Intersection Density for Transitive Permutation Groups

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June 2025

Background Information

Let $G \leq \mathrm{Sym}(V)$ be any finite transitive group where V is a finite non-empty set. A subset $\mathcal{F} \subseteq G$ is *intersecting* if given any $\pi, \sigma \in \mathcal{F}$, there exists $v \in V$ such that $\pi(v) = \sigma(v)$. The *intersection density* of G is the rational number:

$$\rho(G) = \frac{\{|\mathcal{F}| : \mathcal{F} \subseteq G \text{ is intersecting}\}}{|G|/|V|}$$

The goal of our dataset is to provide the intersection density for small transitive permutation groups, or our best known bounds if the exact value is unable to be computed with our methods. Along with this, we provide as much useful group data as we can reasonably compute to (hopefully) aid in understanding intersection density in general.

Libraries and Repositories

The objects upon which we construct our dataset come from the Transitive Groups Library by Alex Hulpke. The source code for the scripts which populate the table are available on GitHub. These scripts are not built to be run on any machine, and there is not a guide provided to help in doing so. If you are interested in trying to get them working on your own machine, please contact Cody Solie by email: codymsolie (at) gmail (dot) com.

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Data Columns

Each row of the table corresponds to an individual group. A group can be accessed in GAP or Sagemath by writing the following command, referencing the `degree` and `gap_id` columns in the table:

```
TransitiveGroup(degree, gap_id)
```

The following are brief descriptions of each of the columns in the data table:

Degree

Type:	Integer
Value:	3 or greater
Desc:	Size of (finite) base set which G acts upon

Gap ID

Type:	Integer
Value:	1 or greater
Desc:	Position of G in Transitive Groups library (Hulpke).

Structure Description

Type:	string
Value:	N/A
Desc:	Provides insight to the structure of G; a function of GAP

Upper (Lower) Bound

Type:	Real
Value:	1 or greater
Desc:	Best known upper (lower) bound on intersection density of G. Upper and lower are equal if the exact value is known.

Intersection Density

Type:	Real
Value:	1 or greater, -1 if unknown
Desc:	As defined above. Takes on value of -1 if the exact value was not able to be computed with our methods.

Transitivity

Type: Integer
Value: 1 or greater (all groups listed at least 1-transitive)
Desc: G is *n-transitive* if for any $(v_1, \dots v_n)$ and $(v_1^{'}, \dots v_n^{'})$ with all $v_i, v_i^{'} \in V$, there exists $g \in G$ such that $g \cdot v_i = v_i^{'}$ for all v_i.

Minimally Transitive

Type: Boolean
Value: true or false
Desc: G is *minimally transitive* if there are no transitive proper subgroups of G.

Union

Type: Boolean
Value: true or false
Desc: Takes on a value of true if the derangement graph of G can be expressed as a (finite) union of smaller components. When this occurs, the multiplicity of the largest eigenvalue is strictly greater than one.

Join

Type: Boolean
Value: true or false
Desc: Takes on a value of true if the derangement graph of G can be expressed as a (finite) join of smaller components. When this occurs, the largest eigenvalue, d, and the smallest eigenvalue τ are such that $d - \tau = |G|$.

Complete Multipartite

Type: Boolean
Value: true or false
Desc: Takes on a value of true if the derangement graph of G is a complete multipartite graph. When this occurs, the eigenvalues of the graph have the form: $\{d^{(m_1)}, 0^{(m_2)}, \tau^{(m_3)}\}$

PM Join

Type: Boolean
Value: true or false
Desc: Takes on a value of true if the derangement graph of G is a join of perfect matchings (named as such to save space in the table). When this occurs, the eigenvalues of the graph have the form: $\{d^{(m_1)}, 1^{(m_2)}, -1^{(m_3)}\tau^{(m_4)}\}$ where d is the degree of the vertices, and τ is the least eigenvalue.

Cograph

Type: Boolean
Value: true or false
Desc: Takes on a value of true if the derangement graph of G is a cograph (or complement-reducible graph). This means we can represent the graph in the form: $\bigcup_{\ell_1} \bigcap_{\ell_2} \bigcup_{\ell_3} \cdots K_1$ with $\ell_i \in \mathbb{N}$

EKR

Type: Small int
Value: true, false, or none
Desc: Takes on a value of true (false) if G has (does not have) the *EKR* property. Takes on a value of none if we are unable to determine true/false definitively with our methods. When G has the *EKR* property, the largest intersecting set within G has the same size as the point stabilizer. This is equivalent to having intersection density equal to one. Inclusion of this column allows for easy 'negative' searching, to find the groups *without* the *EKR* property.

Abelian

Type: Boolean
Value: true or false
Desc: True if the group is abelian.

Nilpotent

Type: Boolean
Value: true or false
Desc: True if the group is nilpotent.

Primitive

Type:	Boolean
Value:	true or false
Desc:	True if the group is primitive.

Eigenvalues

Type:	List of pairs
Value:	(eigenvalue, multiplicity) pairs
Desc:	Clicking the expand button will reveal the eigenvalues of the derangement graph, sorted in decreasing order.