NAME: CODRIN ONECI
SOURCES: 16.90 Lectures & Textbook
COLLABORATORS: None

16.90 PSET 1 SOLUTION

PROBLEM 1.

(a) $\boxed{\frac{\partial f}{\partial u} = -\frac{\rho}{2\beta}\left(|u| + u \operatorname{sign} u\right) = -\frac{\rho}{\beta}|u|}$ (I observed $u \operatorname{sign} u = |u|$)

(b) $f(u) = f(0+u) = f(0) + u\left.\frac{\partial f}{\partial u}\right|_0 = g - \frac{\rho}{\beta}|u|u$

(I observed $t\left.\frac{\partial f}{\partial t}\right|_0 = -\frac{t\rho}{\beta}|u|\left(g - \frac{\rho}{2\beta}u|u|\right)\Big|_0 = 0$)

$$\frac{du}{g - \frac{\rho}{\beta}|u|u} = dt; \quad \frac{\beta}{\rho}\cdot\frac{du}{\frac{g\beta}{\rho} - |u|u} = dt$$

Use $u>0$ for free fall so $\frac{\beta}{\rho}\frac{du}{\frac{g\beta}{\rho} - u^2} = dt$

$$\frac{\beta}{\rho}\left(-\frac{1}{2\sqrt{\frac{g\beta}{\rho}}}\right)\ln\left(\frac{\sqrt{\frac{g\beta}{\rho}} - u}{u + \sqrt{\frac{g\beta}{\rho}}}\right) + C = t$$; obtain $C=0$ from condition $u(0)=0$

$$\frac{\sqrt{g\beta/\rho} - u}{\sqrt{g\beta/\rho} + u} = e^{-2\sqrt{\rho g/\beta}\,t}; \quad \sqrt{\frac{g\beta}{\rho}} - u = e^{-2\sqrt{\frac{\rho g}{\beta}}t}\sqrt{\frac{g\beta}{\rho}} + u\, e^{-2\sqrt{\frac{\rho g}{\beta}}t}$$

$$\boxed{u(t) = \sqrt{\frac{g\beta}{\rho}}\cdot\frac{1 - e^{-2\sqrt{\frac{\rho g}{\beta}}t}}{1 + e^{-2\sqrt{\frac{\rho g}{\beta}}t}}}$$

Physically, this model is an aproximation of the moments immediately following the start of the movement, when speed is small. The exact solution predicts a different steady-state free fall speed (the limit speed) so this linearization is relevant only for $u \approx 0$.

(c) From $\frac{du^*}{dt} = 0$ we get $g - \frac{\rho}{2\beta}u^{*2} = 0 \rightarrow \boxed{u^* = \sqrt{\frac{2\beta g}{\rho}}}$

(d) $\frac{d\tilde{u}}{dt} = f(u^*) + \tilde{u}\left.\frac{\partial f}{\partial u}\right|_{u^*} = g - \frac{\rho}{2\beta}u^{*2} + \tilde{u}\left(-\frac{\rho}{\beta}u^*\right)$

$$\frac{d\tilde{u}}{dt} + \frac{\rho}{\beta} u^* \tilde{u} = g - \frac{\rho}{2\beta} u^{*^2} = 0 \quad \text{(as observed at point (c))}$$

$$\frac{\beta}{\rho} \cdot \frac{d\tilde{u}}{u^*\tilde{u}} = (-dt) \rightarrow \frac{\beta}{\rho u^*}(\ln|\tilde{u}| + c) = -t \quad ; \quad |\tilde{u}| = -\tilde{u}, \; \tilde{u} < 0$$

Since $\tilde{u}(0) = -u^*$ (from $u(0)=0$) I get $c = -\ln u^*$

$$\boxed{\tilde{u}(t) = -u^* e^{-\frac{\rho u^*}{\beta} t}} \qquad u(t) = u^*\left(1 - e^{-\frac{\rho u^*}{\beta} t}\right)$$

PROBLEM 2.

(a) Forward Euler

Zero Stability: When $\Delta t \to 0$ the solution must be bounded. Let $V^n = V_0 z^n$. Use the recurrent equation $V^{n+1} - V^n = 0$ so $V_0 z^n (z-1) = 0$, which has roots $z_1 = 0$ and $z_2 = 1$. Observe that the root 1 is simple (has multiplicity 1) and $|z| \le 1$ is satisfied so F.E. is a zero stable method.

Consistency
$$V^{n+1} = V^n + \Delta t f(V^n)$$
$$\tau = u^{n+1} - u^n - \Delta t f(u^n)$$

One must have $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0$ for consistency.

Since $u^{n+1} = u^n + \Delta t f(u^n) + O(\Delta t)^2$, we get $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = O(\Delta t) = 0$ satisfies the consistency.

By using the Dahlquist Theorem of Equivalence, zero stability and consistency imply convergence. By using $O(\Delta t^2)$ as the order of local truncation error, the rate of convergence for FE is $2-1=1$, so Forward Euler has first order accuracy.

Adams-Bashford

Zero Stability. Observe that FE has the same recurrence equation as FE $V^{n+1} - V^n = 0$ so we know already that it is zero stable.

Consistency $\quad v^{n+1} = v^n + \Delta t \, \frac{1}{2}\left[3f(v^n) - f(v^{n-1})\right]$

$$\tau = u^{n+1} - u^n - \Delta t \, \frac{1}{2}\left[3f(u^n) - f(u^{n-1})\right]$$

$$\tau = u(t^n) + \Delta t \, u_t(t^n) + \frac{1}{2}\Delta t^2 u_{tt}(t^n) + O(\Delta t^3) - u(t^n) +$$

$$- \frac{\Delta t}{2}\left[3u_t(t^n) - u_t(t^n) + \Delta t \, u_{tt}(t^n) - \frac{\Delta t^2}{2} u_{ttt}(t^n) + O(\Delta t^3)\right]$$

$\tau = O(\Delta t^3)$ so $\lim\limits_{\Delta t \to 0} \frac{\tau}{\Delta t} = O(\Delta t^2)$ $\rightarrow$ AB is consistent

Dahlquist Theorem of Equivalence in this conditions shows that Adams-Bashford is also convergent. $\tau = O(\Delta t^3)$ shows that the rate of convergence is $3-1=2$ (second order of accuracy).

(b) $\quad \frac{du}{dt} = f(u) = g - \frac{\rho}{2\beta} u|u|$

$2\beta = \rho$, $g = 10$, $u(0) = 0$, $u \geq 0$ $\forall t$ are given

$$\frac{du}{dt} = 10 - u^2 \quad , \quad \frac{du}{10 - u^2} = dt$$

Integrating the above ODE results in:

$$\frac{1}{2\sqrt{10}} \ln\left(\left|\frac{u+\sqrt{10}}{u-\sqrt{10}}\right|\right) + C = t \quad , \text{ where } C = 0 \text{ because } u(0) = 0$$

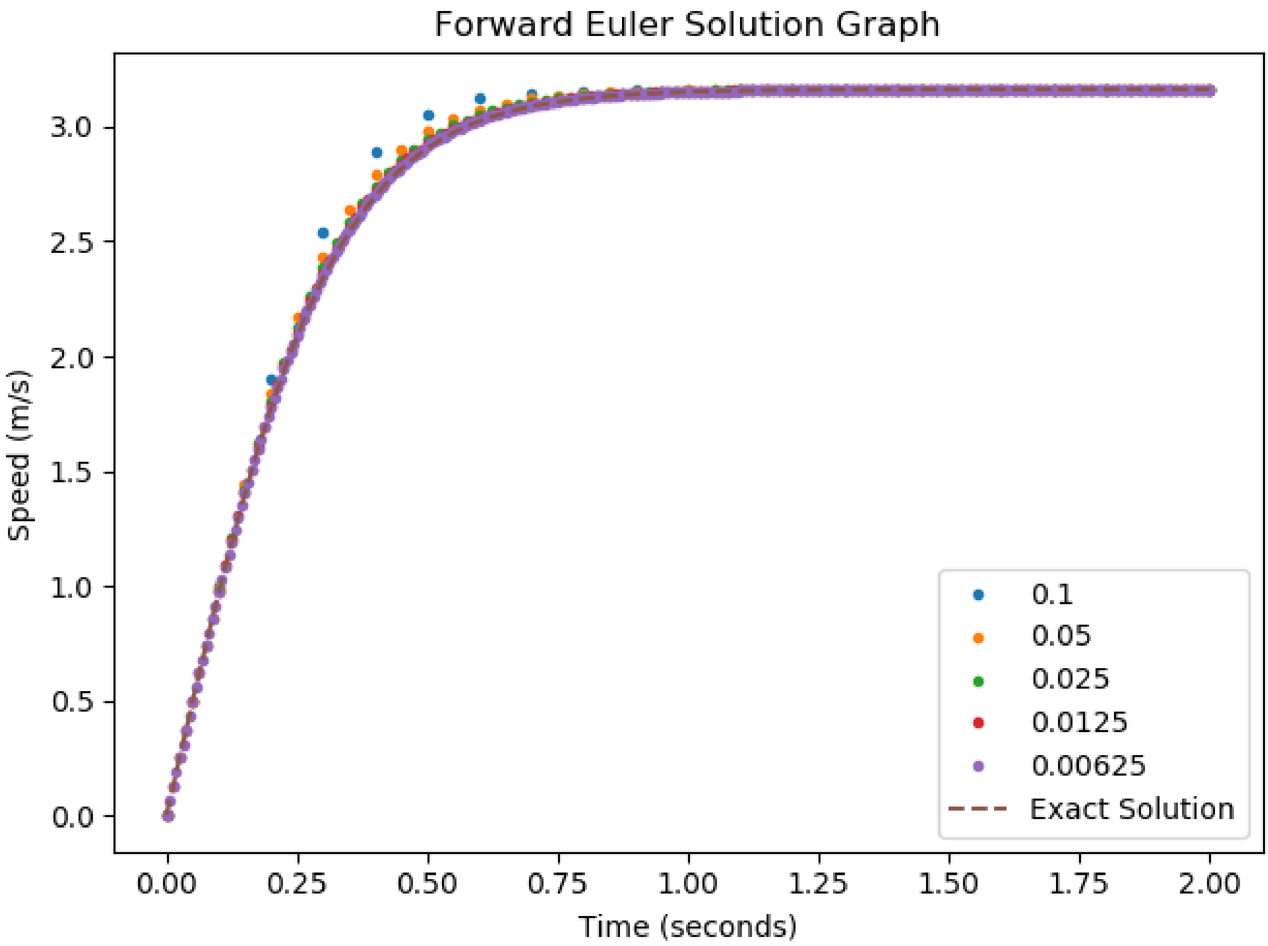
$$\ln\left|\frac{u+\sqrt{10}}{u-\sqrt{10}}\right| = 2\sqrt{10}\,t \quad ; \quad \frac{u+\sqrt{10}}{u-\sqrt{10}} = e^{2\sqrt{10}\,t}$$

Observe $u_{lim} = \sqrt{10}$ as $u \in [0, \sqrt{10}]$. After algebraic manipulations we arrive at:

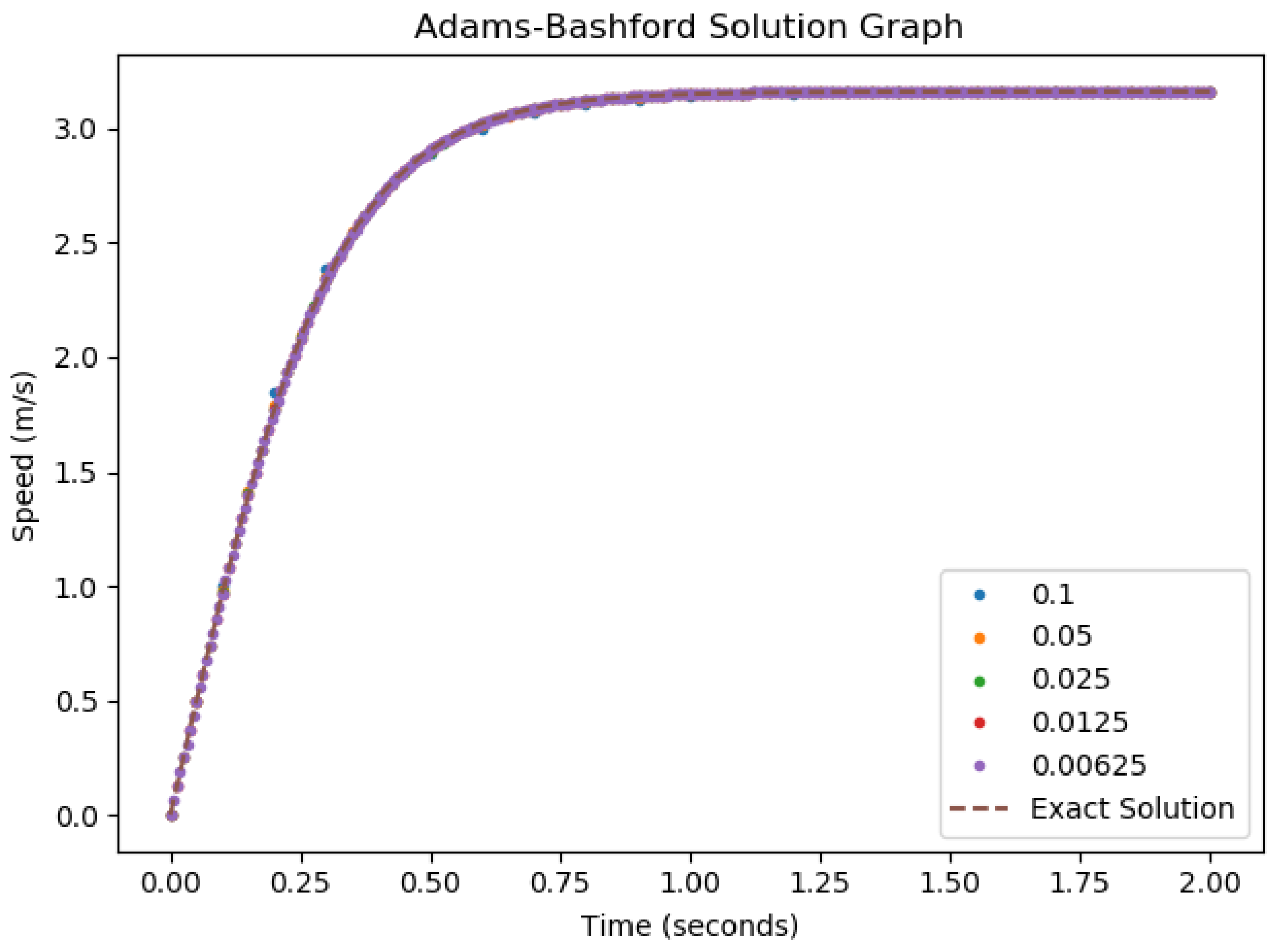
$$\boxed{u = \sqrt{10} \, \frac{e^{2\sqrt{10}t} - 1}{1 + e^{2\sqrt{10}t}}}$$

PROBLEM 2

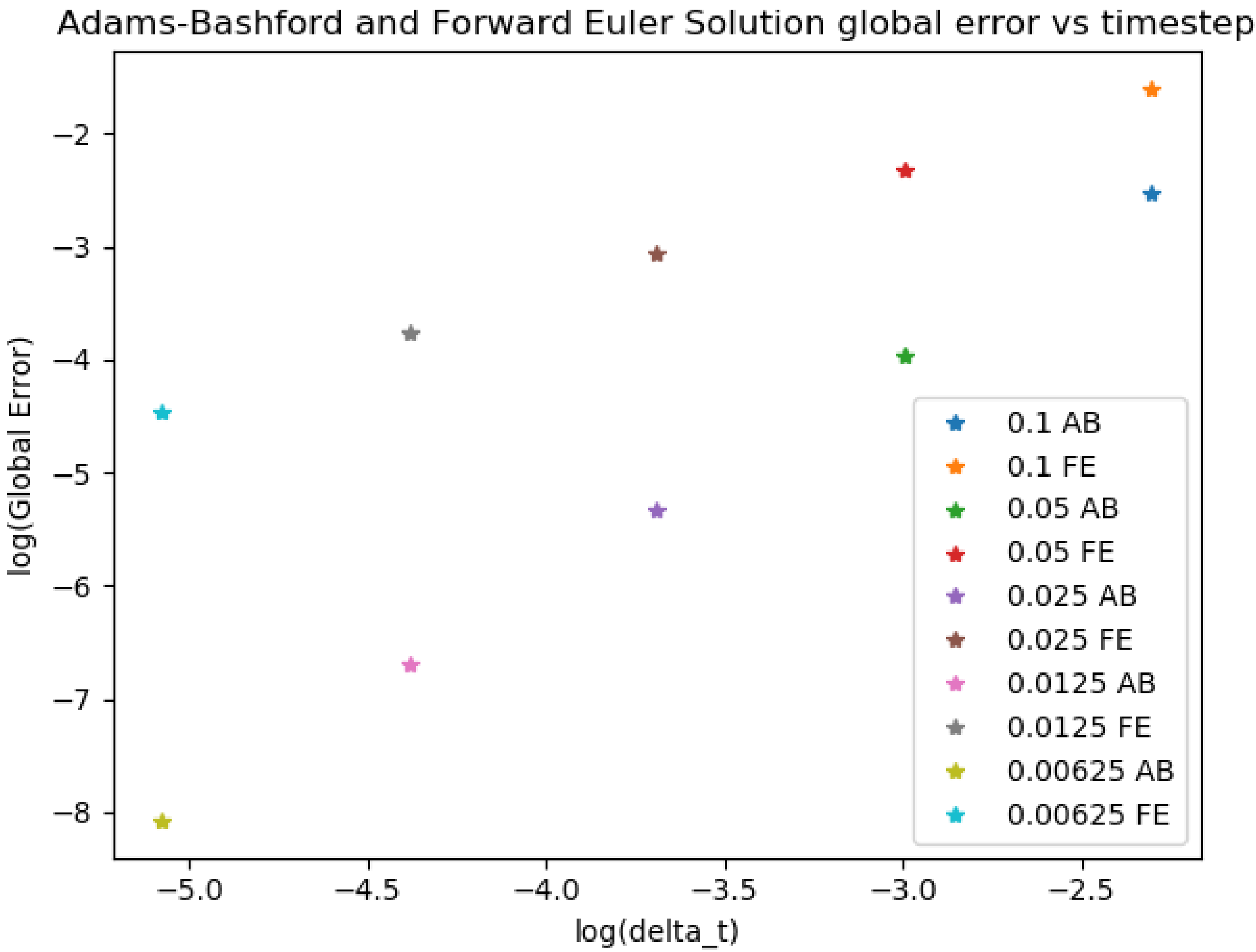
C)



Forward Euler Solution Graph
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Speed (m/s)
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Time (seconds)
0.1
0.05
0.025
0.0125
0.00625
Exact Solution

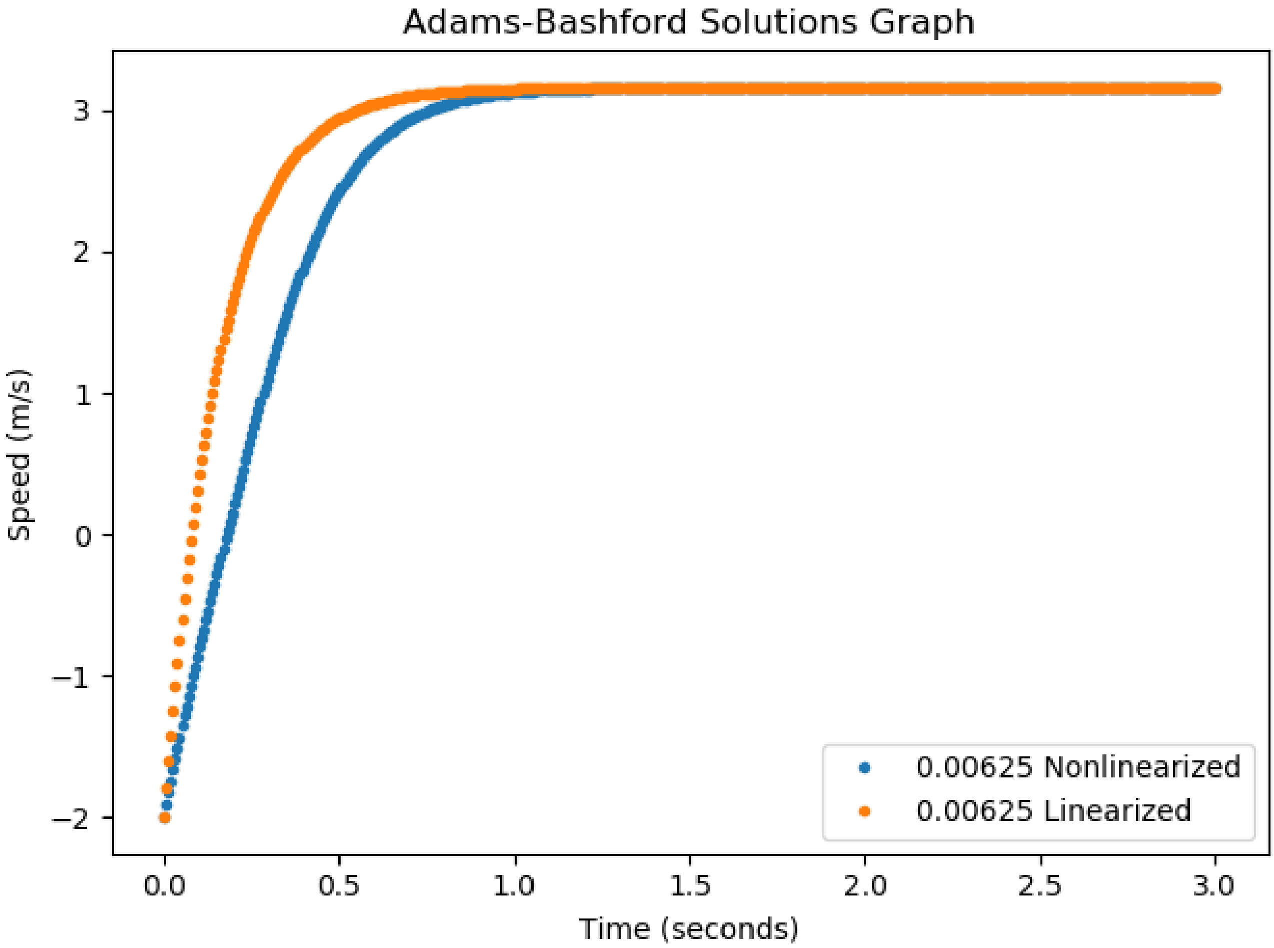

Adams-Bashford Solution Graph
Speed (m/s)
Time (seconds)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
0.1
0.05
0.025
0.0125
0.00625
Exact Solution

D)



As expected, there is a linear trend in the log-log graph for error and timestep for both methods. Specifically, the observed slope of the line for Forward Euler is one, as expected for a method with first order of accuracy. For the Adams-Bashford method we see that the slope is two, as expected for a method that has the second order of accuracy.

E)



The graph above shows that the linearized version of the ODE results in a quicker convergence to the terminal velocity, as compared to the original (nonlinearized) version of the ODE. The main cause of this behavior is neglecting higher order terms that are due to drag effects. The linearization of the ODE was done for speeds close to the terminal one, but in the beginning of the problem we are far away from the terminal velocity regime, so the difference between solutions is more significant there.