MIT 16.90 Spring 2019: Problem Set 1

Due: Wednesday, February 20 at 2:30pm

Problem 1: ODE Linearization

Consider a crude model for a sphere in free fall, in which density, area, and drag coefficient are all assumed constant,

$$D(u) = \frac{1}{2}\rho u|u|C_D A.$$

Here $u|u|$ replaces u^2 so that the direction of drag is always opposite that of the velocity.

Assuming the only forces are drag and gravity, the downward velocity u obeys the ODE

$$\frac{du}{dt} = f(u) = g - \frac{\rho}{2\beta}u|u|, \tag{1}$$

where $\beta = \frac{m}{C_D A}$ is the ballistic coefficient.

(a) Use the chain rule to determine $\frac{\partial f}{\partial u}$. It may help to note that the derivative of the absolute value function is the sign function, $\frac{\partial}{\partial x}|x| = \operatorname{sign} x$. Simplify your final answer, however, so that it does not involve the sign function.

(b) Linearize the ODE about the point $u = 0$. What type of solutions does the resulting linear ODE have? What does this model represent physically?

(c) Determine the *terminal velocity*, u^*, in terms of ρ, g, and β. You should not need to do any simulations or ODE integrations.

(d) Linearize the ODE about the point $u = u^*$. Express your answer as an ODE with dependent variable $\tilde{u}$, where $u = u^* + \tilde{u}$. Describe the solution of the linearized equation starting from the intitial condition $u(0) = 0$.

Problem 2: Integrating ODEs

We consider two explicit numerical integration schemes,

$$v^{n+1} - v^n = \Delta t f(v^n) \qquad \text{(Forward Euler)} \tag{2}$$

$$v^{n+1} - v^n = \Delta t \frac{1}{2}\left[3f(v^n) - f(v^{n-1})\right] \qquad \text{(Adams-Bashford)} \tag{3}$$

(a) Detemine if the above scehems are consistent, zero stable and convergent. If they are convergent, determine the order of accuracy.

(b) Calculate the exact solution to 1 in the previous problem assuming that $u \geq 0$ for all t, and $u(0) = 0$. Assume that the equation is nondimensionalized in such a way that $g = 10$ and $\beta = \rho/2$.

(c) Write a Matlab code to compute the numerical solution for $0 < t \leq T = 2$ and the following five stepsizes $\Delta t \in \{10^{-1}, 10^{-1}/2, 10^{-1}/2^2, 10^{-1}/2^3, 10^{-1}/2^4\}$ using methods 2 and 3. For each scheme, show the solutions for the different timesteps on the same plot. Note: For the first sime step of method 3, you can use method 2 since v^{-1} is not avalaible.

(d) Record the global error

$$\max_{n=[0,T/\Delta t]} |v^n - u(n\Delta t)|$$

for each choice of Δt. Plot the global error vs. the time step on a log-log plot for both schemes on the same plot and comment on the observed slope and its relationship to the global order of accuracy.

(e) Using the integration scheme 3 and $\Delta t = 10^{-1}/2^4$, solve the the nonlinear equation 1 and the linearized equation derived in Problem 1 (d) for the initial condition $u(0) = -2$ (initial upwards velocity) and $0 < t \geq T = 3$. Use the same nondimensionalization as in part (b). Show both solutions on the same plot and comment on the results.