1 Key Concepts

- Continuum Hypothesis
- Viscosity
- Hydrostatic Pressure
- Surface Tension
- First Law of Thermodynamics
- Adiabatic and Reversible Proceses
- Entropy
- Ideal Gases
- Conservation of Mass and Momentum
- Bernoulli's Equation
- Lagrangian vs. Eulerian View
- Stream Lines, Path Lines, Streak lines
- Material Derivative
- Compressible vs. Incompressible flow
- Rotational vs. Irrotational flow
- Reynold's Transport Theorem
- Conservation of Mass, Energy, and Momentum
- Line Integrals
- Integrals in polar, cylindrical, and spherical coordinates
- Navier-Stokes Equations
- Low-Reynolds Number Flows
- Thin-Film Lubrication
- Free Vorticies
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- Free Vorticies
- Kelvin's and Helmholtz's Theorems
- Biot-Savart Law
- Differential Equation Basics
- Potential flow

2 Important Equations

2.1 Basic Definitions

$$\tau = \mu \frac{du}{dx} \qquad \text{(Newton's Law of Viscosity)}$$

$$P = \frac{F}{A} \qquad \text{(Definition of Pressure)}$$

$$h = u + \rho\nu \qquad \text{(Definition of Enthalpy)}$$

$$Tds = dq \qquad \text{(Definition of Entropy)}$$

$$C_v = \left(\frac{du}{dT}\right)\bigg|_V \quad \text{and} \quad C_p = \left(\frac{dh}{dT}\right)\bigg|_P \qquad \text{(Heat Capacities)}$$

2.2 Calculus Rules

$$\frac{Df}{Dt} = \frac{df}{dt} + \overrightarrow{u} \cdot \nabla f \qquad \text{(Material Derivative)}$$

$$\int_V \overrightarrow{\nabla} \cdot \overrightarrow{F} dV = \oint_S \overrightarrow{F} \cdot \overrightarrow{n} dS \qquad \text{(Gauss's Theorem)}$$

$$\oint_C \overrightarrow{u} \cdot \overrightarrow{t} ds = \int_A \left(\overrightarrow{\nabla} \times \overrightarrow{u}\right) \cdot \overrightarrow{n} dA \qquad \text{(Stoke's Theorem)}$$

$$\frac{d}{dt}\int_{V(t)} F\left(\overrightarrow{u}, t\right) dV = \int_{V(t)} \frac{dF}{dt} dV + \int_{S(t)} F\left(\overrightarrow{u}, t\right) \overrightarrow{b} \cdot \overrightarrow{n} dS \qquad \text{(Reynolds Transport Theorem)}$$

$$dA = rdrd\theta \qquad \text{(Differential Surface Area in Cylindical Coordinates)}$$

$$dV = rdrd\theta dz \qquad \text{(Differential Volume in Cylindical Coordinates)}$$

$$dA = \rho^2 sin(\phi) d\phi d\theta \qquad \text{(Differential Surface Area in Spherical Coordinates)}$$

$$dV = \rho^2 sin(\phi) d\phi d\theta d\rho \qquad \text{(Differential Volume in Spherical Coordinates)}$$

$$\overrightarrow{u} \cdot \overrightarrow{\nabla} = u_r \frac{d}{dr} + \frac{u_\theta}{r}\frac{d}{d\theta} + u_z \frac{d}{dz} \tag{1}$$

$$\nabla = \left\langle \frac{d}{dr}, \frac{1}{r}\frac{d}{d\theta}, \frac{d}{dz} \right\rangle \tag{2}$$

$$\nabla^2 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) + \frac{1}{r^2}\frac{d^2}{d\theta^2} + \frac{d^2}{dz^2} \tag{3}$$

2.3 Conservation Equations

$$\frac{dM}{dt} = \int_S \rho \overrightarrow{u} \cdot \overrightarrow{n} dS \qquad \text{(Conservation of Mass)}$$

$$\frac{dM}{dt} = \dot{m}_{in} - \dot{m}_{out} \qquad \text{(Conservation of Mass (simple))}$$

$$de = dq + dw \quad \text{or} \quad \frac{de}{dt} = \dot{w} + \dot{q} \qquad \text{(First Law of Thermodynamics)}$$

$$\int_V \frac{d}{dt} \rho \overrightarrow{u} dV + \int_A \rho \overrightarrow{u} \left(\overrightarrow{u} \cdot \overrightarrow{n} \right) dA = \int_V \rho \overrightarrow{g} dV + \int_A \overrightarrow{f} dA \qquad \text{(Integral Conservation of Momentum)}$$

$$\rho \left(\frac{d\overrightarrow{u}}{dt} + \overrightarrow{u} \cdot \nabla \overrightarrow{u} \right) = \rho \overrightarrow{g} - \nabla P \qquad \text{(Differential Conservation of Momentum)}$$

$$\frac{du_r}{dt} + \left(\overrightarrow{u} \cdot \overrightarrow{\nabla} \right) u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{dP}{dr} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{du_\theta}{d\theta} \right) \tag{4}$$

$$\frac{du_\theta}{dt} + \left(\overrightarrow{u} \cdot \overrightarrow{\nabla} \right) u_\theta + \frac{u_\theta u_r}{r} = -\frac{1}{\rho r} \frac{dP}{d\theta} + \nu \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{du_r}{d\theta} \right) \tag{5}$$

$$\frac{du_z}{dt} + \left(\overrightarrow{u} \cdot \overrightarrow{\nabla} \right) u_z = -\frac{1}{\rho} \frac{dP}{dz} + \nu \nabla^2 u_z \qquad \text{(Navier-Sokes Equations - Cylindrical)}$$

$$\frac{D\overrightarrow{u}}{Dt} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \overrightarrow{u} \qquad \text{(Conservation of Momentum - low Re)}$$

$$u(x,y,t) \cong -\frac{h^2(x,t)}{2\mu} \frac{dp(x,t)}{dx} \frac{y}{h(x,t)} \left(1 - \frac{y}{h(x,t)} \right) + \left(U_h(t) - U_0(t) \right) \frac{y}{h(x,t)} + U_0(t)$$

(Navier-Stokes Thin-Film Velocity Approximation)

2.4 Derived Equations

$$H = \frac{v^2}{2} + \overrightarrow{g} \cdot \overrightarrow{r} + \frac{p}{\rho} \qquad \text{(Bernoulli's Equation)}$$

$$P = \rho R T \quad \text{or} \quad PV = nR_0T \qquad \text{(Ideal Gas Equations)}$$

$$F_1 A_1 = F_2 A_2, \text{ where } z_1 = z_2 \qquad \text{(Pascal's Law)}$$

$$\Delta P = \rho g z \qquad \text{(Change in Hydrostatic Pressure)}$$

$$F_b = \rho g V \qquad \text{(Buoyancy Force)}$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{u} = 0 \qquad \text{(Criteria for Incompressible Flow)}$$

$$\overrightarrow{\nabla} \times \overrightarrow{u} = \overrightarrow{0} \qquad \text{(Criteria for Irrotational Flow)}$$

$$\frac{du_i}{dx_j} = S_{ij} + \frac{1}{2} R_{ij} \qquad \text{(Velocity Gradient Tensor)}$$

$$S_{ij} = \frac{1}{2} \left(\frac{du_i}{dx_j} + \frac{du_j}{dx_i} \right) \qquad \text{(Strain Rate Tensor)}$$

$$R_{ij} = \frac{du_i}{dx_j} - \frac{du_j}{dx_i} \qquad \text{(Rotation Tensor)}$$

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{Streamline Criteria}$$

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \overrightarrow{u}) = 0$$
$$\rho\left(\frac{d\overrightarrow{u}}{dt} + \overrightarrow{u} \cdot \nabla \overrightarrow{u}\right) = \rho \overrightarrow{g} - \nabla P \tag{Euler Equations}$$

$$Re = \frac{\rho u D}{\mu} \tag{Reynolds Number}$$

$$u_\theta(r) = \begin{cases} \frac{\Gamma r}{2\pi R^2} & r \leq R \\ \frac{\Gamma}{2\pi r} & r > R \end{cases} \tag{Rankine Vortex}$$

$$\frac{D\overrightarrow{\omega}}{Dt} = (\overrightarrow{\omega} \cdot \nabla)\overrightarrow{u} + \nu \nabla^2 \overrightarrow{\omega} \tag{Conservative Body Vorticity Equation}$$

$$\overrightarrow{u}(\overrightarrow{x}, t) = \frac{1}{4\pi} \int_V' \frac{\overrightarrow{\omega}(\overrightarrow{x'}, t) \times \left(\overrightarrow{x} - \overrightarrow{x'}\right)}{|\overrightarrow{x} - \overrightarrow{x'}|^3} d^3x' \tag{Biot-Savart Induced Flow}$$

$$\overrightarrow{u}(\overrightarrow{x}, t) = \frac{\Gamma}{4\pi} \int_{vortex} \overrightarrow{e}_\omega \times \frac{\left(\overrightarrow{x} - \overrightarrow{x'}\right)}{|\overrightarrow{x} - \overrightarrow{x'}|^3} dl \tag{Biot-Savart Induced Flow - Thin Vortex}$$

$$\frac{D\Gamma}{Dt} = \int_C \frac{1}{\rho} \frac{d\tau_{ij}}{dx_j} dx_i \tag{Material Derivative of Circulation}$$

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x},\ u = \frac{\partial \phi}{\partial x},\ v = \frac{\partial \phi}{\partial y} \tag{Potential Flow Definitions}$$

$$\nabla^2 \Phi = q(r) \tag{Poisson Equation}$$

$$\nabla^2 \Phi = 0 \tag{Laplace Equation}$$

$$\phi_{point} = \frac{q_s}{2\pi} ln|\overrightarrow{x} - \overrightarrow{x}'| \tag{Potential of a Point-Source Element}$$

$$\phi_{Uniform} = U\overrightarrow{x} \tag{Potential of a Uniform Flow Field}$$

$$H = p + \frac{1}{2}\rho|\nabla\phi|^2 = p + \frac{1}{2}\rho|\nabla\psi|^2 \tag{Bernoulli's Equation with Potential Flows}$$

3 Modeling Approach

1. Write out in standard form what is given (e.g. P, $\overrightarrow{u}$, F, etc.)
2. Write out in standard form what property, if any, is being requested.
3. Sketch the problem. If forces are involved, sketch a free-body diagram including all applied forces and body forces (weight).
4. Based on the given and requested properties, write out the equations that relate the given and requested properties.
   (a) Conservation Equations **always** apply.
   (b) If forces, momentum, or pressure is involved with a flowing fluid $\rightarrow$ Conservation of Momentum Equation, using the free-body diagram to determine the net forces applied
   (c) If forces, momentum, or pressure is involved with a flowing fluid **and** steady, inviscid, and incompressible flow can be assumed between two points $\rightarrow$ Bernoulli's Equation
   (d) If temperature, heat, or work of a fluid is involved $\rightarrow$ First Law of thermodynamics
   (e) Hydrostatic Pressure of still fluid $\rightarrow$ Pascal's Law, Buoyancy Force, Change in Hydrostatic Pressure
   (f) Behavior of Individual particles $\rightarrow$ Lagrangian forms of Newton's Laws, Velocity, Path lines
   (g) Behavior of Flow at a location $\rightarrow$ Eulerian forms of Velocity, Material derivative, and conservation of momentum.
   (h) A time derivative of an extrinsic property $\rightarrow$ Reynolds Transport Theorem of the intrinsic property
   (i) If ideal flow can be assumed and velocity or pressure profiles are needed $\rightarrow$ potential flow
   (j) If a vortex is involved $\rightarrow$ Biot-Savart, vorticity equations, and and circulation equations.
5. Eliminate unnecessary components using appropriate approximations
   (a) Steady flow $\rightarrow \frac{du}{dt} = 0$
   (b) Constant/Incompressible flow $\rightarrow \frac{d\rho}{dt} = 0, \overrightarrow{\nabla} \cdot \overrightarrow{u} = 0$
   (c) Inviscid/Irrotational flow $\rightarrow \mu = 0, \overrightarrow{\nabla} \times \overrightarrow{u} = \overrightarrow{0}$
6. Use algebra and calculus to solve for the requested property with respect to the given properties
   (a) Gauss's Theorem and Stoke's Theorem to convert bounds of integrated differentials
   (b) Conversion of Lagrangian to Eulerian coordinates
   (c) Converting line integrals over curves to Cartesian coordinates

4 Practice Problems

1. (Example 4.9 of *Fluid Mechanics, Sixth Edition*) Write out the Navier-Stokes equations in two-dimensional (x,y)-coordinates when $u = (u, v)$, and simplify them to the one-dimensional flow case when $u = u(x, t)$ and $v = 0$.