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SOURCES: 16.90 Lectures & Textbook
COLLABORATORS: None

16.90 PSET 1 SOLUTION

PROBLEM 1.

(a) $\boxed{\frac{\partial f}{\partial u} = -\frac{\rho}{2\beta}\left(|u| + u\,\text{sign}\,u\right) = -\frac{\rho}{\beta}|u|}$ (I observed $u\,\text{sign}\,u = |u|$)

(b) $f(u) = f(0+u) = f(0) + u\left.\frac{\partial f}{\partial u}\right|_0 = g - \frac{\rho}{\beta}|u|u$

(I observed $t\left.\frac{\partial f}{\partial t}\right|_0 = -\frac{t\rho}{\beta}|u|\left(g - \frac{\rho}{2\beta}u|u|\right)\Big|_0 = 0$)

$$\frac{du}{g - \frac{\rho}{\beta}|u|u} = dt; \quad \frac{\beta}{\rho}\cdot\frac{du}{\frac{g\beta}{\rho} - |u|u} = dt$$

Use $u>0$ for free fall so $\frac{\beta}{\rho}\frac{du}{\frac{g\beta}{\rho} - u^2} = dt$

$$\frac{\beta}{\rho}\left(-\frac{1}{2\sqrt{\frac{g\beta}{\rho}}}\right)\ln\left(\frac{\sqrt{\frac{g\beta}{\rho}} - u}{u + \sqrt{\frac{g\beta}{\rho}}}\right) + c = t$$; obtain $c=0$ from condition $u(0)=0$

$$\frac{\sqrt{g\beta/\rho} - u}{\sqrt{g\beta/\rho} + u} = e^{-2\sqrt{\rho g/\beta}\,t}; \quad \sqrt{\frac{g\beta}{\rho}} - u = e^{-2\sqrt{\frac{\rho g}{\beta}}t}\sqrt{\frac{g\beta}{\rho}} + u\,e^{-2\sqrt{\frac{\rho g}{\beta}}t}$$

$$\boxed{u(t) = \sqrt{\frac{g\beta}{\rho}}\cdot\frac{1 - e^{-2\sqrt{\frac{\rho g}{\beta}}t}}{1 + e^{-2\sqrt{\frac{\rho g}{\beta}}t}}}$$

Physically, this model is an aproximation of the moments immediately following the start of the movement, when speed is small. The exact solution predicts a different steady-state free fall speed (the limit speed) so this linearization is relevant only for $u \approx 0$.

(c) From $\frac{du^*}{dt} = 0$ we get $g - \frac{\rho}{2\beta}u^{*2} = 0 \rightarrow \boxed{u^* = \sqrt{\frac{2\beta g}{\rho}}}$

(d) $$\frac{d\tilde{u}}{dt} = f(u^*) + \tilde{u}\left.\frac{\partial f}{\partial u}\right|_{u^*} = g - \frac{\rho}{2\beta}u^{*2} + \tilde{u}\left(-\frac{\rho}{\beta}u^*\right)$$

$$\frac{d\tilde{u}}{dt} + \frac{\rho}{\beta} u^* \tilde{u} = g - \frac{\rho}{2\beta} u^{*2} = 0 \qquad \text{(as observed at point (c))}$$

$$\frac{\beta}{\rho} \cdot \frac{d\tilde{u}}{u^* \tilde{u}} = -dt \quad \rightarrow \quad \frac{\beta}{\rho u^*}\left(\ln|\tilde{u}| + c\right) = -t \quad ; \quad |\tilde{u}| = -\tilde{u}, \ \tilde{u} < 0$$

Since $\tilde{u}(0) = -u^*$ (from $u(0) = 0$) I get $c = -\ln u^*$

$$\boxed{\tilde{u}(t) = -u^* e^{-\frac{\rho u^*}{\beta} t}} \qquad u(t) = u^*\left(1 - e^{-\frac{\rho u^*}{\beta} t}\right)$$

PROBLEM 2.

(a) Forward Euler

Zero Stability: When $\Delta t \to 0$ the solution must be bounded. Let $V^n = V_0 z^n$. Use the recurrent equation $V^{n+1} - V^n = 0$ so $V_0 z^n (z-1) = 0$, which has roots $z_1 = 0$ and $z_2 = 1$. Observe that the root 1 is simple (has multiplicity 1) and $|z| \leq 1$ is satisfied so F.E. is a zero stable method.

Consistency

$$V^{n+1} = V^n + \Delta t f(V^n)$$

$$\tau = u^{n+1} - u^n - \Delta t f(u^n)$$

One must have $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = 0$ for consistency.

Since $u^{n+1} = u^n + \Delta t f(u^n) + O(\Delta t)^2$, we get $\lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = O(\Delta t) = 0$ satisfies the consistency.

By using the Dahlquist Theorem of Equivalence, zero stability and consistency imply convergence. By using $O(\Delta t^2)$ as the order of local truncation error, the rate of convergence for FE is $2-1=1$, so Forward Euler has first order accuracy.

Adams-Bashford

Zero Stability. Observe that FE has the same recurrence equation as FE $V^{n+1} - V^n = 0$ so we know already that it is zero stable.

Consistency

$$v^{n+1} = v^n + \Delta t \, \tfrac{1}{2}\left[3f(v^n) - f(v^{n-1})\right]$$

$$\tau = u^{n+1} - u^n - \Delta t \, \tfrac{1}{2}\left[3f(u^n) - f(u^{n-1})\right]$$

$$\tau = u(t^n) + \Delta t \, u_t(t^n) + \tfrac{1}{2}\Delta t^2 u_{tt}(t^n) + O(\Delta t^3) - u(t^n) +$$

$$- \frac{\Delta t}{2}\left[3u_t(t^n) - u_t(t^n) + \Delta t \, u_{tt}(t^n) - \frac{\Delta t^2}{2} u_{ttt}(t^n) + O(\Delta t^3)\right]$$

$$\tau = O(\Delta t^3) \quad \text{so} \quad \lim_{\Delta t \to 0} \frac{\tau}{\Delta t} = O(\Delta t^2) \quad \rightarrow \text{AB is consistent}$$

Dahlquist Theorem of Equivalence in this conditions shows that Adams-Bashford is also convergent. $\tau = O(\Delta t^3)$ shows that the rate of convergence is $3-1=2$ (second order of accuracy).

(b) $$\frac{du}{dt} = f(u) = g - \frac{\rho}{2\beta} u|u|$$

$2\beta = \rho$, $g = 10$, $u(0) = 0$, $u \geq 0 \ \forall t$ are given

$$\frac{du}{dt} = 10 - u^2 \ , \quad \frac{du}{10 - u^2} = dt$$

Integrating the above ODE results in:

$$\frac{1}{2\sqrt{10}} \ln\left(\left|\frac{u+\sqrt{10}}{u-\sqrt{10}}\right|\right) + C = t$$, where $C=0$ because $u(0)=0$

$$\ln\left|\frac{u+\sqrt{10}}{u-\sqrt{10}}\right| = 2\sqrt{10}\, t \ ; \quad \frac{u+\sqrt{10}}{u-\sqrt{10}} = e^{2\sqrt{10}\, t}$$

Observe $u_{lim} = \sqrt{10}$ as $u \in [0, \sqrt{10}]$. After algebraic manipulations we arrive at:

$$\boxed{u = \sqrt{10}\, \frac{e^{2\sqrt{10}t} - 1}{1 + e^{2\sqrt{10}t}}}$$