10.4 Testing hypotheses concerning variances

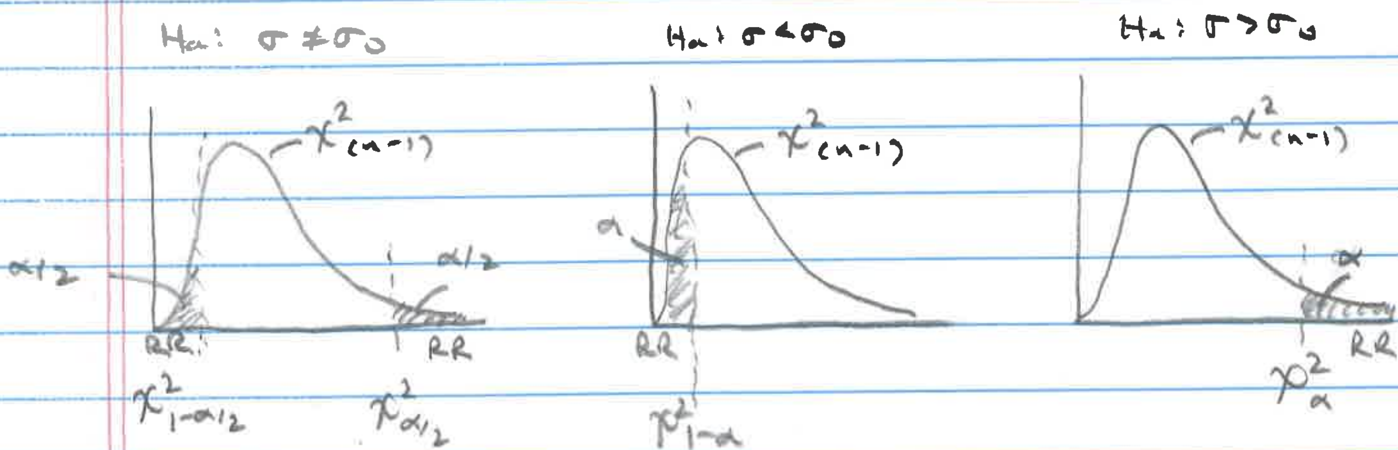
In many applications, the variability for the pop. is the primary RQ.

- e.g. risk in stock mkt
- e.g. variability in filling machines
- e.g. variability in diameter of pipes that must fit together

If we are sampling from one pop we have shown that $y_1, \cdots, y_n$ is a r.s. from $Y \sim N(\mu, \sigma^2)$

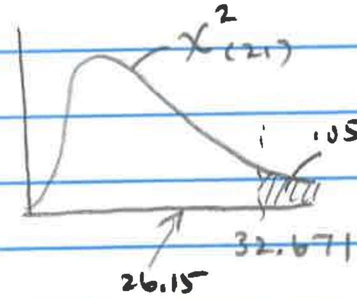
$$W = \frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

This will be the test statistic for $H_0: \sigma = \sigma_0$ vs. $H_a: \sigma \overset{\neq}{\underset{>}{<}} \sigma_0$

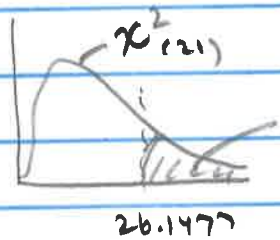


e.g. a filling machine for 64 oz bottles of juice, is calibrated so that $\mu = 64$ oz and $\sigma = .04$ oz. If $\sigma > .04$, then the process is deemed "out-of-control." The quality control mgr. takes a r.s. of $n = 22$ bottles and finds $s = .0446$. Is there evidence at $\alpha = .05$ to conclude the process is out-of-control? note: assume fill amounts are normally dist.

$$\chi^2 = \frac{(22-1)(.0446)^2}{.04^2} = 26.1477$$



$\Rightarrow$ don't reject Ho. i.e. no evidence the process "out-of-control"



p-value = .201 $\Rightarrow$ don't reject Ho

95% lower CI for σ:

$$\sqrt{\frac{(22-1)(.0446)^2}{32.671}} = .036$$

$\Rightarrow$ 95% confident $\sigma > .036$ i.e. can't conclude $\sigma > .04$

$\Rightarrow$ don't reject Ho

e.g. Mutual Fund risk

moderate risk if $\sigma < 4\%$. Mutual fund rating agency does not believe a certain fund has the moderate risk it claims to.

(r.s. of $n = 25$ $s = 3.01\%$)

Is there evidence to justify the moderate risk claim? use $\alpha = .05$

Ho: $\sigma = 4$

Ha: $\sigma < 4$

$$\chi^2 = \frac{(24)(3.01)^2}{4^2} = 13.59$$

$\Rightarrow$ Reject Ho i.e. There is enough evidence at $\alpha = .05$ to conclude $\sigma < 4\%$

(moderate risk is appropriate)

note: p-value = .0446 < α

What if we are comparing variances between 2 pops.?

e.g. Two stocks have similar mean returns, do they have different risks?

e.g. New filling machine is faster, but does it have more variability than old filling machine?

$H_0: \sigma_1 = \sigma_2$ vs. $H_a: \sigma_1^2 \neq \sigma_2^2$

Pop 1: $Y_1 \sim N(\mu_1, \sigma_1^2)$

Let $Y_{11}, Y_{12} \cdots Y_{1n_1}$ be a r.s. of n_1 from Y_1

$$\Rightarrow W_1 = \frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{(n_1-1)}$$

Pop 2: $Y_2 \sim N(\mu_2, \sigma_2^2)$

Let $Y_{21}, Y_{22} \cdots Y_{2n_2}$ be a r.s. of n_2 from Y_2

$$\Rightarrow W_2 = \frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2_{(n_2-1)}$$

Since we have independent samples, W_1 and W_2 are ind. χ^2 RVs. From ch. 7 ...

$$F = \frac{W_1/(n_1-1)}{W_2/(n_2-1)} \sim F_{(n_1-1),(n_2-1)}$$

Under Ho ... $(\sigma_1 = \sigma_2)$

$$F = \frac{\frac{(n_1-1)s_1^2}{\sigma_1^2} \Big/ (n_1-1)}{\frac{(n_2-1)s_2^2}{\sigma_2^2} \Big/ (n_2-1)} = \frac{\sigma_2^2 \cdot s_1^2}{\sigma_1^2 \cdot s_2^2} = \boxed{\frac{s_1^2}{s_2^2}}$$

e.g. Example 10.19

Company: $n_1 = 10$ $s_1^2 = .0003$

competitor: $n_2 = 20$ $s_2^2 = .0001$

Does the competitor have a smaller variability in diameter of parts? Use $\alpha = .05$

$H_0: \sigma_1 = \sigma_2$

$H_a: \sigma_1 > \sigma_2$

$$F = \frac{.0003}{.0001} = 3 \sim F_{9,19}$$

$\Rightarrow$ Reject Ho. Yes there is evidence to conclude the variability in the competitor's process is lower.

note: p-value = .021

note: If we are interested in testing $H_0: \sigma_1 = \sigma_2$ vs. $H_a: \sigma_1 < \sigma_2$, we can equivalently test...

$H_0: \sigma_1 = \sigma_2$

$H_a: \sigma_2 > \sigma_1$

$$F = \frac{s_2^2}{s_1^2} \sim F_{(n_2-1,\, n_1-1)} \qquad RR: \{F > F_{n_2-1,\, n_1-1,\, \alpha}\}$$

What about a two-tail test? Now we actually need the left-tail CV.

Let F^a_b denote an F dist. w/ a d.f. in the numerator and b d.f. in the denominator.

$\Rightarrow P\left(F^{v_1}_{v_2} > F^{v_1}_{v_2,\alpha/2}\right) = \alpha/2 \quad \Rightarrow P\left(1 > \frac{F^{v_1}_{v_2,\alpha/2}}{F^{v_1}_{v_2}}\right) = \alpha/2$

$\Rightarrow P\left(1 > \left(F^{v_1}_{v_2,\alpha/2}\right) \cdot F^{v_2}_{v_1}\right) = \alpha/2$

$\Rightarrow P\left(\left[F^{v_1}_{v_2,\alpha/2}\right]^{-1} > F^{v_2}_{v_1}\right] = \alpha/2$

Therefore, if $H_0: \sigma_1 = \sigma_2$ vs. $H_a: \sigma_1 \neq \sigma_2$

$$F = \frac{s_1^2}{s_2^2} \sim F_{n_1-1,\, n_2-1}$$

$$\boxed{RR: \; F < \left(F^{n_2-1}_{n_1-1,\, \alpha/2}\right)^{-1} \quad \text{or} \quad F > F^{n_1-1}_{n_2-1,\, \alpha/2}}$$

e.g. 10.21

	Males	Females
n	14	10
$\bar{y}$	16.2	14.9
s^2	12.7	26.4

$H_0: \sigma_M = \sigma_F$ vs. $H_a: \sigma_M \neq \sigma_F$

$$F = \frac{s_M^2}{s_F^2} = \frac{12.7}{26.4} \approx .481 \sim F_{13,9}$$

not in Table XII

RR: $F > F_{13,9,.05} = 3.0475$ or $F < \frac{1}{F_{9,13,.05}} = \frac{1}{2.71} = .369$

$.369 < F = .481 < 3.0475 \Rightarrow$ don't reject H_0 i.e. not enough evidence to conclude males / females have different variability in pain threshold.

10.81

$$F = \frac{S_1^2}{S_2^2} \sim F^{v_1}_{v_2} \quad , \; v_1 = n_1 - 1 \; , \; v_2 = n_2 - 1$$

We showed earlier that for $H_0: \sigma_1 = \sigma_2$ $H_a: \sigma_1 \neq \sigma_2$

that $$RR = \{ F > F^{v_1}_{v_2, \alpha/2} \; \overset{\cup}{\text{or}} \; F < (F^{v_2}_{v_1, \alpha/2})^{-1} \}$$

Now,
$$F < (F^{v_2}_{v_1,\alpha/2})^{-1} \Rightarrow \frac{S_1^2}{S_2^2} < (F^{v_2}_{v_1,\alpha/2})^{-1}$$

$$\Rightarrow \frac{S_2^2}{S_1^2} > F^{v_2}_{v_1, \alpha/2}$$

This implies $RR = \{ \frac{S_1^2}{S_2^2} > F^{v_1}_{v_2,\alpha/2}$ or $\frac{S_2^2}{S_1^2} > F^{v_2}_{v_1,\alpha/2} \}$

$\Rightarrow$ We could use $\frac{S_L^2}{S_S^2}$ as our test statistic $(S_L^2 > S_S^2)$

and RR: $\{ \frac{S_L^2}{S_S^2} > F^{v_L}_{v_S, \alpha/2} \}$

note: technically $\frac{S_L^2}{S_S^2}$ does not follow an F dist. because it is always > 1. However, RR is of size α and we only have to worry about a right-tail value.

Back to 10.21 ...

$$F = \frac{26.4}{12.7} = 2.079$$

$F_{9,13,.05} = 2.71$ $\qquad 2.079 < 2.71 \Rightarrow$ don't reject H_0

(same decision)

10.10 Power of Tests and the Neyman-Pearson Lemma

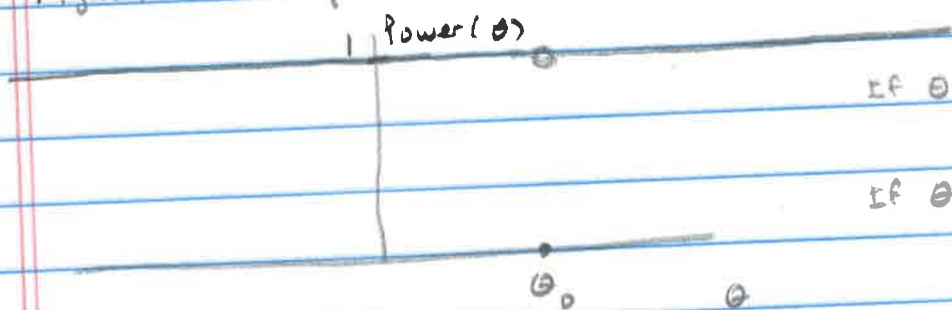
So far, we have spent a lot of time talking about the significance level α, but we also need to discuss the power of the test

Def 10.3 Spse that W is the test stat and RR is the rejection region for a test of a hypothesis involving the value of a parameter θ. Then the power of the test, denoted power(θ), is the prob. that the test will lead to rejection of H_0 when the actual parameter value is θ. That is,

$$\boxed{\text{power}(\theta) = P(W \in RR \mid \theta)}$$

Spse we are testing $H_0: \theta = \theta_0$ vs. $H_a: \theta \neq \theta_0$

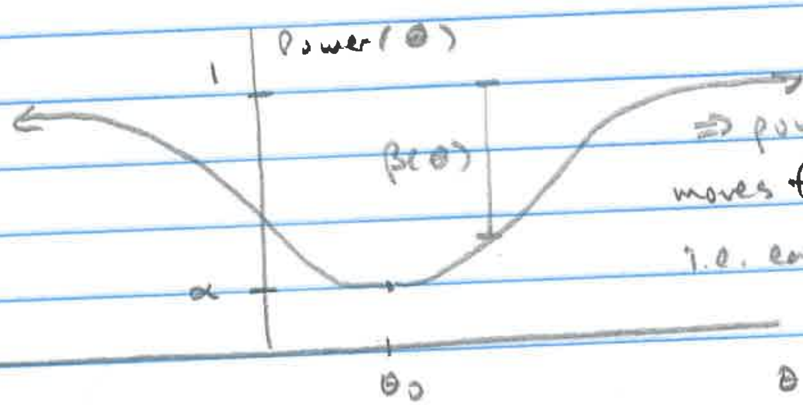
Fig. 14 (Ideal power curve)



If $\theta \neq \theta_0$ always reject H_0

If $\theta = \theta_0$ always accept H_0

i.e. we always make the correct decision. In reality, a typical power curve looks like Fig. 13



$\Rightarrow$ power increases as θ moves further away from θ_0

i.e. easier to reject H_0

<u>note:</u> power$(\theta_0) = \alpha$

$\beta = P($don't reject $H_0 \mid H_0$ is false$)$

$\Rightarrow$ power$(\theta_a) = 1 - \beta(\theta_a)$ for $\theta_a \neq \theta_0$

The main object of this section is to determine the <u>most powerful test possible.</u>

[Recall... H_0: status quo vs. H_a: researcher's claim]

Before we develop the most powerful test, we need to distinguish between simple vs. composite hypotheses

<u>Def 10.4</u> If a r.s. is taken from a dist. w/ parameter θ, a hypothesis is said to be a <u>simple hypothesis</u> if that hypothesis <u>uniquely specifies</u> the dist. of the pop. from which the sample is taken. Any hypothesis that is not a simple hypothesis is called a <u>composite hypothesis</u>.

e.g. $Y_1 \cdots Y_n$ is a r.s. from $Y \sim EXP(\theta)$

$H_0: \theta = 3$ is a <u>simple</u> hypothesis because it uniquely defines dist. of Y

i.e. $f(y) = \frac{1}{3} e^{-y/3}$ $\quad F(y) = 1 - e^{-y/3}$

$H_a: \theta > 3$ is a <u>composite hypothesis</u> because it does not uniquely define the dist. of Y. θ can take on an infinite no. of values under H_a

e.g. $Y_1 \cdots Y_n$ is a r.s. from $Y \sim N(\mu, \sigma^2)$

$H_0: \mu = 100$ is a composite hypothesis because it does not uniquely define the dist. of Y

$$Y \sim N(\mu = 100, \sigma^2 = 10) \quad \text{or} \quad Y \sim N(\mu = 100, \sigma^2 = 15)$$

There are still an infinite no. of possible values for σ^2 under H_0

For a simple vs. simple hypothesis test, we can use the Neyman-Pearson Lemma to find the form of the most powerful test, for a given level α.

Thm 10.1 The Neyman-Pearson Lemma

Spse that we wish to test the simple null hypothesis $H_0: \theta = \theta_0$ vs. the simple alternative hypothesis $H_a: \theta = \theta_a$, based on a r.s. $Y_1 \cdots Y_n$ from a dist. w/ parameter θ.

Let $L(\theta)$ denote the likelihood of the sample when the value of the parameter is θ. Then for a given α, the test that maximizes the power at θ_a has RR determined by

$$\boxed{\frac{L(\theta_0)}{L(\theta_a)} < k}$$

note: The value k is chosen so that the test has the desired value for α. Such a test is the most powerful α-level test for H_0 vs. H_a

Proof: wait until grad school

Intuition:

as the ratio $\frac{L(\theta_0)}{L(\theta_a)}$ becomes smaller and smaller it means that data was more and more likely to have been drawn from a dist. where $\Theta = \Theta_a$ instead of $\Theta = \Theta_0$.

Example 10.22

Spse Y represents a single observation from

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & , 0<y<1 \\ 0 & , \text{———} \end{cases}$$

Find the most powerful test w/ $\alpha = .05$ for $H_0: \Theta = 2$ vs. $H_a: \Theta = 1$

note: simple vs. simple

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{2y^{2-1}}{1 \cdot y^{1-1}} = 2y < k \Rightarrow y < k/2$$

note: $k/2$ is a constant so let $k^* = k/2$

$\Rightarrow$ RR: $\{y < k^*\}$ is the form of the most powerful test

Next, we need to determine k^* so that RR is size $\alpha = .05$

$$\int_0^{k^*} 2 \cdot y \, dy = .05 \Rightarrow y^2 \Big|_0^{k^*} = .05 \Rightarrow (k^*)^2 = .05$$

$\Rightarrow$ RR: $\{y < \sqrt{.05} = .2236\}$ is the most powerful test

What is the power?

$$\text{power}(\theta = 1) = \int_0^{.2236} 1 \cdot y^{1-1} \, dy = y \Big|_0^{.2236} = .2236$$

For most powerful test, $\beta = .7764$!

e.g. Let $Y_1 \cdots Y_n$ be a r.s. from $Y \sim GAM(\alpha = 2, \beta)$

Find the most powerful test of size .10 for $H_0: \beta = 4$ vs. $H_a: \beta = 8$

<u>note:</u> simple vs. simple

$$f(y; \alpha = 2, \beta) = \frac{1}{\Gamma(2)\beta^2} y e^{-y/\beta}, \quad y > 0, \beta > 0$$

$$\frac{L(4)}{L(8)} = \frac{\frac{1}{(\Gamma(2))^n \cdot 4^{2n}} \prod_{i=1}^n y_i e^{-\frac{1}{4}\sum y_i}}{\frac{1}{(\Gamma(2))^n 8^{2n}} \cdot \prod_{i=1}^n y_i e^{-\frac{1}{8}\sum y_i}} = (2)^{2n} \cdot e^{-\frac{1}{8}\sum y_i}$$

$$(2)^{2n} e^{-\frac{1}{8}\sum y_i} \le k \Rightarrow e^{-\frac{1}{8}\sum y_i} \le k(2)^{-2n} \Rightarrow -\frac{1}{8}\sum y_i \le \ln(k(2)^{-2n})$$

$$\sum y_i \ge -8 \cdot \ln\left(\frac{k}{2^{2n}}\right) \quad \text{or} \quad \boxed{\sum y_i \ge c}$$

<u>Intuition:</u> $H_0: E(Y) = \alpha\beta = 2(4) = 8$

$H_a: E(Y) = \alpha\beta = 2(8) = 16$

$\Rightarrow$ It seems reasonable we would reject H_0 as $\sum y_i$ gets large

We need to find the dist. of $\sum y_i$ under H_0 to get RR w/ size .10

$$\sum y_i \sim GAM(\sum \alpha_i, \beta) \quad \text{(method of MGFs)}$$

$$\sim GAM(\alpha = 2n, \beta = 4)$$

from 6.46 If $Y \sim GAM(\alpha, \beta)$ then $W = \frac{2Y}{\beta} \sim \chi^2_{(2\alpha)}$

$$\Rightarrow \frac{2(\sum Y_i)}{4} = \frac{\sum Y_i}{2} \sim \chi^2_{(4n)}$$

If $n=10$ $\frac{\sum Y_i}{2} \sim \chi^2_{(40)}$

$P\left(\frac{\sum Y_i}{2} \geq 51.80\right) = .10$

$\Rightarrow P(\sum Y_i \geq 103.60) = .10$

i.e. $RR: \{\sum Y_i \geq 103.60\}$

Sample values: 7, 13, 22, 6, 11, 17, 13, 5, 10, 13

$\sum y_i = 117$ $\Rightarrow$ Reject H_0 i.e. conclude $\beta = 8$

Simple vs. Composite

$H_0: \theta = \theta_0$ vs. $H_a: \theta > \theta_0$

Neyman-Pearson Lemma does not cover this situation (directly)
However, in many cases the form of RR for the MP test of
$H_0: \theta = \theta_0$ vs. $H_a: \theta > \theta_0$ is the same for every $\theta_a > \theta_0$.
If this is the case, then RR defines the test that is uniformly
most powerful (UMP) for a given level α.

Example 10.23

Spse $Y_1 \cdots Y_n$ is a r.s. from $Y \sim N(\mu, \sigma^2)$ σ^2 known

Find UMP for $H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$ with significance level α

Let $\mu_a > \mu_0$ and consider $H_0: \mu = \mu_0$ vs. $H_a: \mu = \mu_a$ (Simple vs. simple)

$$f(y;\mu) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$$

$$L(\mu) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2}$$

$$\frac{L(\mu_0)}{L(\mu_a)} = e^{-\frac{1}{2\sigma^2}\left[\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_a)^2\right]} \le k$$

$$\Rightarrow -\frac{1}{2\sigma^2}\left[\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_a)^2\right] \le k$$

$$\Rightarrow \Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_a)^2 \ge -2\sigma^2 k$$

$$\Rightarrow \left(\Sigma y_i^2 - 2\mu_0\Sigma y_i + n\mu_0^2\right) - \left(\Sigma y_i^2 - 2\mu_a\Sigma y_i + n\mu_a^2\right) \ge -2\sigma^2 k$$

$$\Rightarrow -2n\cdot\mu_0\cdot\bar{y} + n\mu_0^2 + 2n\mu_a\bar{y} - n\mu_a^2 \ge -2\sigma^2 k$$

$$\Rightarrow \bar{y} \ge \frac{-2\sigma^2 k + n\mu_a^2 - n\mu_0^2}{2n(\mu_a-\mu_0)}$$ constant

$$\Rightarrow \bar{y} \ge c$$ (Has intuition)

<u>note</u>: At this pt. c depends on the value of μ_a

We need to find c so that $P(\bar{y} \ge c \mid \mu = \mu_0) = \alpha$

$$= P\left(\frac{\bar{y}-\mu_0}{\sigma/\sqrt{n}} \ge \frac{c-\mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$= P\left(z \ge \frac{c-\mu_0}{\sigma/\sqrt{n}}\right) = \alpha \quad \Rightarrow \quad \frac{c-\mu_0}{\sigma/\sqrt{n}} = z_\alpha$$

$\Rightarrow C = \mu_0 + z_\alpha \cdot \sigma/\sqrt{n}$

i.e. RR: $\{ \bar{y} \geq \mu_0 + z_\alpha \cdot \sigma/\sqrt{n} \}$

This the same RR for every $\mu_a > \mu_0$ $\Rightarrow$ the test is UMP for significance level α.

note: This is the same test we used in 10.3 (substituted s for σ)

10.94 $Y_1 \cdots Y_n$ is a r.s from $Y \sim POI(\lambda)$

Find the form of the UMP test for $H_0: \lambda = \lambda_0$ vs. $H_a: \lambda > \lambda_0$

$p(y;\lambda) = \dfrac{\lambda^y e^{-\lambda}}{y!}$ $\Rightarrow L(\lambda) = \prod_{i=1}^{n} \dfrac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \dfrac{\lambda^{\Sigma y_i} \cdot e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$

Consider: $H_0: \lambda = \lambda_0$ vs. $H_a: \lambda = \lambda_a$ $(\lambda_a > \lambda_0)$
(simple vs. simple)

$$\Rightarrow \frac{L(\lambda_0)}{L(\lambda_a)} = \frac{\lambda_0^{\Sigma y_i} \cdot e^{-n\lambda_0}}{\lambda_a^{\Sigma y_i} \cdot e^{-n\lambda_a}} = \left(\frac{\lambda_0}{\lambda_a}\right)^{\Sigma y_i} \cdot e^{-n(\lambda_0 - \lambda_a)} \leq k$$

$$\Sigma y_i \cdot \ln\left(\frac{\lambda_0}{\lambda_a}\right) - n(\lambda_0 - \lambda_a) \leq \ln k$$

$$\Rightarrow \Sigma y_i \cdot \ln\left(\frac{\lambda_0}{\lambda_a}\right) \leq \ln k + n(\lambda_0 - \lambda_a)$$

$$\Rightarrow \Sigma y_i \geq \frac{\ln k + n(\lambda_0 - \lambda_a)}{\ln\left(\frac{\lambda_0}{\lambda_a}\right)} \quad \text{or} \quad \boxed{\Sigma y_i \geq C}$$

note: $\frac{\lambda_0}{\lambda_a} < 1$
$\Rightarrow \ln(\lambda_0/\lambda_a) < 0$

note: At this pt. c depends on λ_a

In general if $Y_i \sim POI(\lambda_i)$ then $\sum Y_i \sim POI(\lambda = \sum \lambda_i)$

So under Ho...

$\sum Y_i \sim POI(n \cdot \lambda_0)$

Chose c so that $P(w \geq c) = \alpha$ $(w = \sum Y_i)$. Since c found under this criterion will not depend on λ_a, the test we have developed is UMP

note: For discrete RVs, it is often not possible to find a test with significance exactly α. In these cases, use closest possible value to α without going over.

e.g. $\lambda_0 = 4$ $\quad n = 5$ $\quad \Rightarrow \sum Y_i \sim POI(\lambda = 20)$

$H_0: \lambda = 4$ vs. $H_a: \lambda > 4$

$P(\sum Y_i \geq 28) = .052$ $\quad P(\sum Y_i \geq 29) = .034$

We could run a test at either $\alpha = .034$ or $\alpha = .052$, but not $\alpha = .05$

Two-sided Tests

we rarely can form a UMP test for $H_0: \theta \neq \theta_0$

e.g.

Let $Y_1 \cdots Y_n$ denote a r.s. from $N(\mu, 1)$

Test $H_0: \mu = \mu_0$ vs. $H_a: \mu \neq \mu_0$

Let $\mu_a \neq \mu_0$

⑨

Consider $\dfrac{L(\theta_0)}{L(\theta_1)} = \dfrac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i-\theta_0)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i-\theta_1)^2}} = e^{-\frac{1}{2}\left[\sum(x_i-\theta_0)^2 - \sum(x_i-\theta_1)^2\right]}$

$\cancel{\sum x_i^2} - 2\theta_0\sum x_i + n\theta_0^2 - \cancel{\sum x_i^2} + 2\theta_1\sum x_i - n\theta_1^2 = n(\theta_0-\theta_1)^2 - 2\sum x_i(\theta_0-\theta_1)$

$\Rightarrow \dfrac{L(\theta_0)}{L(\theta_1)} = e^{-\frac{n}{2}(\theta_0-\theta_1)^2 + \sum x_i(\theta_0-\theta_1)}$

$e^{-\frac{n}{2}(\theta_0-\theta_1)^2 + \sum x_i(\theta_0-\theta_1)} \le k$

~~$\Rightarrow -\frac{n}{2}(\theta_0-\theta_1)^2 + \sum x_i(\theta_0-\theta_1) \le \ln k$~~

$(\theta_0-\theta_1)\sum x_i \le \ln k + \frac{n}{2}(\theta_0-\theta_1)^2$

$C_1 : \left\{ \sum x_i \le \dfrac{\ln k + \frac{n}{2}(\theta_0-\theta_1)^2}{\theta_0-\theta_1} \right\}$ for $\theta_0 > \theta_1$

$C_2 : \left\{ \sum x_i \ge \dfrac{\ln k + \frac{n}{2}(\theta_0-\theta_1)^2}{\theta_0-\theta_1} \right\}$ for $\theta_0 < \theta_1$

Now, C_1 is the best critical region for $H_0: \theta=\theta_0$ vs. $H_1: \theta=\theta_1$ $\theta_1<\theta_0$
C_2 " " " " " " " $\theta_1>\theta_0$

Therefore, there is no UMP test of size α for this problem
(Recall, a UMP test has a single critical region C which is the "best" critical region $\forall\, \theta_1 \in \Xi_1$)

Remark If we were to combine these tests as follows

$\phi(x) = \begin{cases} 1 & \sum x_i \ge c_1 \text{ or } \sum x_i \le c_2 \\ 0 & c_1 < \sum x_i < c_2 \end{cases}$

$\frac{L(\mu_0)}{L(\mu_a)} \leq k$ is an MP test for $H_0: \mu = \mu_0$ vs. $H_a: \mu = \mu_a$

$\downarrow$

$$e^{-\frac{1}{2}\left[\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_a)^2\right]} \leq k$$

$$\Sigma(y_i-\mu_0)^2 - \Sigma(y_i-\mu_a)^2 \geq -2\ln k$$

$$-2n\mu_0\cdot\bar{y} + n\mu_0^2 + 2n\mu_a\cdot\bar{y} - n\mu_a^2 \geq -2\ln k$$

$$-2n(\mu_0-\mu_a)\cdot\bar{y} \geq -2\ln k + n(\mu_a^2-\mu_0^2)$$

$$(\mu_0-\mu_a)\cdot\bar{y} \leq \frac{-2\ln k + n(\mu_a^2-\mu_0^2)}{-2n}$$

or $(\mu_0-\mu_a)\cdot\bar{y} \leq c$

Here is where we have a problem....

If $\mu_a > \mu_0$ then the MP test is $\boxed{\bar{y} \geq \frac{c}{(\mu_0-\mu_a)} = c_1}$ RR1

If $\mu_a < \mu_0$ then the MP test is $\boxed{\bar{y} \leq \frac{c}{(\mu_0-\mu_a)} = c_2}$ RR2

Solution: Find c_1 so that $P(\bar{y} \geq c_1) = \alpha/2$ and find c_2 so that $P(\bar{y} \leq c_2) = \alpha/2$

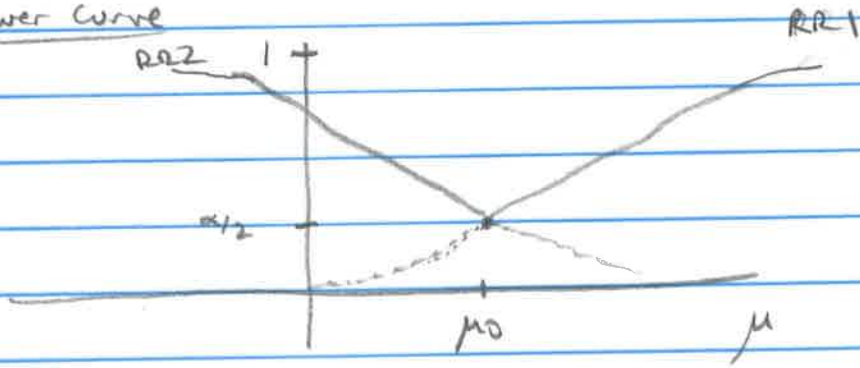
(under H_0)...

Since $\bar{y} \sim N(\mu_0, \sigma^2/n)$ $\Rightarrow$ $c_1 = \mu_0 + z_{\alpha/2}\cdot\sigma/\sqrt{n}$

$\Rightarrow$ $c_2 = \mu_0 - z_{\alpha/2}\cdot\sigma/\sqrt{n}$

(11)

Power curve



10.11 Likelihood ratio tests

The generalized likelihood ratio tests are a more general approach to hypothesis testing than the Neyman-Pearson Lemma (which produced the MP test for simple vs. simple and often a UMP test for simple vs. composite)

The GLR test does not guarantee an MP or UMP test, but it often produces one.

In general, we assume we are sampling from a prob. dist. with parameters $\theta_1\ \theta_2 \cdots \theta_k$, which are denoted as the vector $\Theta = (\theta_1, \dots, \theta_k)$

e.g. $Y \sim N(\mu, \sigma^2)$ $\quad \Theta = (\theta_1 = \mu,\ \theta_2 = \sigma^2)$

$Y \sim POI(\lambda)$ $\quad \Theta = (\theta_1 = \lambda)$

In some cases, we might be interested in testing for just one parameter, and then all other unknown parameters are called nuisance parameters

e.g. $Y \sim N(\mu, \sigma^2)$ $\quad H_0: \mu = \mu_0 \quad H_a: \mu \neq \mu_0$

$\Rightarrow \sigma^2$ is a nuisance parameter

Let Ω_0 be the set of all possible values for Θ under H_0

Let Ω_a " " " " " Θ under H_a

Let $\Omega = \Omega_0 \cup \Omega_a$

e.g. $Y \sim POI(\lambda)$ $H_0: \lambda = \lambda_0$ $H_a: \lambda \neq \lambda_0$

$\Omega_0 = \{\lambda_0\}$ $\Omega_a = \{\lambda : \lambda > 0 \text{ and } \lambda \neq \lambda_0\}$

$\Rightarrow \Omega = \{\lambda : \lambda \geq 0\}$

e.g. $Y \sim POI(\lambda)$ $H_0: \lambda = \lambda_0$ $H_a: \lambda > \lambda_0$

$\Omega_0 = \{\lambda_0\}$ $\Omega_a = \{\lambda : \lambda > \lambda_0\}$

$\Rightarrow \Omega = \{\lambda : \lambda \geq \lambda_0\}$

e.g. $Y \sim N(\mu, \sigma^2)$ $H_0: \mu = \mu_0$ vs. $H_a: \mu > \mu_0$

$\Omega_0 = \{(\mu, \sigma^2) : \mu = \mu_0, \sigma^2 > 0\}$

$\Omega_a = \{(\mu, \sigma^2) : \mu > \mu_0, \sigma^2 > 0\}$

$\Rightarrow \Omega = \{(\mu, \sigma^2) : \mu \geq \mu_0, \sigma^2 > 0\}$

To run a likelihood ratio test, we will need to maximize the likelihood fct under both Ω_0 and Ω. The procedure will be to find (or use) MLEs $\hat{\theta}_1, \cdots \hat{\theta}_k$ as we did in Ch. 9

Let $L(\hat{\Omega}_0)$ denote the maximum of the likelihood fct. for all $\Theta \in \Omega_0$.

Let $L(\hat{\Omega})$ denote the maximum of the likelihood fct. for all $\Theta \in \Omega$

Define λ (the likelihood ratio) by ...

$$\boxed{\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})}} \quad , \; 0 \le \lambda \le 1$$

• If λ is near 0, then the data seems much more likely under H_a than H_0

$\Rightarrow$ $\boxed{\text{Reject } H_0 \text{ if } \lambda \le k}$ (Choose k so that significance level is α)

e.g. Let $Y_1 \cdots Y_n$ be a r.s. from $N(\mu, 1)$

Find the GLR for $H_0: \mu = 3$ vs. $H_a: \mu \neq 3$

$\Omega_0 = \{3\}$ $\quad \Omega = \{\mu : -\infty < \mu < \infty\}$

In general,

$$L(\mu) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(y_i - \mu)^2}$$

$$L(\hat{\Omega}_0) = L(3) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(y_i - 3)^2}$$

For Ω, we found the MLE for μ to be $\bar{Y}$

$$\Rightarrow L(\hat{\Omega}) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(y - \bar{y})^2}$$

$$\Rightarrow \lambda = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(y_i - 3)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(y_i - \bar{y})^2}} = \frac{e^{-\frac{1}{2}\sum(y_i - 3)^2}}{e^{-\frac{1}{2}\sum(y_i - \bar{y})^2}}$$

Consider $\Sigma(y_i-3)^2 = \Sigma\left((y_i-\bar{y})+(\bar{y}-3)\right)^2$

$$= \Sigma(y_i-\bar{y})^2 + 2(\bar{y}-3)\underbrace{\cancel{\Sigma(y_i-\bar{y})}}_{\to 0} + n(\bar{y}-3)^2$$

$$= \Sigma(y_i-\bar{y})^2 + n(\bar{y}-3)^2$$

$$\Rightarrow \lambda = \frac{e^{-\frac{1}{2}\Sigma(y_i-\bar{y})^2 - \frac{n}{2}(\bar{y}-3)^2}}{e^{-\frac{1}{2}\Sigma(y_i-\bar{y})^2}} = e^{-\frac{n}{2}(\bar{y}-3)^2}$$

Now, if $\bar{y}$ is close to 3, λ is near 1

if $\bar{y}$ is not close to 3, λ is near 0 (very small)

$$e^{-\frac{n}{2}(\bar{y}-3)^2} \le k$$

$$\Rightarrow -\frac{n}{2}(\bar{y}-3)^2 \le \ln k$$

$$(\bar{y}-3)^2 \cdot n \ge -2 \cdot \ln k$$

$$\Rightarrow (\bar{y}-3)^2 \cdot n \ge C$$

The question now is what dist. does $(\bar{y}-3)^2 \cdot n$ have under H_0?

$$\frac{(\bar{y}-3)^2}{1/n} = \left(\frac{\bar{y}-3}{1/\sqrt{n}}\right)^2 \sim z^2 \sim \chi^2_{(1)}$$

for $\alpha = .05$ $\quad P(\chi^2_{(1)} > 3.84) = .05$

$\Rightarrow$ If we reject Ho for $\left(\frac{\bar{y}-3}{1/\sqrt{n}}\right)^2 \geq 3.84$, it is equivalent to

$$\frac{\bar{y}-3}{1/\sqrt{n}} \geq 1.96 \quad \text{or} \quad \frac{\bar{y}-3}{1/\sqrt{n}} \leq -1.96$$

This just gave us the z-test for μ with σ known

e.g. If $Y_1 \cdots Y_n$ is a r.s. from $Y \sim N(\mu, \sigma^2)$, find the likelihood ratio test for $H_0: \sigma^2 = \sigma_0^2$ vs. $H_a: \sigma^2 \neq \sigma_0^2$. Identify the dist. for the test statistic under Ho.

$$L(\mu, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2}$$

$$\Omega_0 = \{(\mu, \sigma^2) : -\infty < \mu < \infty,\ \sigma^2 = \sigma_0^2\}$$

$$\Omega = \{(\mu, \sigma^2) : -\infty < \mu < \infty,\ \sigma^2 > 0\}$$

MLEs:	μ	σ^2
Ω_0	$\bar{Y}$	σ_0^2
Ω	$\bar{Y}$	$\hat{\sigma}^2 = \frac{\sum(y_i-\bar{y})^2}{n}$

$$\Rightarrow \lambda = \frac{\left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum(Y_i-\bar{Y})^2}}{\left(\frac{1}{\sqrt{2\pi}\,\hat{\sigma}}\right)^n e^{-\frac{1}{2\hat{\sigma}^2}\sum(Y_i-\bar{Y})^2}} =$$

⑥

$$= \left(\frac{\hat{\sigma}}{\sigma_0}\right)^n \cdot \frac{e^{-\frac{1}{2\sigma_0^2}\Sigma(Y_i-\bar{Y})^2}}{e^{-n/2}}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \cdot e^{-\frac{n\hat{\sigma}^2}{2\sigma_0^2}+n/2}$$ let $u = \frac{\hat{\sigma}^2}{\sigma_0^2}$

$$\Rightarrow \lambda = (u)^{n/2} e^{-\frac{n}{2}(u-1)}$$

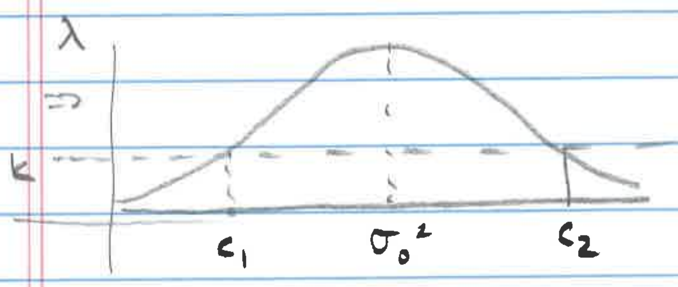
$$\ln\lambda = \frac{n}{2}\cdot\ln u - \frac{n}{2}(u-1)$$

$$\frac{\partial \ln\lambda}{\partial u} = \frac{n}{2u} - \frac{n}{2} = \frac{n}{2}\left[\frac{1}{u}-1\right]$$

$\Rightarrow \lambda$ is increasing if $u<1$ or $\hat{\sigma}^2 < \sigma_0^2$

$\Rightarrow \lambda$ is decreasing if $u>1$ or $\hat{\sigma}^2 > \sigma_0^2$

$\Rightarrow \lambda$ is maximized if $u=1$ or $\hat{\sigma}^2 = \sigma_0^2$



$\Rightarrow$ Reject H_0 if $\lambda \geq c_2$ or $\lambda \leq c_1$. What is the dist. of λ?
We don't know, but we could equivalently use $u = \hat{\sigma}^2/\sigma_0^2$ instead

$$n\cdot\frac{\hat{\sigma}^2}{\sigma_0^2} = \frac{\Sigma(y_i-\bar{y})^2}{\sigma_0^2} = \frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{(n-1)} \text{ under } H_0$$

$\Rightarrow$ Reject Ho if $\frac{(n-1)s^2}{\sigma_0^2} \geq \chi^2_{\alpha/2}$ or $\frac{(n-1)s^2}{\sigma^2} \leq \chi^2_{1-\alpha/2}$

e.g. Let $Y_1 \cdots Y_n$ be a r.s. from $Y \sim POI(\theta)$

Find the GLR test for $H_0: \theta = 3/2$ vs. $H_a: \theta < 3/2$

$\Omega_0 = \{3/2\}$ $\Omega = \{\theta : \theta \leq 3/2\}$

$$L(\hat{\Omega}_0) = \prod_{i=1}^{n} \frac{(3/2)^{y_i} e^{-3/2}}{y_i!}$$

Next, we need to find the MLE under the restriction $\hat{\theta} \leq 3/2$

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} = \frac{e^{-n\theta}\theta^{\sum y_i}}{\prod_{i=1}^{n} y_i!}$$

$$\ln L(\theta) = -n\theta + \sum y_i \ln\theta - \ln\left(\prod_{i=1}^{n} y_i!\right)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = -n + \frac{\sum y_i}{\theta} \overset{SET}{=} 0 \Rightarrow \hat{\theta} = \bar{y}, \ \text{if } \bar{y} \leq 3/2 \text{ (restriction)}$$

$\hat{\theta} > 3/2$ is not a permissable value

If $\bar{y} > 3/2$ then $\frac{\partial \ln L(\theta)}{\partial\theta} = \underbrace{-\frac{n}{\theta}}_{(-)} \times \underbrace{(\theta - \bar{y})}_{(-)} > 0$

$\Rightarrow$ for $\bar{y} > 3/2$, $L(\theta)$ is an increasing fct. in θ.