COMPSCI 527 Homework 2

Yixin Lin, Cody Lieu

September 24, 2015

Problem 1(a)

$$A_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{q_1} = \frac{\mathbf{a_1}}{|\mathbf{a_1}|} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\mathbf{q_2} = normalized(\mathbf{a_2} - proj_{\mathbf{q_1}}\mathbf{a2}) = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$\mathbf{q_3} = normalized(\mathbf{a_3} - proj_{\mathbf{q_1}}\mathbf{a3} - proj_{\mathbf{q_2}}\mathbf{a3}) = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

Problem 1(b)

$$|\mathbf{q_1}| = \sqrt{(\frac{\sqrt{2}}{2})^2 + 0^2 + (\frac{\sqrt{2}}{2})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(-\frac{\sqrt{6}}{6})^2 + (\frac{\sqrt{6}}{3})^2 + (\frac{\sqrt{6}}{6})^2} = 1$$

$$|\mathbf{q_2}| = \sqrt{(\frac{\sqrt{3}}{3})^2 + (\frac{\sqrt{3}}{3})^2 + (-\frac{\sqrt{3}}{3}} = 1$$

$$\mathbf{q_1} \cdot \mathbf{q_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_2} \cdot \mathbf{q_3} = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

$$\mathbf{q_1} \cdot \mathbf{q_3} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} = \mathbf{0}$$

Problem 1(c)

r is equal to the rank of the matrix A.

Problem 1(d)

Yes, since Gram-Schmidt gives us an orthogonal basis for the column space when applied to the column vectors in A, it gives us the dimension of the column space, which is equal to the rank.

Problem 1(e)

First, apply Gram-Schmidt on the set of column vectors of A; then add **b** to the set and continue. If adding **b** increased the number of orthogonormal vectors, that means there is no solution (i.e. there is no linear combination of basis vectors of the column space of A that equals **b**). Otherwise, **b** is a linear combination of the column vectors of A, and there exists a solution.

Problem 1(f)

$$r_{ij} = \mathbf{q_i} \cdot \mathbf{a_j}$$

$$r_{jj} = |\mathbf{a_j'}|$$

Problem 1(g)

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Problem 1(h)

$$Q = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \mathbf{q_3} \end{bmatrix}$$

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & & * \end{bmatrix}$$

In the third iteration of the `for` loop, the `if` statement fails and therefore r is not incremented. The total number of columns in q is r, so there is one less column in Q. This makes sense, since Q should be an orthogonal matrix whose column space is the same as A; if A has linearly dependent vectors, then the number of column vectors in Q will be reduced.

$r_{jj} = |\mathbf{a'_j}| = 0$, so the last element in the diagonal will be 0.

Problem 1(i)

$$Q_{m\times r}$$

$$R_{r\times n}$$

Problem 1(j)

```
function [Q, R] = gs(A)

Q = [];
R = [];

r = 0;
for j = 1:size(A, 2)
    ap = A(:, j);
    for i = 1:r
        R(i, j) = dot(Q(:, i), A(:, j));
        ap = ap - R(i, j).*Q(:, i);
    end
    rjj = norm(ap);
    if rjj > sqrt(eps)
        r = r + 1;
        R(r,j) = rjj;
        Q(:, r) = ap/rjj;
    end
end
```

Problem 1(k)

```
function [Q, R] = ggs(A, Q, R)

if nargin < 3 || isempty(Q) || isempty(R)
    Q = [];
    R = [];
end

[m, n] = size(A);
[r0, n0] = size(R);

if ~isempty(Q) && size(Q, 1) ~= m
    error('A and Q have inconsistent row sizes')
end
```

```
if ~isempty(Q) && size(Q, 2) ~= r0
    error('Q and R have inconsistent sizes')
end

[qr, qc] = size(Q);

% Your code here
r = qc;
for j = 1+qc:n+qc
    ap = A(:, j-qc);
    for i = 1:r
        R(i, j) = dot(Q(:, i), A(:, j-qc));
        ap = ap - R(i, j).*Q(:, i);
    end
    rjj = norm(ap);
    if rjj > sqrt(eps)
        r = r + 1;
        R(r,j) = rjj;
        Q(:, r) = ap/rjj;
    end
end
```

Problem 2(a)

$$\mathbf{c} = Q^{-1}\mathbf{b} = Q^T\mathbf{b}$$

Problem 2(b)

Since Q is an orthogonal matrix, $Q^{-1} = Q^T$. Therefore, we don't require the expensive computation of determining the inverse of Q and can instead take the transpose (which is very efficient).

Problem 2(c)

Since R is a triangular matrix, apply the algorithm of backward substitution pseudo-coded below:

```
x = zeros(i,1);
for i = n:-1:1
    x(i) = c(i);
    for j = i+1:cR
        x(i) = x(i) - x(j)*R(i, j);
    end
    x(i) = x(i)/R(i, i);
end
```

Problem 2(d)

Since some columns of A are linearly independent, the solution has free variables. In order to pick one solution, pick a free variable at random and then back-substitute.

In the code below, we first compute the pivot columns (which are fixed), and we set the free variables to 0. We check if we're on a free column, and if we are we simply ignore it.

```
pivotColumns = [];
for rC = 1:size(R, 2)
    for rR = rC:size(R, 1)
        if abs(R(rR,rC)) > sqrt(eps)
            pivotColumns = [pivotColumns rC];
            break;
        end
    end
end
cR = size(R, 2);
i = cR;
x = zeros(i,1);
while i > 0
    if ismember(i,pivotColumns)
        x(i) = c(i);
        for j = i+1:cR
            x(i) = x(i) - x(j)*R(i, j);
        end
        x(i) = x(i)/R(i, i);
    else
        x(i) = 0;
    end
    i = i-1;
end
```

Problem 2(e)

Since $leftnull(A) = range(A)^{\perp}$, we can find a basis of R^m by using the identity matrix as A and keeping Q and R. Then we remove the vectors that were already in Q to find a basis for the orthogonal complement of $range(A)$.

This works because we know that the identity matrix of size m is definitely a basis for R^M, so using `ggs` on it allows us to definitely get a basis of R^M by adding vecotrs onto Q.

```
[m, n] = size(Q);
[Qn, Rn] = ggs(eye(m), Q, R);
L = Qn(:, n+1:end);
```

Problem 2(f)

```
function [x, N, W, L, Q, R] = solve(A, b)

[Q, R] = ggs(A, [], []);

[m, n] = size(Q);
[Qn, Rn] = ggs(eye(m), Q, R);
L = Qn(:, n+1:end);


[W, Rw] = ggs(A', [], []);
```

```
[r, col] = size(W);
[Qw, Rw] = ggs(eye(r), W, Rw);
N = Qw(:, col+1:end);

Q
b

c = Q'*b;

pivotColumns = [];
for rC = 1:size(R, 2)
    for rR = rC:size(R, 1)
        if abs(R(rR,rC)) > sqrt(eps)
            pivotColumns = [pivotColumns rC];
            break;
        end
    end
end


[Qb, Rb] = ggs(b, Q, R)
if ~isequal(Qb, Q)
    x = []
else
    cR = size(R, 2);
    i = cR;
    x = zeros(i,1);
    while i > 0
        if ismember(i,pivotColumns)
            x(i) = c(i);
            for j = i+1:cR
                x(i) = x(i) - x(j)*R(i, j);
            end
            x(i) = x(i)/R(i, i);
        else
            x(i) = 0;
        end
        i = i-1;
    end
end

end
```

Problem 2(g)

The system in eq. 3 admits infinitely many solutions if the rank of A is less than the number of columns in A (i.e. `size(Q,2) < size(A,2)`) and x exists (i.e. `x~=[]`).

The set of solutions is $\mathbf{n} + \mathbf{x}$, $\forall n \in N$. This is the set of vectors in N, shifted by $\mathbf{x}$, which is an **affine space**.

This is true because of the following theorem, taken verbatim from **this link from Stony Brook**:

Theorem

Let x_p be a *particular solution* to $Ax = b$. Then the *general solution* to $Ax = b$ is given by $x_p + x_n$ where x_n is a vector in the nullspace of A.

Problem 2(h)

Check that $\mathbf{n} \cdot \mathbf{w} = \mathbf{0} \; \forall \mathbf{n} \in N, \forall \mathbf{w} \in W$. Equivalently, `~any(N.' * W)` (Replace N and W with L and Q respectively, to check for L and Q.)

Problem 2(i)

See next page.

Test case 1: A has rank 1. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.32 & -0.05 \\ -0.55 & -0.08 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.48 \\ 0.54 \end{bmatrix}, \quad \mathbf{x} = [], \quad R = \begin{bmatrix} 0.63 & 0.10 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.15 \\ -0.99 \end{bmatrix}, \quad W = \begin{bmatrix} -0.99 \\ -0.15 \end{bmatrix}, \quad L = \begin{bmatrix} 0.86 \\ -0.50 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.50 \\ -0.86 \end{bmatrix}$$

Test case 2: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.93 & -0.56 \\ -0.53 & -0.59 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.00 \\ 0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2.66 \\ -2.66 \end{bmatrix}, \quad R = \begin{bmatrix} 1.07 & 0.78 \\ 0.00 & 0.24 \end{bmatrix}$$

$$N = [], \quad W = \begin{bmatrix} -0.86 & 0.51 \\ -0.51 & -0.86 \end{bmatrix}, \quad L = [], \quad Q = \begin{bmatrix} -0.87 & 0.50 \\ -0.50 & -0.87 \end{bmatrix}$$

Test case 3: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} 0.41 & 0.55 \\ -0.34 & -0.19 \\ -0.55 & -0.44 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.23 \\ -0.05 \\ -0.15 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -0.15 \\ 0.53 \end{bmatrix}, \quad R = \begin{bmatrix} 0.76 & 0.69 \\ 0.00 & 0.22 \end{bmatrix}$$

$$N = [], \quad W = \begin{bmatrix} 0.60 & -0.80 \\ 0.80 & 0.60 \end{bmatrix}, \quad L = \begin{bmatrix} 0.25 \\ -0.73 \\ 0.64 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.54 & 0.80 \\ -0.44 & 0.53 \\ -0.72 & 0.28 \end{bmatrix}$$

Test case 4: A has rank 1. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.30 & -0.31 \\ -0.23 & -0.24 \\ -0.67 & -0.69 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -0.07 \\ -0.06 \\ -0.16 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.12 \\ 0.12 \end{bmatrix}, \quad R = \begin{bmatrix} 0.77 & 0.80 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.72 \\ -0.69 \end{bmatrix}, \quad W = \begin{bmatrix} -0.69 \\ -0.72 \end{bmatrix}, \quad L = \begin{bmatrix} 0.92 & 0.00 \\ -0.13 & 0.95 \\ -0.37 & -0.32 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.39 \\ -0.30 \\ -0.87 \end{bmatrix}$$

Test case 5: A has rank 1. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.52 & -0.19 & -0.47 \\ -0.54 & -0.20 & -0.49 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.29 \\ 0.34 \end{bmatrix}, \quad \mathbf{x} = [], \quad R = \begin{bmatrix} 0.75 & 0.28 & 0.68 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.70 & 0.00 \\ -0.27 & 0.93 \\ -0.66 & -0.38 \end{bmatrix}, \quad W = \begin{bmatrix} -0.72 \\ -0.26 \\ -0.65 \end{bmatrix}, \quad L = \begin{bmatrix} 0.72 \\ -0.70 \end{bmatrix}, \quad Q = \begin{bmatrix} -0.70 \\ -0.72 \end{bmatrix}$$

Test case 6: A has rank 2. Product checks passed. Dimension checks passed.

$$A = \begin{bmatrix} -0.62 & -0.79 & -0.77 \\ -0.06 & -0.11 & 0.40 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1.23 \\ 0.46 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 0.20 \\ 0.19 \\ 1.24 \end{bmatrix}, \quad R = \begin{bmatrix} 0.63 & 0.79 & 0.72 \\ 0.00 & 0.03 & -0.47 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.80 \\ -0.60 \\ -0.04 \end{bmatrix}, \quad W = \begin{bmatrix} -0.49 & -0.34 \\ -0.62 & -0.50 \\ -0.61 & 0.79 \end{bmatrix}, \quad L = [], \quad Q = \begin{bmatrix} -0.99 & 0.10 \\ -0.10 & -0.99 \end{bmatrix}$$
