Some starter notes on trees

James Halverson, Cody Long, Benjamin Sung[1]
[1]*Department of Physics, Northeastern University*
Boston, MA 02115-5000 USA
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Abstract here.

TECHNICAL CONSIDERATIONS

Let B be a smooth weak Fano toric threefold. Such a variety is naturally associated with a fine regular star triangulation (FRST) of a three-dimensional reflexive polytope Δ. The monomials that may appear in the global sections f and g of the Weierstrass model are in one-to-one correspondence with points in the polytopes

$$P_n := \{m \in \mathbb{Z}^3 \mid \langle m, v\rangle + n \geq 0 \;\forall v \in \Delta\}, \tag{1}$$

in the cases $n = 4, 6$, respectively. Δ is a reflexive polytope because its dual $\Delta^\circ := P_1$ is itself a lattice polytope in $\mathbb{Z}^3$. Then if x_i is the toric homogeneous coordinate associated with some point $v_i \in \Delta$, the order of vanishing along $x_i = 0$ of some monomial $m_n \in P_n$ is

$$p_{x_i, m_n} = \langle m, v_i\rangle + n, \tag{2}$$

and the overall monomial associated with m_n is $\prod_i x_i^{\langle m_n, v_i\rangle + n}$. From this the order of vanishings of the monomials along various toric divisors and their intersections can be read off.

(JH:) Filler text will be needed here

Consider a toric variety $\tilde{B}$ obtained via a sequence of smooth blowups from the original toric threefold B associated to an FRST of Δ. The vertex associated to any given exceptional divisor is of the form $v_e = \sum a_i v_i$, with non-negative a_i, where the points v_i all lie on some two-face F of Δ. $\tilde{B}$ also has polytopes P_n, which we will call $\tilde{P}_n$ to distinguish them from the P_n associated to B. Due to the structure of the blowups, for fixed n $\tilde{P}_n$ may be obtained by slicing out upper half planes from P_n.

We will now show that if every exceptional divisor satisfies $\sum a_i \leq 6$, then the generic Weierstrass model over $\tilde{B}$ has no $(4, 6)$ divisors.

First, note that all vertices of $\tilde{B}$ are of the form $v = \sum a_i v_i$ with a_i non-negative; v corresponds to an exceptional divisor if and only if the vector a_i is not a unit vector. This, together with the definition of Δ°, implies that $\langle m, v\rangle = \sum a_i \langle m, v_i\rangle \geq -\sum a_i \geq -6$ for all v. This implies that any $m \in \Delta^\circ$ is also an element of $\tilde{P}_6$; i.e. the associated monomial appears in g of the associated Weierstrass model over $\tilde{B}$. Second, consider the unique $m \in \Delta^\circ$ that is dual to a chosen two-face F; it satisfies $\langle m, v\rangle = -1 \;\forall v \in F$. Then for any $v_e = \sum a_i v_i$ that is the sum of multiple vertices in F, we have $\langle m, v_e\rangle = -\sum a_i < 0$. By the previous argument, $m \in \tilde{P}_6$, and rewriting in terms of the power in the exponent we have

$$6 > p_{e,m} \geq 0. \tag{3}$$

This implies the monomial m prevents $e = 0$ from being a $(4, 6)$ divisor. This m does so for any exceptional divisor built on top of F, and for any F there is such an m. Therefore, none of the exceptional divisors of $\tilde{B} \to B$ are $(4, 6)$ divisors. Third, for any $m \in \Delta^\circ$ and $v \in \Delta$, $m \in \tilde{P}_6$ by the previous argument and $\langle m, v\rangle = -1$ which implies that g vanishes to order 5 along the associated divisor, and thus it cannot be a $(4, 6)$ divisor.

This exhausts the possible types of divisors; therefore any $\tilde{B}$ obtained in this way has no $(4, 6)$ divisors.

We will now show that to any $\tilde{B}$ constructed via our method there is a sequence of smooth toric blowups that together give $\check{B} \to \tilde{B}$ were $\hat{B}$ has no (4,6) points, curves, or divisors. This will be criticall for showing that the general Weierstrass model over $\tilde{B}$ is at finite distance in the moduli space using the Weil-Petersson metric.

Consider any toric curve $C = D_s \cdot D_t \subset \tilde{B}$. Take $v_s = \sum_i a_{i,s} v_i$ and $v_t = \sum_i a_{i,t} v_i$ and define $a := \sum_i a_{i,s}$ and $b := \sum_i a_{i,t}$. Let F be a facet on which or above which v_s and v_t sit; let $m \in \Delta^\circ$ be the dual to F. As an element of $\tilde{P}_4$ the associated monomial may be written

$$s^{\langle m, v_s\rangle + 4} t^{\langle m, v_t\rangle + 4} \times \ldots, \tag{4}$$

and the monomial vanishes to order $\langle m, v_s\rangle + \langle m, v_t\rangle + 8 = -a - b + 8$ along C, which must be ≥ 4 for C to be a $(4, 6)$ curve. Therefore $a + b \leq 4$ is necessary for C to be a $(4, 6)$ curve. Now suppose C is a $(4, 6)$ curve that we blow up via $\hat{B} \to \tilde{B}$ by adding an exceptional divisor $v_e = \sum a_i v_i = v_s + v_t$. Then $\sum a_i = a + b$, which satisfies $\sum a_i \leq 4$ since C is $(4, 6)$, but this condition is sufficient to ensure that $\hat{B}$ has no $(4, 6)$ divisors! If $2a + b \leq 4$ or $a + 2b \leq 4$ then $\hat{B}$ may still have a $(4, 6)$ curve, but this blowup process can be iterated until there are no longer $(4, 6)$ curves. So any $\tilde{B}$ in our construction that has $(4, 6)$ curves admits a sequence of blowups to a smooth toric threefold with no $(4, 6)$ curves or divisors.

We must make a similar argument for $(4, 6)$ points. Consider a point $p = D_s \cdot D_t \cdot D_u \subset \tilde{B}$, with v_s and v_t as before and $v_u = \sum_i a_{i,u} v_i$ with $c := \sum_i a_{i,u}$. There is a unique facet F above which or on which $v_{s,t,u}$ all sit, and again $m \in \Delta^\circ$ is the dual to F. As an element of $\tilde{P}_4$ the associated monomial may be written

$$s^{\langle m,v_s\rangle+4}t^{\langle m,v_t\rangle+4}u^{\langle m,v_u\rangle+4} \times \dots, \tag{5}$$

and the monomial vanishes to order $-a-b-c+12$ at p. Therefore $a+b+c \leq 8$ is a necessary condition for p to be a $(4,6)$ point. If $a+b+c \leq 6$, then the blowup of p with $v_e = v_s+v_t+v_u$ has no $(4,6)$ divisors and this type of blowup can be iterated until either there are no more $(4,6)$ points or there is a $(4,6)$ point with associated a,b,c satisfying $6 < a+b+c \leq 8$. In this case the blowup $v_e = v_s + v_t + v_u$ may have a $(4,6)$ divisor. However, a short calculation shows that any positive a,b,c satisfying this bound admits sequences of curve blowups along $D_s \cdot D_t$, $D_t \cdot D_u$, $D_u \cdot D_s$ that leads to a threefold base with no $(4,6)$ points, curves, or divisors.

Together, these arguments imply that any base $\tilde{B}$ in our construction has a sequence of smooth toric blowups to another base $\hat{B}$ (also present in our construction) that has no $(4,6)$ divisors, curves, or points. This means that the elliptic fibration $\hat{X} \to \hat{B}$ has canonical singularities, which in turn implies the general Weierstrass model $\tilde{X}$ over $\tilde{B}$ also has canonical singularities, since the blowup $\hat{B} \to \tilde{B}$ induces a blowup $\hat{X} \xrightarrow{\phi} \tilde{X}$. To see that the elliptic fibration $\hat{X}$ has only canonical singularities, we employ a modified version of Nakayama's result as presented in Lemma 3.6 in [1]. Consider a point $b \in \hat{B}$ and S a smooth surface through b. We consider the base change $\hat{X} \times_{\hat{B}} S$ defined by the following pullback diagram.

$$\begin{tikzcd}
\hat{X} \times_{\hat{B}} S \arrow[r] \arrow[d] & \hat{X} \arrow[d] \\
S \arrow[r] & \hat{B}
\end{tikzcd}$$

(JH:) Ben can you give the intuitive sentence about the fiber product, again? The one that made me and Cody happy? I already forgot. We may consider $\hat{X} \times_{\hat{B}} S$ as a Weierstrass model over S, which can be thought of as a restriction within $\hat{X}$ that gives an elliptic threefold over S. By hypothesis, since $\hat{B}$ has no (4,6) points, it follows that S has no (4,6) points and hence the pullback is a minimal Weierstrass model and has rational singularities. By a result of [2] on the deformations of rational singularities, it follows that $\hat{X}$ has rational singularities, and since $\hat{B}$ is smooth, it follows that $\hat{X}$ has rational Gorenstein singularities and in particular, at worst canonical singularities. This means that there is a smooth resolution (not necessarily Calabi-Yau) $X_s \xrightarrow{\rho} \hat{X}$ such that $K_{X_s} = \rho^*(K_{\hat{X}}) + \sum_i a_i E_i$ with all $a_i \geq 0$ and where E_i is an exceptional divisor of ρ. Then we also have a smooth resolution $X_s \xrightarrow{\rho} \hat{X} \xrightarrow{\phi} \tilde{X}$ so that $\tilde{X}$ also has canonical singularities. To see this, we consider the following diagram

$$\begin{tikzcd}
X_s \arrow[r, "f"] & \hat{X} \cong \tilde{X} \times_{\tilde{B}} \hat{B} \arrow[r, "\phi"] \arrow[d, "p"] & \tilde{X} \arrow[d, "p'"] \\
& \hat{B} \arrow[r, "\phi'"] & \tilde{B}
\end{tikzcd}$$

where $\hat{B}$ is a resolution of $\tilde{B}$ and we take $\hat{X}$ to be induced weierstrass model over $\hat{B}$ from the above pullback diagram. It suffices to show that $R^\bullet(f \circ \phi)_\star(\mathcal{O}_{X_s}) \cong \mathcal{O}_{\tilde{X}}$. By hypothesis, $R^\bullet f_\star(\mathcal{O}_{X_s}) \cong \mathcal{O}_{\hat{X}}$ and $R^\bullet \phi'_\star(\mathcal{O}_{\hat{B}}) \cong \mathcal{O}_{\tilde{B}}$. Applying the Grothendieck spectral sequence, we have that $R^\bullet(f \circ \phi)_\star(\mathcal{O}_{X_s}) \cong R^\bullet \phi_\star(\mathcal{O}_{\hat{X}})$. Since p' in particular, is a flat morphism, and flatness is stable under base change, p is also a flat morphism, and in particular, we have the induced isomorphism $p'^\star R^\bullet \phi'_\star \mathcal{O}_{\hat{B}} \xrightarrow{\sim} R^\bullet \phi_\star p^\star \mathcal{O}_{\hat{B}}$. It follows that $R^\bullet \phi_\star(\mathcal{O}_{\hat{X}}) \cong R^\bullet \phi_\star p^\star(\mathcal{O}_{\hat{B}})$ Any F-theory model on $\tilde{X}$ in our construction therefore has canonical singularities, and by results of Hayakawa [3] and Wang [4] they are all at finite distance from one another in the Weil-Petersson metric on moduli space.

GAUGE CONFIGURATIONS

We now embark on a combinatorial study of locality of non-Higgsable clusters on intersecting divisors. Indeed, we first observe trivially that sufficiently tall trees on one simplex may induce non-Higgsable clusters on another simplex with some relative height difference that could be quantitatively analyzed. Let v_1, v_2, and v_3 denote three rays forming a smooth, rational, polyhedral cone, and define $v_e \equiv av_1 + bv_2 + cv_3$ with $(a,b,c) \in \mathbb{Z}^3_{\geq 0}$. Define $H_{n,e}$ to be the hyperplane defined by $\langle m, v_e\rangle = -n$ and define $H_{n,i}$ to be the upper half-planes defined by $\langle m, v_i\rangle \geq -n$ for $i = 1,2,3$. We wish to study the region defined by $H \equiv H_{n,e} \bigcap\limits_i H_{n,i}$.

We first show that there exists a finite number of integral points in H. Consider the equation given by $\langle m, v_e\rangle = a\langle m, v_1\rangle + b\langle m, v_2\rangle + c\langle m, v_3\rangle = -n$. As the tuple (a,b,c) are strictly positive, it easily follows from the above inequalities, that $\langle m, v_i\rangle$ are also bounded above. Thus, we consider the integral solution set to the matrix inequality

$$\begin{pmatrix} -n \\ -n \\ -n \end{pmatrix} \leq \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} m \end{pmatrix} \leq \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{6}$$

As the vectors v_i form a basis for $\mathbb{Z}^3$ by hypothesis, it follows easily that there exists only finitely many integral solutions to the above matrix inequality, and hence we are done.

We now give a sufficient condition for bounding hyperplanes to eliminate points on the hyperplane $H_{n,e}$. Let v_e and the v_i be as defined above, and define $v_i' \equiv a_i v_1 + b_i v_2 + c_i v_3$ for positive integers a_i, b_i, and c_i such that $v_1' + v_2' + v_3' = v_e$. We construct three smooth torus-equivariant blowups along the toric curves given by the edges (v_e, v_i') for $i = 1, 2, 3$. The resulting hyperplane inequalities given by the vertices $v_e + v_i'$ yield the condition $\langle m, 3v_e + v_1' + v_2' + v_3' \rangle = \langle m, 4v_e \rangle \geq -3n$. Thus, we find that $\langle m, v_e \rangle \geq -\frac{3}{4}n$ and hence there does not exist integral points on the hyperplane $H_{n,e}$.

Minimal Vanishing From Resolution

(CL:) to discuss notation A blow-up on an n-face $F_n \subset \Delta^\circ$ $(n = 0, 1, 2)$ will induce a minimal order of vanishing in f and g along the divisors corresponding to the points involved in the blow-up, as well as any of the points interior to any higher codimension faces bounded by F_n. In this section we derive this by consider blow-ups along faces of each dimension.

Points interior to facets

First, we consider a point p interior to a facet F, with corresponding divisor D_p, and homogenous coordinate x_p. To check for non-Higgsable 7-branes on D_p, we check for the absence of any monomials m_n such that $\langle m_n, p \rangle = -n$, for $n = 4, 6$. Define the lattice polytope corresponding to the sections of degree $\mathcal{O}(-nK_B)$ as Δ_n. We note that all vertices of Δ_n take the form nm_i, where the m_i are the vertices of Δ. Recall that there exists a vertex $m \subset \Delta$ dual to F : $\langle m, F \rangle = -1$. Let the vertices that are not m be $\hat{m}_j$. Since $\langle m, p \rangle = -1$, and $\langle \hat{m}_j, p \rangle \geq 0$, it is clear that $\langle n\, m, p \rangle = -n$, and $\langle n\, \hat{m}_j, p \rangle \geq 0$. Therefore the linear function $\langle p, \cdot \rangle$ is minimized at m, and is strictly increasing when moving from m to any other vertex, and therefore strictly increasing when moving away from m in any direction. Therefore, m is the only monomial satisfying $\langle p, \cdot \rangle = -n$. Therefore, before resolution there is a single monomial in f (g), which is degree zero in x_p, and corresponds to a vertex in Δ_4 (Δ_6). The existence of these monomials obstructs the existence of non-Higgsable 7-branes along D_p.

Now let us consider a resolution, with exceptional divisor $D_e = \sum a_i D_i$, where $a_i \in \{0, 1\}$, and the D_i correspond to points v_i on F, which can be vertices, points interior to edges that bound F, or points strictly interior to F. For the blow-up to be non-trivial either two or three of the a_i are non-zero. This blow-up introduces a hyperplane that further cuts Δ_n to $\tilde{\Delta}_n$. In particular, we consider the effect on the monomial $n\,m$. First, let us note that if we have an $m \in \Delta$ such that $\langle m, p \rangle = -1$, where p is in the strict interior of a facet F, then $\langle m, v_i \rangle = -1$ for all other $v_i \in F$. To see this, consider a triangulation of F using the vertices of F as vertices in the triangulation. Then all other points lie within the simplices of the triangulation. Let p lie within a simplex generated by the points $\{v_1, v_2, v_3\}$. Then we can write $p = a_1 v_1 + a_2 v_2 + (1 - a_1 - a_2) v_3$, where $0 < a_i < 1$. Therefore, we have $-1 = a_1 \langle m, v_1 \rangle + a_2 \langle m, v_2 \rangle + (1 - a_1 - a_2) \langle m, v_3 \rangle$. From the reflexivity of Δ° we have $\langle v_i, m \rangle \geq -1$. Assume that $\langle m, v_1 \rangle > -1$. We then have $-1 > -a_1 + a_2 \langle m, v_2 \rangle + (1 - a_1 - a_2) \langle m, v_3 \rangle > -a_1 - a_2 - (1 - a_1 - a_2) = -1$, which is a contradiction. One can repeat this argument to see that all of the the $\langle m, v_i \rangle = -1$.

Now we take the inner product of $n\,m$ with $v \equiv \sum_i a_i v_i$, we have $\langle n\,m, v \rangle = -1$. Then $\langle m, v \rangle - n \sum_i v_i$, which must satisfy $\langle m, v \rangle - n \sum_i v_i \geq -n$ for m to be an allowed point in $\tilde{\Delta}_n$. However, a non-trivial blow-up have $\sum_i v_i > 1$, and therefore the blowup cuts out the point m from $\tilde{\Delta}_n$. Applying this to Δ_f and Δ_g, we can see that after the resolution both f and g will vanish to at least order one along x_p. Therefore any blow-up along any points in F will automatically introduce at least a type II singularity along all divisors corresponding to points in the strict interior of F.

Points interior to edges

We now perform a similar analysis for a divisor D_p corresponding to a point p with homogeneous coordinate x_p, and is interior to an edge E. In this case, the dual to p in Δ under $\langle p, \cdot \rangle = -1$ is an entire edge of points $E_\Delta \in \Delta$, bounded by two vertices which we label m_1 and m_2. m_1 and m_2 are vertices in Δ, and therefore nm_1 and nm_2 are vertices in Δ_n, which bound an edge $E_n \subset \Delta_n$. Every point $m^i_{E_n} \in E_n$ then satisfies $\langle m^i_{E_n}, p \rangle = -n$. Since all other vertices $\hat{m}_j \neq m_1, m_2$ in Δ satisfy $\langle \hat{m}_j, p \rangle \geq 0$, then all vertices $n\hat{m}_j \neq nm_1, nm_2$ of Δ_n satisfy $\langle n\hat{m}_j, p \rangle \geq 0$. The linear function $\langle p, \cdot \rangle$ is then minimized along E_n and strictly increasing when moving away from E_n, and therefore the only points m_n in Δ_n satisfying $\langle p, \cdot \rangle = -n$ are the $m^i_{E_n}$. In a similar fashion to the $p \in F$ case one can show that $\langle m^i_{E_n}, v_i \rangle = -n$ for all points v_i on E.

We now perform a blow-up using two points v_1 and v_2 on E, introducing a new ray $v = v_1 + v_2$ and corresponding exceptional divisor $D_v = D_1 + D_2$, where the D_i correspond to the v_i. Again, this resolution will cut points out of Δ_n by introducing an additional hyperplane constraint. We consider the effect on the points $m^i_{E_n}$, which correspond to the non-trivial monomials in $\mathcal{O}(-nK_B)$ that are degree zero in x_p. We have $\langle m^i_{E_n}, v \rangle = -2n$. However, this violates $\langle m^i_{E_n}, v \rangle \geq -n$, and therefore the resolution cuts out all degree zero monomials in x_p. Therefore any resolution along E will introduce at least a type II singularity along each divisor corresponding to a point on the strict interior of E. (CL:) stopping here, I don't think there's a direct continuation of this argument to vertices

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