Some starter notes on trees

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Abstract here.

TECHNICAL CONSIDERATIONS

Let B be a smooth weak Fano toric threefold. Such a variety is naturally associated with a fine regular star triangulation (FRST) of a three-dimensional reflexive polytope Δ. The monomials that may appear in the global sections f and g of the Weierstrass model are in one-to-one correspondence with points in the polytopes

$$P_n := \{m \in \mathbb{Z}^3 \mid \langle m, v\rangle + n \geq 0 \ \forall v \in \Delta\}, \tag{1}$$

in the cases $n = 4, 6$, respectively. Δ is a reflexive polytope because its dual $\Delta^\circ := P_1$ is itself a lattice polytope in $\mathbb{Z}^3$. Then if x_i is the toric homogeneous coordinate associated with some point $v_i \in \Delta$, the order of vanishing along $x_i = 0$ of some monomial $m_n \in P_n$ is

$$p_{x_i,m_n} = \langle m, v_i\rangle + n, \tag{2}$$

and the overall monomial associated with m_n is $\prod_i x_i^{\langle m_n,v_i\rangle+n}$. From this the order of vanishings of the monomials along various toric divisors and their intersections can be read off.

(JH:) Filler text will be needed here

Consider a toric variety $\tilde{B}$ obtained via a sequence of smooth blowups from the original toric threefold B associated to an FRST of Δ. The vertex associated to any given exceptional divisor is of the form $v_e = \sum a_i v_i$, with non-negative a_i, where the points v_i all lie on some two-face F of Δ. $\tilde{B}$ also has polytopes P_n, which we will call $\tilde{P}_n$ to distinguish them from the P_n associated to B. Due to the structure of the blowups, for fixed n $\tilde{P}_n$ may be obtained by slicing out upper half planes from P_n.

We will now show that if every exceptional divisor satisfies $\sum a_i \leq 6$, then the generic Weierstrass model over $\tilde{B}$ has no $(4,6)$ divisors.

First, note that all vertices of $\tilde{B}$ are of the form $v = \sum a_i v_i$ with a_i non-negative; v corresponds to an exceptional divisor if and only if the vector a_i is not a unit vector. This, together with the definition of Δ°, implies that $\langle m, v\rangle = \sum a_i \langle m, v_i\rangle \geq -\sum a_i \geq -6$ for all v. This implies that any $m \in \Delta^\circ$ is also an element of $\tilde{P}_6$; i.e. the associated monomial appears in g of the associated Weierstrass model over $\tilde{B}$. Second, consider the unique $m \in \Delta^\circ$ that is dual to a chosen two-face F; it satisfies $\langle m, v\rangle = -1 \ \forall v \in F$. Then for any $v_e = \sum a_i v_i$ that is the sum of multiple vertices in F, we have $\langle m, v_e\rangle = -\sum a_i < 0$. By the previous argument, $m \in \tilde{P}_6$, and rewriting in terms of the power in the exponent we have

$$6 > p_{e,m} \geq 0. \tag{3}$$

This implies the monomial m prevents $e = 0$ from being a $(4,6)$ divisor. This m does so for any exceptional divisor built on top of F, and for any F there is such an m. Therefore, none of the exceptional divisors of $\tilde{B} \to B$ are $(4,6)$ divisors. Third, for any $m \in \Delta^\circ$ and $v \in \Delta$, $m \in \tilde{P}_6$ by the previous argument and $\langle m, v\rangle = -1$ which implies that g vanishes to order 5 along the associated divisor, and thus it cannot be a $(4,6)$ divisor.

This exhausts the possible types of divisors; therefore any $\tilde{B}$ obtained in this way has no $(4,6)$ divisors.

We will now show that to any $\tilde{B}$ constructed via our method there is a sequence of smooth toric blowups that together give $\check{B} \to \tilde{B}$ were $\hat{B}$ has no (4,6) points, curves, or divisors. This will be criticall for showing that the general Weierstrass model over $\tilde{B}$ is at finite distance in the moduli space using the Weil-Petersson metric.

Consider any toric curve $C = D_s \cdot D_t \subset \tilde{B}$. Take $v_s = \sum_i a_{i,s} v_i$ and $v_t = \sum_i a_{i,t} v_i$ and define $a := \sum_i a_{i,s}$ and $b := \sum_i a_{i,t}$. Let F be a facet on which or above which v_s and v_t sit; let $m \in \Delta^\circ$ be the dual to F. As an element of $\tilde{P}_4$ the associated monomial may be written

$$s^{\langle m,v_s\rangle+4} t^{\langle m,v_t\rangle+4} \times \dots, \tag{4}$$

and the monomial vanishes to order $\langle m, v_s\rangle + \langle m, v_t\rangle + 8 = -a - b + 8$ along C, which must be ≥ 4 for C to be a $(4,6)$ curve. Therefore $a + b \leq 4$ is necessary for C to be a $(4,6)$ curve. Now suppose C is a $(4,6)$ curve that we blow up via $\hat{B} \to \tilde{B}$ by adding an exceptional divisor $v_e = \sum a_i v_i = v_s + v_t$. Then $\sum a_i = a + b$, which satisfies $\sum a_i \leq 4$ since C is $(4,6)$, but this condition is sufficient to ensure that $\hat{B}$ has no $(4,6)$ divisors! If $2a + b \leq 4$ or $a + 2b \leq 4$ then $\hat{B}$ may still have a $(4,6)$ curve, but this blowup process can be iterated until there are no longer $(4,6)$ curves. So any $\tilde{B}$ in our construction that has $(4,6)$ curves admits a sequence of blowups to a smooth toric threefold with no $(4,6)$ curves or divisors.

We must make a similar argument for $(4,6)$ points. Consider a point $p = D_s \cdot D_t \cdot D_u \subset \tilde{B}$, with v_s and v_t as before and $v_u = \sum_i a_{i,u} v_i$ with $c := \sum_i a_{i,u}$. There is a unique facet F above which or on which $v_{s,t,u}$ all sit, and again $m \in \Delta^\circ$ is the dual to F. As an element of $\tilde{P}_4$ the associated monomial may be written

$$s^{\langle m,v_s\rangle+4}t^{\langle m,v_t\rangle+4}u^{\langle m,v_u\rangle+4} \times \ldots, \tag{5}$$

and the monomial vanishes to order $-a-b-c+12$ at p. Therefore $a+b+c \leq 8$ is a necessary condition for p to be a $(4,6)$ point. If $a+b+c \leq 6$, then the blowup of p with $v_e = v_s+v_t+v_u$ has no $(4,6)$ divisors and this type of blowup can be iterated until either there are no more $(4,6)$ points or there is a $(4,6)$ point with associated a,b,c satisfying $6 < a+b+c \leq 8$. In this case the blowup $v_e = v_s+v_t+v_u$ may have a $(4,6)$ divisor. However, a short calculation shows that any positive a,b,c satisfying this bound admits sequences of curve blowups along $D_s \cdot D_t$, $D_t \cdot D_u$, $D_u \cdot D_s$ that leads to a threefold base with no $(4,6)$ points, curves, or divisors.

Together, these arguments imply that any base $\tilde{B}$ in our construction has a sequence of smooth toric blowups to another base $\hat{B}$ (also present in our construction) that has no $(4,6)$ divisors, curves, or points. This means that the elliptic fibration $\hat{X} \to \hat{B}$ has canonical singularities, which in turn implies the general Weierstrass model $\tilde{X}$ over $\tilde{B}$ also has canonical singularities, since the blowup $\hat{B} \to \tilde{B}$ induces a blowup $\hat{X} \xrightarrow{\phi} \tilde{X}$. To see that the elliptic fibration $\hat{X}$ has only canonical singularities, we employ a modified version of Nakayama's result as presented in Lemma 3.6 in [1]. Consider a point $b \in \hat{B}$ and S a smooth surface through b. We consider the base change $\hat{X} \times_{\hat{B}} S$ defined by the following pullback diagram.

$$\begin{tikzcd} \hat{X} \times_{\hat{B}} S \arrow[r] \arrow[d] & \hat{X} \arrow[d] \\ S \arrow[r] & \hat{B} \end{tikzcd}$$

(JH:) Ben can you give the intuitive sentence about the fiber product, again? The one that made me and Cody happy? I already forgot. We may consider $\hat{X} \times_{\hat{B}} S$ as a Weierstrass model over S, which can be thought of as a restriction within $\hat{X}$ that gives an elliptic threefold over S. By hypothesis, since $\hat{B}$ has no (4,6) points, it follows that S has no (4,6) points and hence the pullback is a minimal Weierstrass model and has rational singularities. By a result of [2] on the deformations of rational singularities, it follows that $\hat{X}$ has rational singularities, and since $\hat{B}$ is smooth, it follows that $\hat{X}$ has rational Gorenstein singularities and in particular, at worst canonical singularities. This means that there is a smooth resolution (not necessarily Calabi-Yau) $X_s \xrightarrow{\rho} \hat{X}$ such that $K_{X_s} = \rho^*(K_{\hat{X}}) + \sum_i a_i E_i$ with all $a_i \geq 0$ and where E_i is an exceptional divisor of ρ. Then we also have a smooth resolution $X_s \xrightarrow{\rho} \hat{X} \xrightarrow{\phi} \tilde{X}$ so that $\tilde{X}$ also has canonical singularities. To see this, we consider the following diagram

$$\begin{tikzcd} X_s \arrow[r, "f"] & \hat{X} \cong \tilde{X} \times_{\tilde{B}} \hat{B} \arrow[r, "\phi"] \arrow[d, "p"] & \tilde{X} \arrow[d, "p'"] \\ & \hat{B} \arrow[r, "\phi'"] & \tilde{B} \end{tikzcd}$$

where $\hat{B}$ is a resolution of $\tilde{B}$ and we take $\hat{X}$ to be induced weierstrass model over $\hat{B}$ from the above pullback diagram. It suffices to show that $R^\bullet(\phi \circ f)_*(\mathcal{O}_{X_s}) \cong \mathcal{O}_{\tilde{X}}$. By hypothesis, $R^\bullet f_*(\mathcal{O}_{X_s}) \cong \mathcal{O}_{\hat{X}}$ and $R^\bullet \phi'_*(\mathcal{O}_{\hat{B}}) \cong \mathcal{O}_{\tilde{B}}$. Applying the Grothendieck spectral sequence, we have that $R^\bullet(\phi \circ f)_*(\mathcal{O}_{X_s}) \cong R^\bullet \phi_*(\mathcal{O}_{\hat{X}})$. Since p' in particular, is a flat morphism, and flatness is stable under base change, p is also a flat morphism, and in particular, we have the induced isomorphism $p'^* R^\bullet \phi'_* \mathcal{O}_{\hat{B}} \xrightarrow{\sim} R^\bullet \phi_* p^* \mathcal{O}_{\hat{B}}$. It follows that $R^\bullet \phi_*(\mathcal{O}_{\hat{X}}) \cong R^\bullet \phi_* p^*(\mathcal{O}_{\hat{B}}) \cong p'^* R^\bullet \phi'_* \mathcal{O}_{\hat{B}} \cong p'^* \mathcal{O}_{\tilde{B}} \cong \mathcal{O}_{\tilde{X}}$ as desired. Thus, it follows that $\tilde{X}$ also has rational gorenstein singularities. Any F-theory model on $\tilde{X}$ in our construction therefore has canonical singularities, and by results of Hayakawa [3] and Wang [4] they are all at finite distance from one another in the Weil-Petersson metric on moduli space.

GAUGE CONFIGURATIONS

We now embark on a combinatorial study of locality of non-Higgsable clusters on intersecting divisors. Indeed, we first observe trivially that sufficiently tall trees on one simplex may induce non-Higgsable clusters on another simplex with some relative height difference that could be quantitatively analyzed. Let v_1, v_2, and v_3 denote three rays forming a smooth, rational, polyhedral cone, and define $v_e \equiv av_1 + bv_2 + cv_3$ with $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$. Define $H_{n,e}$ to be the hyperplane defined by $\langle m, v_e\rangle = -n$ and define $H_{n,i}$ to be the upper half-planes defined by $\langle m, v_i\rangle \geq -n$ for $i = 1,2,3$. We wish to study the region defined by $H \equiv H_{n,e} \bigcap_i H_{n,i}$.

We first show that there exists a finite number of integral points in H. Consider the equation given by $\langle m, v_e\rangle = a\langle m, v_1\rangle + b\langle m, v_2\rangle + c\langle m, v_3\rangle = -n$. As the tuple (a,b,c) are strictly positive, it easily follows from the above inequalities, that $\langle m, v_i\rangle$ are also bounded above. Thus, we consider the integral solution set to the matrix inequality

$$\begin{pmatrix} -n \\ -n \\ -n \end{pmatrix} \leq \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} m \end{pmatrix} \leq \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{6}$$

As the vectors v_i form a basis for $\mathbb{Z}^3$ by hypothesis, it follows easily that there exists only finitely many integral solutions to the above matrix inequality, and hence we are done.

We now give a sufficient condition for bounding hyperplanes to eliminate points on the hyperplane $H_{n,e}$. Let v_e and the v_i be as defined above, and define $v_i' \equiv a_i v_1 + b_i v_2 + c_i v_3$ for positive integers a_i, b_i, and c_i such that $v_1' + v_2' + v_3' = v_e$. We construct three smooth torus-equivariant blowups along the toric curves given by the edges (v_e, v_i') for $i = 1, 2, 3$. The resulting hyperplane inequalities given by the vertices $v_e + v_i'$ yield the condition $\langle m, 3v_e + v_1' + v_2' + v_3' \rangle = \langle m, 4v_e \rangle \geq -3n$. Thus, we find that $\langle m, v_e \rangle \geq -\frac{3}{4}n$ and hence there does not exist integral points on the hyperplane $H_{n,e}$.

Minimal Vanishing From Resolution

(CL:) to discuss notation A blow-up on an n-face $F_n \subset \Delta^\circ$ $(n = 0, 1, 2)$ will induce a minimal order of vanishing in f and g along the divisors corresponding to the points involved in the blow-up, as well as any of the points interior to any higher codimension faces bounded by F_n. In this section we derive this by consider blow-ups along faces of each dimension.

Points interior to facets

First, we consider a point p interior to a facet F, with corresponding divisor D_p, and homogenous coordinate x_p. To check for non-Higgsable 7-branes on D_p, we check for the absence of any monomials m_n such that $\langle m_n, p \rangle = -n$, for $n = 4, 6$. Define the lattice polytope corresponding to the sections of degree $\mathcal{O}(-nK_B)$ as Δ_n. We note that all vertices of Δ_n take the form nm_i, where the m_i are the vertices of Δ. Recall that there exists a vertex $m \subset \Delta$ dual to $F : \langle m, F \rangle = -1$. Let the vertices that are not m be $\hat{m}_j$. Since $\langle m, p \rangle = -1$, and $\langle \hat{m}_j, p \rangle \geq 0$, it is clear that $\langle n\, m, p \rangle = -n$, and $\langle n\, \hat{m}_j, p \rangle \geq 0$. Therefore the linear function $\langle p, \cdot \rangle$ is minimized at m, and is strictly increasing when moving from m to any other vertex, and therefore strictly increasing when moving away from m in any direction. Therefore, m is the only monomial satisfying $\langle p, \cdot \rangle = -n$. Therefore, before resolution there is a single monomial in f (g), which is degree zero in x_p, and corresponds to a vertex in Δ_4 (Δ_6). The existence of these monomials obstructs the existence of non-Higgsable 7-branes along D_p.

Now let us consider a resolution, with exceptional divisor $D_e = \sum a_i D_i$, where $a_i \in \{0, 1\}$, and the D_i correspond to points v_i on F, which can be vertices, points interior to edges that bound F, or points strictly interior to F. For the blow-up to be non-trivial either two or three of the a_i are non-zero. This blow-up introduces a hyperplane that further cuts Δ_n to $\tilde{\Delta}_n$. In particular, we consider the effect on the monomial $n\, m$. First, let us note that if we have an $m \in \Delta$ such that $\langle m, p \rangle = -1$, where p is in the strict interior of a facet F, then $\langle m, v_i \rangle = -1$ for all other $v_i \in F$. To see this, consider a triangulation of F using the vertices of F as vertices in the triangulation. Then all other points lie within the simplices of the triangulation. Let p lie within a simplex generated by the points $\{v_1, v_2, v_3\}$. Then we can write $p = a_1 v_1 + a_2 v_2 + (1 - a_1 - a_2) v_3$, where $0 < a_i < 1$. Therefore, we have $-1 = a_1 \langle m, v_1 \rangle + a_2 \langle m, v_2 \rangle + (1 - a_1 - a_2) \langle m, v_3 \rangle$. From the reflexivity of Δ° we have $\langle v_i, m \rangle \geq -1$. Assume that $\langle m, v_1 \rangle > -1$. We then have $-1 > -a_1 + a_2 \langle m, v_2 \rangle + (1 - a_1 - a_2) \langle m, v_3 \rangle > -a_1 - a_2 - (1 - a_1 - a_2) = -1$, which is a contradiction. One can repeat this argument to see that all of the the $\langle m, v_i \rangle = -1$.

Now we take the inner product of $n\, m$ with $v \equiv \sum_i a_i v_i$, we have $\langle n\, m, v \rangle = -1$. Then $\langle m, v \rangle - n \sum_i v_i$, which must satisfy $\langle m, v \rangle - n \sum_i v_i \geq -n$ for m to be an allowed point in $\tilde{\Delta}_n$. However, a non-trivial blow-up have $\sum_i v_i > 1$, and therefore the blowup cuts out the point m from $\tilde{\Delta}_n$. Applying this to Δ_f and Δ_g, we can see that after the resolution both f and g will vanish to at least order one along x_p. Therefore any blow-up along any points in F will automatically introduce at least a type II singularity along all divisors corresponding to points in the strict interior of F.

Points interior to edges

We now perform a similar analysis for a divisor D_p corresponding to a point p with homogeneous coordinate x_p, and is interior to an edge E. In this case, the dual to p in Δ under $\langle p, \cdot \rangle = -1$ is an entire edge of points $E_\Delta \in \Delta$, bounded by two vertices which we label m_1 and m_2. m_1 and m_2 are vertices in Δ, and therefore nm_1 and nm_2 are vertices in Δ_n, which bound an edge $E_n \subset \Delta_n$. Every point $m^i_{E_n} \in E_n$ then satisfies $\langle m^i_{E_n}, p \rangle = -n$. Since all other vertices $\hat{m}_j \neq m_1, m_2$ in Δ satisfy $\langle \hat{m}_j, p \rangle \geq 0$, then all vertices $n\hat{m}_j \neq nm_1, nm_2$ of Δ_n satisfy $\langle n\hat{m}_j, p \rangle \geq 0$. The linear function $\langle p, \cdot \rangle$ is then minimized along E_n and strictly increasing when moving away from E_n, and therefore the only points m_n in Δ_n satisfying $\langle p, \cdot \rangle = -n$ are the $m^i_{E_n}$. In a similar fashion to the $p \in F$ case one can show that $\langle m^i_{E_n}, v_i \rangle = -n$ for all points v_i on E.

We now perform a blow-up using two points v_1 and v_2 on E, introducing a new ray $v = v_1 + v_2$ and corresponding exceptional divisor $D_v = D_1 + D_2$, where the D_i correspond to the v_i. Again, this resolution will cut points out of Δ_n by introducing an additional hyperplane constraint. We consider the effect on the points $m^i_{E_n}$, which correspond to the non-trivial monomials in $\mathcal{O}(-nK_B)$ that are degree zero in x_p. We have $\langle m^i_{E_n}, v \rangle = -2n$. However, this violates $\langle m^i_{E_n}, v \rangle \geq -n$, and therefore the resolution cuts out all degree zero monomials in x_p. Therefore any resolution along E will introduce at least a type II singularity along each divisor corresponding to a point on the strict interior of E. (CL:) stopping here, I don't think there's a direct continuation of this argument to vertices

Monomial Chopping

For many of the analyses in this paper we use small lemmas about when monomials of certain types are chopped out. We would like to prove them here.

Proof. Suppose $h_v \leq 6/n$, $\forall v = av_1 + bv_2 + cv_3 \bar{\in} \Delta^\circ$, where the leaf height $h_v = a + b + c$. Then since $m \cdot v_i \geq -1 \forall v \in \Delta^\circ$, nm $\square$

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