

- $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)}$ is infinite if the degree of $f(x)$ is greater than the degree of $g(x)$.

$$\text{Example:} \qquad \lim_{x\to\infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$$

- $\lim_{x\to\pm\infty} \dfrac{f(x)}{g(x)}$ is finite if the degree of $f(x)$ is equal to the degree of $g(x)$. The limit will be equal to the ratio of the leading coefficients of $f(x)$ to $g(x)$.

$$\text{Example:} \qquad \lim_{x\to\infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$$

2.4.2 Remarkable Limits

See The Later Section.

2.5 Intermediate Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. As a more specific result, if f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval $[a, b]$.

3 Derivatives

3.1 Notation of Derivatives

The derivative of the function $y = f(x)$ is commonly written as any of the following:

$$f'(x) \qquad [f(x)]' \qquad y'$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

Higher order derivatives can be written in a number of ways. Below are the respective notations for second, fourth, and nth order derivatives in three different styles.

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \qquad \frac{\mathrm{d}^4y}{\mathrm{d}x^4} \qquad \frac{\mathrm{d}^ny}{\mathrm{d}x^n}$$
$$f''(x) \qquad f^{IV}(x) \qquad f^{N}(x)$$
$$f^{(2)}(x) \qquad f^{(4)}(x) \qquad f^{(n)}(x)$$

3.2 Rate of Change

If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the **average rate of change** of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}$$

If (x_0, y_0) is a point on the graph of $y = f(x)$, then the **instantaneous rate of change** of y with respect to x at x_0 is $f'(x_0)$.

3.3 Definition of a Derivative

The derivative of a function $f(x)$ at the point $x = a$ can be defined in either of two ways:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(a) = \lim_{b\to a} \frac{f(b) - f(a)}{b - a}$$

If this limit exists at a, then $f(x)$ is said to be **differentiable** at a.

If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

3.4 Derivatives of Inverse Functions

If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and $g'(f(x)) = \frac{1}{f'(x)}$.

3.5 Finding Maxima and Minima

To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist
2. the end points, if any, on the domain of $f(x)$

These are the only candidates for the value of x where where $f(x)$ may have a maximum or a minimum.

3.6 Monotonicity

Let f be differentiable for $a < x < b$ and continuous for $a \leq x \leq b$.

1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

3.7 Concavity

Suppose that $f''(x)$ exists on the interval (a, b).

1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).
2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

3.8 Evaluating Derivatives

For more details on evaluating derivatives of specific functions, see the later section.

The derivative of any constant function is 0.

$$\frac{\mathrm{d}}{\mathrm{d}x} a = 0$$

Constant coefficients can be factored out of derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x} A f(x) = A \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

The derivative of a sum is the sum of the derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x} [f(x) + g(x)] = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{\mathrm{d}}{\mathrm{d}x} g(x)$$

3.8.1 Power Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$$

3.8.2 Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} f \cdot g = f \cdot g' + g \cdot f'$$

3.8.3 Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f}{g} = \frac{g \cdot f' - f \cdot g'}{g^2}$$

3.8.4 Chain Rule

The derivative of a composite function with respect to x is equal to the derivative of the product of the derivative of the outer function with respect to the inner function and the derivative of the inner function.

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x)$$

3.8.5 Implicit Differentiation

When an equation in y and x is not written in the form $y = f(x)$, we say the function is defined implicitly. To differentiate an implicit function,

1. From the given equation construct a function $F(x, y) = 0$
2. Use the formula

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

   In the numerator, differentiate F with respect to x, treating y as constant, and in the denominator, differentiate F with respect to y, treating x as constant.

Example: Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ of $y = x^2 + 3\sin y$.

$$F(x, y) = y - x^2 - 3\sin y = 0$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} &= -\frac{-2x}{1 - 3\cos y} \\ &= \frac{2x}{1 - 3\cos y} \end{aligned}$$

3.9 Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on $(a.b)$ such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

Note that this is a special case of the Mean Value Theorem.

3.10 Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \Rightarrow f(b) - f(a) = f'(c)(b - a)$$

3.11 Extreme Value Theorem

If f is continuous on $[a, b]$, then $f(x)$ has both a maximum and a minimum on $[a, b]$.

3.12 Linear Approximation

The linear approximation to $f(x)$ near $x = x_0$ is given by

$$y = f(x_0) + f'(x_0)(x - x_0)$$

for x sufficiently close to x_0.

3.13 Newton's Method

Let f be a differentiable function and suppose r is a real zero of f. If x_n is an approximation to r, then the next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided $f'(x_n) \neq 0$. Successive approximations can be found using this method.

3.14 L'Hôpital's Rule

If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

4 Integrals

4.1 Area Approximation Methods

There are a few similar ways to approximate the area under a curve. They all involve dividing the total area into smaller sections which approximate the shape of the enclosed region but which have easily computable areas. The interval on the x-axis $[a, b]$ is typically divided into n subintervals, each with length $\frac{b-a}{n} = \Delta x$.

4.1.1 Left Riemann Sum

$$\begin{aligned}
\text{Area} &\approx \Delta x f(a) + \Delta x f(a + \Delta x) + \Delta x f(a + 2\Delta x) + \ldots + \Delta x f(b - \Delta x) \\
&\approx \Delta x[f(a) + f(a + \Delta x) + f(a + 2\Delta x) + \ldots + f(b - \Delta x)] \\
&\approx \Delta x \sum_{k=0}^{n-1} f(a + k\Delta x)
\end{aligned}$$

This approximation will be an underestimation if f is strictly increasing on $[a, b]$, and an overestimation if f is strictly decreasing.

4.1.2 Right Riemann Sum

$$\begin{aligned}\text{Area} &\approx \Delta x f(a+\Delta x) + \Delta x f(a+2\Delta x) + \ldots + \Delta x f(b)\\ &\approx \Delta x[f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(b)]\\ &\approx \Delta x \sum_{k=1}^{n} f(a+k\Delta x)\end{aligned}$$

This approximation will be an overestimation if f is strictly increasing on $[a, b]$, and an underestimation if f is strictly decreasing.

4.1.3 Midpoint Rule

$$\begin{aligned}\text{Area} &\approx \Delta x f\left(a+\frac{\Delta x}{2}\right) + \Delta x f\left(a+\frac{3\Delta x}{2}\right) + \ldots + \Delta x f\left(b-\frac{\Delta x}{2}\right)\\ &\approx \Delta x\left[f\left(a+\frac{\Delta x}{2}\right) + f\left(a+\frac{3\Delta x}{2}\right) + \ldots + f\left(b-\frac{\Delta x}{2}\right)\right]\\ &\approx \Delta x \sum_{k=1}^{n} f\left(a+\frac{(2n-1)\Delta x}{2}\right)\end{aligned}$$

4.1.4 Trapezoidal Rule

$$\text{Area} \approx$$

4.2 Definition of the Definite Integral as the Limit of a Sum

Lorem Ipsum