# Universality in an Exponentially Large Ensemble of F-theory Compactifications

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(Dated: April 24, 2017)

We study universality of gauge sectors in the string landscape in the context of F-theory compactifications. A finite time construction algorithm for $\frac{4}{3} \times 1.3998 \times 10^{755}$ F-theory geometries is presented. High probability geometric assumptions uncover universal structures in the ensemble without explicitly constructing it. For example, non-Higgsable clusters of seven-branes with intricate gauge sectors occur with probability above $1 - 1.07 \times 10^{-755}$, and the geometric gauge group rank is above 160 with probability .9999994. In the latter case there are at least 10 $E_8$ factors, the structure of which fixes the gauge groups on certain nearby seven-branes. Visible sectors may arise from $E_6$ or $SU(3)$ seven-branes, which occur in certain random samples with probability $\simeq 1/200$.

**I. Introduction.** String theory is a theory of quantum gravity that naturally gives rise to interesting gauge and cosmological sectors. As such, it is a candidate unified theory. However, it gives rise to a vast landscape of four-dimensional metastable vacua that may realize different laws of physics, making predictions difficult.

Other areas of physics signal a possible way forward, though, since a large ensemble of states may exhibit universality. For large ensembles of string vacua, such as the oft-quoted $O(10^{500})$ type IIb flux vacua [1], studying universality via explicit construction is impractical. In such cases it seems progress must necessarily be made by formulating a precise algorithm that allows for the extraction of universal physical data but does not require explicit construction of the ensemble.

We present such an algorithm in the context of $4d$ F-theory [2] compactifications. The ensemble is a collection of $4/3 \times 1.3998 \times 10^{755}$ six-manifolds, that serve as the extra dimensions of space. Their topological structure determines the $4d$ gauge group that arises geometrically from configurations of seven-branes that form a network of so-called non-Higgsable clusters (NHC) [3]. We establish that non-Higgsable clusters arise with probability above $1-1.07\times10^{-755}$ in this ensemble and demonstrate that certain minimal and rich gauge structure arises with high probability. We also present results from random samples that are potentially relevant for visible sectors.

A number of recent results suggest that NHC are very important in the $4d$ F-theory landscape. These configurations exist for generic scalar field (complex structure moduli) expectation values, and therefore obtaining gauge symmetry does not require [4] stabilization on subloci in moduli space, which can have high codimension [5]. Standard model structure may arise naturally [4], strong coupling is generic [6], and $4d$ NHC may exhibit features [7] (such as loops and branches) not present in $6d$. NHC arise in the F-theory geometry with the largest number of flux vacua [8], and they arise universally in known ensembles [9] that are closely related to ours. NHC in $6d$ have been classified [3] and studied extensively [10].

In Sec. II we review non-Higgsable clusters. In Sec. III we present our ensemble. In Sec. IV we study universality. In Sec. V we discuss our results.

**II. Seven-Branes and Non-Higgsable Clusters.** A $4d$ F-theory geometry is a Calabi-Yau elliptic fibration $X$ over a threefold base space $B$ defined by the equation

$$y^2 = x^3 + fx + g \tag{1}$$

where $f$ and $g$ are homogeneous polynomials in the coordinates of $B$; technically $f \in \Gamma(-4K_B)$, $g \in \Gamma(-6K_B)$, with $K_B$ the canonical line bundle on $B$. $B$ provides extra spatial dimensions, and seven-branes are localized on the discriminant locus $\Delta = 0 \subset B$, where $\Delta = 4f^3+27g^2$.

Upon compactification, the gauge group structure of seven-branes gives rise to four-dimensional gauge sectors. It is controlled by $f$ and $g$, and for a typical $B$ the most general $f, g$ take the form $f = \tilde{f} \prod_i x_i^{l_i}$, $g = \tilde{g} \prod_i x_i^{m_i}$, so

$$\Delta = \tilde{\Delta} \prod_i x_i^{\min(3l_i, 2m_i)} =: \tilde{\Delta} \prod_i x_i^{n_i}, \tag{2}$$

and therefore $f$, $g$ and $\Delta$ vanish along $x_i = 0$ to $ord_{x_i}(f, g, \Delta) = (l_i, m_i, n_i)$. This seven-brane carries a gauge group $G_i$ given in Table I according to the Kodaira classification. In some cases further geometric data is necessary to uniquely specify $G_i$, see e.g. [9] for conditions, but this data always exists for fixed $B$. For generic $f$ and $g$, a seven-brane on $x_i = 0$ requires $(l_i, m_i) \geq (1, 1)$.

Such a seven-brane is called a geometrically non-Higgsable seven-brane (NH7) because it carries a gauge group that cannot be removed by deforming $f$ or $g$. A NH7 may have geometric gauge group

$$G \in \{E_8, E_7, E_6, F_4, SO(8), G_2, SU(3), SU(2)\}, \tag{3}$$

which could be broken by turning on particular fluxes. A typical base $B$, as we will show in the strongest generality to date, has many non-Higgsable seven-branes that often intersect in pairs, giving rise jointly charged matter. Such a cluster of seven-brane is a geometrically non-Higgsable cluster (NHC). For brevity, we henceforth drop the adjectives geometric and geometrically.

| $F_i$ | $l_i$ | $m_i$ | $n_i$ | Sing. | $G_i$ |
|---|---|---|---|---|---|
| $I_0$ | $\geq 0$ | $\geq 0$ | 0 | none | none |
| $I_n$ | 0 | 0 | $n \geq 2$ | $A_{n-1}$ | $SU(n)$ or $Sp(\lfloor n/2 \rfloor)$ |
| $II$ | $\geq 1$ | 1 | 2 | none | none |
| $III$ | 1 | $\geq 2$ | 3 | $A_1$ | $SU(2)$ |
| $IV$ | $\geq 2$ | 2 | 4 | $A_2$ | $SU(3)$ or $SU(2)$ |
| $I_0^*$ | $\geq 2$ | $\geq 3$ | 6 | $D_4$ | $SO(8)$ or $SO(7)$ or $G_2$ |
| $I_n^*$ | 2 | 3 | $n \geq 7$ | $D_{n-2}$ | $SO(2n-4)$ or $SO(2n-5)$ |
| $IV^*$ | $\geq 3$ | 4 | 8 | $E_6$ | $E_6$ or $F_4$ |
| $III^*$ | 3 | $\geq 5$ | 9 | $E_7$ | $E_7$ |
| $II^*$ | $\geq 4$ | 5 | 10 | $E_8$ | $E_8$ |

TABLE I. Kodaira fiber $F_i$, singularity, and gauge group $G_i$ on the seven-brane at $x_i = 0$ for given $l_i$, $m_i$, and $n_i$.

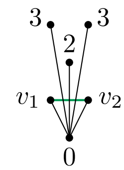
### III. Large Landscapes of Geometries from Trees.

We now introduce our construction, which utilizes building blocks in toric varieties that we call trees to systematically build up F-theory geometries. After describing the geometric setup and defining terms that simplify the discussion, we will present a criterion, classify all trees satisfying it, and build F-theory geometries.

Our construction begins with a simple starting point, a smooth weak-Fano toric threefold $B_i$, and then builds structure on top of it. These geometries $B_i$ are determined by fine regular star triangulations (FRST) of one of the 4319 (c) $3d$ reflexive polytopes, and it has been estimated [5] that there are $O(10^{15})$ such geometries. The $2d$ faces of the $3d$ polytope are known as facets, and a triangulated polytope will have triangulated facets. None of these geometries have non-Higgsable seven-branes.

Consider such a $B_i$ determined by an FRST of a $3d$ reflexive polytope $\Delta^\circ$. Consider also a triangulated facet $F$ in it and an edge between two points in $F$ with edge vertices $v_1$ and $v_2$, which have associated homogeneous coordinates $x_1$ and $x_2$. Since $v_1$ and $v_2$ are vertices of an edge they generate a $2d$ cone $\sigma_{12}$ in the fan description of $B_i$, which means that $x_1 = x_2 = 0$ defines an algebraic curve in $B_i$. This curve can be blown up by adding a new ray $v_e = v_1 + v_2$ and appropriately subdividing the unique $3d$ cones that intersect at $\sigma_{12}$, giving a new smooth toric variety $B$. A coordinate $e$ is associated to $v_e$ and $e = 0$ defines the exceptional divisor in $B$. This process can be iterated, for example blowing up along $e = x_1 = 0$ which would add a new ray $v_e + v_1$.

After a number of iterations the associated toric variety will have a collection of exceptional divisors with associated rays $v_{e_i} = a_i v_1 + b_i v_2$, which will appear to have formed a *tree* above the *ground* that connects $v_1$ and $v_2$ in $F$. Each $v_{e_i}$ is a *leaf* with *height* $h_{e_i} = a_i + b_i$, and we will refer to trees built on edges within $F$ as *edge trees*. The height of a tree is the height of its highest leaf. As an example, $\{v_1 + v_2, 2v_1 + v_2, v_1 + 2v_2\}$ appears as




where the green line is the edge (ground) in $F$, 0 is the origin of $\Delta^\circ$, and the new rays are labelled by their heights.

Similarly, one can also build *face trees* by beginning with a face on $F$, with face vertices $v_1, v_2, v_3$ having associated $x_1, x_2, x_3$ and $3d$ cone $\sigma_{123}$. Adding $v_e = v_1 + v_2 + v_3$ and subdividing $\sigma_{123}$ blows up at the point $x_1 = x_2 = x_3 = 0$ and produces a new toric variety. Again such blowups can be iterated. This process builds a collection of leaves $v_{e_i} = a_i v_1 + b_i v_2 + c_i v_3$ with $a_i, b_i, c_i > 0$ of height $h_{e_i} = a_i + b_i + c_i$ that comprise a *face tree*. Face trees are built above the interior of the face due to the strict inequality in the definition. Note if one leaf coefficient was zero the associated leaf would be above an edge of the face, not above the face interior.

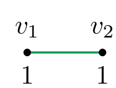
Geometries can be systematically constructed by adding a face tree to each face in each triangulated facet of $\Delta^\circ$, and then an edge tree to each edge. The associated smooth toric threefold $B$ has a collection of rays $v$, each of which can be written $v = av_1 + bv_2 + cv_3$ with $v_i$ $3d$ cone vertices in $B_i$. If $(a, b, c) = (1, 0, 0)$ or some permutation thereof, $v \in \Delta^\circ$ and this height $h_v = 1$ "leaf" is more appropriately a root, since it is on the ground.

A natural question in systematically building up geometries is whether there is a maximal tree height. For a toric variety $B$ to be an allowed F-theory base it must not have any $(4, 6)$ divisors, which we ensure by a simple height criterion proven in Prop. 1:

If $h_v \leq 6$ for all leaves $v \in B$,
then there are no $(4, 6)$ divisors.

This condition is simple and sufficient, but not necessary for the absence of $(4, 6)$ divisors. Nevertheless, it will allow us to build a large class of geometries.

The task is now clear: we must systematically build all topologically distinct edge trees and face trees of height $\leq$ 6. Since the combinatorics are daunting, let us exemplify the problem for $h \leq 3$ trees. Viewing the facet head on, an edge appears as

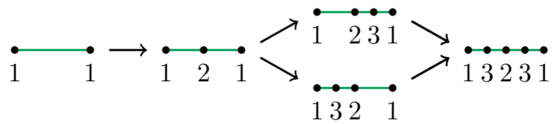



where the vertices are labelled, as well as their height 1. With our definitions a ray on the ground (i.e. in a facet) is technically a height 1 leaf. Adding $v_1 + v_2$ subdivides the edge, which can then be further subdivided

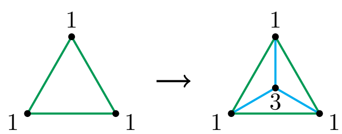
| $N$ | # Edge Trees | # Face Trees |
|---|---|---|
| 3 | 5 | 2 |
| 4 | 10 | 17 |
| 5 | 50 | 4231 |
| 6 | 82 | 41, 439, 964 |

| $h_v$ | Probability |
|---|---|
| 3 | .99999998 |
| 4 | .999995 |
| 5 | .999997 |
| 6 | .9999898 |

TABLE II. *Left:* The number of edge trees and face trees with of height $h \leq N$. *Right:* The probability that a face tree with $h \leq 6$ has a leaf $v$ with a given height $h_v$.




where we have dropped the vertex labels and kept the heights. The trees emerge out of the page, but visualization is made easier by projecting on to the edge. We see there are five edge trees with height $\leq 3$. Similarly we have form face trees via




from which we see that there are 2 face trees of height $\leq 3$. In this image we have denoted the new edges by blue lines since they do not sit in the facet. With our definitions edge trees are built above an edge in the facet, whereas higher leaves in face trees may be built on new edges that don't sit in the facet. For example, a height 4 leaf could be added on any of the blue lines above.

A tedious but straightforward calculation shows that the number of edge or face trees with $h \leq N$ grows rapidly. See Table II for specifics of this growth, and note that the number of trees with $h \leq 6$, as relevant in our construction, is large. Rotations and flips should not be taken into account now; symmetries will enter later.

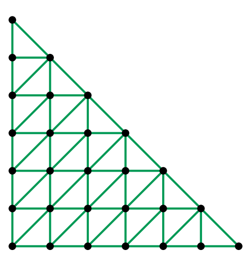


FIG. 1. The largest facet in one of the two 3d reflexive polytopes with the most number of interior points. Presented is one triangulation, from which we see $\#\tilde{E} = 63$ and $\#\tilde{F} = 36$.

Having classified the number of $h \leq 6$ face trees and edge trees, we now estimate the number of toric varieties that may arise from putting trees on an FRST of $\Delta^\circ$. We construct an ensemble $S_{\Delta^\circ}$ of geometries by systematically putting $h \leq 6$ face trees on all faces $\tilde{F}$ of a triangulated facet $F$ and then putting $h \leq 6$ edge trees on all edges $\tilde{E}$ in $F$. The number of ways of doing this overcounts due to the fact that some $\tilde{E}$ lie in two facets, i.e. on an edge $E$ of $\Delta^\circ$. Accounting for this and using Table II we obtain an equation for the size of $S_{\Delta^\circ}$

$$|S_{\Delta^\circ}| = \sum_F 82^{\#\tilde{E}\text{ on }F}(4.2\times 10^6)^{\#\tilde{F}\text{ on }F} - \sum_E 82^{\#\tilde{E}\text{ on }E}, \tag{4}$$

which may be simplified since on a triangulated facet $F$

$$\#\tilde{E} = 2n_B + 3n_I - 3 \qquad \#\tilde{F} = n_B + 2n_I - 2, \tag{5}$$

where $n_I$ and $n_B$ are the number of facet interior and boundary points, respectively.

There are two polytopes that give a far larger number $|S_{\Delta^\circ}|$ than the others. They are the convex hulls $\Delta_i^\circ := \mathrm{Conv}(S_i), i = 1, 2$ of the vertex sets

$$\begin{aligned} S_1 &= \{(-1,-1,-1),(-1,-1,5),(-1,5,-1),(1,-1,-1)\} \\ S_2 &= \{(-1,-1,-1),(-1,-1,11),(-1,2,-1),(1,-1,-1)\}. \end{aligned} \tag{6}$$

$S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$ have the same number of edges and faces. The largest facet of $\Delta_1^\circ$ is displayed in 1, which has $\#\tilde{E} = 63$ and $\#\tilde{F} = 36$. Taking into account symmetry arguments that we will make later, a direct computation yields

$$|S_{\Delta_1^\circ}| = \frac{1.4}{3} \times 10^{755} \qquad |S_{\Delta_2^\circ}| = 1.4 \times 10^{755}, \tag{7}$$

where the factor of 1/3 is due to the symmetries. All other polytopes $\Delta^\circ$ contribute negligibly, $|S_{\Delta^\circ}| \leq 1.65 \times 10^{692}$ configurations. This gives a lower bound

$$\#\text{ 4d F-theory Geometries} \geq \frac{4}{3} \times 1.4 \times 10^{755}, \tag{8}$$

which undercounts due to the fact that we choose to do face blowups followed by edge blowups to simplify the subdivision combinatorics.

## IV. Universality and Non-Higgsable Clusters.

We now study universality in the dominant sets of F-theory geometries $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$. We prove non-Higgsable cluster universality, minimal gauge group universality, and discuss initial results from random sampling.

**Non-Higgsable Cluster Universality.** We wish to establish the likelihood that an F-theory base in $S_{\Delta_1^\circ}$ or $S_{\Delta_2^\circ}$ give rise to non-Higgsable seven-branes. The result arises from Prop. 2.

Summarizing it, if there is a tree anywhere on $F$, even a single leaf, there is a non-Higgsable seven-brane on the divisor associated to each point $p$ interior to $F$. For any $S_{\Delta^\circ}$ only one configuration has no trees, and therefore

$$P(\text{NHC in } S_{\Delta^\circ}) = 1 - \frac{1}{|S_{\Delta^\circ}|}. \tag{9}$$

This is always very close to one, and in particular

$$P(\text{NHC in } S_{\Delta_1^\circ}) = 1 - 2.14\times 10^{-755}$$
$$P(\text{NHC in } S_{\Delta_2^\circ}) = 1 - .71\times 10^{-755}, \qquad (10)$$

for the dominant ensembles in our construction. Non-Higgsable clusters are universal in these ensembles.

**Gauge Group Universality.** We now wish to study physics in our ensemble. Consider a geometric assumption $A_i$ and a physical property $P_i$ such that $A_i \to P_i$. Our goal is to determine high probability assumptions that lead to interesting physical properties, computing $P(A_i)$ since $A_i \to P_i$ ensures $P(P_i) \geq P(A_i)$. We will focus $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$ since these dominate the ensemble.

Consider first $S_{\Delta_1^\circ}$ and let $A_1$ be the assumption that any simplex in an FRST of $\Delta_1^\circ$ containing a vertex of $\Delta_1^\circ$ has an $h \geq 3$ face tree on it. For the 3 symmetric facets of $\Delta_1^\circ$ there are 17 ways to choose simplices containing the vertices, and 1796 ways for its largest facet. The maximum number of simplices containing vertices is 24, and $P(h \geq 3 \text{ tree on simplex}) = .9999998$ from Table II

$$P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .9999998^{24} = .9999994. \qquad (11)$$

There are $17^3\times 1796$ ways to choose simplices that contain the vertices, all of which have $G \geq F_4^{18}\times E_6^{10}\times U^9$ where $U \in \{G_2, F_4, E_6\}$ depending on details and all of these factors arise on the ground; generally there will be many more factors from non-Higgsable seven-branes in the leaves. Here $E_6^{10}$ arises from an $E_6$ on every interior point of the large facet in $\Delta_1^\circ$, see Fig. 1. This set of statements defines physical property $P_1$, and since $A_1 \to P_1$ we deduce $P(P_1 \text{ in } S_{\Delta_1^\circ}) \geq P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .9999994$.

Let $A_2$ be the assumption that there exists a $h = 5$ face tree somewhere on the large facet $F$ in $\Delta_1^\circ$. Knowing $\tilde{F} = 36$ on $F$ and using Table II, we compute $P(A_2 \text{ in } S_{\Delta_1^\circ}) = (1-(1-.999997)^{36}) \simeq 1-10^{-199}$. Let $A_3$ be that $A_1$ and $A_2$ hold, so $P(A_3) = P(A_1)P(A_2) \simeq P(A_1)$. Then given $A_3$ a short calculation shows that the $h = 5$ tree on $F$ enhances $E_6$ in $P_1$ to $E_8$, giving 10 $E_8$'s on the ground. $P_1$ with this enhancement defines $P_3$.

Similar results hold for for $S_{\Delta_2^\circ}$. Let $A_1$ be the assumption that any simplex in an FRST of $\Delta_2^\circ$ containing a vertex of $\Delta_2^\circ$ has an $h \geq 3$ face tree on it. This ensures that $G \geq F_4^{15}\times E_6^7\times U^{12}$. However, this is quickly enhanced to $G \geq F_4^{18}\times E_6^{10}\times U^9$, via a $h = 5$ face tree on each face, and a $1 - 6.55\times 10^{-8}$ probability blow-up along an edge connecting the point $\{-1, 2, -1\}$ to one of the points $\{-1, 1, n\}$, where $n = -1 \ldots 3$. The existence of these edges is independent of triangulation.

The geometric gauge group on a leaf $v$ in a tree built above $E_8$'s is determined by the leaf height $h_v$, as proven in Prop. 3 in the Appendix. The result is that a $h_v = 1, 2, 3, 4, 5, 6$ leaf above $E_8$ roots has Kodaira fiber $F_v = II^*, IV^*_{ns}, I^*_{0,ns}, IV_{ns}, II, -$ with gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.

This leads to a high probability result about the structure of the geometric gauge group. Since $A_3 \to P_3$, which has at least 10 $E_8$ factors nearby one another, $P_3$ also has

$$G \geq E_8^{10}\times F_4^{18}\times U^9\times F_4^{H_2}\times G_2^{H_3}\times A_1^{H_4}, \qquad (12)$$

where $H_i$ is the number of height $i$ leaves in trees built on $E_8$ roots, and $rk(G) \geq 160 + 4H_2 + 2H_3 + H_4$. There are $H_6$ Kodaira type $II$ seven-branes. The first $F_4$ and also the $U$ factors may enhance, but the are fixed. The probability of this physical property is $P(P_3 \text{ in } S_{\Delta_1^\circ}) \geq P(A_3 \text{ in } S_{\Delta_1^\circ}) \simeq .9999994$ and $P(P_3 \text{ in } S_{\Delta_2^\circ}) \geq P(A_3 \text{ in } S_{\Delta_2^\circ}) \simeq .9999994$ . This non-trivial minimal gauge structure is universal in our large ensemble given by $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$.

**Random Samples and Geometric Visible Sectors.** It may be possible to accomodate visible sectors from flux breaking these gauge sectors, but it is also interesting to study whether gauge factors $E_6$ and / or $SU(3)$ arise with high probability. We have not yet discovered a high probability simple geometric assumption that leads to $E_6$ or $SU(3)$. However, it is possiblethat they arise regularly, but due to a complex geometric assumption.

This idea can be tested by random sampling. Let $B$ be an F-theory base obtained by adding face trees at random, followed by edge trees at random, to the pushing triangulation of $\Delta_1^\circ$. We studied an ensemble $S_r$ of $10^6$ such random samples and found $P(SU(3) \,\text{or}\, E_6 \text{ in } S_r) \simeq 1/200$, and that at least 36 of the points in $\Delta_1^\circ$ carried $E_8$, a significant enhancement beyond assumption 3. Furthermore, in our sample we found that $E_6$ only arose on the point $(1, -1, -1)$, which is the only vertex of $\Delta_1^\circ$ that isn't in the largest facet. Similar results and probabilities also hold using this techniques on $\Delta_2^\circ$. It would be interesting to study random samples of other triangulations, or see if new simple geometric assumptions imply these enhancements. We leave this, and the systematic study of geometric visible sectors, to future work.

**V. Discussion.**

We have presented a large ensemble of geometries for $4d$ F-theory compactifications.

**A. Appendix: Technical Subtleties.** We now address technical subtleties that are important for establishing, but not understanding, results in the main text.

In equation (7) we have included a factor of 1/3 relative to the count one would obtain directly from the algorithm. This takes into account an overcounting of geometries due to toric equivalences, which arise when there is a $GL(3,\mathbb{Z})$ transformation on the toric rays that preserves the cone structure of the fan. In general, there may be many such equivalences between elements of two ensembles $S_{\Delta_i^\circ}$ and $S_{\Delta_j^\circ}$, where $\Delta_{i,j}^\circ$ are any two $3d$ reflexive polytopes. However, to ensure that the count (7) is accurate, we only need to consider whether there are equivalences between two elements in $S_{\Delta_1^\circ}$, two in $S_{\Delta_2^\circ}$, or one in $S_{\Delta_1^\circ}$ to $S_{\Delta_2^\circ}$. It is sufficient to consider $GL(3,\mathbb{Z})$ actions on the ground, i.e. on the facets. This follows from the fact that rays of different height cannot be exchanged under automorphisms of the fan. First note that points in a hyperplane remain in a hyperplane after a $GL(3,\mathbb{Z})$ transformation, and points in a line remain in a line. Facet must therefore map to facets. The big facets in $\Delta_1^\circ$ and $\Delta_2^\circ$ cannot map to other facets by point counting, and therefore they must map to themselves. There is no non-trivial map taking the big facet in $\Delta_2^\circ$ to itself, but there is a $\mathbb{Z}_3$ rotation taking the big facet in $\Delta_1^\circ$ to itself, giving a factor of 1/3 in $S_{\Delta_1^\circ}$. There is no non-trivial map between the big facets in $\Delta_1^\circ$ and $\Delta_2^\circ$, and therefore $S_{\Delta_2^\circ} \cap S_{\Delta_1^\circ} = \emptyset$. Together, these establish (7).

In discussing what constitutes a good $4d$ F-theory geometry $X \to B$, we mentioned certain criteria on orders of vanishing that we now elaborate on. In [13, 14] it was shown that if a Calabi-Yau variety has at worst canonical singularities, it is at finite distance from the bulk of the moduli space in the Weil-Petersson metric. This criterion is general and therefore applies to elliptic fibrations such as $X$. The reason that it is physically relevant is that if $X$ has worse singularities than a nearby Calabi-Yau $X'$ that is known to represent a physical configuration, and $X$ is at finite distance in the moduli space from $X'$, we should expect that $X$ is also a physical configuration. This criterion, which we refer to as the Hayakawa-Wang criterion, gives a related criterion by studying elliptic fibrations: if $ord_C(f,g) \leq (4,6)$ and $ord_p(f,g) \leq (8,12)$ for all curves $C \subset B$ and points $p \subset B$ then $X$ has at worst canonical singularities and is at finite distance in the moduli space due to the Hayakawa-Wang criterion[1]. This latter criterion, which we refer to as the weak OOV (orders of vanishing) criterion, is sufficient but not necessary for at worst canonical singularities; for example a rich set of $6d$ $N=1$ SCFTs [15] arises from relaxing the $(4,6)$ condition on codimension two loci in the base. Furthermore, if $B$ is such that $X \to B$ does not satisfy the weak OOV criterion, but there is a blowup $B' \to B$ such that $X' \to B'$ does, then $X$ also has at worst canonical singularities and is at finite distance in moduli space.

For geometries in our construction to satisfy the Hayakawa-Wang criterion it suffices to show that if $X \to B$ does not satisfy weak OOV, then there is a blowup $B' \to B$ such that $X' \to B'$ does. This is straightforward. Consider any toric curve $C = D_s \cdot D_t \subset B$. Take $v_s = \sum_i a_{i,s} v_i$ and $v_t = \sum_i a_{i,t} v_i$ and define $a := \sum_i a_{i,s}$ and $b := \sum_i a_{i,t}$. Let $F$ be a facet on which or above which $v_s$ and $v_t$ sit, with $m_F$ the facet dual. As an element of $\Delta_g$ the associated monomial may be written $s^{\langle m,v_s\rangle+6} t^{\langle m,v_t\rangle+6} \times \dots$, and the monomial vanishes to order $\langle m,v_s\rangle + \langle m,v_t\rangle + 12 = -a-b+12$ along $C$, which must be $\geq 6$ for $C$ to be a $(4,6)$ curve. Therefore $a+b \leq 6$ is necessary for $C$ to be a $(4,6)$ curve. Now suppose $C$ is a $(4,6)$ curve that we blow up via $B' \to B$ by adding an exceptional divisor $v_e = \sum a_i v_i = v_s + v_t$. Then $\sum a_i = a+b$, which satisfies $\sum a_i \leq 6$ since $C$ is $(4,6)$, but this condition is sufficient to ensure that $\hat{B}$ has no $(4,6)$ divisors! If $2a+b \leq 6$ or $a+2b \leq 6$ then $B'$ may still have a $(4,6)$ curve, but this blowup process can be iterated until there are no longer $(4,6)$ curves or divisors. A similar argument holds for each $(4,6)$ point $p = D_s \cdot D_t \cdot D_u \subset B$, with $v_s$ and $v_t$ as before and $v_u = \sum_i a_{i,u} v_i$ with $c := \sum_i a_{i,u}$. Considering it as an element of $\Delta_g$ leads to the condition that $a+b+c \leq 6$ is necessary for $p$ to be a $(4,6)$ point, and as before we can perform blowups until there no longer are $(4,6)$ points. Performing iterative blowups of points and curves in this way gaurantee the existence of a map $B' \to B$ such that the weak OOV criterion is satisfied.

**Proposition 1.** *Suppose each leaf $v \in B$ has height $h_v \leq 6$. Then $B$ has no $(4,6)$ divisors.*

*Proof.* Consider a facet $F$, which has a unique associated point $m_F$ satisfying $(m_F, \tilde{v}) = -1 \; \forall \tilde{v} \in F$; furthermore since $m_F \in \Delta$, $(m_F, v) \geq -1 \; \forall v \in \Delta^\circ$. Now suppose $h_v \leq 6/n$, $n \in \mathbb{N}$ for all rays $v = av_1 + bv_2 + cv_3$ in $B$, with $v_i$ $3d$ cone vertices in $B_i$ . Then $(nm_F, v) \geq -n(a+b+c) = -nh_v \geq -6$ for all rays $v$ and therefore $nm_F \in \Delta_g$. If $h_v \leq 6 \; \forall v$, then $m_F \in \Delta_g$. This monomial has order of vanishing $(v, m_F) + 6 = 5$ for any $v$ in or above $F$, which protects $v$ from being a $(4,6)$ divisor. If $h_v \leq 6 \; \forall v$ then $m_F \in \Delta_g \; \forall F$ and there is a monomial that prevents each divisor from being $(4,6)$. □

**Proposition 2.** *Suppose $\exists \, v$ in or above a facet $F$,i.e. $v = av_1 + bv_2 + cv_3$ with $v_i$ simplex vertices in $F$, such that $h_v \geq 2$. Then there is a non-Higgsable seven-brane on the divisor associated to each interior point of $F$.*

*Proof.* Then $(6m_F, v) = -6h_v \leq -12$ implies $6m_F \notin \Delta_g$. Similarly, $4m_F \notin \Delta_f$. Since any point $p$ interior to $F$ has $(m,p) = -1 \leftrightarrow m = m_F$ and reflexive polytopes of dimension three are normal, i.e. any $m_f \in \Delta_f$ ($m_f \in \Delta_g$) has $m_f = \sum_i m_i, m_i \in \Delta$ ($m_g = \sum_i m_i, m_i \in \Delta$), it follows that $(m_f, p) = -4 \leftrightarrow m_f = 4m_F$ and $(m_g, p) = -6 \leftrightarrow m_g = 6m_F$. Therefore is there is any tree on $F$ then $4m_F \notin \Delta_f$ and $6m_F \notin \Delta_g$. By normality, for any $p$ interior to $F$ this implies $\nexists m_f \in \Delta_f | (m_f, p) = -4$ and $\nexists m_g \in \Delta_g | (m_g, p) = -6$, and therefore $ord_p(f,g) > (0,0)$, which implies there is a non-Higgsable seven-brane on the divisor associated to $p$. □

**Proposition 3.** *Let $v$ be a leaf $v = av_1 + bv_2 + cv_3$ with $v_i$ simplex vertices in $F$. If the associated the divisors $D_{1,2,3}$ carry a non-Higgsable $E_8$ seven-brane, then if $v$ has height $h_v = 1,2,3,4,5,6$ it also has Kodaira fiber $F_v = II^*, IV^*_{ns}, I^*_{0,ns}, IV_{ns}, II, -$ and gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.*

[1] We thank D. Morrison for discussions on this and related points.

*Proof.* Recall that the height criterion gives $ord_v(g) \leq 6-h_v$. If $v = av_1+bv_2+cv_3$ with $v_i$ each carrying $E_8$, then $(m_f, v_i) \geq 0, (m_g, v_i) \geq -1$ ,$\forall m_f \in \Delta_f$ and $\forall m_g \in \Delta_g$. This gives $(m_f, v) \geq 0$, $(m_g, v) \geq -(a+b+c) = -h_v$. Together, we see $ord_v(f) \geq 4$, $ord_v(g) = 6-h_v$. For $h_v = 1, 5, 6$ this fixes $G_v$, but to determine $G_v$ for $h_v = 2, 3, 4$ we must study the split condition. A necessary condition is that there is one monomial $m_g \in \Delta_g$ such that $(m_g, v) + 6 = 6 - h_v$, and since $m_F \in \Delta_g$ always, where $F$ is the facet in which $v_i$ lie, then $m_g = m_F$. Morever, the monomial $m$ in $g$ associated to $m_F$ must be a perfect square; since $(m_F, v_i) + 6 = 5$, $m \sim x_i^5$ and $m$ is not a perfect square. Therefore the fibers are all non-split. This establishes the result. □

**[ben:** `This is where your proof goes with right-derived functors, etc. Please write it as concisely as possible in a paragraph here, not longer than the previous paragraph. Thanks!` *-jh-*]

**Acknowledgements.** We thank Tina Eliassi-Rad, Dima Krioukov, Brent Nelson, Wati Taylor, and especially Dave Morrison for useful discussions. J.H. is supported by NSF Grant PHY-1620526.


[1] R. Bousso and J. Polchinski, JHEP **06**, 006 (2000), arXiv:hep-th/0004134 [hep-th]; S. Ashok and M. R. Douglas, *ibid.* **01**, 060 (2004), arXiv:hep-th/0307049 [hep-th]; F. Denef and M. R. Douglas, *ibid.* **05**, 072 (2004), arXiv:hep-th/0404116 [hep-th].

[2] C. Vafa, Nucl. Phys. **B469**, 403 (1996), arXiv:hep-th/9602022 [hep-th]; D. R. Morrison and C. Vafa, *ibid.* **B476**, 437 (1996), arXiv:hep-th/9603161 [hep-th].

[3] D. R. Morrison and W. Taylor, Central Eur. J. Phys. **10**, 1072 (2012), arXiv:1201.1943 [hep-th].

[4] A. Grassi, J. Halverson, J. Shaneson, and W. Taylor, JHEP **01**, 086 (2015), arXiv:1409.8295 [hep-th].

[5] A. P. Braun and T. Watari, JHEP **01**, 047 (2015), arXiv:1408.6167 [hep-th]; T. Watari, *ibid.* **11**, 065 (2015), arXiv:1506.08433 [hep-th]; J. Halverson and J. Tian, Phys. Rev. **D95**, 026005 (2017), arXiv:1610.08864 [hep-th].

[6] J. Halverson, Nucl. Phys. **B919**, 267 (2017), arXiv:1603.01639 [hep-th].

[7] D. R. Morrison and W. Taylor, JHEP **05**, 080 (2015), arXiv:1412.6112 [hep-th].

[8] W. Taylor and Y.-N. Wang, JHEP **12**, 164 (2015), arXiv:1511.03209 [hep-th].

[9] J. Halverson and W. Taylor, JHEP **09**, 086 (2015), arXiv:1506.03204 [hep-th]; W. Taylor and Y.-N. Wang, *ibid.* **01**, 137 (2016), arXiv:1510.04978 [hep-th].

[10] D. R. Morrison and W. Taylor, Fortsch. Phys. **60**, 1187 (2012), arXiv:1204.0283 [hep-th]; W. Taylor, JHEP **08**, 032 (2012), arXiv:1205.0952 [hep-th]; D. R. Morrison and W. Taylor, (2014), arXiv:1404.1527 [hep-th]; G. Martini and W. Taylor, *ibid.* **06**, 061 (2015), arXiv:1404.6300 [hep-th]; S. B. Johnson and W. Taylor, *ibid.* **10**, 23 (2014), arXiv:1406.0514 [hep-th]; W. Taylor and Y.-N. Wang, (2015), arXiv:1504.07689 [hep-th].

[11] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, Phys. Rev. **D68**, 046005 (2003), arXiv:hep-th/0301240 [hep-th].

[12] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, JHEP **03**, 007 (2005), arXiv:hep-th/0502058 [hep-th].

[13] Y. Hayakawa, *Degeneration of Calabi-Yau manifold with Weil-Petersson metric* (1994) p. 44, thesis (Ph.D.)–University of Maryland, College Park.

[14] C.-L. Wang, Math. Res. Lett. **4**, 157 (1997).

[15] J. J. Heckman, D. R. Morrison, and C. Vafa, JHEP **05**, 028 (2014), [Erratum: JHEP06,017(2015)], arXiv:1312.5746 [hep-th].