Starting with the elastic wave equation in first order form on the domain $[0, 1]$

$$\begin{aligned} \rho v_t &= \sigma_x \\ \sigma_t &= \mu v_x \end{aligned}$$

we write

$$\begin{bmatrix} v \\ \sigma \end{bmatrix}_t = \begin{bmatrix} 0 & \frac{1}{\rho} \\ \mu & 0 \end{bmatrix} \begin{bmatrix} v \\ \sigma \end{bmatrix}_x.$$

This may be diagonalized as

$$\begin{bmatrix} v \\ \sigma \end{bmatrix}_t = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1} \begin{bmatrix} v \\ \sigma \end{bmatrix}_x.$$

where $\mathbf{P} = \begin{bmatrix} k & k \\ 1 & -1 \end{bmatrix}$, $\mathbf{P}^{-1} = \frac{1}{2k} \begin{bmatrix} 1 & k \\ 1 & -k \end{bmatrix}$, $\mathbf{\Lambda} = \begin{bmatrix} c_s & 0 \\ 0 & -c_s \end{bmatrix}$, for $c_s = \sqrt{\frac{\mu}{\rho}}$ the shear wave speed and $k = (\sqrt{\mu\rho})^{-1}$. By defining the transformation

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{P}^{-1} \begin{bmatrix} v \\ \sigma \end{bmatrix}$$

and multiplying by $\mathbf{P}^{-1}$ on the left we get the decoupled system

$$\mathbf{z}_t = \mathbf{\Lambda}\mathbf{z}_x.$$

This decoupled system

$$\begin{aligned} (z_1)_t - c_s(z_1)_x &= 0 \\ (z_2)_t + c_s(z_2)_x &= 0 \end{aligned}$$

tells us that a solution z_1 will be a left traveling wave and z_2 will be a right-traveling wave. So we must specify a boundary condition for z_1 at $x = 1$ and a boundary condition for z_2 at $x = 0$:

$$\begin{aligned} z_1(1, t) &= g_1 \quad t > 0 \\ z_2(0, t) &= g_2 \quad t > 0. \end{aligned}$$

Semidiscrete problem

$$\mathbf{z}_t = (\boldsymbol{\Lambda} \otimes \boldsymbol{D})\mathbf{z} + \boldsymbol{\Sigma}\mathbf{z}$$

where $\boldsymbol{\Lambda}$ 2×2 diagonal matrix of eigenvalues, $\boldsymbol{D}$ is a $N \times N$ SBP finite difference approximation to the first derivative. z is a $2N \times 1$ vector consisting of z_1 stacked on top of z_2, that is

$$\mathbf{z} = \begin{bmatrix} z_{11} \\ z_{12} \\ \vdots \\ z_{1N} \\ z_{21} \\ z_{22} \\ \vdots \\ z_{2N} \end{bmatrix}.$$

Finally, take $\boldsymbol{H}$ to be the quadrature matrix associated with $\mathbf{D}$ and $\mathcal{H} = I_2 \otimes \boldsymbol{H}$. We proceed by multiplying $\mathbf{z}_t$ by $\mathbf{z}^T\mathcal{H}$ and adding the transpose:

$$\begin{aligned} \mathbf{z}^T\mathcal{H}\mathbf{z}_t + (\mathbf{z}^T\mathcal{H}\mathbf{z}_t)^T &= \mathbf{z}^T\mathcal{H}(\boldsymbol{\Lambda} \otimes \boldsymbol{D})\mathbf{z} + \mathbf{z}^T\mathcal{H}\boldsymbol{\Sigma}\boldsymbol{z} + \mathbf{z}^T(\boldsymbol{\Lambda} \otimes \boldsymbol{D})^T\mathcal{H}\mathbf{z} + \mathbf{z}^T\boldsymbol{\Sigma}^T\mathcal{H}\boldsymbol{z} \\ &= \mathbf{z}^T\left[(\boldsymbol{\Lambda} \otimes \boldsymbol{HD}) + (\boldsymbol{\Lambda} \otimes (\boldsymbol{HD})^T)\right]\mathbf{z} + \mathbf{z}^T\left[\mathcal{H}\boldsymbol{\Sigma} + (\mathcal{H}\boldsymbol{\Sigma})^T\right]\mathbf{z} \\ &= \mathbf{z}^T\left[\boldsymbol{\Lambda} \otimes (\boldsymbol{Q} + \boldsymbol{Q}^T)\right]\mathbf{z} + \mathbf{z}^T\left[\mathcal{H}\boldsymbol{\Sigma} + (\mathcal{H}\boldsymbol{\Sigma})^T\right]\mathbf{z} \\ &= \mathbf{z}^T\left[\boldsymbol{\Lambda} \otimes (\boldsymbol{E}_N - \boldsymbol{E}_0)\right]\mathbf{z} + \mathbf{z}^T\left[\mathcal{H}\boldsymbol{\Sigma} + (\mathcal{H}\boldsymbol{\Sigma})^T\right]\mathbf{z} \\ &= \lambda_1(z_{1N}^2 - z_{11}^2) + \lambda_2(z_{2N}^2 - z_{21}^2) + \mathbf{z}^T\left[\mathcal{H}\boldsymbol{\Sigma} + (\mathcal{H}\boldsymbol{\Sigma})^T\right]\mathbf{z}. \end{aligned}$$

Though I neglected to mention it earlier, the penalty matrix $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} \alpha_1\boldsymbol{H}^{-1}\boldsymbol{E}_N & -\alpha_1\boldsymbol{H}^{-1}R_N\boldsymbol{E}_N \\ -\alpha_2\boldsymbol{H}^{-1}R_0\boldsymbol{E}_0 & \alpha_2\boldsymbol{H}^{-1}\boldsymbol{E}_0 \end{bmatrix}$$

and we compute

$$\begin{aligned} \mathcal{H}\boldsymbol{\Sigma} &= (I_2 \otimes \boldsymbol{H})\begin{bmatrix} \alpha_1\boldsymbol{H}^{-1}\boldsymbol{E}_N & -\alpha_1\boldsymbol{H}^{-1}R_N\boldsymbol{E}_N \\ -\alpha_2\boldsymbol{H}^{-1}R_0\boldsymbol{E}_0 & \alpha_2\boldsymbol{H}^{-1}\boldsymbol{E}_0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1\boldsymbol{E}_N & -\alpha_1R_N\boldsymbol{E}_N \\ -\alpha_2R_0\boldsymbol{E}_0 & \alpha_2\boldsymbol{E}_0 \end{bmatrix}. \end{aligned}$$

Finally we have

$$\begin{aligned} \mathbf{z}^T\mathcal{H}\mathbf{z}_t + (\mathbf{z}^T\mathcal{H}\mathbf{z}_t)^T = \lambda_1 z_{1N}^2 - \lambda_1 z_{11}^2 + \lambda_2 z_{2N}^2 - \lambda_2 z_{21}^2 + 2\alpha_1 z_{1N}^2 + 2\alpha_2 z_{21}^2 \\ - 2\alpha_2R_0z_{11}z_{21} - 2\alpha_1R_Nz_{1N}z_{2N}. \end{aligned}$$

To deal with the final two terms in the previous equation, we can use the fact that for $a, b \in \mathbb{R}$

$$2ab \leq a^2 + b^2,$$

and in our case

$$\begin{aligned} 2(R_0 z_{11})(z_{21}) &\leq R_0^2 z_{11}^2 + z_{21}^2 \\ 2(R_N z_{2N})(z_{1N}) &\leq R_N^2 z_{2N}^2 + z_{1N}^2. \end{aligned}$$

Now we employ the fact that $\alpha_1, \alpha_2 < 0$ and thus $-\alpha_1 > 0$ and $-\alpha_2 > 0$. Note that because we also have that $-2ab \leq a^2 + b^2$ then if α_1 or α_2 turned out to be a positive number we can still use this approach by replacing 2 with -2 in the above inequalities. Multiplying by $-\alpha_1, -\alpha_2$

$$\begin{aligned} 2(-\alpha_2)(R_0 z_{11})(z_{21}) &\leq -\alpha_2 R_0^2 z_{11}^2 - \alpha_2 z_{21}^2 \\ 2(-\alpha_1)(R_N z_{2N})(z_{1N}) &\leq -\alpha_1 R_N^2 z_{2N}^2 - \alpha_1 z_{1N}^2. \end{aligned}$$

Returning to $\mathbf{z}^T \mathcal{H} \mathbf{z}_t + (\mathbf{z}^T \mathcal{H} \mathbf{z}_t)^T$ we can now write

$$\begin{aligned} \mathbf{z}^T \mathcal{H} \mathbf{z}_t + (\mathbf{z}^T \mathcal{H} \mathbf{z}_t)^T &= \lambda_1 z_{1N}^2 - \lambda_1 z_{11}^2 + \lambda_2 z_{2N}^2 - \lambda_2 z_{21}^2 + 2\alpha_1 z_{1N}^2 + 2\alpha_2 z_{21}^2 \\ &\quad - 2\alpha_2 R_0 z_{11} z_{21} - 2\alpha_1 R_N z_{1N} z_{2N} \\ &\leq \lambda_1 z_{1N}^2 - \lambda_1 z_{11}^2 + \lambda_2 z_{2N}^2 - \lambda_2 z_{21}^2 + 2\alpha_1 z_{1N}^2 + 2\alpha_2 z_{21}^2 \\ &\quad - \alpha_2 R_0^2 z_{11}^2 - \alpha_2 z_{21}^2 - \alpha_1 R_N^2 z_{2N}^2 - \alpha_1 z_{1N}^2 \\ &= z_{1N}^2(\lambda_1 + 2\alpha_1 - \alpha_1) + z_{21}^2(2\alpha_2 - \lambda_2 - \alpha_2) \\ &\quad + z_{11}^2(-\alpha_2 R_0^2 - \lambda_1) + z_{2N}^2(-\alpha_1 R_N^2 + \lambda_2) \\ &= z_{1N}^2(\lambda_1 + \alpha_1) + z_{21}^2(\alpha_2 - \lambda_2) \\ &\quad + z_{11}^2(-\alpha_2 R_0^2 - \lambda_1) + z_{2N}^2(-\alpha_1 R_N^2 + \lambda_2). \end{aligned}$$

From the first two terms we can gather that $\alpha_1 \leq -\lambda_1$ and $\alpha_2 \leq \lambda_2 = -\lambda_1$. Moreover, if we choose $\alpha_1 = \alpha_2 = -\lambda_1$ we have

$$\begin{aligned} \mathbf{z}^T \mathcal{H} \mathbf{z}_t + (\mathbf{z}^T \mathcal{H} \mathbf{z}_t)^T &\leq z_{11}^2(\lambda_1 R_0^2 - \lambda_1) + z_{2N}^2(\lambda_1 R_N^2 - \lambda_1) \\ &= \lambda_1 \left[z_{11}^2(R_0^2 - 1) + z_{2N}^2(R_N^2 - 1) \right] \end{aligned}$$

Energy Method for the Continuous Problem

Consider the continuous form of the diagonalized system we derived from the first-order wave equation

$$\begin{aligned} \eta_t &= C_s \eta_x \\ \xi_t &= -C_s \xi_x \end{aligned}$$

$$\begin{aligned} \eta(1) &= R_N \xi(1) & \qquad |R_N| \leq 1 \\ \xi(0) &= R_0 \eta(0) & \qquad |R_0| \leq 1 \end{aligned}$$

Now let $\mathbf{z} = \begin{bmatrix} \eta \\ \xi \end{bmatrix}$ and integrate $\mathbf{z}^T \mathbf{z}_t$ over the domain $[0, 1]$. This yields

$$\begin{aligned} \int_0^1 \mathbf{z}^T \mathbf{z}_t \, dx &= \int_0^1 \eta\eta_t + \xi\xi_t \, dx & &= \int_0^1 C_s \eta\eta_x - C_s \xi\xi_x \, dx \\ &= \int_0^1 \frac{\partial}{\partial t}\left(\frac{\eta^2}{2} + \frac{\xi^2}{2}\right) dx & &= C_s \left[\frac{\eta^2}{2}\bigg|_0^1 - \frac{\xi^2}{2}\bigg|_0^1 \right] \\ &= \frac{1}{2}\frac{\partial}{\partial t} \int_0^1 |\mathbf{z}|^2 \, dx & &= \frac{C_s}{2}\Big[\eta^2(1) - \eta^2(0) - \xi^2(1) + \xi^2(0)\Big] \\ &= \frac{1}{2}\frac{\partial}{\partial t} \|\mathbf{z}\|_2^2 & &= \frac{C_s}{2}\Big[R_N^2 \xi^2(1) - \eta^2(0) - \xi^2(1) + R_0^2 \eta^2(0)\Big] \\ &= \frac{1}{2}\frac{\partial}{\partial t} \|\mathbf{z}\|_2^2 & &= \frac{C_s}{2}\Big[\eta^2(0)\big(R_0^2 - 1\big) + \xi^2(1)\big(R_N^2 - 1\big)\Big], \end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$ and $\|\cdot\|_2$ is the $\mathcal{L}^2$-norm.