Exam 1 - Practice Questions

Rules:

- Show all your work and reasoning how you arrive at your answer. Just writing an answer is not sufficient for most problems.
- When in doubt, *draw the picture!!!*

Problems:

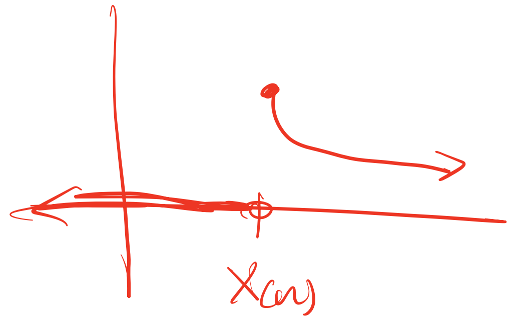
1. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution $\mathrm{U}[-1, \theta]$, where $\theta > -1$.
   a) Obtain the MLE for EX_1.
   b) Obtain the MLE for $\mathrm{var}(X_1)$.
   c) Obtain two Method of Moments estimators for $\mathrm{var}(X_1)$.
   d) Now suppose we know that $\theta \geq 0$. Obtain the MLE for θ.

2. You participate in a study that aims to develop a new vaccine to protect against Covid-19 infection. The new vaccine has been applied in two independent trials. The first trial has led to a sample of size n from a Bernoulli(θ_1) model and the second trial to a sample of size m from a Bernoulli(θ_2) model, where θ_1, θ_2 are the success probabilities that the vaccine is effective.
   a) Obtain the MLEs $\hat{\theta}_1, \hat{\theta}_2$ for θ_1, θ_2.
   b) Provide arguments why θ_1, θ_2 might be the same or might be different. What about $\hat{\theta}_1, \hat{\theta}_2$?
   c) It is now assumed that $\theta_1 = \theta_2$. Calculate $\mathrm{var}(\hat{\theta}_1)$, $\mathrm{var}(\hat{\theta}_2)$.
   d) Under this assumption, a scientist suggests to combine the two MLEs in a suitable way to obtain an improved estimator that takes all of the available information into account. For that, one considers combinations $\hat{\theta}_c = c\hat{\theta}_1 + (1-c)\hat{\theta}_2$ for $0 \leq c \leq 1$. Find the best c such that $\mathrm{var}(\hat{\theta}_c)$ is minimized.

3. Consider a sample of size n from an exponential distribution with parameter $\beta > 0$. Assume the r.v. X has this distribution.
   a) Find a method of moments estimator for β.
   b) Find a method of moments estimator for $P(0 \leq X \leq 2)$.
   c) Find a method of moments estimator for the median of the distribution.

1. $X_i \overset{iid}{\sim} U[-1, \theta]$

a. MLE for θ:



$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \frac{1}{(\theta+1)^n} I(X_{(n)} < \theta)$$

The likelihood is monotone decreasing after $\hat{\theta} = X_{(n)} = \max_i X_i \Rightarrow$

$\hat{\theta} = X_{(n)} = \max_i X_i$ is the MLE of θ.

Then,

$$E(X) = \frac{\theta - 1}{2} = g(\theta).$$

By invariance of MLEs,

$$\widehat{E(X)} = g(\hat{\theta}) = \frac{X_{(n)} - 1}{2}$$

b. $Var(X) = \dfrac{(\theta - (-1))^2}{12} = \dfrac{(\theta+1)^2}{12} = q(\theta)$

$\Rightarrow$ Invariance $\Rightarrow$

$$\widehat{Var(X)} = \frac{(\hat{\theta}+1)^2}{12} = \frac{(X_{(n)}+1)^2}{12}.$$

c. Method of Moment est for θ:

$\boxed{1}$ $E(X) = \dfrac{\theta-1}{2} \overset{!}{=} m_1 = \bar{X}$

$\Rightarrow \hat{\theta}_{mom} = 2\bar{X}+1$

Then $Var(X) = q(\theta) = \dfrac{(\theta+1)^2}{12}$

$$\Rightarrow \widehat{Var(X)} = q(\hat{\theta}_{mom}) = \left(\frac{2\bar{X}+2}{12}\right)^2$$

$$= \left(\frac{\bar{X}+1}{6}\right)^2$$

12 Another:

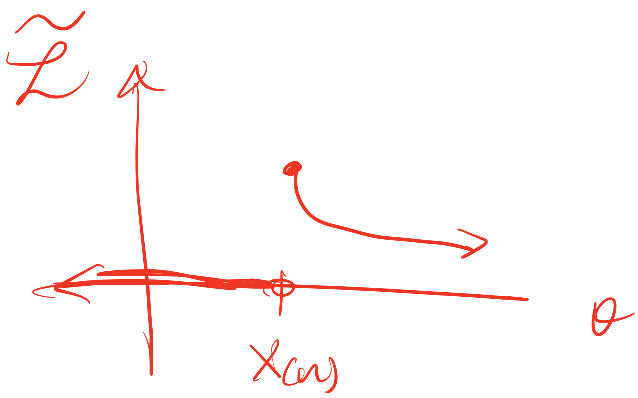
$$Var(X) = h(\mu_1, \mu_2) = \mu_2 - \mu_1^2$$

replace population moments w/ sample versions:

$$\widehat{Var(X)} = m_2 - \mu_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2.$$

d. Now $\Omega = [0, \infty)$,

if $X_{(n)} > 0$ then no problem

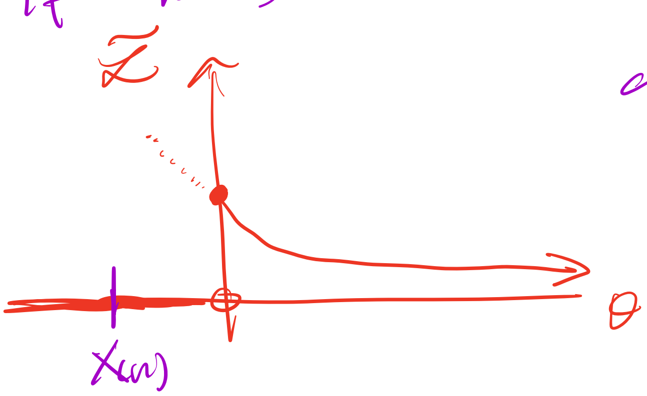


$\Rightarrow \hat{\theta} = X_{(n)}$.

$\in \Omega$:)

If not, then the MLE is actually at $\hat{\theta} = 0$.

$\in \Omega$:)



2. Let sample 1 be:

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(\theta_1)$

& Sample 2 be:

$Y_1, \ldots, Y_m \overset{iid}{\sim} Bern(\theta_2)$.

a. Sample 1:

$$\mathcal{L}(\theta_1) = \prod_{i=1}^{n} \theta_1^{x_i} (1-\theta)^{1-x_i}$$

$$= \theta_1^{\sum_i x_i} (1-\theta_1)^{n-\sum_i x_i}$$

$$\ell(\theta) = \left(\sum_i x_i\right) \log \theta_1 + \left(n - \sum_i x_i\right) \log(1-\theta_1)$$

$$\frac{\partial \ell}{\partial \theta_1} = \frac{\sum_i x_i}{\theta_1} + \frac{n - \sum_i x_i}{1-\theta_1}(-1) \overset{!}{=} 0$$

$$\Rightarrow \sum_i x_i - \left(\sum_i x_i\right)\theta_1 = n\theta_1 - \left(\sum_i x_i\right)\theta_1$$

$$\Rightarrow \hat{\theta}_1 = \frac{\sum_i x_i}{n} = \bar{x}$$

2nd deriv check:

$$\frac{\partial^2 \ell}{\partial \theta_1^2} = \frac{-\sum_i x_i}{\theta_1^2} - \frac{(n - \sum_i x_i)}{(1-\theta_1)^2} < 0 \quad \forall \theta \in \Omega$$

$\Rightarrow$ Yes. Maximizer.

By an identical argument,

the MLE for θ_2 is

$$\hat{\theta}_2 = \frac{\sum_{j=1}^{m} y_j}{m} = \bar{y}$$

b. The parameters θ_1 & θ_2 may be diff b/c for instance the 2 studies used 2 distinct vaccines which have different efficacies.

On the other hand, regardless of if $\theta_1 = \theta_2$ or not, it's very possible (likely rather) that $\hat{\theta}_1 \neq \hat{\theta}_2$ as they are both random quantities!

c. Call $\theta = \theta_1 = \theta_2$.

Then

$$Var(\hat{\theta}_1) = Var\left(\frac{1}{n}\sum_i X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_i X_i\right)$$

(II)

$$= \frac{1}{n^2}\sum_i Var(X_i)$$

$$= \frac{1}{n^2}\sum_i \theta(1-\theta) = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}$$

Similar argument gives:

$$Var(\hat{\theta}_2) = \frac{\theta(1-\theta)}{m}$$

d. If $\hat{\theta}_c = c\hat{\theta}_1 + (1-c)\hat{\theta}_2$ + we assume $X_i \perp\!\!\!\perp Y_j \; \forall \, i,j$, then

$$Var(\hat{\theta}_c) = Var(c\hat{\theta}_1 + (1-c)\hat{\theta}_2)$$

$$= c^2 Var(\hat{\theta}_1) + (1-c)^2 Var(\hat{\theta}_2)$$

$$= \frac{c^2\theta(1-\theta)}{n} + \frac{(1-c)^2\theta(1-\theta)}{m}$$

$$= \theta(1-\theta)\left[\frac{c^2}{n} + \frac{(1-c)^2}{m}\right] = f(c)$$

To minimize $Var(\hat{\theta}_c) = f(c)$, let's take derivatives wrt c:

$$\frac{\partial f}{\partial c} = \theta(1-\theta)\left[\frac{2c}{n} - \frac{2(1-c)}{m}\right] \overset{!}{=} 0$$

$$\Rightarrow \quad \frac{c}{n} = \frac{1}{m} - \frac{c}{m}$$

$$\Rightarrow c\left(\frac{1}{n}+\frac{1}{m}\right)=\frac{1}{m}$$

$$\Rightarrow c=\frac{m^{-1}}{n^{-1}+m^{-1}}=\frac{1}{mn^{-1}+1}=\frac{n}{m+n}$$

Verify $\frac{\partial^2 f}{\partial c^2}<0$ on your own ☺

3. $X_i \overset{iid}{\sim} \text{Exp}(\beta)$

a) Find a method of moments estimator for β.
b) Find a method of moments estimator for $P(0 \le X \le 2)$.
c) Find a method of moments estimator for the median of the distribution.

a. MoM for β

$$E(X)=\frac{1}{\beta}\overset{!}{=}M_1=\overline{X}$$

$$\Rightarrow \hat{\beta}=\frac{1}{\overline{X}}$$

b $P(0\le X\le 2)=\int_0^2 f(x)\,dx$

$$=\int_0^2 \beta e^{-\beta x}dx=-e^{-\beta x}\Big|_0^2$$

$$=1-e^{-2\beta}=g(\beta)$$

$$\Rightarrow \hat{P}(0\le X\le 2)=g(\hat{\beta})=1-e^{-2/\overline{X}}.$$

c. let M denote the median.

$\Rightarrow$ M is such that:

$$\int_0^M \beta e^{-\beta x} dx = 1/2$$

$$\Leftrightarrow \quad -e^{-\beta x}\Big|_0^M = 1/2$$

$$\Leftrightarrow \quad 1 - e^{-\beta M} = 1/2$$

$$\Leftrightarrow \quad \frac{1}{2} = e^{-\beta M}$$

$$\Leftrightarrow \quad \log(2^{-1}) = -\beta M$$

$$\Leftrightarrow \quad M = \frac{-\log(2)}{-\beta} = \frac{\log(2)}{\beta} = g(\beta).$$

$\Rightarrow$ A MoM est for M is

$$\hat{M} = g(\hat{\beta}) = \frac{\log(2)}{(1/\bar{x})} = \bar{x}\log(2).$$