1 Key Concepts

- Continuum Hypothesis
- Viscosity
- Hydrostatic Pressure
- Surface Tension
- First Law of Thermodynamics
- Adiabatic and Reversible Proceses
- Entropy
- Ideal Gases

2 Important Equations

$$\tau = \mu \frac{du}{dx} \quad \text{(Newton's Law of Viscosity)}$$

$$P = \frac{F}{A} \quad \text{(Definition of Pressure)}$$

$$F_1 A_1 = F_2 A_2, \text{ where } z_1 = z_2 \quad \text{(Pascal's Law)}$$

$$\Delta P = \rho g z \quad \text{(Change in Hydrostatic Pressure)}$$

$$F_b = \rho g V \quad \text{(Buoyancy Force)}$$

$$\int_V \overrightarrow{\nabla} \cdot \overrightarrow{F} dV = \oint_S \overrightarrow{F} \cdot \overrightarrow{n} dS \quad \text{(Gauss's Theorem)}$$

$$de = dq + dw \quad \text{or} \quad \frac{de}{dt} = \dot{w} + \dot{q} \quad \text{(First Law of Thermodynamics)}$$

$$h = u + \rho\nu \quad \text{(Definition of Enthalpy)}$$

$$T ds = dq \quad \text{(Definition of Entropy)}$$

$$P = \rho R T \quad \text{or} \quad PV = nR_0 T \quad \text{(Ideal Gas Equations)}$$

$$C_v = \left(\frac{du}{dT}\right)\bigg|_V \quad \text{and} \quad C_p = \left(\frac{dh}{dT}\right)\bigg|_P \quad \text{(Heat Capacities)}$$

3 Practice Problems

1. Derive the mean free path of a particle in a fluid.
2. Derive the buoyant force using Gauss's Theorem.
3. Suppose you have a wine barrel of height H, on the top of which is attached a long transparent tube in which water is poured. The barrel is expected to fail with a hoop stress of σ_θ which is a function of the barrel center-line pressure. If the barrel is expected to fail (burst) at hoop stress corresponding to a center-line pressure of P_f, what is height of the water within the transparent tube at the point of failure?
   - Suppose the 3 foot tall (0.9 m) barrel fails at 30 psia (2.1e5 Pa). Assume a standard atmospheric pressure of 14.5 psi (100 kPa), water density of 1000 kg/m3, and acceleration of gravity as 9.8 m/s2.
   - Suppose the width of the tube was doubled. How does the burst height change?
4. What fraction of an iceberg is above the water? Assume an ice density of 917 kg/m3.
5. A 300 Watt pressure cooker containing 1 kg of food operates for 1 minute under sealed, insulated conditions. If the food within was initially 27 degrees C, what is its temperature after one minute? You may assume no boiling occurs, and the fluid has a constant $C_v = 4.13$ kJ/kg K.
6. Last summer you purchased a Lamborghini Aventador, and made sure the tires were inflated with nitrogen to the recommended 33 psi (230 kPa) pressure. It is now winter and you no longer have any money to purchase a thermometer, but you do still have the pressure gage that came with the car. If the summer temperature was 85 F (30 C), and the tire pressure is currently 23 psi (158 kPa), then what is the current temperature? You may assume that it is acceptable to approximate the change in temperature as a reversible, adiabatic process with $\gamma = 1.4$. (Note that this is **not** an adiabatic process, however we can get a useful approximation this way.)
7. Derive the equation: $Tds = dh - \rho\nu$
8. (*) In a 3000 MW (3.0e9 W) nuclear boiling water reactor, the average 12 ft (3.7 m) tall channel produces 5 MW (5.0e6 W) of heat. Assuming that a saturated mixture of water is flowing through the channel at 0.14 Mlbm/hr (17.3 kg/s). You may assume that changes in potential and kinetic energy are negligible compared to the rise in water enthalpy, and you may assume that mass is conserved. If the inlet enthalpy is 1228.0 kJ/kg, what is the exit enthalpy of the steam at the top of the channel?

4 Practice Solutions

4.1

Derive the mean free path of a particle in a fluid.
We first start with a single fluid molecule modeled as a sphere with diameter d traveling with speed v through a fluid. This sphere can only interact with other spheres within an area $A = \pi d^2$. If the distance between molecule centers is greater than d then no interaction occurs, and if closer than d, then the molecule collide.
We can define the mean free path λ (expected distance traveled before a collision) as the reciprocal of the macroscopic cross section Σ (expected number of collisions per unit distance) or

$$\lambda = \frac{1}{\Sigma}$$

The macroscopic cross section is easier to heuristically derive. Suppose a molecule is traveling with an average *relative* velocity $\overline{v_{rel}}$ over some time Δt, then the number of expected collisions over this time will be

$$\begin{aligned}\Sigma &= \frac{\text{(No. collisions in path)}}{\text{(length of path)}}\\ &= \frac{\text{(No. of interactions in the path volume per atom)*(No. of mol. per unit volume)}}{\text{(length of path)}}\\ &= \frac{\text{(No. of interactions in the unit area per atom)*(length of path)*(No. of mol. per unit volume)}}{\text{(length of path)}}\end{aligned}$$

The length of the interacting path will be $\overline{v_{rel}}\Delta t$, the absolute travel length will be $v\Delta t$, the number of molecules per unit volume is known as the *number density* n of the fluid, and the interacting area will be the $A = \pi d^2$ that we calculated earlier. Incorporating these definitions:

$$\begin{aligned}\Sigma &= \frac{\pi d^2 n \overline{v_{rel}} \Delta t}{v \Delta t}\\ &= \frac{\pi d^2 n \overline{v_{rel}}}{v}\end{aligned}$$

The average relative velocity can be expressed in terms of the molecule velocity. The definition of the relative velocity of the particle to the colliding particle velocity v_c is defined as the vector $\overrightarrow{v_{rel}} = \overrightarrow{v} - \overrightarrow{v_c}$. The relative square speed can be taken as the norm of the relative velocity to itself:

$$v_{rel}^2 = \overrightarrow{v_{rel}} \cdot \overrightarrow{v_{rel}}$$

We can now substitute in the definition of the relative velocity and take the average:

$$\begin{aligned}v_{rel}^2 &= (\overrightarrow{v} - \overrightarrow{v_c}) \cdot (\overrightarrow{v} - \overrightarrow{v_c})\\ &= \overrightarrow{v} \cdot \overrightarrow{v} - 2\overrightarrow{v} \cdot \overrightarrow{v_c} + \overrightarrow{v_c} \cdot \overrightarrow{v_c}\\ \overline{v_{rel}^2} &= \overline{\overrightarrow{v} \cdot \overrightarrow{v}} - 2\overline{\overrightarrow{v} \cdot \overrightarrow{v_c}} + \overline{\overrightarrow{v_c} \cdot \overrightarrow{v_c}}\end{aligned}$$

Assuming all molecule interactions are isotropic (independent of angle), then $\overline{\overrightarrow{v} \cdot \overrightarrow{v_c}} = |v||v_c| \int_0^{2\pi} \theta cos(\theta) d\theta = 0$ and that molecules have the same average velocity, then we can approximate the relative velocity as

$$\overline{v_{rel}} = \sqrt{2} v$$

. Finally, we can substitute the relation into the macroscopic cross section to get

$$\Sigma = \sqrt{2}\pi d^2 n$$

And finally by taking the inverse to can obtain the mean free path

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n}$$

4.2

Derive the buoyant force using Gauss's Theorem.
Recall that the hydrostatic pressure force on a small surface dA of a an object located at a depth z is given by

$$\begin{aligned} P(z) &= P_0 - \rho g z \\ F_p(z) &= P_0 dA - \rho g z dA \end{aligned}$$

Because the hydrostatic force is applied over 90 degrees from the normal vector of the objects surface, the resulting force vector respect to the normal unit vector will be negative

$$\begin{aligned} \overrightarrow{P}(z) &= (\rho g z - P_0)\,\overrightarrow{e_f} \\ \overrightarrow{F_p}(z) &= (\rho g z dA - P_0 dA)\,\overrightarrow{e_f} \end{aligned}$$

Recalling that the hydrostatic pressure force is applied normal to the surface of the object, we can determine sum up the force from each individual surface area dA by integrating over all the normal forces interfacing the object's surface S.

$$\begin{aligned} F_b &= \oint_S \overrightarrow{F_p}(z) \cdot \overrightarrow{n} \\ &= \oint_S \overrightarrow{P}(z) \cdot \overrightarrow{n}\, dA \\ &= \int_V \overrightarrow{\nabla} \cdot \overrightarrow{P}(z) dV \end{aligned}$$

Because the internal pressure forces of the object only vary in the z axis per Pascal's Law

$$\begin{aligned} F_b &= \int_V \frac{dP(z)}{dz} dV \\ &= \int_V \rho g dV \\ &= \rho g \int_V dV \\ &= \rho g V \end{aligned}$$

4.3

Pascal's Barrel
From the equation for the change in hydrostatic pressure, we know that the pressure at the center-line of the barrel will be equal to the hydrostatic pressure of the water within the top half of the

barrel, the water height H_t of the tube, and the atmospheric pressure at the top of the water P_0 level within the tube:

$$P_B = \frac{1}{2}\rho g H + \rho g H_t + P_0$$

Knowing that the barrel will fail at a pressure P_f, then we can solve for the tube water height:

$$H_t = \frac{P_f - \frac{1}{2}\rho g H - P_0}{\rho g}$$

Pascal's Barrel - Numerical
Around 11 m.
Pascal's Barrel - size
There is no change. Pressure depends only on the height of the liquid; this is the fundamental principle to the operation of hydraulic mechanisms.

4.4

What fraction of an iceberg is above the water? Assume an ice density of 917 kg/m3.
First we start with a force balance: the weight of the iceberg is equal to the buoyancy force.

$$\begin{aligned} W_{iceberg} &= F_b \\ \rho_{ice} g V_{iceberg} &= \rho_{water} g V_{disp} \\ V_{disp} &= \frac{V_{iceberg}\rho_{ice}}{\rho_{water}} \end{aligned}$$

If the fraction remaining above the surface is $f = \frac{V_{iceberg} - V_{disp}}{V_{iceberg}}$ then

$$\begin{aligned} f &= \frac{V_{iceberg} - V_{disp}}{V_{iceberg}} \\ &= \frac{V_{iceberg} - \frac{V_{iceberg}\rho_{ice}}{\rho_{water}}}{V_{iceberg}} \\ &= 1 - \frac{\rho_{ice}}{\rho_{water}} \\ &\simeq 10\% \end{aligned}$$

4.5

Pressure Cooker
Because no mass is exchanged, no heat is diffused, and the volume maintains constant, we can use the simple heat capacity relation to approximate the temperature change:

$$m\Delta h = mC_v \Delta T = \dot{q}\Delta t$$
$$T_f = T_0 + \frac{\dot{q}\Delta t}{C_v m}$$

T_f = 32.8 degrees C (Note: This example is limited to only temperatures within the range of $C_v(T) \approx C_v$)

4.6

Lambo Blues
We begin with the pressure ratio equation for an ideal gas undergoing a reversible, adiabatic process:

$$\frac{P_f}{P_i} = \left(\frac{T_f}{T_i}\right)^{\frac{\gamma}{\gamma-1}}$$
$$T_f = \frac{T_i}{\frac{\gamma-1}{\gamma} ln\left(\frac{P_f}{P_i}\right)}$$

The temperature is approximately 0 degrees C.

4.7

Derive the equation: $Tds = dh - \rho\nu$
Start with the definition of enthalpy:

$$\begin{aligned} h &= u + p\nu \\ dh &= d(u + p\nu) \text{ (take derivative)} \\ &= du + dp\nu + d\nu p \text{ (chain rule)} \end{aligned}$$

From the First Law of Thermodynamics for a reversible process:

$$\begin{aligned} dq &= du + pd\nu \\ &= (dh - dp\nu - d\nu p) + pd\nu \\ &= dh - \rho\nu \end{aligned}$$

From the definition of Entropy:

$$\begin{aligned} Tds &= dq \\ Tds &= dh - \rho\nu \end{aligned}$$

4.8

Nuclear Fuel Channel
We can start with the First Law of Thermodynamics:

$$d\dot{E} = d\dot{Q} + d\dot{W}$$

We can integrate over the volume of the channel:

$$\int_V d\dot{E} = \int_V d\dot{Q} + \int_V d\dot{W}$$

We can assume that the fluid thermodynamic properties are approximately constant over the width of the channel (1D channel approximation), leaving only an integration over the height:

$$\begin{aligned} \int_H d\dot{E} &= \int_H d\dot{Q} + \int_H d\dot{W} \\ \dot{E}_{outlet} - \dot{E}_{inlet} &= \int_H d\dot{Q} + \int_H d\dot{W} \end{aligned}$$

Assuming the work imposed upon the reactor internals is minimal, then the work generated by the fluid will be the pressure work at the inlet and outlet:

$$\dot{E}_{outlet} - \dot{E}_{inlet} = \int_H d\dot{Q} + p_{inlet}\nu_{inlet}\dot{m} - p_{outlet}\nu_{outlet}\dot{m}$$

The integrated heat over the bundle height will be the energy produced by the bundle q_{bundle}, which was given to be 5MW:

$$\dot{E}_{outlet} - \dot{E}_{inlet} = q_{bundle} + p_{inlet}\nu_{inlet}\dot{m} - p_{outlet}\nu_{outlet}\dot{m}$$

The energy of the fluid will consist of three parts: kinetic energy, potential energy, and internal energy. Per the given assumption, we can disregard the change in and potential energy.

$$u_{outlet}\dot{m} - u_{inlet}\dot{m} = q_{bundle} + p_{inlet}v_{inlet}\dot{m} - p_{outlet}\dot{m}\nu_{outlet}$$
$$u_{outlet}\dot{m} + p_{outlet}\nu_{outlet}\dot{m} - u_{inlet}\dot{m} - p_{inlet}\nu_{inlet}\dot{m} = q_{bundle}$$

Substituting in the definition of enthalpy:

$$h_{outlet}\dot{m} - h_{inlet}\dot{m} = q_{bundle} + \rho_{inlet}gz_{inlet} - \rho_{outlet}gz_{outlet}$$
$$h_{outlet} = h_{inlet} + \frac{q_{bundle}}{\dot{m}}$$

Using the given values, the exit enthalpy is 1523 kJ/kg