Project 1: Aerodonetics

Due: Wednesday, March 6 at 2:30pm

Note: *Projects are meant to be open-ended and to allow some flexibility and creativity. Therefore it is important for you to show all relevant steps, numerical plots, and justifications for the choices made in your work.*

In his book *Aerodonetics* published in 1908, Frederick Lanchester described a model for the flight dynamics of a glider. Although the model is outmoded, it is straightforward to derive and produces interesting dynamics including oscillations and stalls. In this project, we will explore Lanchester's model through numerical simulation.

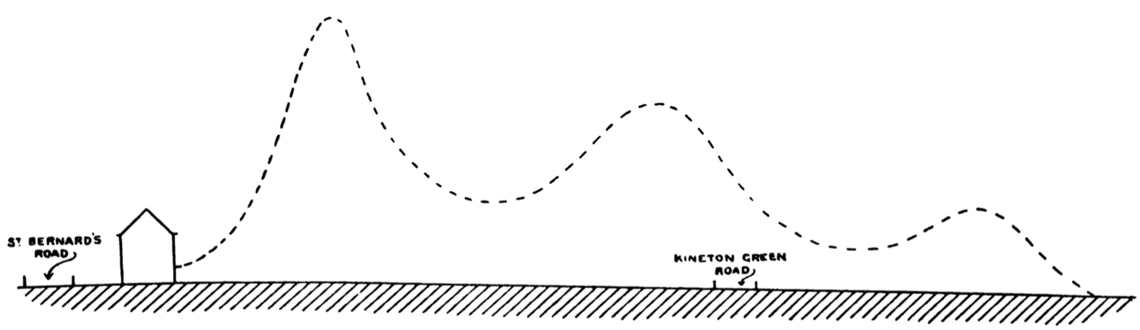

Figure 1: The trajectory of a glider launched from a catapult *Aerodonetics* (1908)

Lanchester set out to determining whether a passive glider could achieve stable flight at high speed. He constructed his own gliders (see fig. 3) and launched them using a catapult in his backyard. He recorded the resulting trajectories, one of which is shown in fig. 1.

The model Lanchester developed can be written as a system of two nonlinear differential equations in a frame of reference relative to the glider. Let $v(t)$ be the forward velocity of the glider at time t, and $\theta(t)$ be the angle of attack. The forces acting on the glider are illustrated in fig. 2.

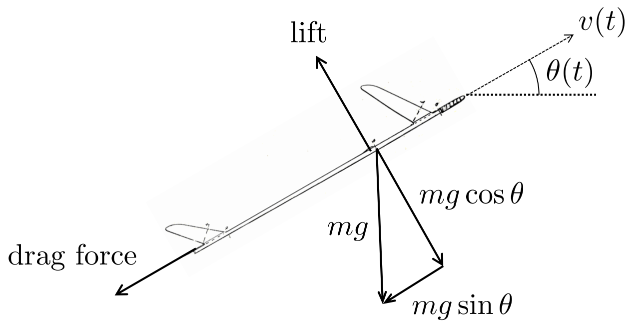

Figure 2: Forces acting on the glider (solid arrows)

Applying Newton's second law to the forces in the forward direction of the glider, we get

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg\sin\theta - \underbrace{\frac{1}{2}\rho v^2 C_D S}_{\text{drag force}},$$

where ρ is the density of air, C_D is the drag coefficient, and S is the wing area. The forces in the perpendicular direction balance the centripetal acceleration, giving

$$mv\frac{\mathrm{d}\theta}{\mathrm{d}t} = -mg\cos\theta + \underbrace{\frac{1}{2}\rho v^2 C_L S}_{\text{lift}},$$

where C_L is the coefficient of lift. For this project, we make the approximation that C_D and C_L are constant. We can absorb the the physical constants into $R_D = \frac{1}{2m}\rho C_D S$ and $R_L = \frac{1}{2m}\rho C_L S$, which gives us the following system of ODEs:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g\sin\theta - R_D\, v^2, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = R_L\, v - \frac{g\cos\theta}{v}.$$

It is difficult to visualize the glider's trajectory based on only the velocity and angle of attack, so we will also introduce the glider's displacement in the horizontal direction (x) and the vertical direction (y). We can define $\mathrm{d}x/\mathrm{d}t = v\cos\theta$ and $\mathrm{d}y/\mathrm{d}t = v\sin\theta$, so we can augment the original system to obtain a fourth-order system with the four state variables $u(t) = [v(t), \theta(t), x(t), y(t)]$:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g\sin\theta - R_D\, v^2, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = R_L\, v - \frac{g\cos\theta}{v}, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = v\cos\theta, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = v\sin\theta.$$

Using this system, we can specify our initial conditions as $u(0) = [v_0, \theta_0, x_0, y_0]$.

Constant	Value	Units
g	9.81	$\mathrm{m/s^2}$
ρ	1.22	$\mathrm{kg/m^3}$
m	0.65	kg
S	0.06	$\mathrm{m^2}$
C_D	0.10	
C_L	1.20	

Table 1: Based on the original drawings in *Aerodonetics* and other historical sources (see fig. 3), we can find approximate measurements for the geometry and weight of the glider, from which we can estimate values for the physical constants in R_L and R_D.

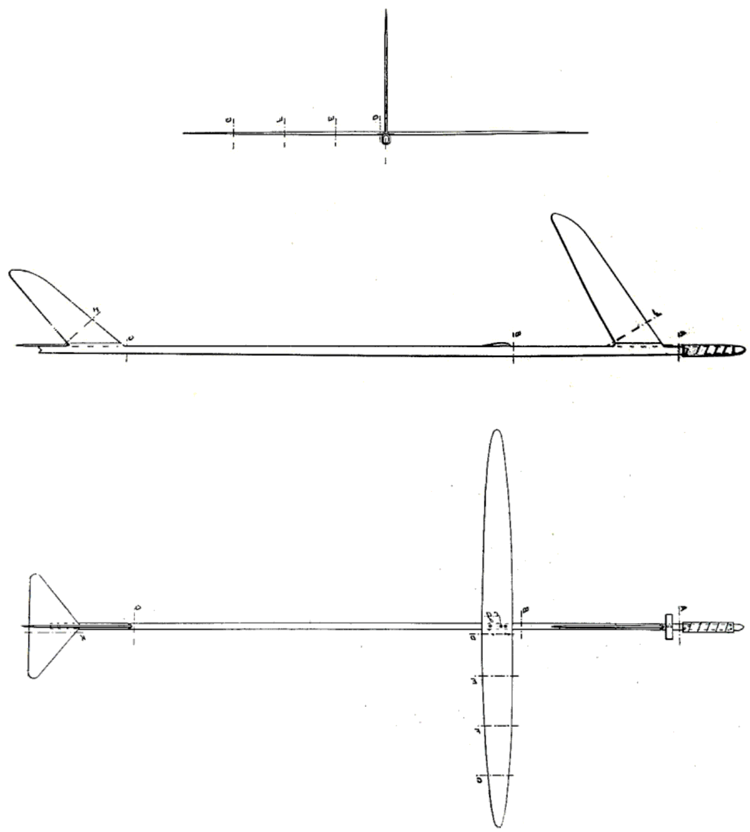

Figure 3: Lanchester's 1894 glider as depicted in general-arrangement drawing in *Aerodonetics*. Lanchester's gliders (or 'aerodones', as he called them) were distinctive in several respects. They were very long in relation to their span, the 'backbone' being some 6ft 2in long, whereas the narrow-chord elliptical wing spanned only 3ft 4in. It had a high aspect ratio of 13.3 and an area of 0.65 ft^2. The model was ballasted by a strip lead wound around the glider's nose. Immediately aft of this weight and just in front of the tailplane were two sweptback vertical fins, the front one having an area of 0.3 ft^2 and the rear one an area of 0.2 ft^2. The glider's center of gravity was located about an inch in front of the wing leading edge, and the distance between the centres of pressure of the wing and tailplane was approximately 3.9 ft. A complete glider weighed approximately 1 lb 7 oz.

From Jarrett, Philip, "Lanchester and the great divide". *Journal of Aeronautical History*, 2014.

Tasks

Your project submission should be a brief but professional self-contained report. In answering the questions below, include informative plots of your numerical results, accompanied by clear written arguments and explanations. If you're making an assertion about how well a method performs or why the system behaves in a certain way, please support this assertion with numerical results and plots. Be sure to carefully label all figures and refer to them in the text.

For submission, create a compressed folder named with your Kerberos ID followed by an underscore followed by "P1". For instance, `peraire_P1`. In that folder, you will create a script entitled `main.m` that runs and plots sequentially everything (each in a new `figure` in MATLAB) that you include in your report.

1. Consider the following initial condition which mimics Lanchester's experiments:

$$v(0) = 22\,\text{m/s}, \qquad \theta(0) = 0, \qquad x(0) = 0\,\text{m}, \qquad y(0) = 5\,\text{m}.$$

   Compute a reference solution using MATLAB's `ode45` function making sure to terminate the integration when $y(t) \leq 0$. This can be done by using the 'Events' option of the `odeset` function, e.g., by defining the following event function:

```
function [position, isterminal, direction] = ground_intersection(t, u)
    position = u(4);   % event triggers when position = 0
    isterminal = 1;    % halt integration
    direction = -1;    % trigger when event function is decreasing
end
```

   Plot the reference solution both in the x-y plane, and in the θ-v plane. Comment on the the relationship between the two plots by matching features in one plot with those from the other plot.

2. Numerically integrate the system of ODEs using your own implementation of Forward Euler and 4-stage Runge-Kutta. How closely do your solutions resemble the reference solution computed using `ode45`?

   Make sure to explain your choice of timestep for each scheme, and if you encountered unstable solutions when using larger timesteps.

3. Verify the order of accuracy you expect for each of your integration schemes. Since there is no closed-form solution to this system, you will need to create an "exact" solution either by decreasing the tolerance of `ode45` or by using a numerical solution computed using RK4 with a very small timestep.

   We suggest using timesteps $\Delta t \in \{0.05, 0.01, 0.005, 0.001\}$ and calculating the error at time $t = 10$ defined as $\|v(t) - u(t)\|_\infty$, where $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$.

4. Compute the trajectories using your choice of integration scheme (your own implementation, not ode45) and describe the types of behavior you observe when starting from the following initial conditions:
   (a) $u_A = [\,29, 0, 0, 10\,]$
   (b) $u_B = [\,23.1, 0, 0, 10\,]$
   (c) $u_C = [\,12.0223, -0.0831, 0, 10\,]$
   (d) $u_D = [\,6, 0, 0, 10\,]$
5. Find the point (v^*, θ^*) where $\frac{\mathrm{d}v}{\mathrm{d}t}(v^*, \theta^*) = 0$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}(v^*, \theta^*) = 0$. This point corresponds to the solution where the angle of attack $\theta(t) = \theta^*$ and velocity $v(t) = v^*$ are constant. You should be able to express v^* and θ^* in terms of R_D, R_L and g.

   Investigate the behavior near this stable solution by linearizing the ODE about $u^* = (v^*, \theta^*)$. Classify the solution behavior as a sink, source, or saddle, and comment on the long-term behavior of solutions near u^* and the effect on the resulting trajectories in the x-y plane.
6. Since $\frac{\mathrm{d}v}{\mathrm{d}t}$ and $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ do not explicitly depend on t, we only need to know the values of θ and v to find the tangent vector to any solution passing through the point (θ, v). We can get an idea of how solutions behave by plotting the vector $\langle \frac{\mathrm{d}\theta}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t} \rangle$ at regularly-spaced points (θ, v) in the following ranges:

   $$-\pi \le \theta \le 3\pi, \qquad 1 \le v \le 30.$$

   This can be done using the MATLAB function `quiver` in conjunction with `meshgrid` function which generates the grid of (θ, v) values.

   By drawing a path from an initial condition while following the arrows, you can predict the behavior of the system. Comment on the expected solution behavior when starting from $\theta(0) = 0$ and $v(0) = 0$, which corresponds to u_A from problem 4.
7. Upload a zip file containing your (commented) codes to Stellar.
8. (Optional) A simple (and fun!) way to validate this model is to build a paper glider yourself and try to reproduce the modes you plotted in problem 4 by launching the glider at different velocities and angles. If you plan on doing a quantitative analysis, you should replace the values in Table 1 with those that are more appropriate for your paper glider. We suggest constructing the Barnaby Flyer, which won the aerobatics category at the 1967 International Paper Airplane Contest (see Figure 4).

Figure 4: Acrobatic maneuvers using the Barnaby flyer from *How To Make And Fly Paper Airplanes*, Ralph Barnaby (1972)