On Algorithmic Universality in F-theory Geometries

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We study universality of gauge sectors in the string landscape in the context of F-theory compactifications. A finite time construction algorithm for $\frac{4}{3} \times 1.3998 \times 10^{755}$ F-theory geometries is presented. High probability geometric assumptions uncover universal structures in the ensemble without explicitly constructing it. For example, non-Higgsable clusters of seven-branes with intricate gauge sectors occur with probability above $1 - 1.07 \times 10^{-755}$, and the geometric gauge group rank is above 160 with probability .9999994. In the latter case there are at least 10 E_8 factors, the structure of which fixes the gauge groups on certain nearby seven-branes. Visible sectors may arise from E_6 or $SU(3)$ seven-branes, which occur in certain random samples with probability $\simeq 1/200$.

I. Introduction. String theory is a theory of quantum gravity that naturally gives rise to interesting gauge and cosmological sectors. As such, it is a candidate unified theory. However, it gives rise to a vast landscape of four-dimensional metastable vacua that may realize different laws of physics, making predictions difficult.

Other areas of physics signal a possible way forward, though, since a large ensemble may exhibit universality. For large ensembles of string vacua, such as the oft-quoted $O(10^{500})$ type IIb flux vacua [1], studying universality via explicit construction is impractical and complex [12]. However, it may be possible to derive universality from a precise construction algorithm, rather than the constructed ensemble. This algorithmic universality is one promising way forward in the string landscape.

We present such an algorithm in the context of $4d$ F-theory [2] compactifications. The ensemble is a collection of $4/3 \times 1.3998 \times 10^{755}$ six-manifolds, perhaps the largest set of string geometries to date, that serve as the extra dimensions of space. Their topological structure determines the $4d$ gauge group that arises geometrically from configurations of seven-branes that form a network of so-called non-Higgsable clusters (NHC) [3]. We establish that non-Higgsable clusters arise with probability above $1 - 1.07 \times 10^{-755}$ in this ensemble and demonstrate that certain minimal and rich gauge structure arises with high probability. We also present results from random samples that are potentially relevant for visible sectors.

A number of recent results suggest that NHC are very important in the $4d$ F-theory landscape. These configurations exist for generic scalar field (complex structure moduli) expectation values, and therefore obtaining gauge symmetry does not require [4] stabilization on subloci in moduli space, which can have high codimension [5]. Standard model structure may arise naturally [4], strong coupling is generic [6], and $4d$ NHC may exhibit features [7] (such as loops and branches) not present in $6d$. NHC arise in the F-theory geometry with the largest number of flux vacua [8], and they arise universally in known ensembles [9] that are closely related to ours. NHC in $6d$ have been studied extensively [3, 10].

In Sec. II we review non-Higgsable clusters. In Sec. III we present our ensemble. In Sec. IV we study universality. In Sec. V we discuss our results.

II. Seven-Branes and Non-Higgsable Clusters. A $4d$ F-theory geometry is a Calabi-Yau elliptic fibration X over a threefold base space B defined by the equation

$$y^2 = x^3 + fx + g \tag{1}$$

where f and g are homogeneous polynomials in the coordinates of B; technically $f \in \Gamma(-4K_B)$, $g \in \Gamma(-6K_B)$, with K_B the canonical line bundle on B. B provides extra spatial dimensions, and seven-branes are localized on the discriminant locus $\Delta = 0 \subset B$, where $\Delta = 4f^3 + 27g^2$.

Upon compactification, the gauge group structure of seven-branes gives rise to four-dimensional gauge sectors. It is controlled by f and g, and for a typical B the most general f, g take the form $f = \tilde{f} \prod_i x_i^{l_i}$, $g = \tilde{g} \prod_i x_i^{m_i}$, so

$$\Delta = \tilde{\Delta} \prod_i x_i^{\min(3l_i, 2m_i)} =: \tilde{\Delta} \prod_i x_i^{n_i}, \tag{2}$$

and therefore f, g and Δ vanish along $x_i = 0$ to $ord_{x_i}(f, g, \Delta) = (l_i, m_i, n_i)$. This seven-brane carries a gauge group G_i given in Table I according to the Kodaira classification. In some cases further geometric data is necessary to uniquely specify G_i, see e.g. [9] for conditions, but this data always exists for fixed B. For generic f and g, a seven-brane on $x_i = 0$ requires $(l_i, m_i) \geq (1, 1)$.

Such a seven-brane is called a geometrically non-Higgsable seven-brane (NH7) because it carries a gauge group that cannot be removed by deforming f or g. A NH7 may have geometric gauge group

$$G \in \{E_8, E_7, E_6, F_4, SO(8), G_2, SU(3), SU(2)\}, \tag{3}$$

which could be broken by turning on particular fluxes. A typical base B, as we will show in the strongest generality to date, has many non-Higgsable seven-branes that often intersect in pairs, giving rise jointly charged matter. Such a cluster of seven-brane is a geometrically non-Higgsable cluster (NHC). For brevity, we henceforth drop the adjectives geometric and geometrically.

F_i	l_i	m_i	n_i	Sing.	G_i
I_0	≥ 0	≥ 0	0	none	none
I_n	0	0	$n \geq 2$	A_{n-1}	$SU(n)$ or $Sp(\lfloor n/2 \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$SU(2)$
IV	≥ 2	2	4	A_2	$SU(3)$ or $SU(2)$
I_0^*	≥ 2	≥ 3	6	D_4	$SO(8)$ or $SO(7)$ or G_2
I_n^*	2	3	$n \geq 7$	D_{n-2}	$SO(2n-4)$ or $SO(2n-5)$
IV^*	≥ 3	4	8	E_6	E_6 or F_4
III^*	3	≥ 5	9	E_7	E_7
II^*	≥ 4	5	10	E_8	E_8

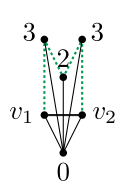
TABLE I. Kodaira fiber F_i, singularity, and gauge group G_i on the seven-brane at $x_i = 0$ for given l_i, m_i, and n_i.

III. Large Landscapes of Geometries from Trees. We now introduce our construction, which utilizes building blocks in toric varieties that we call trees to systematically build up F-theory geometries. After describing the geometric setup and defining terms that simplify the discussion, we will present a criterion, classify all trees satisfying it, and build F-theory geometries.

Our construction begins with a smooth weak-Fano toric threefold B_i, and then builds structure on top of it. These geometries B_i are determined by fine regular star triangulations (FRST) of one of the 4319 $3d$ reflexive polytopes [11]; there are an estimated $O(10^{15})$ such geometries [5]. The $2d$ faces of the $3d$ polytope are known as facets, and a triangulated polytope will have triangulated facets. Such B_i do not have non-Higgsable clusters.

Consider such a B_i determined by an FRST of a $3d$ reflexive polytope Δ°, a triangulated facet F in Δ°, and an edge between two points v_1 and v_2 in F with associated homogeneous coordinates x_1 and x_2. Since $v_{1,2}$ are connected by an edge, $x_1 = x_2 = 0$ defines a Riemann surface (algebraic curve) in B_i, which can be "blown up" using a new ray $v_e = v_1 + v_2$ and subdividing cones using standard toric techniques. This is a topological transition that introduces a new ("exceptional") divisor $e = 0$ in B where e is the coordinate associated to v_e. This process can be iterated, for example blowing up along $e = x_1 = 0$ which would add a new ray $v_e + v_1 = 2v_1 + v_2$.

After a number of iterations the associated toric variety will have a collection of exceptional divisors with associated rays $v_{e_i} = a_i v_1 + b_i v_2$, which will appear to have formed a *tree* above the *ground* that connects v_1 and v_2 in F. Each v_{e_i} is a *leaf* with *height* $h_{e_i} = a_i + b_i$, and we will refer to trees built on edges within F as *edge trees*. The height of a tree is the height of its highest leaf. As an example, $\{v_1 + v_2, 2v_1 + v_2, v_1 + 2v_2\}$ appears as



where the v_1 to v_2 line is the edge (ground) in F, dashed green lines are above the ground, 0 is the origin of Δ°, and the new rays are labelled by their heights.

Similarly, one can also build *face trees* by beginning with a face on F, with face vertices v_1, v_2, v_3 having associated x_1, x_2, x_3. Adding $v_e = v_1 + v_2 + v_3$ and subdividing appropriately blows up at the point $x_1 = x_2 = x_3 = 0$ and produces a new toric variety. Again such blowups can be iterated. This process builds a collection of leaves $v_{e_i} = a_i v_1 + b_i v_2 + c_i v_3$ with $a_i, b_i, c_i > 0$ of height $h_{e_i} = a_i + b_i + c_i$ that comprise a *face tree*. Face trees are built above the interior of the face due to the strict inequality in the definition. Note if one leaf coefficient was zero the associated leaf would be above an edge of the face, not above the face interior.

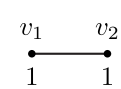
Geometries can be systematically constructed by adding a face tree to each face in each triangulated facet of Δ°, and then an edge tree to each edge. The associated smooth toric threefold B has a collection of rays v, each of which can be written $v = av_1 + bv_2 + cv_3$ with v_i $3d$ cone vertices in B_i. If $(a, b, c) = (1, 0, 0)$ or some permutation thereof, $v \in \Delta^\circ$ and this height $h_v = 1$ "leaf" is more appropriately a root, since it is on the ground.

A natural question in systematically building up geometries is whether there is a maximal tree height. For a toric variety B to be an allowed F-theory base it must not have any so-called $(4, 6)$ divisors (see Appendix), which we ensure by a simple height criterion proven in Prop. 1:

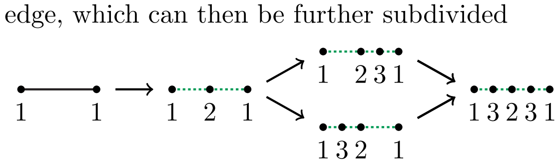
If $h_v \leq 6$ for all leaves $v \in B$,
then there are no $(4, 6)$ divisors.

This condition is simple and sufficient, but not necessary for the absence of $(4, 6)$ divisors. Nevertheless, it will allow us to build a large class of geometries.

The task is now clear: we must systematically build all topologically distinct edge trees and face trees of height $\leq$ 6. Since the combinatorics are daunting, let us exemplify the problem for $h \leq 3$ trees. Viewing the facet head on, an edge in F appears as



where the vertices are labelled, as well as their height 1. With our definitions a ray on the ground (i.e. in a facet) is technically a height 1 leaf. Adding $v_1 + v_2$ subdivides the edge, which can then be further subdivided



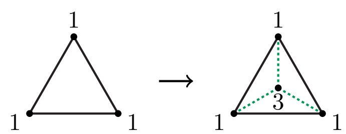
where we have dropped the vertex labels and kept the heights. The trees emerge out of the page, but visualization is made easier by projecting on to the edge; the

N	# Edge Trees	# Face Trees
3	5	2
4	10	17
5	50	4231
6	82	41, 439, 964

h_v	Probability
3	.99999998
4	.999995
5	.999997
6	.9999898

TABLE II. *Left:* The number of edge trees and face trees with of height $h \leq N$. *Right:* The probability that a face tree with $h \leq 6$ has a leaf v with a given height h_v.

right-most tree is the one previously presented vertically. There are five edge trees with height ≤ 3. Similarly,



shows that there are 2 face trees of height ≤ 3. Here we have denoted the new edges by blue lines since they do not sit in the facet. With our definitions edge trees are built above an edge in the facet, whereas higher leaves in face trees may be built on new edges that don't sit in the facet. For example, a height 4 leaf could be added on any of the blue lines above.

A (tedious) straightforward calculation shows that the number of edge or face trees with $h \leq N$ grows rapidly, see Table II; note the large number of $h \leq 6$ trees.

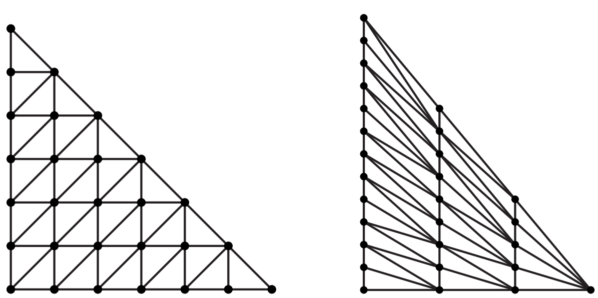


FIG. 1. The largest facets in the two 3d reflexive polytopes Δ_1° and Δ_2° with the most number of interior points. Presented is one triangulation of each, from which we see $\#\tilde{E} = 63$ edges and $\#\tilde{F} = 36$ faces in both facets.

Having classified the number of $h \leq 6$ face trees and edge trees, we now estimate the number of toric varieties that may arise from putting trees on an FRST of Δ°. We construct an ensemble S_{Δ° of geometries by systematically putting $h \leq 6$ face trees on all faces $\tilde{F}$ of a triangulated facet F and then putting $h \leq 6$ edge trees on all edges $\tilde{E}$ in F. The number of ways of doing this overcounts due to the fact that some $\tilde{E}$ lie in two facets, i.e. on an edge E of Δ°. Accounting for this and using Table II we obtain an equation for the size of S_{Δ°,

$$|S_{\Delta^\circ}| = \sum_F 82^{\#\tilde{E} \text{ on } F} (4.2 \times 10^6)^{\#\tilde{F} \text{ on } F} - \sum_E 82^{\#\tilde{E} \text{ on } E}, \tag{4}$$

which may be simplified since on a triangulated facet F

$$\#\tilde{E} = 2n_B + 3n_I - 3 \qquad \#\tilde{F} = n_B + 2n_I - 2, \tag{5}$$

where n_I and n_B are the number of facet interior and boundary points, respectively.

Two $3d$ reflexive polytopes give a far larger number $|S_{\Delta^\circ}|$ than the others. They are the convex hulls $\Delta_i^\circ := \text{Conv}(S_i), i = 1, 2$ of the vertex sets

$$\begin{aligned} S_1 &= \{(-1,-1,-1), (-1,-1,5), (-1,5,-1), (1,-1,-1)\} \\ S_2 &= \{(-1,-1,-1), (-1,-1,11), (-1,2,-1), (1,-1,-1)\}. \end{aligned} \tag{6}$$

$S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$ have the same number of edges and faces. The largest facets of Δ_1° and Δ_2° are displayed in 1, both of which has $\#\tilde{E} = 63$ and $\#\tilde{F} = 36$. We compute

$$|S_{\Delta_1^\circ}| = \frac{1.4}{3} \times 10^{755} \qquad |S_{\Delta_2^\circ}| = 1.4 \times 10^{755}, \tag{7}$$

where the factor of 1/3 is due to the symmetries discussed in the Appendix. All other polytopes Δ° contribute negligibly, $|S_{\Delta^\circ}| \leq 1.65 \times 10^{692}$ configurations. This gives

$$\# \text{ 4d F-theory Geometries} \geq \frac{4}{3} \times 1.4 \times 10^{755}, \tag{8}$$

which undercounts due to the fact that we choose to do face blowups followed by edge blowups to simplify the subdivision combinatorics. See the discussion.

IV. Universality and Non-Higgsable Clusters.

We now study universality in the dominant sets of F-theory geometries $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$. We prove non-Higgsable cluster universality, minimal gauge group universality, and discuss initial results from random sampling.

Algorithmic Universality and Gauge Groups. We wish to establish the likelihood that an F-theory base in $S_{\Delta_1^\circ}$ or $S_{\Delta_2^\circ}$ give rise to non-Higgsable seven-branes. The result arises from Prop. 2: if there is a tree anywhere on F, even a single leaf, there is a non-Higgsable seven-brane on all divisors associated to F interior points. For any S_{Δ° only one configuration has no trees, and therefore

$$P(\text{NHC in } S_{\Delta^\circ}) = 1 - \frac{1}{|S_{\Delta^\circ}|}. \tag{9}$$

This is always very close to one, and in particular

$$\begin{aligned} P(\text{NHC in } S_{\Delta_1^\circ}) &= 1 - 2.14 \times 10^{-755} \\ P(\text{NHC in } S_{\Delta_2^\circ}) &= 1 - .71 \times 10^{-755}, \end{aligned} \tag{10}$$

for the dominant ensembles in our construction. Non-Higgsable clusters are universal in these ensembles.

We now wish to study physics in our ensemble. Consider a geometric assumption A_i and a physical property P_i such that $A_i \rightarrow P_i$. Our goal is to determine high probability assumptions that lead to interesting physical properties, computing $P(A_i)$ since $A_i \rightarrow P_i$ ensures $P(P_i) \geq P(A_i)$. We will focus $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$ since these dominate the ensemble.

Consider first $S_{\Delta_1^\circ}$ and let A_1 be the assumption that any simplex in an FRST of Δ_1° containing a vertex of Δ_1° has an $h \geq 3$ face tree on it. For the 3 symmetric facets of Δ_1° there are 17 ways to choose simplices containing the vertices, and 1796 ways for its largest facet. The maximum number of simplices containing vertices is 24, and $P(h \geq 3 \text{ tree on simplex}) = .9999998$ from Table II

$$P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .9999998^{24} = .999995. \qquad (11)$$

There are $17^3 \times 1796$ ways to choose simplices that contain the vertices, all of which have $G \geq F_4^{18} \times E_6^{10} \times U^9$ where $U \in \{G_2, F_4, E_6\}$ depending on details and all of these factors arise on the ground; generally there will be many more factors from non-Higgsable seven-branes in the leaves. Here E_6^{10} arises from an E_6 on every interior point of the large facet in Δ_1°, see Fig. 1. This set of statements defines physical property P_1, and since $A_1 \rightarrow P_1$ we deduce $P(P_1 \text{ in } S_{\Delta_1^\circ}) \geq P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .9999994$.

Let A_2 be the assumption that there exists a $h = 5$ face tree somewhere on the large facet F in Δ_1°. Knowing $\tilde{F} = 36$ on F and using Table II, we compute $P(A_2 \text{ in } S_{\Delta_1^\circ}) = (1-(1-.999997)^{36}) \simeq 1-10^{-199}$. Let A_3 be that A_1 and A_2 hold, so $P(A_3) = P(A_1)P(A_2) \simeq P(A_1)$. Then given A_3 a short calculation shows that the $h = 5$ tree on F enhances E_6 in P_1 to E_8, giving 10 E_8's on the ground. P_1 with this enhancement defines P_3.

Similar results hold for for $S_{\Delta_2^\circ}$. Let A_1 be the assumption that any simplex in an FRST of Δ_2° containing a vertex of Δ_2° has an $h \geq 3$ face tree on it. This ensures that $G \geq F_4^{15} \times E_6^7 \times U^{12}$. However, this is quickly enhanced to $G \geq F_4^{18} \times E_6^{10} \times U^9$, via a $h = 5$ face tree on each face, and a $1 - 6.55 \times 10^{-8}$ probability blow-up along an edge connecting the point $\{-1, 2, -1\}$ to one of the points $\{-1, 1, n\}$, where $n = -1 \ldots 3$. The existence of these edges is independent of triangulation.

Summarizing, the probability that a geometry in our set has $G \geq F_4^{18} \times E_8^{10} \times U^9$ on the ground is $\geq .999995$.

It is natural to ask whether this structure on the ground constrains the gauge structure in the trees. In Prop. 3 it is shown that the geometric gauge group on a leaf v in a tree built above E_8's on the ground is determined by the leaf height h_v. The result is that a $h_v = 1, 2, 3, 4, 5, 6$ leaf above E_8 roots has Kodaira fiber $F_v = II^*, IV^*_{ns}, I^*_{0,ns}, IV_{ns}, II, -$ with gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.

This leads to a high probability result about the structure of the geometric gauge group. Since $A_3 \rightarrow P_3$, which has at least 10 E_8 factors nearby one another, P_3 also has

$$G \geq E_8^{10} \times F_4^{18} \times U^9 \times F_4^{H_2} \times G_2^{H_3} \times A_1^{H_4}, \qquad (12)$$

where H_i is the number of height i leaves in trees built on E_8 roots, and $rk(G) \geq 160 + 4H_2 + 2H_3 + H_4$. There are H_6 Kodaira type II seven-branes that do not carry a gauge group but realize Argyres-Douglas theories on D3 probes. The first F_4 and also the U factors may enhance, but the other factors are fixed. The probability of this physical property is $P(P_3 \text{ in } S_{\Delta_1^\circ}) \geq P(A_3 \text{ in } S_{\Delta_1^\circ}) \simeq .999995$ and $P(P_3 \text{ in } S_{\Delta_2^\circ}) \geq P(A_3 \text{ in } S_{\Delta_2^\circ}) \simeq .999995$. This non-trivial minimal gauge structure is universal in our large ensemble given by $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$.

Random Samples and Geometric Visible Sectors. It may be possible to accomodate visible sectors from flux breaking these gauge sectors, but it is also interesting to study whether gauge factors E_6 and / or $SU(3)$ arise with high probability. We have not yet discovered a high probability simple geometric assumption that leads to E_6 or $SU(3)$. However, it is possible that they arise regularly, but due to a complex geometric assumption.

This idea can be tested by random sampling. Let B be an F-theory base obtained by adding face trees at random, followed by edge trees at random, to the "pushing" triangulation of Δ_1°. We studied an ensemble S_r of 10^6 such random samples and found $P(SU(3) \,\text{or}\, E_6 \text{ in } S_r) \simeq 1/200$, and that at least 36 of the points in Δ_1° carried E_8, a significant enhancement beyond assumption 3. Furthermore, in our sample we found that E_6 only arose on the point $(1, -1, -1)$, which is the only vertex of Δ_1° that isn't in the largest facet. Similar results and probabilities also hold using this techniques on Δ_2°. It would be interesting to study random samples of other triangulations, or see if new simple geometric assumptions imply these enhancements. We leave this, and the systematic study of geometric visible sectors, to future work.

V. Discussion. We have presented a construction algorithm for $\frac{4}{3} \times 1.3998 \times 10^{755}$ geometries for $4d$ F-theory compactifications. This number is only a lower bound and may be may be enlarged in at least three ways: by relaxing the requirement of edge blowups after face blowups, by taking into account the $O(10^{15})$ FRS triangulations of 3d reflexive polytopes, and by considering blow-ups of non-toric intersections of seven-branes.

We have initiated the study of this ensemble by focusing on the geometric gauge group. Using knowledge of the construction algorithm, we derived the existence of universality properties for the minimal geometric gauge group on non-Higgsable clusters. High rank groups are generic, as are the existence of at least 10 E_8 factors on the ground. The gauge group on leaves above these E_8 factors on the ground are fixed entirely by their height.

There are many directions for future work. For example, it would be interesting to study how flux could break or further enhance the geometric gauge group, perform a systematic statistical analysis of the gauge group in the leaves, or analyze physics that arises from blow-ups of non-toric intersections of seven-branes. Perhaps most pressing is that, though we have demonstrated that the gauge group is generically high rank and have reviewed some realization of the standard model discussed in [4], it is not yet clear whether the standard model is realized with high probability in our ensemble.

We believe that this is the first time that such a large ensemble has been systematically studied in string theory. In our view, the critical ingredient that made the results possible is what we call algorithmic universality: derivation of universality from a construction algorithm, rather than an explicitly constructed ensemble. Given the plethora of large ensembles in string theory and the infeasibility of constructing all of them, universality of this sort may play a critical role in making the string landscape tractable.

[cl: `Tightened the text a bit, and only removed ideas that we might not want to let out of the bag.` *-jh-*]

A. Appendix: Technical Subtleties. We now address technical subtleties that are important for establishing, but not understanding, results in the main text.

In equation (7) we have included a factor of 1/3 relative to the count one would obtain directly from the algorithm. This takes into account an overcounting of geometries due to toric equivalences, which arise when there is a $GL(3,\mathbb{Z})$ transformation on the toric rays that preserves the cone structure of the fan. In general, there may be many such equivalences between elements of two ensembles $S_{\Delta_i^\circ}$ and $S_{\Delta_j^\circ}$, where $\Delta_{i,j}^\circ$ are any two $3d$ reflexive polytopes. However, to ensure that the count (7) is accurate, we only need to consider whether there are equivalences between two elements in $S_{\Delta_1^\circ}$, two in $S_{\Delta_2^\circ}$, or one in $S_{\Delta_1^\circ}$ to $S_{\Delta_2^\circ}$. It is sufficient to consider $GL(3,\mathbb{Z})$ actions on the ground, i.e. on the facets. This follows from the fact that rays of different height cannot be exchanged under automorphisms of the fan. First note that points in a hyperplane remain in a hyperplane after a $GL(3,\mathbb{Z})$ transformation, and points in a line remain in a line. Facet must therefore map to facets. The big facets in Δ_1° and Δ_2° cannot map to other facets by point counting, and therefore they must map to themselves. There is no non-trivial map taking the big facet in Δ_2° to itself, but there is a $\mathbb{Z}_3$ rotation taking the big facet in Δ_1° to itself, giving a factor of 1/3 in $S_{\Delta_1^\circ}$. There is no non-trivial map between the big facets in Δ_1° and Δ_2°, and therefore $S_{\Delta_2^\circ} \cap S_{\Delta_1^\circ} = \emptyset$. Together, these establish (7).

In discussing what constitutes an allowed $4d$ F-theory geometry $X \to B$, we mentioned certain criteria on orders of vanishing that we now elaborate on. In [13, 14] it was shown that if a Calabi-Yau variety has at worst canonical singularities, it is at finite distance from the bulk of the moduli space in the Weil-Petersson metric. This criterion is general and therefore applies to elliptic fibrations such as X. The reason that it is physically relevant is that if X has worse singularities than a nearby Calabi-Yau X' that is known to represent a physical configuration, and X is at finite distance in the moduli space from X', we should expect that X is also a physical configuration. This criterion, which we refer to as the Hayakawa-Wang criterion, gives a related criterion by studying elliptic fibrations: if $ord_C(f,g) \leq (4,6)$ and $ord_p(f,g) \leq (8,12)$ for all curves $C \subset B$ and points $p \subset B$, respectively, then X has at worst canonical singularities and is at finite distance in the moduli space due to the Hayakawa-Wang criterion[1]. This latter criterion, which we refer to as the weak OOV (orders of vanishing) criterion, is sufficient but not necessary for at worst canonical singularities; for example a rich set of $6d$ $N=1$ SCFTs [15] arises from relaxing the $(4,6)$ condition on codimension two loci in the base. Furthermore, if B is such that $X \to B$ does not satisfy the weak OOV criterion, but there is a blowup $B' \to B$ such that $X' \to B'$ does, then X' has at worst canonical singularities and therefore X also has at worst canonical singularities by Proposition 5. By the Hayakawa-Wang criterion, it is at finite distance in moduli space.

For geometries in our construction to satisfy the Hayakawa-Wang criterion it suffices to show that if $X \to B$ does not satisfy weak OOV, then there is a blowup $B' \to B$ such that $X' \to B'$ does. This is straightforward. Consider any toric curve $C = D_s \cdot D_t \subset B$. Take $v_s = \sum_i a_{i,s} v_i$ and $v_t = \sum_i a_{i,t} v_i$ and define $a := \sum_i a_{i,s}$ and $b := \sum_i a_{i,t}$. Let F be a facet on which or above which v_s and v_t sit, with m_F the facet dual. As an element of Δ_g the associated monomial may be written $s^{\langle m, v_s\rangle + 6} t^{\langle m, v_t\rangle + 6} \times \dots$, and the monomial vanishes to order $\langle m, v_s\rangle + \langle m, v_t\rangle + 12 = -a - b + 12$ along C, which must be ≥ 6 for C to be a $(4,6)$ curve. Therefore $a+b \leq 6$ is necessary for C to be a $(4,6)$ curve. Now suppose C is a $(4,6)$ curve that we blow up via $B' \to B$ by adding an exceptional divisor $v_e = \sum a_i v_i = v_s + v_t$. Then $\sum a_i = a+b$, which satisfies $\sum a_i \leq 6$ since C is $(4,6)$, but this condition is sufficient to ensure that $\hat{B}$ has no $(4,6)$ divisors! If $2a+b \leq 6$ or $a+2b \leq 6$ then B' may still have a $(4,6)$ curve, but this blowup process can be iterated until there are no longer $(4,6)$ curves or divisors. A similar argument holds for each $(4,6)$ point $p = D_s \cdot D_t \cdot D_u \subset B$, with v_s and v_t as before and $v_u = \sum_i a_{i,u} v_i$ with $c := \sum_i a_{i,u}$. Considering it as an element of Δ_g leads to the condition that $a+b+c \leq 6$ is necessary for p to be a $(4,6)$ point, and as before we can perform blowups until there no longer are $(4,6)$ points. Performing iterative blowups of points and curves in this way gaurantee the existence of a map $B' \to B$ such that the weak OOV criterion is satisfied.

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[1] We thank D. Morrison for discussions on this and related points.

Proposition 1. *Suppose each leaf $v \in B$ has height $h_v \leq 6$. Then B has no $(4,6)$ divisors.*

Proof. Consider a facet F, which has a unique associated point m_F satisfying $(m_F, \tilde{v}) = -1 \; \forall \tilde{v} \in F$; furthermore since $m_F \in \Delta$, $(m_F, v) \geq -1 \; \forall v \in \Delta^\circ$. Now suppose $h_v \leq 6/n$, $n \in \mathbb{N}$ for all rays $v = av_1 + bv_2 + cv_3$ in B, with v_i $3d$ cone vertices in B_i . Then $(nm_F, v) \geq -n(a+b+c) = -nh_v \geq -6$ for all rays v and therefore $nm_F \in \Delta_g$. If $h_v \leq 6 \; \forall v$, then $m_F \in \Delta_g$. This monomial has order of vanishing $(v, m_F) + 6 = 5$ for any v in or above F, which protects v from being a $(4,6)$ divisor. If $h_v \leq 6 \; \forall v$ then $m_F \in \Delta_g \; \forall F$ and there is a monomial that prevents each divisor from being $(4,6)$. □

Proposition 2. *Suppose $\exists v$ in or above a facet F,i.e. $v = av_1 + bv_2 + cv_3$ with v_i simplex vertices in F, such that $h_v \geq 2$. Then there is a non-Higgsable seven-brane on the divisor associated to each interior point of F.*

Proof. Then $(6m_F, v) = -6h_v \leq -12$ implies $6m_F \notin \Delta_g$. Similarly, $4m_F \notin \Delta_f$. Since any point p interior to F has $(m,p) = -1 \leftrightarrow m = m_F$ and reflexive polytopes of dimension three are normal, i.e. any $m_f \in \Delta_f$ ($m_f \in \Delta_g$) has $m_f = \sum_i m_i, m_i \in \Delta$ ($m_g = \sum_i m_i, m_i \in \Delta$), it follows that $(m_f, p) = -4 \leftrightarrow m_f = 4m_F$ and $(m_g, p) = -6 \leftrightarrow m_g = 6m_F$. Therefore is there is any tree on F then $4m_F \notin \Delta_f$ and $6m_F \notin \Delta_g$. By normality, for any p interior to F this implies $\nexists m_f \in \Delta_f | (m_f, p) = -4$ and $\nexists m_g \in \Delta_g | (m_g, p) = -6$, and therefore $ord_p(f,g) > (0,0)$, which implies there is a non-Higgsable seven-brane on the divisor associated to p. □

Proposition 3. *Let v be a leaf $v = av_1 + bv_2 + cv_3$ with v_i simplex vertices in F. If the associated the divisors $D_{1,2,3}$ carry a non-Higgsable E_8 seven-brane, then if v has height $h_v = 1,2,3,4,5,6$ it also has Kodaira fiber $F_v = II^*, IV^*_{ns}, I^*_{0,ns}, IV_{ns}, II, -$ and gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.*

Proof. Recall that the height criterion gives $ord_v(g) \leq 6 - h_v$. If $v = av_1 + bv_2 + cv_3$ with v_i each carrying E_8, then $(m_f, v_i) \geq 0, (m_g, v_i) \geq -1$,$\forall m_f \in \Delta_f$ and $\forall m_g \in \Delta_g$. This gives $(m_f, v) \geq 0$, $(m_g, v) \geq -(a+b+c) = -h_v$. Together, we see $ord_v(f) \geq 4$, $ord_v(g) = 6 - h_v$. For $h_v = 1,5,6$ this fixes G_v, but to determine G_v for $h_v = 2,3,4$ we must study the split condition. A necessary condition is that there is one monomial $m_g \in \Delta_g$ such that $(m_g, v) + 6 = 6 - h_v$, and since $m_F \in \Delta_g$ always, where F is the facet in which v_i lie, then $m_g = m_F$. Morever, the monomial m in g associated to m_F must be a perfect square; since $(m_F, v_i) + 6 = 5$, $m \sim x_i^5$ and m is not a perfect square. Therefore the fibers are all non-split. This establishes the result. □

Proposition 4. *Let B be a smooth toric threefold and $W(K_B, a, b) \to B$ be a Weierstrass model over B. Suppose that $ord_D(f) < 4$ or $ord_D(g) < 6$ for all prime divisors D, $ord_C(f) \leq 4$ or $ord_C(g) \leq 6$ for all irreducible curves C and that $ord_p(f) \leq 8$ or $ord_p(g) \leq 12$ at all points p in B. Then W has canonical singularities.* **[all:** `I believe the weakest condition for finite distance we could possibly have is` $ord_C(f) < 8$ `or` $ord_C(g) < 12$ `and` $ord_p(f) < 12$ `or` $ord_p(g) < 18$ `which is probably not true in general. Any higher order of vanishing guarantees infinite distance, but these conditions as stated (weak OOV) is sufficient for our purposes.` *-ben-*]

Proof. As ω_W is locally free, in fact trivial, W having canonical singularities is equivalent to W having rational singularities. By Corollary 2.4 in [16], W has rational singularities if and only if (K_W, a, b) is minimal and the reduced subscheme of the discriminant divisor $div(4a^3 + 27b^2)_{red}$ has only normal crossing singularities. Thus, it suffices to show that the reduced subscheme of the discriminant locus has only normal crossing singularities. **[all:** `This is where I have to think a bit more carefully. In Nakayama's paper, he shows that if all points and curves on a surface base are (4,6) then the the weierstrass model has rational singularities. The strategy he used was to base change to a curve, since he assumed all points are (4,6), then all divisors inside this curve are (4,6) and hence this is a minimal weierstrass model. In particular, any divisor of a curve is normal crossings, and so this pullback has rational singularities, and then you can use the deformation result to conclude that the original elliptic threefold has rational singularities.`

`In our case, we are weakening the assumption on vanishing along higher codimension loci. I think it follows since if the curves are (8,12), then they correspond to intersecting divisors. In particular, if the vanishing orders were higher than (8,12), then these curves cannot be complete intersections (in other words correspond to a regular ambient ring), and so in this case we wouldn't have rational singularities.` *-ben-*] □

Proposition 5. *Let B be a smooth toric threefold and $p\colon W(K_B, a, b) \to B$ be a Weierstrass model over B. Let $f\colon B' \to B$ be a proper birational morphism, and consider the base change of W defined by $W' = W \times_B B'$ with the induced projections $f' : W' \to W$ and $p' : W' \to B'$. If W' has canonical singularities then W has canonical singularities.*

Proof. This is straightforward. Consider the following diagram

$$\begin{tikzcd} W'' \arrow[r, "g"] & W' \arrow[r, "f'"] \arrow[d, "p'"] & W \arrow[d, "p"] \\ & B' \arrow[r, "f"] & B \end{tikzcd}$$

where W'' is a resolution of singularities of W' such that $K_{W''} \cong g^*(K_{W'}) + \sum_i a_i E_i$ with E_i exceptional divisors and $a_i \geq 0$ for all i; this is the hypothesis that W' has canonical singularities. As f is proper and birational, and p is surjective, f' is also proper and birational. Since W' and W are Calabi-Yau, it follows that $f'^*(K_W) \cong K_{W'}$. Since the composition of proper birational morphisms is also proper and birational, it follows that $f' \circ g \colon W'' \to W$ is indeed a resolution of singularities such that $(f' \circ g)^*(K_W) \cong K_{W'} + \sum_i a_i E_i$ and hence the result follows. **[bs:** `The point is that this composition map then provides a resolution` $W'' \to W$ `that gaurantees canonicality of` W`, right? If so, please write this sentence. Thanks!` *-jh,cl-*] $\square$

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