For comparison purposes: $n = 100$ $\Sigma y_i = 40$

$$\hat{p} = .40 \qquad \hat{\sigma}_{\hat{p}} = \sqrt{\frac{(.4)(.6)}{100}} = .049$$

$$\hat{p}_B = .357 \qquad \hat{\sigma}_{\hat{p}_B} = .04$$ (using prior beta(10, 30))

Find the Bayes estimator for $\sigma^2 = p(1-p)$

$$\widehat{p(1-p)}_B = E\left[p(1-p) \mid y_1 \cdots y_n\right]$$

$$= \int_0^1 p(1-p) \cdot \frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*) \cdot \Gamma(\beta^*)} \cdot p^{\alpha^* - 1} (1-p)^{\beta^* - 1} \, dp$$

$$= \int_0^1 \frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} \cdot p^{\alpha^*} (1-p)^{\beta^*} \, dp$$

$$= \frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} \cdot \frac{\Gamma(\alpha^* + 1)\Gamma(\beta^* + 1)}{\Gamma(\alpha^* + \beta^* + 2)}$$

$$= \frac{\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} \cdot \frac{\alpha^* \Gamma(\alpha^*) \cdot \beta^* \Gamma(\beta^*)}{(\alpha^* + \beta^* + 1)(\alpha^* + \beta^*) \cdot \Gamma(\alpha^* + \beta^*)}$$

$$= \frac{\alpha^* \beta^*}{(\alpha^* + \beta^* + 1)(\alpha^* + \beta^*)} = \boxed{\frac{(\Sigma y_i + \alpha)(n - \Sigma y_i + \beta)}{(n + \alpha + \beta + 1)(n + \alpha + \beta)}}$$

$\hat{\sigma}^2 = \hat{p}(1-\hat{p}) = .4(.6) = .24$

$$\widehat{p(1-p)}_B = \frac{(40+10)(100-40+30)}{(100+10+30+1)(100+10+30)} = .228$$

One more look at $\hat{p}_B$:

$$\hat{p}_B = \frac{\sum Y_i + \alpha}{n+\alpha+\beta} = \frac{\sum Y_i}{n+\alpha+\beta} + \frac{\alpha}{n+\alpha+\beta}$$

$$= \frac{n}{n+\alpha+\beta} \cdot \frac{\sum Y_i}{n} + \frac{\alpha+\beta}{n+\alpha+\beta} \cdot \frac{\alpha}{\alpha+\beta}$$

$$= \underbrace{\frac{n}{n+\alpha+\beta}}_{\text{wt. on sample est.}} \cdot \underbrace{\hat{p}}_{\text{sample estimate of } p} + \underbrace{\frac{\alpha+\beta}{n+\alpha+\beta}}_{\text{wt. on prior est.}} \cdot \underbrace{\frac{\alpha}{\alpha+\beta}}_{\text{assumed prior mean of } p}$$

- as $n \uparrow$, we wt. more towards the sample estimate, and less on our prior estimate.

e.g. 16.11 $Y_1 \cdots Y_n$ is a r.s. from $Y \sim POI(\lambda)$

- We have shown before that $U = \sum Y_i$ is a suff. stat for λ and $U \sim POI(n\lambda)$
- Use a conjugate $GAM(\alpha, \beta)$ prior for λ to:

a) Find the joint likelihood of U and λ

$$L(u|\lambda) = \frac{(n\lambda)^u e^{-n\lambda}}{u!} \qquad g(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

$$\Rightarrow f(u,\lambda) = \frac{(n\lambda)^u e^{-n\lambda}}{u!} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

$$= \frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha} \lambda^{u+\alpha-1} e^{-\lambda(n+1/\beta)}$$

$$= \boxed{\frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha} \lambda^{u+\alpha-1} e^{-\lambda/\left(\frac{\beta}{\beta n+1}\right)}}$$

b) find the marginal dist. of u

$$m(u) = \int_0^\infty \frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha} \cdot \lambda^{u+\alpha-1} e^{-\lambda/\left(\frac{\beta}{\beta n+1}\right)} \cdot d\lambda$$

$$= \frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha} \cdot \Gamma(u+\alpha) \cdot \left(\frac{\beta}{\beta n+1}\right)^{u+\alpha} \cdot \int_0^\infty \frac{1}{\Gamma(u+\alpha)\left(\frac{\beta}{\beta n+1}\right)^{u+\alpha}} \cdot \lambda^{(u+\alpha)-1} e^{-\lambda/\left(\frac{\beta}{\beta n+1}\right)} d\lambda$$

$$= \boxed{\frac{n^u \cdot \Gamma(u+\alpha)}{u!\,\Gamma(\alpha)\beta^\alpha} \cdot \left(\frac{\beta}{\beta n+1}\right)^{u+\alpha}}$$

c) Find the posterior density of $\lambda | u$

$$g^*(\lambda|u) = \frac{\cancel{\frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha}} \lambda^{u+\alpha-1} e^{-\lambda/\left(\frac{\beta}{\beta n+1}\right)}}{\cancel{\frac{n^u}{u!\,\Gamma(\alpha)\beta^\alpha}}\,\Gamma(u+\alpha) \left(\frac{\beta}{\beta n+1}\right)^{u+\alpha}}$$

(10)

$$= \frac{1}{\Gamma(u+\alpha)\left(\frac{\beta}{\beta n+1}\right)^{u+\alpha}} \cdot \lambda^{u+\alpha-1} \cdot e^{-\lambda / \left(\frac{\beta}{\beta n+1}\right)}, \quad \lambda > 0$$

$$\Rightarrow \lambda | u \sim GAM\left(\alpha^* = u+\alpha, \ \beta^* = \frac{\beta}{\beta n+1}\right)$$

d) Find the Bayesian estimator of λ

$$\hat{\lambda}_B = E[\lambda | u] = \alpha^* \cdot \beta^*$$

$$= \frac{(u+\alpha)\cdot\beta}{\beta n+1} = \boxed{\frac{(\Sigma Y_i + \alpha)\cdot\beta}{\beta n+1}}$$

e) $$\hat{\lambda}_B = \frac{(\Sigma Y_i + \alpha)\cdot\beta}{\beta n+1} = \frac{n\bar{y}\cdot\beta + \alpha\beta}{\beta n+1}$$

$$= \left(\frac{\beta n}{\beta n+1}\right)\cdot\bar{y} + \frac{1}{\beta n+1}(\alpha\beta)$$ → prior mean of λ

note: $\lim_{n\to\infty} \frac{\beta n}{\beta n+1} = 1, \quad \lim_{n\to\infty} \frac{1}{\beta n+1} = 0$

$\Rightarrow$ as $n \uparrow$ we put more wt. on $\bar{Y}$ and less wt. on the prior estimate for λ

f)

$$E[\hat{\lambda}_B] = \frac{\beta n}{\beta n+1} E(\bar{y}) + \frac{\alpha\beta}{\beta n+1}$$

$$= \frac{\beta n}{\beta n+1} \cdot \lambda + \frac{\alpha\beta}{\beta n+1}$$

$$\lim_{n\to\infty} \left(E[\hat{\lambda}_B] - \lambda\right) = \lim_{n\to\infty} \lambda \left(\frac{\beta n}{\beta n+1} - 1\right) + \lim_{n\to\infty} \frac{\alpha\beta}{\beta n+1}$$

$$= \lim_{n\to\infty} \lambda \left(\frac{-1}{\beta n+1}\right) = 0$$

$\Rightarrow \hat{\lambda}_B$ is asymptotically unbiased for λ

Now, $V(\hat{\lambda}_B) = \left(\frac{\beta n}{\beta n+1}\right)^2 var(\bar{y})$

$$= \left(\frac{\beta n}{\beta n+1}\right)^2 \cdot \frac{\lambda^2}{n}$$

$$\lim_{n\to\infty} V(\hat{\lambda}_B) = \lim_{n\to\infty} \frac{\beta^2 n^2 \lambda^2}{(\beta^2 n^2 + 2\beta n + 1)\cdot n}$$

$$= \lim_{n\to\infty} \frac{\beta^2 n^2 \lambda^2}{(\beta^2 n^3 + 2\beta n^2 + n)} = \lim_{n\to\infty} \frac{\frac{\beta^2\lambda^2}{n} \to 0}{\beta^3 + \frac{2\beta}{n} + \frac{1}{n^2} \to 0} = 0$$

$\Rightarrow \hat{\lambda}_B$ is a consistent estimator for λ