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Efficient L_2 Approximation by Splines

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Abstract. Let $S_N^k(t)$ be the linear space of k-th order splines on $[0,1]$ having the simple knots t_i determined from a fixed function t by the rule $t_i = t(i/N)$. In this paper we introduce sequences of operators $\{Q_N\}_{N=1}^{\infty}$ from $C^k[0,1]$ to $S_N^k(t)$ which are computationally simple and which, as $N \to \infty$, give essentially the best possible approximations to f and its first $k-1$ derivatives, in the norm of $L_2[0,1]$. Precisely, we show that $N^{k-i}(\|(f-Q_N f)^{(i)}\| - \mathrm{dist}_2(f^{(i)}, S_N^{k-i}(t))) \to 0$ for $i = 0, 1, \ldots, k-1$. Several numerical examples are given.

Subject Classifications: primary 41 A 15, secondary 41 A 50.

1. Introduction

This paper examines a method of approximating smooth functions by splines which is simultaneously computationally simple and essentially best possible. To illustrate the method we will present in the Introduction an example of approximation by linear splines. We begin with some notation and background information that will be necessary to explain the method.

The linear space of spline functions $S_N^k(t)$ was introduced by the authors in [1] as a means of investigating approximation by splines having a large number of variable knots. This space is defined as follows: let t be a function in $C^1[0,1]$ satisfying

$$t' \geqq \delta > 0, \qquad t(0) = 0, \qquad t(1) = 1; \tag{1.1}$$

let N and k be positive integers; then $S_N^k(t) = \{$splines of order k having the simple knots $t_i = t(i/N)$, $i = 0, 1, \ldots, N\}$. That is, $s \in S_N^k(t)$ provided $s \in C^{k-2}[0,1]$

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and s restricted to each subinterval (t_i, t_{i+1}) is a polynomial of degree at most $k-1$. An essential feature of these spaces is that, as N increases, the knots are increased in such a regular way as to permit a fairly precise description of the approximation properties of the spaces $S_N^k(t)$. For example, as regards $L_2[0,1]$ approximation, we have the following result from [1].

Theorem A. *Let* $f \in C^k[0,1]$. *Then*

$$\lim_{N\to\infty} N^k \operatorname{dist}_2(f, S_N^k(t)) = \left(\frac{|B_{2k}|}{(2k)!}\int_0^1 |f^{(k)}(t(x))|^2 (t'(x))^{2k+1}\,dx\right)^{1/2}, \tag{1.2}$$

where B_{2k} *denotes the* $2k$*-th Bernoulli number.*

As discussed in [1], one can now choose the *knot distribution function* t to minimize the right side of (1.2) and obtain asymptotic estimates for the error of best L_2 approximation by splines with *variable* knots. This paper, however, will not treat the variable knot case, but will assume the function t to be fixed. We will discuss certain approximation schemes which will be shown, via Theorem A, to be asymptotically best possible.

Since an uncerstanding of the Bernoulli polynomials $B_k(\cdot)$ is essential to the rest of the paper, we will briefly review some of their properties. They can be defined inductively as follows:

$$B_0(x)=1, \qquad B_k(x)=x^k+\ldots+B_k=\int_0^x kB_{k-1}(x)\,dx+B_k,$$

where the constant B_k, called the k-th Bernoulli number, is chosen so that $\int_0^1 B_k(x)\,dx=0$. One can readily check that the first few Bernoulli polynomials are $B_1(x)=x-1/2$, $B_2(x)=x^2-x+1/6$, $B_3(x)=x^3-(3/2)x^2+(1/2)x$. For reference, we list the following properties, for $k\geqq 1$:

$$\int_0^1 B_k(x)\,dx=0; \tag{1.3}$$

$$B_k'(x)=kB_{k-1}(x); \tag{1.4}$$

$$B_k^{(j)}(0)=B_k^{(j)}(1), \qquad j=0,1,\ldots,k-2 \quad \text{and} \quad k\geqq 2. \tag{1.5}$$

A Bernoulli *monospline* [cf. 6] having the knot sequence $t_i=i$, for $i=0,1,\ldots,N$, is a function $\overline{B_k}(\tau)$ such that for $t_i\leqq\tau<t_{i+1}$, $\overline{B_k}(\tau)=B_k(\tau-t_i)$. According to (1.5), $\overline{B_k}(\cdot)\in C^{k-2}$ and hence is of the form $B_k(\tau)=\tau^k+$(*a* k-th order spline with simple knots). Schoenberg has shown in [6] that such a monospline is the error function for the best approximation (in $L_2[0,N]$) to τ^k among all k-th order splines having the knots $\{i\}_{i=0}^N$ and with the error satisfying periodic boundary conditions. This fact led us to the conjecture that (1.2) was true, where we supposed the error $f-P_Nf$ (P_N is the orthogonal projection onto $S_N^k(t)$) to have approximately the shape, on each subinterval $[t_i,\ t_{i+1})$, of a scaled Bernoulli polynomial. Precisely, we showed that

$$\lim_{N\to\infty} N^k \|(f-P_N f)-M_N\|=0$$

where (with $h_i=t_{i+1}-t_i$)

$$M_N(\tau)=h_i^k B_k\left(\frac{\tau-t_i}{h_i}\right) f^{(k)}(t_i)/k!, \qquad t_i \leqq \tau < t_{i+1}.$$

Here and subsequently, $\|f\|$ means $\left(\int_0^1 f^2(\tau)\,d\tau\right)^{1/2}$. The above considerations motivated us to seek easily computable approximation schemes which, on small subintervals of $[0,1]$ and for N large, would have an error function with nearly the shape of a Bernoulli monospline.

Before giving our general results in Sect. 2, we first consider the following illustration. Let a function t satisfying (1.1) be given, and for given positive integers k and N, let $t_i=t(i/N)$, $i=-k+1,\ldots,N+k-1$; we have assumed t has been extended outside the interval $[0,1]$, for convenience. (Throughout this paper we will assume that knot sequences have been determined in this manner from a fixed function t.) Now let $k=2$, and let $\{N_{i,2}(\cdot)\}_{i=-1}^{N-1}$ be the basis for $S_N^2(t)$ of normalized B-splines (we will assume the reader is familiar with the concepts of B-splines as discussed for example in [2]).

Let $f\in C^2[0,1]$. We seek approximations

$$Q_N f=\sum_{i=-1}^{N-1} a_i N_{i,2}$$

which will satisfy $\lim_{N\to\infty} N^2[\|f-Q_N f\|-\operatorname{dist}_2(f,S_N^2(t))]=0$. Let t_i be in $[0,1)$. Then on the interval $[t_i,t_{i+1})$, $Q_N f$ depends only on the coefficients a_{i-1} and a_i, since only $N_{i-1,2}$ and $N_{i,2}$ are non-zero in that interval. Let $\bar{f}(\tau)=f(t_i)+f'(t_i)(\tau-t_i)+f''(t_i)(\tau-t_i)^2/2$. We seek to determine a_{i-1} and a_i so that on $[t_i, t_{i+1})$,

$$(\bar{f}-Q_N f)(\tau)=\frac{f''(t_i)h_i^2}{2}B_2\left(\frac{\tau-t_i}{h_i}\right)=\frac{f''(t_i)h_i^2}{2}\left[\left(\frac{\tau-t_i}{h_i}\right)^2-\left(\frac{\tau-t_i}{h_i}\right)+\frac{1}{6}\right].$$

Solving for $Q_N f(\tau)$, we obtain

$$Q_N f(\tau)=(f'(t_i)+f''(t_i)h_i/2)(\tau-t_i)+(f(t_i)-f''(t_i)h_i^2/12). \tag{1.6}$$

Now, on $[t_i,t_{i+1})$, we have

$$\tau-t_i=h_i N_{i,2}(\tau) \quad\text{and}\quad 1=N_{i-1,2}(\tau)+N_{i,2}(\tau). \tag{1.7}$$

Substituting (1.7) into (1.6) yields

$$a_{i-1}=f(t_i)-f''(t_i)h_i^2/12, \tag{1.8}$$

$$\begin{aligned} a_i&=h_i(f'(t_i)+f''(t_i)h_i/2)+f(t_i)-f''(t_i)h_i^2/12\\ &=f(t_{i+1})-f''(t_{i+1})h_{i+1}^2/12+o(h^2), \end{aligned} \tag{1.9}$$

where $h = \min h_j$. We choose, for reasons of symmetry, to replace (1.9) by

$$a_i = f(t_{i+1}) - f''(t_{i+1})\, h_{i+1}^2/12, \tag{1.9'}$$

which is the expression we would have obtained for a_i by applying the above procedure on the interval $[t_{i+1}, t_{i+2})$. Moreover, since $0 \leqq N_{i,2}(t) \leqq 1$, such a modification in a_i causes only a $o(h^2)$ change in $Q_N f$, and hence will not affect the asymptotic behavior we are seeking. In fact, we may even replace the second derivative in (1.9) by a difference approximation of sufficient accuracy that a_i is again perturbed by $o(h^2)$. Such a modification yields

$$a_i = (7f(t_{i+1}) - (h_{i+1} f(t_i) + h_i f(t_{i+2}))/(h_i + h_{i+1}))/6. \tag{1.9''}$$

We will use the equation (1.9″) to determine all the coefficients of $Q_N f$. The results of a numerical computation using this *local* spline approximation method are given in Table 1. For these examples we are using $f_1(\tau) = \tau^2/2$ with knot distribution function $t_1(x) = (x^2 + x)/2$, and $f_2(\tau) = \tau^4/4!$ with $t_2(x) = x^2$.

The numbers in the columns headed by $N^2 \|f_i - Q_N f_i\|$ are seen to be converging to the number in the row labeled "theoretical limit", which is

$$\left(\frac{|B_4|}{4!} \int_0^1 |f_i''(t_i(x))|^2\, |t_i'(x)|^5\, dx\right)^{1/2} = \lim N^2 \operatorname{dist}_2(f_i, S_N^2(t_i)).$$

In addition, the other two columns show that the difference of the corresponding first derivatives is tending in norm to

$$\left(\frac{|B_2|}{2!} \int_0^1 |f_i''(t_i(x))|^2\, |t_i'(x)|^3\, dx\right)^{1/2} = \lim N \operatorname{dist}_2(f_i', S_N^1(t_i)).$$

Thus, not only is $Q_N f_i$ an asymptotically best possible approximation to f_i, but $(Q_N f_i)'$ is likewise an asymptotically best possible approximation to f_i'! This phenomenon is a consequence of the fact that the derivative of a Bernoulli monospline gives (a multiple of) the Bernoulli monospline of next lower order, as follows from (1.4). It should also be noted that the function $t_2(x)$ doesn't satisfy (1.1); the numerical example therefore suggests that the hypotheses of Theorem A, and also of the theorems given in this paper, might be relaxed somewhat.

Table 1

N	$N^2 \|f_1 - Q_N f_1\|$	$N \|f_1' - (Q_N f_1)'\|$	$N^2 \|f_2 - Q_N f_2\|$	$N \|f_2' - (Q_N f_2)'\|$
4	0.052173	0.32342	0.040596	0.12103
8	0.051533	0.32292	0.030239	0.11812
16	0.051369	0.32279	0.028591	0.11789
32	0.051328	0.32276	0.028270	0.11786
64	0.051317	0.32275	0.028196	0.11785
128	0.051315	0.32275	0.028178	0.11785
Theoretical limit	0.051314	0.32275	0.028172	0.11785

In Sect. 2 we will discuss generalizations of the above example, and prove the convergence we have observed numerically. We conclude in Sect. 3 with a derivation of formulas analogous to (1.9′) in the cases of quadratic and cubic splines. Numerical examples using these schemes are also given.

2. k-Efficient Approximation Methods

In this section we discuss certain local spline approximations to a function f in $C^k[0,1]$ by elements of $S_N^k(t)$. These approximations will be shown to be k-*efficient*, in the following sense.

Definition. Let $\{A_N\}_{N=1}^{\infty}$ be a sequence of operators, not necessarily linear, from $C^k[0,1]$ to $S_N^k(t)$. This sequence will be called *k-efficient* provided that for each $f \in C^k[0,1]$ and for $i=0,1,\dots,k-1$,

$$\lim_{N\to\infty} N^{k-i}[\|f^{(i)}-(A_N f)^{(i)}\| - \operatorname{dist}_2(f^{(i)}, S_N^{k-i}(t))]=0. \tag{2.1}$$

We next construct a sequence of operators which will be shown to be k-efficient. Let $B_k(\tau)=\tau^k+\sum_{j=0}^{k-1} b_{j,k}\tau^j$ be the k-th Bernoulli polynomial. Let $\{N_{l,k}\}_{l=1-k}^{N-1}$ be the normalized B-spline basis for $S_N^k(t)$. For notational simplicity we will delete the explicit reference to the fixed integer k, letting $b_j=b_{j,k}$ and $N_l=N_{l,k}$. Let α be a fixed integer, $0\leqq\alpha<k$. For certain linear functionals $\{\lambda_{l,N}^{\alpha}\}_{l=-k+1}^{N-1}$ to be defined presently, we introduce the family of operators $\{Q_N^{\alpha}\}$ defined by

$$Q_N^{\alpha} f=\sum_{l=-k+1}^{N-1} \lambda_{l,N}^{\alpha}(f)\, N_l \equiv \sum_l a_l^{\alpha} N_l. \tag{2.2}$$

The linear functional $\lambda_{l,N}^{\alpha}$ will have support near the knot t_l, and hence the Q_N^{α} will be *local*.

Let i be an integer, $0\leqq i<N$, such that $i=\alpha+mk$ for some integer m. On the interval $I_i=[t_i,t_{i+1})$, we have

$$f(\tau)=\sum_{j=0}^{k} f^{(j)}(t_i)(\tau-t_i)^j/j!+o(h^k)\equiv \bar f(\tau)+o(h^k). \tag{2.3}$$

Denote $f^{(j)}(t_i)/j!$ by f_i^j. The spline $Q_N^{\alpha} f$ will be characterized by the requirement that on I_i,

$$(\bar f - Q_N^{\alpha} f)(\tau)\equiv E_i(\tau)=f_i^k B_k((\tau-t_i)/h_i)\, h_i^k. \tag{2.4}$$

Hence, on I_i,

$$\begin{aligned} Q_N^{\alpha} f(\tau) &= \sum_{l=i-k+1}^{i} a_l^{\alpha} N_l(\tau)=\bar f(\tau)-E_i(\tau)\\ &=\sum_{j=0}^{k-1}(f_i^j-f_i^k h_i^{k-j} b_j)(\tau-t_i)^j. \end{aligned} \tag{2.5}$$

Now, on I_i,

$$(\tau - t_i)^j = \sum_{l=i-k+1}^{i} \xi_{l,i}^{j+1} N_l(\tau), \tag{2.6}$$

where $\xi_{l,i}^j = (-1)^{j-1} \dfrac{(j-1)!}{(k-1)!} \phi_l^{(k-j)}(t_i)$ and $\phi_l(\tau) = \prod_{s=1}^{k-1} (\tau - t_{l+s})$. The identity (2.6) follows from [2, (19)]:

$$(s-\tau)^{k-1} = \sum_l \phi_l(s) N_l(\tau). \tag{2.7}$$

Substituting (2.6) into (2.5), we obtain

$$a_l^\alpha = \sum_{j=0}^{k-1} \xi_{l,i}^{j+1} (f_i^j - f_i^k b_j h_i^{k-j}), \qquad l = i-k+1, \ldots, i. \tag{2.8}$$

As i takes on sufficiently many of the values such that $i \equiv \alpha \pmod k$, then (2.8) will determine all the coefficients a_l^α occuring in (2.2).

To give an indication of why one might expect such a scheme to produce the desired error, we make the following observation. Suppose $p(\tau) = \tau^k + s(\tau)$ is a monospline of order k having simple knots at the integers. Assume that the coefficients of the spline $s(\cdot)$, with respect to the basis of normalized B-splines, have been determined by the requirement that on intervals of the form $[jk, jk+1)$, $p(\tau) = B_k(\tau - jk)$. Then it must be that $p(\tau) = \bar{B}_k(\tau)$, the Bernoulli monospline, and hence $p(\tau) = B_k(\tau - j)$ on *every* interval $[j, j+1)$.

We now state and prove the central theorem of this paper.

Theorem 1. *The sequence of operators $\{Q_N^\alpha\}_{N=1}^\infty$ is k-efficient.*

According to Theorem A, this theorem is equivalent to the statement that for any f in $C^k[0,1]$ and for $i = 0, 1, \ldots, k-1$,

$$\lim_{N\to\infty} N^{k-i} \|(f - Q_N^\alpha f)^{(i)}\| = \left(\frac{|B_{2(k-i)}|}{(2(k-i))!} \int_0^1 |f^{(k)}(t(x))|^2 (t'(x))^{2(k-i)+1} \, dx \right)^{1/2}. \tag{2.9}$$

The proof of the theorem will follow easily from the next Lemma. We use the notation $\|g\|_{[a,b]}$ for the supremum norm on the interval $[a,b]$.

Lemma 1. *Let $f \in C^k[0,1]$, and let $0 \leqq \bar{t} < 1$. Let j be chosen so that $t_j \leqq \bar{t} < t_{j+1}$ and let $h_j = t_{j+1} - t_j$. With $x = t^{-1}(\bar{t})$, let*

$$R_N(\tau; \bar{t}) = N^k (f - Q_N^\alpha f)(t_j + \tau h_j),$$

and

$$P(\tau; \bar{t}) = (t'(x))^k f^{(k)}(\bar{t}) \bar{B}_k(\tau)/k!$$

Then there exists a sequence of positive constants $\{\varepsilon_N\}_{N=1}^{\infty}$ *tending to zero, and which may be chosen independently of* $\bar{t}$, *such that*

$$\|(R_N(\cdot;\bar{t})-P(\cdot;\bar{t}))^{(i)}\|_{[0,1]}<\varepsilon_N$$

for $i=0,1,\ldots,k-2$. *The above inequality is also true for* $i=k-1$ *if the norm is changed to that of* $L_2[0,1]$.

The Lemma says, in essence, that for N sufficiently large, the error function $f-Q_N^{\alpha}f$ is nearly equal to a properly scaled Bernoulli polynomial on each subinterval $[t_j,t_{j+1})$. Assuming for the present the truth of Lemma 1, we complete the proof of the Theorem. Fix an integer i, $0\leqq i\leqq k-1$. Then

$$\begin{aligned}N^{2(k-i)}\|(f-Q_N^{\alpha}f)^{(i)}\|^2&=N^{2(k-i)}\int_0^1((f-Q_N^{\alpha}f)^{(i)}(\tau))^2\,d\tau\\&=N^{-2i}\sum_{j=0}^{N-1}\int_0^1(R_N^{(i)}(\tau;t_j))^2\,h_j^{1-2i}\,d\tau.\end{aligned}$$

By Lemma 1, this equals

$$\begin{aligned}&N^{-1}\sum_{j=0}^{N-1}\{(Nh_j)^{1-2i}[(t'(j/N))^k f^{(k)}(t(j/N))]^2\\&\quad\cdot\left(\int_0^1(B_k^{(i)}(\tau)/k!)^2\,d\tau+r_{j,N}\right)\Bigg\},\end{aligned}$$

where $|r_{j,N}|<C\varepsilon_N$, for some constant C depending only on k and f. We next note that $Nh_j=N(t_{j+1}-t_j)=N(t((j+1)/N)-t(j/N))=t'(\xi_j)=t'(j/N)+s_{j,N}$, where $|s_{j,N}|\leqq\omega(N^{-1},t')$ and $\omega(\cdot,t')$ is the modulus of continuity of t'. Hence, we have

$$\begin{aligned}&N^{2(k-i)}\|(f-Q_N^{\alpha}f)^{(i)}\|^2\\&\quad=\left[\int_0^1(B_k^{(i)}(\tau)/k!)^2\,d\tau\right]\frac{1}{N}\sum_{j=0}^{N-1}(t'(j/N))^{2k-2i+1}\\&\qquad\cdot[f^{(k)}(t(j/N))]^2+o(1).\end{aligned}$$

Theorem 1 now follows, since the sum is a Riemann sum for the integral in (2.9), and

$$\int_0^1(B_k^{(i)}(\tau)/k!)^2\,d\tau=|B_{2(k-i)}|/(2(k-i))!,$$

as follows from integrating by parts.

We now turn to the proof of Lemma 1. It will be convenient to replace f by its Taylor expansion $\bar{f}(\tau;\bar{t})\equiv\sum_{j=0}^{k}f^{(j)}(\bar{t})(\tau-\bar{t})^j/j!$. To see that such a replacement is legitimate we argue as follows. First, since $f^{(k)}$ is continuous, it can be shown easily by induction that for $i=0,1,\ldots,k$,

$$(f(\tau)-\bar{f}(\tau;\bar{t}))^{(i)}=o(|\tau-\bar{t}|^{k-i})\quad\text{as}\quad\tau\to\bar{t},\tag{2.10}$$

uniformly in $\bar{t}$. Hence,

$$\begin{aligned}N^k(f-Q_N^\alpha f)(t_j+\tau h_j)&=N^k(\bar{f}-Q_N^\alpha \bar{f})(t_j+\tau h_j)\\&\quad+N^k((f-\bar{f})-Q_N^\alpha(f-\bar{f}))(t_j+\tau h_j)\\&=N^k(\bar{f}-Q_N^\alpha \bar{f})(t_j+\tau h_j)+o(1)\end{aligned}$$

as $N\to\infty$, for $-L\leqq\tau\leqq L$, where L is fixed but arbitrary. The fact that $Q_N^\alpha(f-\bar{f})$ is appropriately small follows from (2.8) and (2.10), if we first note that $\xi_{l,i}^{j+1}=O(h^j)$, uniformly in l.

We have just shown that without loss of generality (as far as Lemma 1 is concerned), we may assume that f is a polynomial of degree at most k. Hence, both R_N and P are monosplines of order k (i.e., of the form $a\tau^k+s(\tau)$, s a spline). P has knots at the integers and R_N has knots nearly at the integers for large N. In addition, in every k-th knot interval R_N is exactly a k-th degree Bernoulli polynomial having nearly the same leading term $(N^k h_j^k f^{(k)}(\bar{t})\,\tau^k/k!)$ as P has $((t'(x))^k f^{(k)}(\bar{t})\,\tau^k/k!)$. Consequently, it should come as no surprise that R_N and P are close on $[0,1]$.

Two technical lemmas are needed for the proof. These lemmas will say precisely what is needed for a sequence of splines with distinct knot sequences to converge to a given spline.

Lemma 2. *Let* $\underline{\tau}:=\tau_{-k+1}<\ldots<\tau_k$ *be a knot sequence, and let* $\underline{\tau}^N:=\tau_{-k+1}^N<\ldots<\tau_k^N$ *be other knot sequences satisfying* $\tau_l^N\to\tau_l$ *as* $N\to\infty$. *Let*

$$S=\sum_{l=-k+1}^{0} A_l N_{l,k}(\underline{\tau}) \tag{2.11}$$

and

$$S_N=\sum_{l=-k+1}^{0} A_l^N N_{l,k}(\underline{\tau}^N) \tag{2.12i}$$

be splines with knot sequences $\underline{\tau}$ *and* $\underline{\tau}^N$, *respectively. Suppose that* $[\alpha,\beta]\subset(\tau_0,\tau_1)$ *and* $\|S_N-S\|_{[\alpha,\beta]}\to 0$ *as* $N\to\infty$. *Then there is a positive sequence* $\{\delta_N\}_{N=1}^\infty$ *tending to zero such that* $|A_l^N-A_l|<\delta_N$, $l=-k+1,\ldots,0$, *and each* δ_N *depends only on* α, β, $\|S-S_N\|_{[\alpha,\beta]}$, $\max_l|\tau_l-\tau_l^N|$, k *and* $\underline{\tau}$.

Lemma 3. *Let* $\underline{\tau}$, $\underline{\tau}^N$, S, S_N *be as in Lemma 2. Suppose that* $|A_l^N-A_l|<\delta_N$ *for* $l=-k+1,\ldots,k$ *and some sequence* $\{\delta_N\}$ *tending to zero. Then there is a sequence* ε_N *tending to zero such that*

$$\|(S_N-S)^{(i)}\|_{[\tau_0,\tau_1]}<\varepsilon_N,\qquad i=0,1,\ldots,k-2,$$

and

$$\|(S_N-S)^{(k-1)}\|_{L_2[\tau_0,\tau_1]}<\varepsilon_N.$$

Each ε_N *depends only on* δ_N, $\max|\tau_l^N-\tau_l|$, k *and* $\underline{\tau}$.

Lemma 3 is an easy consequence of the differentiation formulas given in [2]. As for Lemma 2, proofs may be readily constructed from information found in [4, 5], especially [4, p. 127–128].

We are now prepared to prove Lemma 1. As noted earlier, we can assume that f is a polynomial of degree at most k, $f(\tau)=\sum_{j=0}^{k} f^{(j)}(\bar{t})(\tau-\bar{t})/j!$. Hence

$$\begin{aligned} R_N(\tau;\bar{t}) &= [R_N(\tau;\bar{t})-f^{(k)}(\bar{t})(N h_j)^k \tau^k/k!] \\ &\quad -[P(\tau;\bar{t})-(t'(x))^k f^{(k)}(\bar{t})\tau^k/k!]-f^{(k)}(\bar{t})\tau^k[(t'(x))^k-(N h_j)^k] \\ &\equiv S_N(\tau;\bar{t})-S(\tau;\bar{t})+r(\tau;\bar{t}). \end{aligned} \tag{2.13}$$

The first bracketed term is a spline of order k with knots nearly at the integers, while the second term has knots at the integers. The third term goes to zero, uniformly in $\bar{t}$, for bounded τ. We will show that the first spline converges to the second by showing that the left side of (2.13) tends to zero on certain intervals. Recall that Q_N^α is defined by the requirement that $f-Q_N^\alpha f$ is a (scaled) Bernoulli polynomial on subintervals of the form $[t_{mk+\alpha}, t_{mk+\alpha+1})$. For each N, let $\bar{j}\equiv mk+\alpha$ be the largest such integer so that $\bar{j}\leqq j$; (recall that j was chosen so that $t(j/N)\leqq\bar{t}<t((j+1)/N)$). Clearly $j-\bar{j}$ may take any of the values $0,1,\ldots,k-1$. However, it will be notationally convenient to assume that $j-\bar{j}\equiv i=$constant, independent of N. This assumption involves no loss of generality, since it could be achieved by passing to subsequences. Denote the knots of R_N by $\underline{\tau}^N$, where $\tau_0^N=0$, and those of P by $\underline{\tau}$, where $\tau_n=n$. It is clear that $\tau_n^N\to\tau_n$ as $N\to\infty$, uniformly for n bounded.

Now, from the way $\bar{j}$ and $i=j-\bar{j}$ where chosen, we have that R_N is a (scaled) Bernoulli polynomial on the intervals $[\tau_{-i}^N,\tau_{-i+1}^N)$ and $[\tau_{-i+k}^N,\tau_{-i+k+1}^N)$. Furthermore, there are fixed intervals (α,β), (γ,δ) so that for sufficiently large N, $(\alpha,\beta)\subset[\tau_{-i}^N,\tau_{-i+1}^N)\cap[-i,-i+1)$ and $(\gamma,\delta)\subset[\tau_{-i+k}^N,\tau_{-i+k+1}^N)\cap[-i+k,-i+k+1)$. On these two intervals, R_N converges to P in the uniform norm, and the convergence is uniform in $\bar{t}$. Writing

$$S_N(\cdot;\bar{t})=\sum A_l^N N_{l,k}(\underline{\tau}^N)$$

and

$$S(\cdot;\bar{t})=\sum A_l N_{l,k}(\underline{\tau}),$$

we conclude via Lemma 2 that $(A_{-i-k+1}^N,\ldots,A_{-i+k}^N)\to(A_{-i-k+1},\ldots,A_{-i+k})$, uniformly in $\bar{t}$. Hence regardless of the value of i, $0\leqq i\leqq k-1$, we have $(A_{-k+1}^N,\ldots,A_0^N)\to(A_{-k+1},\ldots,A_0)$, uniformly in $\bar{t}$; Lemma 3 now yields the conclusion of Lemma 1, and the proof of Theorem 1 is complete.

We conclude this section with a theorem which states in essence that the k-efficient property is stable with respect to small perturbations of the B-spline coefficients. This result will be used in Sect. 3 where we construct and use certain k-efficient operators; the example in the Introduction used such perturbations to obtain a k-efficient family of operators for which the derivatives of the function being approximated, used in the calculation of the B-spline coefficients, were replaced by difference approximations. We begin with the following

Lemma 4. *Let $\{P_N^i\}_{N=1}^{\infty}$, $i=1,2$, be sequences of mappings from $C^k[0,1]$ into $S_N^k(t)$, with $P_N^i f=\sum_l b_{l,N}^i N_{l,k}$. Let $\underline{b}_N^i$ denote the coefficient sequence $\{b_{l,N}^i\}$, $i=1,2$. Suppose that $\{P_N^1\}$ is a k-efficient sequence. Then $\{P_N^2\}$ is a k-efficient sequence if and only if for each $f\in C^k[0,1]$,*

$$\lim_{N\to\infty} N^k \|\underline{b}_N^1-\underline{b}_N^2\|=0. \tag{2.14}$$

The sequence norm used above is $\|\underline{a}\|^2=\sum_l a_l^2(t_{l+k}-t_l)/k$.

Proof. As shown in [3], there is a positive constant C_k depending only on k such that

$$C_k\|\underline{a}\|\leqq\|\sum_l a_l N_{l,k}\|\leqq\|\underline{a}\|. \tag{2.15}$$

First, assume $\{P_N^2\}$ is k-efficient. Let P_N be the operator of orthogonal projection onto $S_N^k(t)$, and let $f\in C^k[0,1]$. Then by orthogonality, $\|f-P_N^i f\|^2=\|f-P_N f\|^2+\|P_N f-P_N^i f\|^2$, and hence we deduce using (2.1) that $\|P_N f-P_N^i f\|=o(N^{-k})$, from which follows $N^k\|P_N^1 f-P_N^2 f\|\to 0$ as $N\to\infty$. Hence, by (2.15), $N^k C_k\|\underline{b}_N^1-\underline{b}_N^2\|\leqq N^k\|P_N^1 f-P_N^2 f\|\to 0$ and (2.14) holds.

Now let (2.14) hold for an arbitrary f in $C^k[0,1]$. By definition, this means

$$N^{2k}\sum_l (b_{l,N}^1-b_{l,N}^2)^2(t_{l+k}-t_l)/k\to 0.$$

We claim that

$$N^{k-1}\|(P_N^1 f)'-(P_N^2 f)'\|\to 0. \tag{2.16}$$

Let $(P_N^i f)'=\sum_l b_{l,N}^{i,1} N_{l,k-1}$. According to [2, (13)],

$$b_{l,N}^{i,1}=(b_{l,N}^i-b_{l-1,N}^i)/(t_{l+k-1}-t_l).$$

Hence

$$\begin{aligned} N^{2k-2}\|\underline{b}_N^{1,1}-\underline{b}_N^{2,1}\|^2 &= N^{2k}\sum_l \frac{(b_{l,N}^1-b_{l-1,N}^1-b_{l,N}^2+b_{l-1,N}^2)^2}{N^2(t_{l+k-1}-t_l)^2}(t_{l+k-1}-t_l)/(k-1)\\ &\leqq 2N^{2k}\sum \frac{((b_{l,N}^1-b_{l,N}^2)^2+(b_{l-1,N}^1-b_{l-1,N}^2)^2)}{N^2(t_{l+k-1}-t_l)^2}\\ &\quad\cdot\frac{(t_{l+k}-t_l)}{k}\left(\frac{k}{k-1}\right)\left(\frac{t_{l+k-1}-t_l}{t_{l+k}-t_l}\right)\to 0 \end{aligned}$$

as $N\to\infty$. Thus, using (2.15) with k replaced by $k-1$, (2.16) follows. By similar reasoning, we can prove that $N^{k-j}\|(P_N^1 f-P_N^2 f)^{(j)}\|\to 0$ for $j=0,1,\dots,k-1$ and this implies that $\{P_N^2\}$ is k-efficient. This completes the proof of Lemma 4. We are now prepared to prove

Theorem 2. *Let $\{Q_N\}$ be a sequence of mappings from $C^k[0,1]$ into $S_N^k(t)$, and let $\{P_N\}$ denote the orthogonal projections. For f in $C^k[0,1]$, let*

$$Q_N f = \sum_l q_l N_{l,k}$$

and

$$P_N f = \sum_l p_l N_{l,k}.$$

Then $\{Q_N\}$ is a k-coefficient sequence if and only if for each $f \in C^k[0,1]$,

$$\lim_{N\to\infty} N^k \|\underline{p} - \underline{q}\| = 0. \tag{2.17}$$

In particular, $\{P_N\}$ is k-efficient.

Proof. If $\{Q_N\}$ is k-efficient, we have for each f,

$$N^{2k}\|Q_N f - P_N f\|^2 = N^{2k}\|f - Q_N f\|^2 - N^{2k}\|f - P_N f\|^2 \to 0.$$

Hence, using (2.15), (2.17) holds. We can now establish that $\{P_N\}$ is k-efficient. In Lemma 4, let $\{P_N^1\} = \{Q_N^\alpha\}$, a k-efficient sequence by Theorem 1, and let $\{P_N^2\} = \{P_N\}$. Then by the above, we see that (2.14) holds, and hence $\{P_N\}$ is k-efficient by Lemma 4.

For the converse, let (2.17) hold. Then, again by Lemma 4, $\{Q_N\}$ is k-efficient, and the proof of Theorem 2 is complete.

3. Some Examples

In this section we will derive some specific formulas for k-efficient approximation, then apply them to a numerical example. The methods will be slight perturbations of the Q_N^α discussed in Sect. 2. The modifications are based on the intuitive notion that one could obtain a better approximation than that given by Q_N^α by "spreading out" the information obtained from f, rather than using information concentrated in the intervals $[t_i, t_{i+1})$, where $i \equiv \alpha \pmod k$. We propose the operator Q_N: $C^k[0,1] \to S_N^k(t)$ defined by

$$Q_N f = \sum_{l=-k+1}^{N-1} a_l N_{l,k}, \tag{3.1}$$

where a_l is the l-th coefficient of Q_N^α, where $\alpha \equiv (l + [k/2]) \bmod k$. (Here $[x]$ denotes the greatest integer less than or equal to x.) In other words, each a_l is given by (2.8), where i is related to l by

$$i = l + [k/2]. \tag{3.2}$$

Thus, each a_l will depend on information taken from f near the center of the support of $N_{l,k}$. That such a scheme will still yield a k-efficient method follows from Theorem 2.

We now derive the formulae for the Q_N (i.e., the a_l) for $k=3$ and 4.

$k=3$. $B_3(\tau)=\tau^3-\frac{3}{2}\tau^2+\frac{1}{2}\tau$, so $b_0=0$, $b_1=\frac{1}{2}$, $b_2=-\frac{3}{2}$. For a given i, we have $l=i-[k/2]=i-1$, and $\phi_l(\tau)=(\tau-t_i)(\tau-t_{i+1})$. A calculation gives $\xi^1_{l,i}=1$, $\xi^2_{l,i}=h_i/2$, $\xi^3_{l,i}=0$. Therefore from (2.8),

$$a_l=\xi^1_{l,i}(f_i^0-f_i^3 b_0 h_i^3)+\xi^2_{l,i}(f_i^1-f_i^3 b_1 h_i^2)+\xi^3_{l,i}(f_i^2-f_i^3 b_2 h_i),$$

or

$$a_{i-1}=f(t_i)+h_i f'(t_i)/2-h_i^3 f'''(t_i)/24. \tag{3.3}$$

$k=4$. $B_4(\tau)=\tau^4-2\tau^3+\tau^3-\frac{1}{30}$, giving $b_3=-2$, $b_2=1$, $b_1=0$, $b_0=-\frac{1}{30}$. In this case $l=i-2$, $\phi_l(\tau)=(\tau-t_{i-1})(\tau-t_i)(\tau-t_{i+1})$, and we obtain

$$\xi^1_{l,i}=1, \quad \xi^2_{l,i}=(h_i-h_{i-1})/2, \quad \xi^3_{l,i}=-h_i h_{i-1}/3, \quad \xi^4_{l,i}=0.$$

Hence, from (2.8),

$$\begin{aligned} a_{i-2}=&f(t_i)+(h_i-h_{i-1})f'(t_i)/3-h_i h_{i-1} f''(t_i)/6\\ &+f''''(t_i)\,h_i^3(h_i/30+h_{i-1}/3)/24. \end{aligned} \tag{3.4}$$

We remark that the derivatives of f occuring in these formulas can be replaced by difference approximations, as was done for the example in the Introduction, provided that the resulting perturbation to the B-spline coefficients is of order $o(h^k)$.

In Table 2 we give the results of using (3.4) to compute Q_N with $k=4$, where $f(\tau)=\tau^4/4!$ and $t(x)=x^2$.

We conclude with an observation which suggests that the "Bernoulli shape" (shifted vertically) for the error is to be found in most of the regular approximation and interpolation schemes. Let I_N be a projection from $C[0,1]$ onto $S_N^k(t)$; for instance I_N might be interpolation at $\alpha t_i+(1-\alpha)t_{i+1}$ with $0<\alpha\leqq 1$ and appropriate boundary conditions. If t satisfies (1.1) it is easy to see that for large N the knots $\{t(i/N)\}_{i=0}^N$ will be locally uniformly distributed (i.e., $(t_{i+1}-t_i)/(t_i-t_{i-1})$ is nearly 1). Let f be in $C^k[0,1]$; then on small intervals near a point x,

Table 2

N	$N^4\|f-Q_Nf\|$	$N^3\|(f-Q_Nf)'\|$	$N^2\|(f-Q_Nf)''\|$	$N\|(f-Q_Nf)'''\|$
8	1.9652 (−2)	2.3988 (−2)	8.6905 (−2)	4.0882 (−1)
16	1.0991 (−2)	2.3064 (−2)	8.6276 (−2)	4.0838 (−1)
32	7.8050 (−3)	2.3000 (−2)	8.6119 (−2)	4.0828 (−1)
64	6.8442 (−3)	2.3000 (−2)	8.6079 (−2)	4.0826 (−1)
128	6.5909 (−3)	2.3001 (−2)	8.6070 (−2)	4.0825 (−1)
Theoretical limit	6.5060 (−3)	2.3002 (−2)	8.6066 (−2)	4.0825 (−1)

Table 3

N	$N^4 \|f-I_N f\|$	$N^3 \|(f-I_N f)'\|$	$N^2 \|(f-I_N f)''\|$	$N \|(f-I_N f)'''\|$
4	5.8383 (−3)	2.1369 (−2)	1.1683 (−1)	4.2829 (−1)
8	8.6083 (−3)	2.2347 (−2)	1.0460 (−1)	4.1732 (−1)
16	1.0407 (−2)	2.2859 (−2)	9.6122 (−2)	4.1230 (−1)
32	1.1223 (−2)	2.3004 (−2)	9.1295 (−2)	4.1013 (−1)
64	1.1576 (−2)	2.3025 (−2)	8.8731 (−2)	4.0915 (−1)

f is essentially a polynomial of degree k or less, say

$$f(\tau)=C_k(\tau-x)^k+\ldots+C_0+o(|\tau-x|^k).$$

Since I_N is a projection, the error $f-I_N f$ should have nearly the same shape on each interval (t_i, t_{i+1}), for t_i in a neighborhood of x and N large. That is, we suspect for $t_i \leqq \tau < t_{i+1}$ and $|\tau-x|$ small,

$$\begin{aligned}(f-I_N f)(t_i+\tau(t_{i+1}-t_i)) &= \gamma_k \tau^k+\gamma_{k-1}\tau^{k-1}+\ldots+\gamma_0+o(|\tau-x|^k)\\ &= p_k(\tau)+o(|\tau-x|^k).\end{aligned}$$

Now $(f-I_N f)\in C^{k-2}$ so that p_k should satisfy the periodicity conditions

$$p_k^{(j)}(0)=p_k^{(j)}(1), \quad j=0,1,\ldots,k-2. \tag{3.5}$$

Condition (3.5) implies that $p_k(\tau)=(B_k(\tau)+C)\gamma_k$ where $B_k(\cdot)$ is the k-th Bernoulli polynomial and C is a constant determined by the projection I_N.

Although we cannot make the above argument precise we do have numerical evidence to support the conjecture that the error $f-I_N f$ has the scaled "Bernoulli shape" for many regular projections I_N. In Table 3 we let $f(\tau) = \tau^4/4!$ and $t(x)=x^2$ just as in Table 2. Let I_N be the interpolation operator from $C[0,1]$ to $S_N^4(t)$ defined by interpolation at the knots $\{t_i\}_{i=0}^N$ and at $(t_0+t_1)/2$ and $(t_N+t_{N-1})/2$. Note that the last three columns are converging to the theoretical limits listed at the bottom of the corresponding columns of Table 2, while the first column is tending to a somewhat larger number than the limit in Table 2. This behavior is consistent with the above heuristic argument that the error function looks like a vertically shifted Bernoulli monospline. In fact, when the error function for the above example was graphed, it was observed to be nonnegative with double zeros at the knots, looking locally like the monospline $\bar{B}(\tau)+C$.

These calculations were performed on the AMDAHL 470 computer at the Data Processing Center at Texas A & M University, using double precision arithmetic in FORTRAN which carries about 16 significant digits.

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