Homework 2

M371: Mathematical Statistics

Due: Friday, 2/16 11:59p

Playlists: Here are a couple of playlists from me to help you get started on the assignment:

Psychedelic Bass Mix: `https://shorturl.at/gEQ38`

Disco Then/Now: `https://shorturl.at/iEKQ4`

Problem 1.

How would you respond to a friend who asks you, "How can we say that the sample mean is a random variable when it is just a number, like the population mean? For example, a simple random sample of size 50 produced $\bar{x} = 938.5$; how can the number 938.5 be a random variable?"

Problem 2.

Suppose that we have a random sample $X_1, .., X_n \sim_{\text{iid}} N(\mu, \sigma^2)$.

(a) Show that the maximum likelihood estimator for μ is $\hat{\mu} = \bar{X}$.
(b) Show that the maximum likelihood estimator for σ^2 is $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
(c) Show that the expected value of $\hat{\sigma}^2$ in part (b) is $\frac{n-1}{n}\sigma^2$.

Problem 3.

Suppose we have $X_1, ..., X_n \sim_{\text{iid}} \text{Beta}(\alpha, 1)$.

(a) Find a method of moments estimator for α.
(b) Find the maximum likelihood estimator of α. (Hint: $B(\alpha, \beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$.)

Problem 4.

Suppose we have $X_1, ..., X_n \sim_{\text{iid}}$ Beta$(1, \beta)$.

(a) Find a method of moments estimator for β.
(b) Find the maximum likelihood estimator of β.
(c) Find the maximum likelihood estimator of β^3.

Problem 5.

Suppose we have $X_1, ..., X_n \sim_{\text{iid}}$ Exp(λ).

(a) Find a method of moments estimator for λ based on the first moment.
(b) Find a method of moments estimator for λ based on the second moment.
(c) What does this tell you about the uniqueness of method of moments estimators? Which estimator do you prefer and why?
(d) Show that the Exp(λ) distribution is a special case of the Gamma distribution by choosing speciific values for the Gamma parameters α and β.
(e) Find the MLE of λ.
(f) Consider the reparametrized Exponential distribution, which uses the parameter $\theta = g(\lambda) = \frac{1}{\lambda}$. In other words, the pdf is now:

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad \text{for} \quad x > 0$$

Find the MLE of θ. What is the relationship between it and the MLE of λ you found in part (f)?
(Bonus:) Generalize the pattern you saw betwen parts (e) and (f) to prove the invariance property of the MLE for the Exponential distribution.

Problem 6.

(a) Suppose that X is a random variable following the Poisson distribution with rate parameter λ. Show that $E[X] = \lambda$.

Hint: You may find the following fact useful:

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a$$

(b) Suppose that we obtained the following count data:

Count	Frequency
0	24
1	30
2	17
3	19
4	7
5	0
6	3

Fit a Poisson distribution to the data using the Method of Moments to estimate the parameter.

(c) Suppose that X is a random variable that follows the Poisson distribution that you fit in part (b). Compute $P(X = 0)$, $P(X = 1)$, and $P(X = 2)$. Are these similar to the sample probabilities from the data in part (b)? Based on these probabilities alone, comment on whether the Poisson distribution is a good fit to the above data.