Homework 1

Math 4610

Cody Frisby

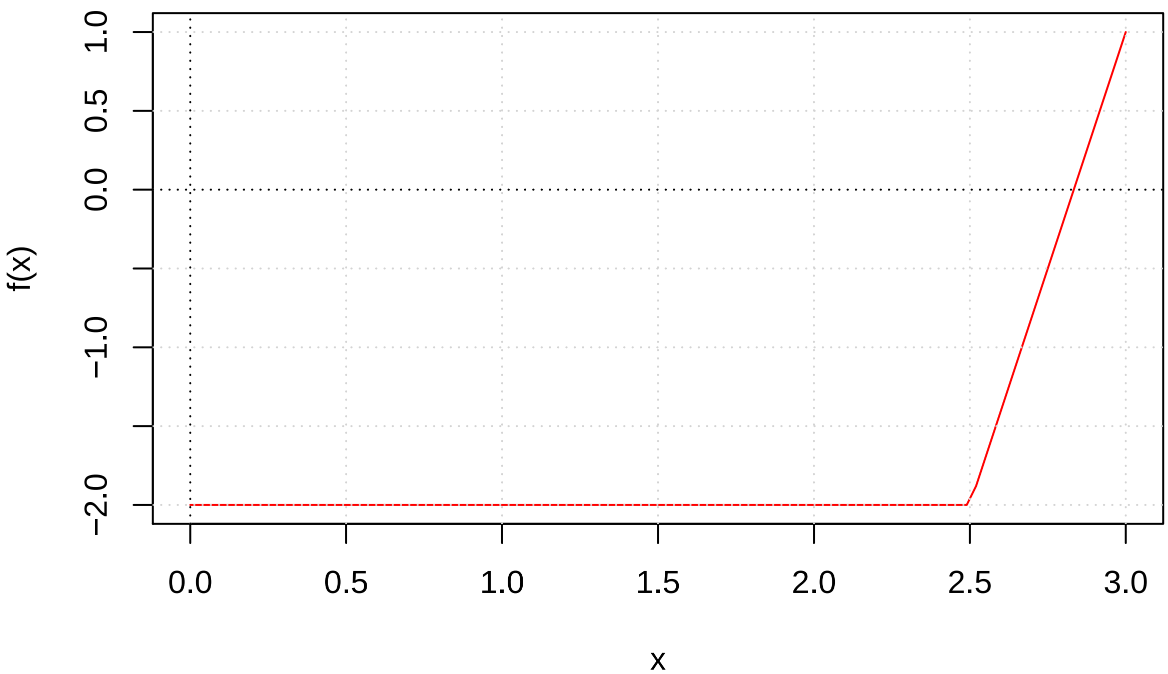
9/16/2017

1. Let

$$f(x) = -2 \quad for \quad x \leq 2.5$$

$$f(x) = -2 + 6(x - 2.5) \quad for \quad 2.5 < x$$

A sketch is included for reference.



Of the 4 methods used, and with a similar set tolerance for each method, Newton's converges the quickest, needing just two iterations to find the root, which is approximately 2.833333. First guess for each method is 3. If a method requires two initial points then the second one is 2.

guess	fx	dfx	error
3.000000	1	6	0.1666667
2.833333	0	6	0.0000000

Next fastest was the false position method. It needed 3 iterations to converge on the solution.

guess	a	b	iteration
2.666667	3.000000	2.000000	1
2.833333	3.000000	2.666667	2
2.833333	2.666667	2.833333	3

And third is the secant method, needing 4.

guess	iteration	e
2.666667	1	0.3333333
3.333333	2	0.2500000
2.833333	3	0.1500000
2.833333	4	0.0000000

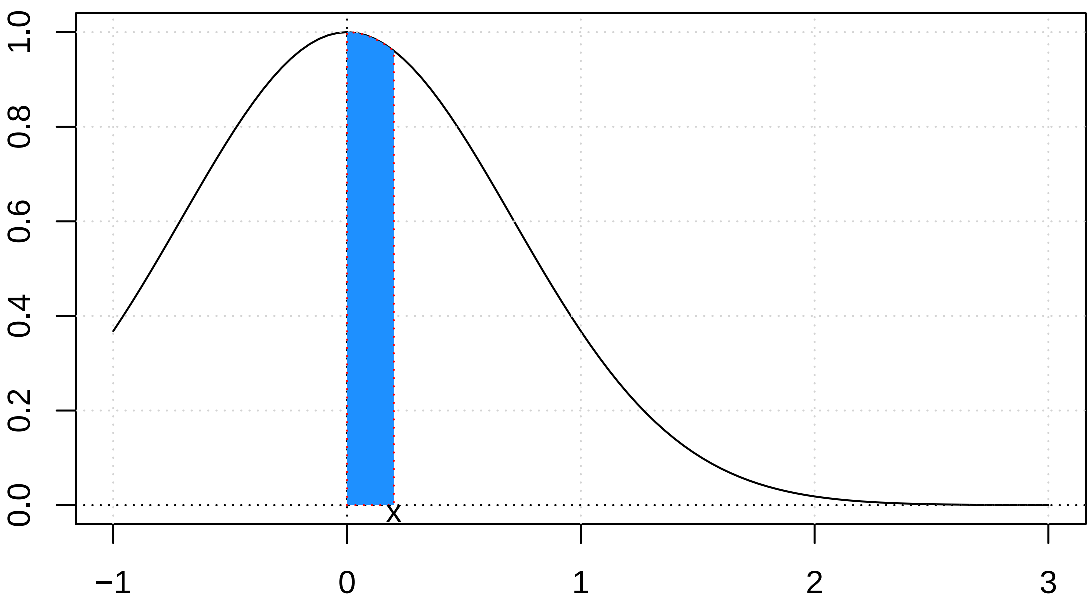
For the sake of brevity, I do not include all the iterates of the bisection method. It took 28 to converge on the root, being very slow at the end.

guess	a	b	iteration
2.500000	3.000000	2.000000	1
2.750000	3.000000	2.500000	3
2.833984	2.835938	2.832031	10
2.833252	2.833496	2.833008	13
2.833333	2.833333	2.833333	27
2.833333	2.833333	2.833333	28

2. Write down a quadratically convergent method to solve the following equation for x.

$$\int_0^x e^{-t^2} dt = 0.1$$

Visually, the solution is the area under the curve that is equal to 0.1 from 0 to x.



I do not know of a way to find the indefinite integral $\int e^{-x^2}$. But, we could take the Taylor series expansion of e^{-x^2} out to two terms, integrate, and then solve using a technique that converges quadratically.

$$\int_0^x 1 - t^2 \; dt = 0.1$$

$$x - \frac{x^3}{3} - 0.1 = 0$$

And since newton's method converges *at least* quadratically, we can apply it using the second function above and its derivative, which after 3 iterations we get

$$x = 0.1003367$$

guess	fx	dfx	error
0.0000000	-0.1000000	1.0000000	0.1000000
0.1000000	-0.0003333	0.9900000	0.0003367
0.1003367	0.0000000	0.9899325	0.0000000

In summary, using Newton's method, my solution looks like this:

$$x_{n+1} = x_n - \frac{x_n - \frac{x_n^3}{3} - 0.1}{1 - x_n^2}$$

In class the method indicated could be

$$x_{n+1} = x_n - \frac{\int_0^x e^{-t^2} dt - 0.1}{e^{-t^2}}$$

3. Find a third order convergent method to approximate a solution of $f(x) = 0$.

A sequence that has cubic convergence could look something like

$$P_n = 10^{-3^n}$$

and a method that has this same behavior approximating $f(x) = 0$ is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{f(x_n)^2 f^2(x_n)}{f'(x_n)^3}$$

which when tested on, say, $f(x) = cos(x) - x = 0$ converges to a given tolerance one iteration quicker than Newton's method.

Below are the results comparing two methods. The initial guess is $x_0 = 1$ for both methods.

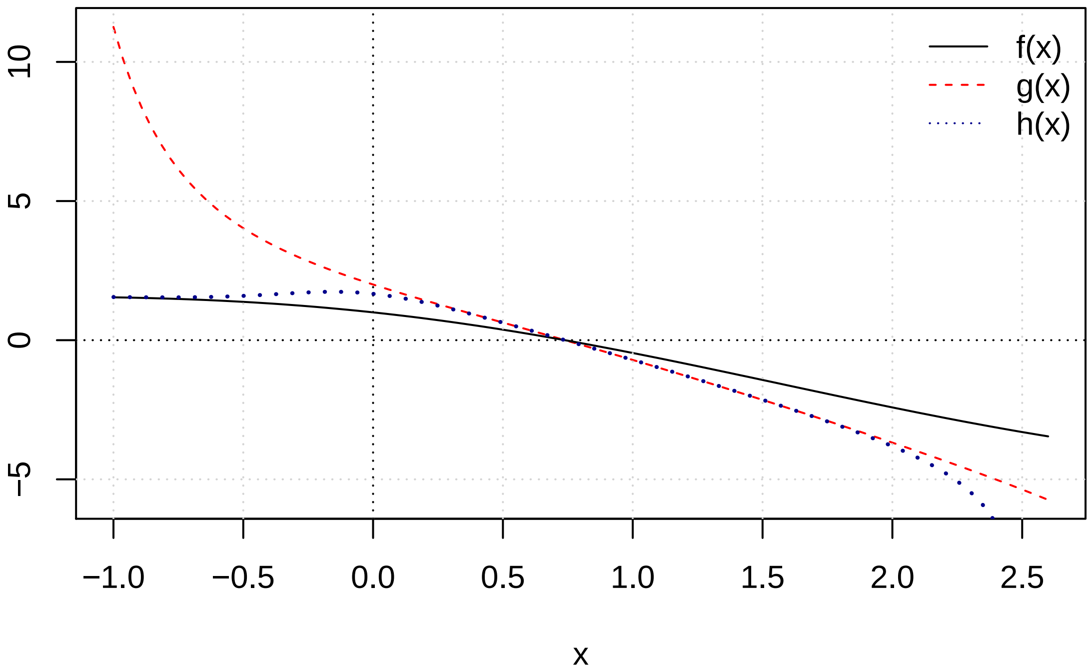
Table 6: Newton's Method for $cos(x) - x = 0$

n	guess	error
1	0.0000000	1.0000000
2	1.0000000	0.2496361
3	0.7503639	0.0112510
4	0.7391129	0.0000278
5	0.7390851	0.0000000

Table 7: Third Order Convergent Method for $cos(x) - x = 0$

n	guess	error
1	0.0000000	0.5000004
2	0.5000004	0.2359022
3	0.7359026	0.0031825
4	0.7390851	0.0000000

It is interesting to visualize the three iterations for $f(x) = cos(x) - x$.



$$g(x) = cos(x) - x - \frac{cos(x) - x}{-sin(x) - 1}$$

$$h(x) = cos(x) - x - \frac{cos(x) - x}{-sin(x) - 1} - \frac{(cos(x) - x)^2 - cos(x)}{2(-sin(x) - 1)^3}$$

4. Write Muller's method in the form

$$x_{n+1} = x_n + []f(x_n)$$

where the expression in the brackets is an approximation of $\frac{-1}{f'(r)}$.

I have no idea how to do this one. Sorry.

Muller's method can be written, with initial guesses p_0, p_1, p_2, as

$$x_{n+1} = x_n - (x_n - x_{n-1})\frac{2c}{max(b \pm \sqrt{b^2 - 4ac})}$$

where a, b, and c are found using

$$c = f(p_2)$$

$$b = \frac{(p_0 - p_2)^2[f(p_1) - f(p_2)] - (p_1 - p_2)^2[f(p_0) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}$$

$$a = \frac{(p_1 - p_2)[f(p_0) - f(p_2)] - (p_0 - p_2)[f(p_1) - f(p_2)]}{(p_0 - p_2)(p_1 - p_2)(p_0 - p_1)}$$

……

5. When does the method $x_{n+1} = x_n + f(x_n)$ converge to a soluton r o

f $f(x) = 0$?

This occurs when $|(x_n + f(x_n))'| < 1$. This can converge very rapidly or very slowly (like with $sin(x) - x = 0$) depending on how closely the values of x_{n+1} and $|(x_n + f(x_n))'|$ are to each other.

6.

$x_0 = 2.251616$. The results of Newton's method, using this value for my starting point are here as well.

guess	fx	dfx	error
2.251616	2.0687747	4.503232	0.4593978
1.792218	0.2110463	3.584436	0.0588785
1.733340	0.0034667	3.466680	0.0010000

For the Newton's method of the problem, I define the following functions:

$$f(x) = x^2 - 3 = 0$$
$$f'(x) = 2x$$

Here's is the code of my function:

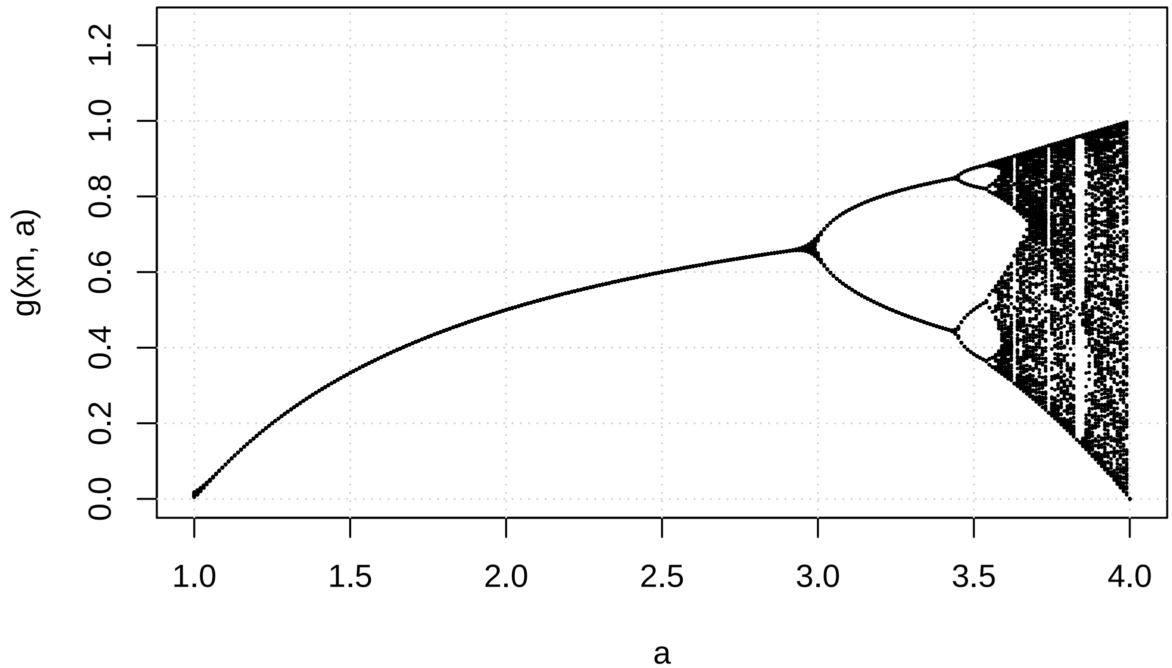
```
err <- function(p0 = 1, p1 = 4, n = 30, e = 0.001) {
  q0 <- newton2(x0 = p0, n = 3)[3, 4] - e
  q1 <- newton2(x0 = p1, n = 3)[3, 4] - e
  r <- p1 - q1 * ((p1 - p0) / (q1 - q0))
  for(i in 1:n) {
    q0 <- newton2(x0 = p0, n = 3)[3, 4] - e
    q1 <- newton2(x0 = p1, n = 3)[3, 4] - e
    r <- p1 - q1 * ((p1 - p0) / (q1 - q0))
    if (abs(q1) <= 1e-14) {
      names(r) <- NULL
      break
    }
    i <- i + 1
    p0 <- p1
    q0 <- q1
    p1 <- r
    q1 <- newton2(x0 = r)[3, 4] - 0.001
  }
  names(r) <- NULL
```

```r
  CHECK <- newton2(x0 = r, n = 3)
  RESULT <- r
  return(list(x0 = r, check = CHECK))
}
```

7.

$$x_{n+1} = g(x, a) = ax - ax^2$$

For the below plot, a starts at 1 and goes to 4 by 0.01 and initial point $x_0 = 0.5$.



Thanks for walking me through this problem, professor.