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SOURCES: 16.90 textbook

16.90 PSET3 SOLUTION

PROBLEM 1

$$u_i = U_i$$

$$u_{i+1} = u_i + \Delta x U_{xi} + \frac{1}{2}\Delta x^2 U_{xxi} + \frac{1}{6}\Delta x^3 U_{xxxi} + O(\Delta x^4)$$

$$u_{i+2} = u_i + 2\Delta x U_{xi} + \frac{1}{2} 4\Delta x^2 U_{xxi} + \frac{1}{6} 8\Delta x^3 U_{xxxi} + O(\Delta x^4)$$

$$\frac{1}{\Delta x}(\alpha u_i + \beta U_{i+1} + \gamma u_{i+2}) = \frac{1}{\Delta x}(\alpha+\beta+\gamma)u_i + (\beta+2\gamma)U_{xi} + \frac{1}{2}(\beta+4\gamma)U_{xxi}\Delta x +$$
$$+ \frac{1}{6}(\beta+8\gamma)U_{xxxi}\Delta x^2 + O(\Delta x^3)$$

We can choose α, β, γ such that the first 3 terms are:

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta+2\gamma &= 1 \\ \tfrac{1}{2}\beta + 2\gamma &= 0 \end{aligned}\right\} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & \frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \begin{pmatrix} -1.5 \\ 2 \\ -0.5 \end{pmatrix}}$$

Observe $\frac{1}{\Delta x}(-1.5U_i + 2u_{i+1} - 0.5u_{i+2}) - U_{xi} = -\frac{1}{3}U_{xxxi}\Delta x^2 + O(\Delta x^3)$ so this approximation is second order accurate.

PROBLEM 2

(a) $V>0 \to |V| = V \longrightarrow F(U_R, U_L) = \frac{V}{2}2U_L = VU_L$

$$\Delta x \frac{du_i}{dt} + Vu_i - VU_{i-1} = 0 \longrightarrow \boxed{\frac{du_i}{dt} + V\frac{u_i - U_{i-1}}{\Delta x} = 0}$$

This is equivalent to the backward space approximation finite difference discretization $U_{ix} \approx \delta_x^- U_i = \frac{u_i - U_{i-1}}{\Delta x}$, observing $\frac{\partial F}{\partial x} = VU_{xi}$.

(b) $$\Delta x\frac{dU_i}{dt} + F(U_{i+1}, U_i) - F(U_i, U_{i-1}) = \Delta x\frac{dU_i}{dt} + \frac{V}{2}(U_{i+1} + U_i) - \frac{V}{2}(U_i + U_{i-1}) =$$
$$= \boxed{\Delta x\frac{dU_i}{dt} + V\frac{U_{i+1} - U_{i-1}}{2}}$$

The equivalent finite difference approach uses $U_{xi} \approx \frac{U_{i+1} - U_{i-1}}{2\Delta x}$ and the corresponding discretization is:

$$\frac{dU_i}{dt} + V\frac{U_{i+1} - U_{i-1}}{2\Delta x} \approx \frac{\partial U}{\partial t} + V\frac{\partial U}{\partial x}$$

PROBLEM 3

a) $\tau = U \dfrac{U_{i+1} - U_{i-1}}{2\Delta x} - \nu \dfrac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2}$

$U_i = U_i$; $U_{i+1} = U_i + \Delta x U_{xi} + \frac{1}{2}\Delta x^2 U_{xxi} + \frac{1}{6}\Delta x^3 U_{xxxi} + \frac{1}{24}\Delta x^4 U_{xxxxi} + O(\Delta x^5)$

$U_{i-1} = U_i - \Delta x U_{xi} + \frac{1}{2}\Delta x^2 U_{xxi} - \frac{1}{6}\Delta x^3 U_{xxxi} + \frac{1}{24}\Delta x^4 U_{xxxxi} - O(\Delta x^5)$

$$\frac{U_{i+1} - U_{i-1}}{2\Delta x} = \frac{2\Delta x U_{xi} + \frac{1}{3}\Delta x^3 U_{xxxi}}{2\Delta x} = U_{xi} + \frac{1}{6}\Delta x^2 U_{xxxi}$$

$$U_{i+1} - 2U_i + U_{i-1} = 2 \cdot \frac{1}{2}\Delta x^2 U_{xxi} + 2 \cdot \frac{1}{24}\Delta x^4 U_{xxxxi} + O(\Delta x^5)$$

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} = U_{xxi} + \frac{1}{12}\Delta x^2 U_{xxxxi}$$

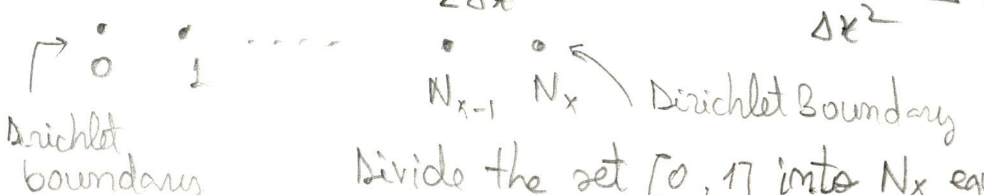
$$\tau = U\left(U_{xi} + \frac{1}{6}\Delta x^2 U_{xxxi}\right) - \nu\left(U_{xxi} + \frac{1}{12}\Delta x^2 U_{xxxxi}\right)$$

Observe $U U_{xi} - \nu U_{xxi} = 0$ because $U \dfrac{\partial U}{\partial x} = \nu \dfrac{\partial^2 U}{\partial x^2}$ was given.

$$\boxed{\tau = \Delta x^2 \left(\frac{U}{6} U_{xxxi} - \frac{\nu}{12} U_{xxxxi}\right)}$$

The local error suggests that this scheme is FIRST order accurate.

b) Write $U \dfrac{U_{i+1} - U_{i-1}}{2\Delta x} - \nu \dfrac{U_{i+1} - 2U_i + U_{i-1}}{\Delta x^2} = 0$



Divide the set $[0, 1]$ into N_x equal cardinality compact sets

The state vector is $U_{bc} = (U_0, U_1, \dots U_{N_x})^T$ but we know $U_0 = 1$ and $U_{N_x} = 0$ in all times, so I create $U = (U_1, \dots U_{N_x-1})^T$ that contains variable data. For $i \in \overline{1, N_{x-1}}$ we impose the discretized ODE so we can create an $(N_x - 1, N_{x-1})$ matrix A with entries resulting from

$$\left(\frac{U}{2\Delta x} + \frac{\nu}{\Delta x^2}\right)U_{i-1} + \left(\frac{-2\nu}{\Delta x^2}\right)U_i + \left(\frac{-U}{2\Delta x} + \frac{\nu}{\Delta x^2}\right)U_{i+1} = 0$$

So $\boxed{A_{i,i-1} = -\dfrac{U}{2\Delta x} - \dfrac{\nu}{\Delta x^2} \; ; \; A_{i,i} = \dfrac{2\nu}{\Delta x^2} \; ; \; A_{i,i+1} = \dfrac{U}{2\Delta x} - \dfrac{\nu}{\Delta x^2}}$

We will write $Ax = b$ with $b_i = 0$ for $i = \overline{1, N_x - 1}$

The boundary conditions must be incorporated into b_0 and b_{N_x}:

$$b_1 = \left(-\frac{u}{2\Delta x} - \frac{\nu}{\Delta x^2}\right)\underset{\overset{\uparrow}{1}}{U_0}, \quad b_{N_x-1} = \left(\frac{u}{2\Delta x} - \frac{\nu}{\Delta x^2}\right)\underset{\overset{\uparrow}{0}}{U_{N_x}}$$

c) Put $N_x = [1/\Delta x]$, where $[\alpha] \equiv$ integer part of α

Even tough is not the most efficient method, we may create matrix A in the memory and then $u = A^{-1} b$ (A^{-1} was found with Gaussian elimination)