MIT 16.90 Spring 2019: Problem Set 3

Due: Friday, March 22 at 2:30pm

Problem 1: One-sided finite differences

Suppose we want to find a second-order accurate finite difference approximation of $\partial u/\partial x$ at $x = 0$ with a spatial discretization $x_i = (i-1)\Delta x$. The forward difference approximation is given by

$$\left.\frac{\partial u}{\partial x}\right|_{x_1} = \frac{u_2 - u_1}{\Delta x} + O(\Delta x).$$

However, backward/central difference approximations need u_0, which is not available. Instead, we seek a one-sided approximation of the form

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} \approx \frac{\alpha u_i + \beta u_{i+1} + \gamma u_{i+2}}{\Delta x}.$$

Find values for α, β, and γ that will make this one-sided finite difference approximation second-order accurate.

Problem 2: Relating Finite Volume and Finite Difference Methods

Consider the one-dimensional convection problem,

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0, \qquad F = vU$$

where v is a constant. Assume a mesh with spacing Δx. The finite volume discretization for this problem is

$$\Delta x \frac{\mathrm{d}U_i}{\mathrm{d}t} + F(U_{i+1}, U_i) - F(U_i, U_{i-1}) = 0,$$

where the flux is approximated as

$$F(U_R, U_L) = \frac{1}{2} v\left(U_R + U_L\right) - \frac{1}{2}|v|\left(U_R - U_L\right). \tag{1}$$

Note: in this problem, do not discretize the time derivative and leave it as $\frac{\mathrm{d}U_i}{\mathrm{d}t}$. Here, we only focus on the approximation of the spatial derivative.

(a) Consider the case where $v > 0$. Write out the finite volume discretization in terms of U_{i-1}, U_i, and U_{i+1}. What is the equivalent finite difference discretization for this finite volume discretization?

(b) Now, instead of the flux given in Equation (1), consider using a flux which simply averages the states, i.e.

$$F(U_R, U_L) = \frac{1}{2} v \left(U_R + U_L\right)$$

Write out the finite volume discretization in terms of U_{i-1}, U_i, and U_{i+1}. What is the equivalent finite difference discretization for this finite volume discretization?

Problem 3: Finite Differences for the Steady Convection-Diffusion Equation

Consider the steady convection-diffusion equation on the domain $x \in [0, 1]$:

$$U \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{2}$$

with boundary conditions

$$u(0) = 1, \qquad u(1) = 0$$

where $U > 0$ is the convection velocity and $\nu > 0$ is the diffusivity.

(a) To solve this equation numerically, we use the central difference approximation:

$$U \delta_{2x} u_i - \nu \delta_x^2 u_i = 0$$

Determine the local truncation error for this approximation.

(b) Write the finite difference scheme in matrix form, i.e. construct a linear system $Au = b$ that can be solved to give $u(x)$. What does the ith row of this matrix look like? *Hint:* For the ith row, you only need to find values for $A_{i,i-1}$, $A_{i,i}$, and $A_{i,i+1}$.

How do you handle the boundary conditions at the left and right ends of the domain?

(c) Implement the finite difference scheme in MATLAB using the values $U = 1.0$, $\nu = 0.1$ and plot the error versus Δx in a log-log plot for $\Delta x \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$. Use the infinity norm of the difference between the numerical and exact solutions to quantify the error. The exact solution can be found by treating the PDE as a second order ODE with constant coefficients, which has the solution

$$u(x) = c_1 e^{Ux/\nu} + c_2, \quad \text{with} \quad c_1 = \frac{1}{1 - e^{U/\nu}}, \quad c_2 = 1 - \frac{1}{1 - e^{U/\nu}}.$$

Does your error plot agree with the analysis performed earlier?