On Algorithmic Universality in F-theory Compactifications

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We study universality of geometric gauge sectors in the string landscape in the context of F-theory compactifications. A finite time construction algorithm is presented for $\frac{4}{3} \times 2.96 \times 10^{755}$ F-theory geometries that are connected by a network of topological transitions in a connected moduli space. High probability geometric assumptions uncover universal structures in the ensemble without explicitly constructing it. For example, non-Higgsable clusters of seven-branes with intricate gauge sectors occur with probability above $1 - 1.01 \times 10^{-755}$, and the geometric gauge group rank is above 160 with probability .999995. In the latter case there are at least 10 E_8 factors, the structure of which fixes the gauge groups on certain nearby seven-branes. Visible sectors may arise from E_6 or $SU(3)$ seven-branes, which occur in certain random samples with probability $\simeq 1/200$.

I. Introduction. String theory is a consistent theory of quantum gravity that naturally gives rise to interesting gauge and cosmological sectors. As such, it is a promising candidate for a unified theory. However, there is a vast landscape of four-dimensional metastable vacua that may realize different physics, making predictions difficult.

A possible way forward, as in many areas of physics, is to demonstrate universality in large ensembles. For string vacua, such as the oft-quoted $O(10^{500})$ type IIb flux vacua [1], studying universality via explicit construction is complex and impractical [2]. However, it may be possible to derive universality from a precise construction algorithm, rather than from the constructed ensemble. We refer to this as algorithmic universality, and find it a promising way forward in the string landscape.

We present such an algorithm in the context of $4d$ F-theory [3] compactifications. The ensemble is a collection of $4/3 \times 2.96 \times 10^{755}$ six-manifolds, perhaps the largest set of string geometries to date, that serve as the extra spatial dimensions. Their topological structure determines the $4d$ gauge group that arises geometrically from configurations of seven-branes that form a network of so-called non-Higgsable clusters (NHC) [4]. We establish that non-Higgsable clusters arise with probability above $1 - 1.01 \times 10^{-755}$ in this ensemble, and demonstrate that a rich minimal gauge structure arises with high probability. We also present results from random sampling that are potentially relevant for visible sectors.

A number of recent results suggest that NHC are important in the $4d$ F-theory landscape. They exist for generic vacuum expectation values of scalar fields (complex structure moduli), and therefore gauge symmetry does not require stabilization on subloci in moduli space [5], which can have high codimension [6]. Standard model structures may arise naturally [5], strong coupling is generic [7], and $4d$ NHC may exhibit features [8] (such as loops and branches) not present in $6d$. NHC arise in the geometry with the largest number of flux vacua [9], and universally in known ensembles [10] closely related to ours. $6d$ NHC have been studied extensively [4, 11].

In Sec. II we review non-Higgsable clusters. In Sec. III we present our ensemble. In Sec. IV we exhibit universality. In Sec. V we discuss our results.

II. Seven-Branes and Non-Higgsable Clusters. A $4d$ F-theory geometry is a Calabi-Yau elliptic fibration X over six extra spatial dimensions described by a complex threefold base space B defined by the equation

$$y^2 = x^3 + fx + g \tag{1}$$

where f and g are polynomials in the coordinates of B; technically, $f \in \Gamma(\mathcal{O}(-4K_B))$, $g \in \Gamma(\mathcal{O}(-6K_B))$, with K_B the canonical class. Seven-branes are localized on the discriminant locus $\Delta = 4f^3 + 27g^2 = 0 \subset B$.

Upon compactification the gauge group structure of seven-branes gives rise to four-dimensional gauge sectors. It is controlled by f and g, and for a typical B the most general f, g take the form $f = \tilde{f} \prod_i x_i^{l_i}$, $g = \tilde{g} \prod_i x_i^{m_i}$, so

$$\Delta = \tilde{\Delta} \prod_i x_i^{\min(3l_i, 2m_i)} =: \tilde{\Delta} \prod_i x_i^{n_i}, \tag{2}$$

and therefore f, g and Δ vanish along $x_i = 0$ to $mult_{x_i=0}(f, g, \Delta) = (l_i, m_i, n_i)$. This seven-brane carries a gauge group G_i given in Table I according to the Kodaira classification. In some cases further geometric data is necessary to uniquely specify G_i (see e.g. [10] for conditions) but this data always exists for fixed B. For generic f and g, a seven-brane on $x_i = 0$ requires $(l_i, m_i) \geq (1, 1)$.

Such a seven-brane is called a geometrically non-Higgsable seven-brane (NH7) because it carries a gauge group that cannot be removed by deforming f or g. A NH7 may have geometric gauge group

$$G \in \{E_8, E_7, E_6, F_4, SO(8), SO(7), G_2, SU(3), SU(2)\},$$

which could be broken by fluxes. We assume fluxes can be turned on in a large fraction of our geometries. A typical base B, as we will show in the strongest generality to date, has many non-Higgsable seven-branes that often intersect in pairs, giving rise to jointly charged matter.

F_i	l_i	m_i	n_i	Sing.	G_i
I_0	≥ 0	≥ 0	0	none	none
I_n	0	0	$n \geq 2$	A_{n-1}	$SU(n)$ or $Sp(\lfloor n/2 \rfloor)$
II	≥ 1	1	2	none	none
III	1	≥ 2	3	A_1	$SU(2)$
IV	≥ 2	2	4	A_2	$SU(3)$ or $SU(2)$
I_0^*	≥ 2	≥ 3	6	D_4	$SO(8)$ or $SO(7)$ or G_2
I_n^*	2	3	$n \geq 7$	D_{n-2}	$SO(2n-4)$ or $SO(2n-5)$
IV^*	≥ 3	4	8	E_6	E_6 or F_4
III^*	3	≥ 5	9	E_7	E_7
II^*	≥ 4	5	10	E_8	E_8

TABLE I. Kodaira fiber F_i, singularity, and gauge group G_i on the seven-brane at $x_i = 0$ for given l_i, m_i, and n_i.

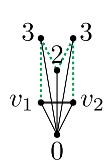
This is a geometrically non-Higgsable cluster (NHC). For brevity, we henceforth drop geometric and geometrically.

III. Large Landscapes of Geometries from Trees. We now introduce our construction, which utilizes building blocks in toric varieties that we call trees to systematically build up F-theory geometries. After describing the geometric setup and defining terms that simplify the discussion, we will present a criterion, classify all trees satisfying it, and build the F-theory geometries.

Our construction begins with a smooth weak-Fano toric threefold B_i, and then builds structure on top. These geometries B_i are determined by fine regular star triangulations (FRST) of one of the 4319 $3d$ reflexive polytopes [12]; there are an estimated $O(10^{15})$ such geometries [6]. The $2d$ faces of the $3d$ polytope are known as facets, and a triangulated polytope will have triangulated facets. Such B_i do not support NHC.

Consider such a B_i determined by an FRST of a $3d$ reflexive polytope Δ°, a triangulated facet F in Δ°, and an edge between two points v_1 and v_2 in F with associated homogeneous coordinates x_1 and x_2. Since $v_{1,2}$ are connected by an edge, $x_1 = x_2 = 0$ defines a Riemann surface (algebraic curve) in B_i, which can be "blown up" using a new ray $v_e = v_1 + v_2$ and subdividing cones using standard toric techniques. This is a topological transition that introduces a new ("exceptional") divisor $e = 0$ in B, where e is the coordinate associated to v_e. This process can be iterated, for example blowing up along $e = x_1 = 0$, which would add a new ray $v_e + v_1 = 2v_1 + v_2$.

After a number of iterations the associated toric variety will have a collection of exceptional divisors with associated rays $v_{e_i} = a_i v_1 + b_i v_2$, which will appear to have formed a *tree* above the *ground* that connects v_1 and v_2 in F. Each v_{e_i} is a *leaf* with *height* $h_{e_i} = a_i + b_i$, and we will refer to trees built on edges within F as *edge trees*. The height of a tree is the height of its highest leaf. As an example, $\{v_1 + v_2, 2v_1 + v_2, v_1 + 2v_2\}$ appears as



where the v_1 to v_2 line is the edge (ground) in F, dashed green lines are above the ground, 0 is the origin of Δ°, and the new rays are labelled by height. Such large gauge sectors motivate dark glueballs; see e.g. [13].

Similarly, one can also build *face trees* by beginning with a face on F, with vertices v_1, v_2, v_3 associated to x_1, x_2, x_3. Adding $v_e = v_1 + v_2 + v_3$ and subdividing appropriately blows up the point $x_1 = x_2 = x_3 = 0$ and produces a new toric variety. Again such blowups can be iterated. This process builds a collection of leaves $v_{e_i} = a_i v_1 + b_i v_2 + c_i v_3$ with $a_i, b_i, c_i > 0$ of height $h_{e_i} = a_i + b_i + c_i$ that comprise a *face tree*. Face trees are built above the interior of the face due to the strict inequality in the definition. Note if one leaf coefficient was zero the associated leaf would be above an edge of the face, not above the face interior.

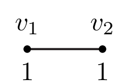
Geometries can be systematically constructed by adding a face tree to each face in each triangulated facet of Δ°, and then an edge tree to each edge. The associated smooth toric threefold B has a collection of rays v, each of which can be written $v = av_1 + bv_2 + cv_3$ with v_i $3d$ cone vertices in B_i. If $(a, b, c) = (1, 0, 0)$ or some permutation thereof, $v \in \Delta^\circ$ and this height $h_v = 1$ "leaf" is more appropriately a root, since it is on the ground.

A natural question in systematically building up geometries is whether there is a maximal tree height. For a toric variety B to be an allowed F-theory base it must not have any so-called $(4, 6)$ divisors (see Appendix), which we ensure by a simple height criterion proven in Prop. 1:

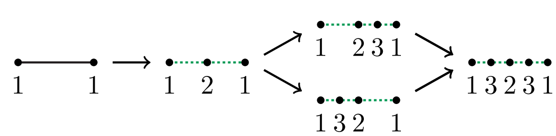
If $h_v \leq 6$ for all leaves $v \in B$,
then there are no $(4, 6)$ divisors.

This condition is simple and sufficient, but not necessary for the absence of $(4, 6)$ divisors. Nevertheless, it will allow us to build a large class of geometries.

The task is now clear: we must systematically build all topologically distinct edge trees and face trees of height ≤ 6. Since the combinatorics are daunting, let us exemplify the problem for $h \leq 3$ trees. Viewing the facet head on, an edge in F appears as



with the vertices and their heights labelled. Adding $v_1 + v_2$ subdivides the edge, and further subdivision gives



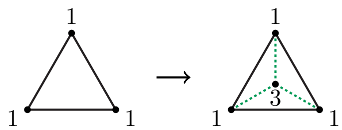
where we have dropped the vertex labels and kept the heights. The trees emerge out of the page, but visualization is made easier by projecting on to the edge; the

N	# Edge Trees	# Face Trees
3	5	2
4	10	17
5	50	4231
6	82	41,873,645

h_v	Probability
3	.99999998
4	.999995
5	.999997
6	.999899

TABLE II. *Left:* The number of edge trees and face trees with height $h \leq N$. *Right:* The probability that a face tree with $h \leq 6$ has a leaf v with a given height h_v.

right-most tree is the one previously presented vertically. There are five edge trees with height ≤ 3. Similarly,



shows that there are 2 face trees of height ≤ 3. Here we have denoted the new edges by green lines since they do not sit in the facet. With our definitions, edge trees are built above an edge in the facet, whereas higher leaves in face trees may be built on new edges that do not sit in the facet. For example, a height 4 leaf could be added on any of the green lines above. A (tedious) straightforward calculation shows that the number of edge or face trees with $h \leq N$ grows rapidly, as in Table II.

Having classified the number of $h \leq 6$ face trees and edge trees, we now give a lower bound for the number of F-theory geometries that arise from building trees on an FRST of Δ°, denoted $\mathcal{T}(\Delta^\circ)$. We construct an ensemble S_{Δ° of geometries by systematically putting $h \leq 6$ face trees on all faces $\tilde{F}$ of $\mathcal{T}(\Delta^\circ)$ and then putting $h \leq 6$ edge trees on all edges $\tilde{E}$ of $\mathcal{T}(\Delta^\circ)$. Using Table II, the size of S_{Δ° is

$$|S_{\Delta^\circ}| = 82^{\#\tilde{E}\,\mathrm{on}\,\mathcal{T}(\Delta^\circ)} \times (4.19 \times 10^6)^{\#\tilde{F}\,\mathrm{on}\,\mathcal{T}(\Delta^\circ)}. \qquad (3)$$

$\#\tilde{E}$ and $\#\tilde{F}$ are triangulation-independent and are entirely determined by Δ° [14].

Two $3d$ reflexive polytopes give a far larger number $|S_{\Delta^\circ}|$ than the others. They are the convex hulls $\Delta_i^\circ := \mathrm{Conv}(S_i), i = 1, 2$ of the vertex sets

$$S_1 = \{(-1,-1,-1), (-1,-1,5), (-1,5,-1), (1,-1,-1)\},$$
$$S_2 = \{(-1,-1,-1), (-1,-1,11), (-1,2,-1), (1,-1,-1)\}.$$

$\mathcal{T}(\Delta_1^\circ)$ and $\mathcal{T}(\Delta_2^\circ)$ have the same number of edges and faces. Their largest facets are displayed in Fig. 1 and have $\#\tilde{E} = 63$ and $\#\tilde{F} = 36$. We compute

$$|S_{\Delta_1^\circ}| = \frac{2.96}{3} \times 10^{755} \qquad |S_{\Delta_2^\circ}| = 2.96 \times 10^{755}, \qquad (4)$$

where the factor of 1/3 is due to the symmetries discussed in the Appendix. All other polytopes Δ° contribute negligibly: $|S_{\Delta^\circ}| \leq 3.28 \times 10^{692}$ configurations. This gives

$$\#\ \text{4d F-theory Geometries} \geq \frac{4}{3} \times 2.96 \times 10^{755}, \qquad (5)$$

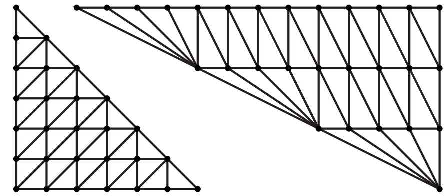


FIG. 1. The largest facets in the two 3d reflexive polytopes Δ_1° and Δ_2° with the most number of interior points. Presented is one triangulation of each, from which we see $\#\tilde{E} = 63$ edges and $\#\tilde{F} = 36$ faces in both facets.

which undercounts due to the facts that we choose to do face blowups followed by edge blowups to simplify the subdivision combinatorics, and that we have not taken into account the $O(10^{15})$ FRSTs of Δ_2° and Δ_1°.

IV. Universality and Non-Higgsable Clusters.

We now study universality in the dominant sets of F-theory geometries $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$. We prove non-Higgsable cluster universality, minimal gauge group universality, and discuss results from random sampling.

Algorithmic Universality and Gauge Groups. We wish to establish the likelihood that an F-theory base in $S_{\Delta_1^\circ}$ or $S_{\Delta_2^\circ}$ give rise to non-Higgsable seven-branes. The result arises from Prop. 2: if there is a tree anywhere on F, even a single leaf, there is a non-Higgsable seven-brane on all divisors associated to interior points of F. For any S_{Δ° only one configuration has no trees, and therefore

$$P(\text{NHC in } S_{\Delta^\circ}) \geq 1 - \frac{1}{|S_{\Delta^\circ}|}. \qquad (6)$$

This is always very close to one, and in particular

$$P(\text{NHC in } S_{\Delta_1^\circ}) \geq 1 - 1.01 \times 10^{-755}$$
$$P(\text{NHC in } S_{\Delta_2^\circ}) \geq 1 - .338 \times 10^{-755}. \qquad (7)$$

We see that NHC are universal in these ensembles.

We now wish to study physics in our ensemble. Consider a geometric assumption A_i and a physical property P_i such that $A_i \implies P_i$. Our goal is to determine high probability assumptions that lead to interesting physical properties, computing $P(A_i)$ since $A_i \implies P_i$ ensures $P(P_i) \geq P(A_i)$. We will focus on $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$ since these dominate the ensemble.

Consider first $S_{\Delta_1^\circ}$ and let A_1 be the assumption that any simplex in an FRST of Δ_1° containing a vertex of Δ_1° has an $h \geq 3$ face tree on it. For the 3 symmetric facets of Δ_1° there are 17 ways to choose simplices containing the vertices, and 1796 ways for its largest facet. The maximum number of simplices containing vertices is 24. Using $P(h \geq 3 \text{ tree on simplex})$ from Table II,

$$P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .9999998^{24} = .999995. \qquad (8)$$

There are $17^3 \times 1796$ ways to choose simplices that contain the vertices, all of which yield $G \geq F_4^{18} \times E_6^{10} \times U^9$

where $U \in \{G_2, F_4, E_6\}$, depending on details. All of these factors arise on the ground, and generally there will be many more factors from non-Higgsable seven-branes in the leaves. Here E_6^{10} arises from an E_6 on every interior point of the large facet in Δ_1°, see Fig. 1. This set of statements defines physical property P_1, and since $A_1 \implies P_1$ we deduce $P(P_1 \text{ in } S_{\Delta_1^\circ}) \geq P(A_1 \text{ in } S_{\Delta_1^\circ}) \geq .999995$.

Let A_2 be the assumption that there exists a $h = 5$ face tree somewhere on the large facet F in Δ_1°. Knowing $\tilde{F} = 36$ on F and using Table II, we compute $P(A_2 \text{ in } S_{\Delta_1^\circ}) = (1 - (1 - .999997)^{36}) \simeq 1 - 10^{-199}$. Let A_3 be that A_1 and A_2 hold, so $P(A_3) = P(A_1)P(A_2) \simeq P(A_1)$. Then given A_3 a short calculation shows that the $h = 5$ tree on F enhances E_6 in P_1 to E_8, giving 10 E_8's on the ground. P_1 with this enhancement defines P_3.

Similar results hold for for $S_{\Delta_2^\circ}$. Let A_1 be the assumption that any simplex in an FRST of Δ_2° containing a vertex of Δ_2° has an $h \geq 3$ face tree on it. This ensures that $G \geq F_4^{15} \times E_6^7 \times U^{12}$. However, this is quickly enhanced to $G \geq F_4^{18} \times E_8^{10} \times U^9$, via a $h = 5$ face tree on each face, and a $1 - 6.55 \times 10^{-8}$ probability blow-up along an edge connecting the point $\{-1, 2, -1\}$ to one of the points $\{-1, 1, n\}$, where $n = -1 \ldots 3$. The existence of these edges is independent of triangulation. Summarizing, the probability that a geometry in our set has $G \geq F_4^{18} \times E_8^{10} \times U^9$ on the ground is $\geq .999995$. This minimal group for P_3 on $S_{\Delta_2^\circ}$ matches that of P_3 on $S_{\Delta_1^\circ}$.

It is natural to ask whether this structure on the ground constrains the gauge structure in the trees. In Prop. 3 it is shown that the gauge group on a leaf v in a tree built above E_8's on the ground is determined by the leaf height h_v. The result is that a $h_v = 1, 2, 3, 4, 5, 6$ leaf above E_8 roots has Kodaira fiber $F_v = II^*, IV_{ns}^*, I_{0,ns}^*, IV_{ns}, II, -$ with gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.

This leads to a high probability result about the structure of the gauge group. Since $A_3 \implies P_3$, which has at least 10 E_8 factors nearby one another, P_3 also has

$$G \geq E_8^{10} \times F_4^{18} \times U^9 \times F_4^{H_2} \times G_2^{H_3} \times A_1^{H_4}, \qquad (9)$$

where H_i is the number of height i leaves in trees built on E_8 roots, and $rk(G) \geq 160 + 4H_2 + 2H_3 + H_4$. There are H_6 Kodaira type II seven-branes that do not carry a gauge group but realize Argyres-Douglas theories on D3 probes. The first F_4 and also the U factors may enhance, but the other factors are fixed. The probability of this physical property is $P(P_3 \text{ in } S_{\Delta_1^\circ}) \geq P(A_3 \text{ in } S_{\Delta_1^\circ}) \simeq .999995$ and $P(P_3 \text{ in } S_{\Delta_2^\circ}) \geq P(A_3 \text{ in } S_{\Delta_2^\circ}) \simeq .999995$. This non-trivial minimal gauge structure is universal in our large ensembles given by $S_{\Delta_1^\circ}$ and $S_{\Delta_2^\circ}$.

Random Samples and Geometric Visible Sectors. It may be possible to accomodate visible sectors from flux breaking these gauge sectors, but it is also interesting to study whether gauge factors E_6 and/or $SU(3)$ arise with high probability. We have not yet discovered a high probability simple geometric assumption that leads to E_6 or $SU(3)$. However, it is possible that they arise regularly, but due to a complex geometric assumption.

This idea can be tested by random sampling. Let B be an F-theory base obtained by adding face trees then edge trees at random, followed by edge trees at random, to the "pushing" triangulation [14] of Δ_1°. We studied an ensemble S_r of 10^6 such random samples and found $P(SU(3) \,\text{or}\, E_6 \text{ in } S_r) \simeq 1/200$, and that at least 36 of the points in Δ_1° carried E_8, a significant enhancement beyond P_3. Furthermore, in our sample we found that E_6 only arose on the point $(1, -1, -1)$, which is the only vertex of Δ_1° that isn't in the largest facet. Similar results and probabilities also hold using these techniques on Δ_2°. It would be interesting to study random samples of other triangulations, or to see if other geometric assumptions imply these enhancements. We leave the systematic study of geometric visible sectors to future work.

V. Discussion. We have presented a construction algorithm for $\frac{4}{3} \times 2.96 \times 10^{755}$ geometries for $4d$ F-theory compactifications. This number is only a lower bound and may be enlarged in at least three ways: by relaxing the requirement of edge blowups after face blowups, by taking into account the $O(10^{15})$ FRS triangulations of 3d reflexive polytopes, and by considering blow-ups of non-toric intersections of seven-branes.

We have initiated the study of this ensemble by focusing on the geometric gauge group. Using knowledge of the construction algorithm, we derived the existence of universal properties for the minimal geometric gauge group on non-Higgsable clusters. High rank groups are generic, as is the existence of at least 10 E_8 factors on the ground. The gauge group on leaves above these E_8 factors on the ground is fixed entirely by their height.

There are many directions for future work. For example, it would be interesting to study the number of consistent fluxes per geometry and how they alter the gauge group, to perform a statistical analysis of the gauge group in the leaves, or to analyze physics that arises from blow-ups of non-toric intersections of seven-branes. Perhaps most pressing is that, though we have demonstrated that the gauge group is generically high rank and have reviewed some possible realizations of the standard model discussed in [5], it is not yet clear whether the standard model is realized with high probability in our ensemble.

We believe that this is the first time that such a large ensemble has been systematically studied in string theory. In our view, the crucial ingredient that made the results possible are what we call algorithmic universality: derivation of universality from a construction algorithm, rather than an explicitly constructed ensemble or random sampling. Given the plethora of large ensembles in string theory and the infeasibility of constructing all of them, universality of this sort may play a critical role in making the string landscape tractable.

A. Appendix: Technical Subtleties. We now address technical subtleties that are important for establishing, but not understanding, results in the main text.

Polytope Symmetries and Toric Morphisms. In equation (4) we have included a factor of 1/3 relative to the count one would obtain directly from the algorithm. This takes into account an overcounting of geometries due to toric equivalences, which arise when there is a $GL(3,\mathbb{Z})$ transformation on the toric rays that preserves the cone structure of the fan. In general, there may be many such equivalences between elements of two ensembles $S_{\Delta_i^\circ}$ and $S_{\Delta_j^\circ}$, where $\Delta_{i,j}^\circ$ are any two $3d$ reflexive polytopes. However, to ensure that the count (4) is accurate, we only need to consider whether there are equivalences between two elements in $S_{\Delta_1^\circ}$, two in $S_{\Delta_2^\circ}$, or one in $S_{\Delta_1^\circ}$ to $S_{\Delta_2^\circ}$. It is sufficient to consider $GL(3,\mathbb{Z})$ actions on the ground, i.e. on the facets. This follows from the fact that rays of different height cannot be exchanged under automorphisms of the fan. First note that points in a hyperplane remain in one after a $GL(3,\mathbb{Z})$ transformation, and points in a line remain in a line by linearity. Facets must therefore map to facets. The big facets in Δ_1° and Δ_2° cannot map to other facets by point counting, and therefore they must map to themselves. There is no non-trivial map taking the big facet in Δ_2° to itself, but there is a $\mathbb{Z}_3$ rotation taking the big facet in Δ_1° to itself, giving a factor of 1/3 in $S_{\Delta_1^\circ}$. There is no non-trivial map between the big facets in Δ_1° and Δ_2°, and therefore $S_{\Delta_2^\circ} \cap S_{\Delta_1^\circ} = \varnothing$. Together, these establish (4).

Multiplicites of Vanishing and Resolutions. In discussing what constitutes an allowed $4d$ F-theory geometry $X \to B$, we mentioned certain criteria on multiplicities of vanishing that we now elaborate on. In [15, 16] it was shown that if a Calabi-Yau variety has at worst canonical singularities, then it is at finite distance from the bulk of the moduli space in the Weil-Petersson metric. This criterion is general and therefore applies to elliptic fibrations such as X. The reason that it is physically relevant is that if X has worse singularities than a nearby Calabi-Yau X' that is known to represent a physical configuration, and X is at finite distance in the moduli space from X', we should expect that X is also a physical configuration. This criterion, which we refer to as the Hayakawa-Wang criterion, gives a related criterion by studying elliptic fibrations [17–19]: if $mult_D(f,g) < (4,6)$, $mult_C(f,g) < (8,12)$, and $ord_p(f,g) < (12,18)$ for all divisors $D \subset B$, curves $C \subset B$, and points $p \subset B$, respectively, then X has at worst canonical singularities and is at finite distance in the moduli space due to the Hayakawa-Wang criterion[1]. Here "less than" means that at least one of the multiplicities or orders is strictly less than the given multiplicity or order. We now translate this multiplicity of vanishing (MOV) condition to constraints on the height of the tree.

Proposition 1. *Suppose each leaf $v \in B$ has height $h_v \leq 6$. Then B has no $(4,6)$ divisors.*

Proof. Consider a facet F, which has a unique associated point m_F satisfying $(m_F, \tilde{v}) = -1\ \forall \tilde{v} \in F$; furthermore since $m_F \in \Delta$, $(m_F, v) \geq -1\ \forall v \in \Delta^\circ$. Now suppose $h_v \leq 6/n$, $n \in \mathbb{N}$ for all rays $v = av_1 + bv_2 + cv_3$ in B, with v_i $3d$ cone vertices in B_i . Then $(nm_F, v) \geq -n(a+b+c) = -nh_v \geq -6$ for all rays v and therefore $nm_F \in \Delta_g$. Here we denote Δ_f, Δ_g as the polytopes corresponding to $\Gamma(\mathcal{O}(-4K_B)), \Gamma(\mathcal{O}(-6K_B))$, respectively. If $h_v \leq 6\ \forall v$, then $m_F \in \Delta_g$. This monomial has multiplicity of vanishing $(v, m_F)+6 = 5$ for any v in or above F, which protects v from being a $(4,6)$ divisor. If $h_v \leq 6\ \forall v$ then $m_F \in \Delta_g\ \forall F$ and there is a monomial that prevents each divisor from being $(4,6)$. □

It is also simple to see that in our ensemble, f and g can only vanish to multiplicities less than $(8,12)$ along curves and orders less than $(12,18)$ at points, respectively. Consider any toric curve $C = D_s \cdot D_t \subset B$. Take $v_s = \sum_i a_{i,s} v_i$ and $v_t = \sum_i a_{i,t} v_i$ and define $a := \sum_i a_{i,s}$ and $b := \sum_i a_{i,t}$. Let F be a facet on which or above which v_s and v_t sit, with m_F the dual facet. As an element of Δ_g the associated monomial may be written $s^{\langle m,v_s\rangle+6} t^{\langle m,v_t\rangle+6} \times \dots$, and the monomial vanishes to multiplicity $\langle m,v_s\rangle + \langle m,v_t\rangle + 12 = -a-b+12$ along C. For g to vanish to multiplicity 12 along a curve, this requires $a+b<0$, which cannot happen. A similar argument shows that g cannot vanish to order 18 or higher at points. On the other hand, our ensemble is generated by a series of repeated blowups along curves and points, and a crepant resolution only exists if the MOV is $\geq (4,6)$ for a curve, and $\geq (8,12)$ for a point. One can achieve the required MOV by tuning in complex structure moduli space, but one has to ensure that no infinite distance singularities (as in the above) are introduced in the process. However, it is simple to see that the desired MOV can be achieved, without introducing any disallowed singularities, by simple tuning without turning off the monomial corresponding to m_F, for all F.

7-Branes and Gauge Enhancement. We now prove some useful results that allow us to determine a universal minimal gauge sector in our ensemble, as well as show that NH 7-branes are ubiquitous.

Proposition 2. *Suppose $\exists\ v$ in or above a facet F,i.e. $v = av_1 + bv_2 + cv_3$ with v_i simplex vertices in F, such that $h_v \geq 2$. Then there is a non-Higgsable seven-brane on the divisor associated to each interior point of F.*

Proof. Then $(6m_F, v) = -6h_v \leq -12$ implies $6m_F \notin \Delta_g$. Similarly, $4m_F \notin \Delta_f$. Since any point p interior to F

[1] We thank D. Morrison for discussions on this and related points.

has $(m,p) = -1 \iff m = m_F$ and reflexive polytopes of dimension three are normal, i.e. any $m_f \in \Delta_f$ ($m_f \in \Delta_g$) has $m_f = \sum_i m_i, m_i \in \Delta$ ($m_g = \sum_i m_i, m_i \in \Delta$), it follows that $(m_f,p) = -4 \iff m_f = 4m_F$ and $(m_g,p) = -6 \iff m_g = 6m_F$. Therefore, if there is any tree on F then $4m_F \notin \Delta_f$ and $6m_F \notin \Delta_g$. By normality, for any p interior to F this gives $\nexists m_f \in \Delta_f | (m_f,p) = -4$ and $\nexists m_g \in \Delta_g | (m_g,p) = -6$, and therefore $ord_p(f,g) > (0,0)$, which implies there is a non-Higgsable seven-brane on the divisor associated to p. □

Proposition 3. *Let v be a leaf $v = av_1 + bv_2 + cv_3$ with v_i simplex vertices in F. If the associated divisors $D_{1,2,3}$ carry a non-Higgsable E_8 seven-brane, and if v has height $h_v = 1,2,3,4,5,6$ it also has Kodaira fiber $F_v = II^*, IV^*_{ns}, I^*_{0,ns}, IV_{ns}, II, -$ and gauge group $G_v = E_8, F_4, G_2, SU(2), -, -$, respectively.*

Proof. The height criterion gives $mult_v(g) \le 6 - h_v$. If $v = av_1+bv_2+cv_3$ with v_i each carrying E_8, then $(m_f, v_i) \ge 0, (m_g, v_i) \ge -1$, $\forall m_f \in \Delta_f$ and $\forall m_g \in \Delta_g$. This gives $(m_f, v) \ge 0$, $(m_g, v) \ge -(a+b+c) = -h_v$. Together, we see $mult_v(f) \ge 4$, $mult_v(g) = 6 - h_v$. For $h_v = 1,5,6$ this fixes G_v, but to determine G_v for $h_v = 2,3,4$ we must study the split condition. A necessary condition is that there is one monomial $m_g \in \Delta_g$ such that $(m_g, v) + 6 = 6 - h_v$, and since $m_F \in \Delta_g$ always, where F is the facet in which v_i lie, then $m_g = m_F$. Morever, the monomial m in g associated to m_F must be a perfect square; since $(m_F, v_i) + 6 = 5$, $m \sim x_i^5$ and m is not a perfect square. Therefore the fibers are all non-split. This establishes the result. □

Blowdowns and Oda's Factorization Conjectures. We have obtained all trees from a sequence of blow-ups from an initial triangle on the ground. It may be possible to arrive at additional consistent tree configurations via blowing down at intermediate steps. We did not consider such possibilities, for combinatorial reasons.

However, such questions about mixing blow-ups and blow-downs are the subject of Oda's Weak and Strong Factorization conjectures. The former states that any proper birational morphism $X \dashrightarrow Y$ of complete, non-singular varieties in characteristic zero factors into a sequence of smooth blow-ups and blow-downs. The latter conjectures that the morphism factors into a sequence of successive blow-ups followed by a sequence of successive blow-downs; it is open in dimension 3 and higher.

An interesting physical question arises in this context. By weak Oda, two trees are related by a sequence of blow-ups and blow-downs. However, if each sequence between fixed X and Y gives rise to an intermediate variety X_i with a $(4,6)$ divisor, then the moduli space of four-dimensional F-theory compactifications is disconnected.

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