Exercise Solutions for Math 20

Sum, Difference, Cofunction, Double Measure Identities

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December 6, 2024

1 Evaluate the following without using a calculator.

1.1 $\sin\left(\frac{19\pi}{12}\right)$

$\Rightarrow \sin\left(\frac{10\pi}{12} + \frac{9\pi}{12}\right)$	
$\Rightarrow \sin\left(\frac{5\pi}{6} + \frac{3\pi}{4}\right)$	
$\Rightarrow \sin\left(\frac{5\pi}{6}\right)\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{5\pi}{6}\right)\sin\left(\frac{3\pi}{4}\right)$	$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
$\Rightarrow (\frac{1}{2})(-\frac{\sqrt{2}}{2}) + (-\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2})$	
$\Rightarrow -\frac{\sqrt{2}}{4} - \frac{\sqrt{3}\sqrt{2}}{4}$	
$\Rightarrow -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$	
$\Rightarrow -\frac{\sqrt{2}+\sqrt{6}}{4}$	Final answer.
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1.2 $\cos(33°)\cos(27°) - \sin(33°)\sin(27°)$

$\Rightarrow \cos(33° + 27°)$	$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
$\Rightarrow \cos(60°)$	
$\Rightarrow \frac{1}{2}$	Final answer.
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2 If $\cot(\theta) = -\frac{5}{12}$ and $\theta \in (-\frac{\pi}{2}, 0)$, find $\cos\left(\theta + \frac{\pi}{3}\right)$.

$\Rightarrow O = -12, A = 5$	$\cot(\theta) = \frac{A}{O}$, and since $\theta \in (-\frac{\pi}{2}, 0)$, we are in QIV. Therefore, $O < 0$ and $A > 0$.
$\Rightarrow H = \sqrt{(-12)^2 + 5^2}$	$H = \sqrt{O^2 + A^2}$
$\Rightarrow H = \sqrt{144 + 25}$	
$\Rightarrow H = \sqrt{169}$	
$\Rightarrow H = 13$	
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \cos(\theta)\cos\left(\frac{\pi}{3}\right) - \sin(\theta)\sin\left(\frac{\pi}{3}\right)$	$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \frac{5}{13}\cos\left(\frac{\pi}{3}\right) - \sin(\theta)\sin\left(\frac{\pi}{3}\right)$	$\cos(\theta) = \frac{A}{H}$
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \frac{5}{13}\cos\left(\frac{\pi}{3}\right) + \frac{12}{13}\sin\left(\frac{\pi}{3}\right)$	$\sin(\theta) = \frac{O}{H}$
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = (\frac{5}{13})(\frac{1}{2}) + (\frac{12}{13})(\frac{\sqrt{3}}{2})$	
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \frac{5}{26} + \frac{12\sqrt{3}}{26}$	
$\Rightarrow \cos\left(\theta + \frac{\pi}{3}\right) = \frac{5+12\sqrt{3}}{26}$	
	$\blacksquare$

3 Given $\cos(\alpha) = \frac{3}{5}$ where α lies in the interval $(\frac{\pi}{2}, 2\pi)$, find the following.

3.1 $\sin(2\alpha)$

$\Rightarrow A = 3, H = 5$	$\cos(\theta) = \frac{A}{H}$, and since $\cos(\alpha) > 0$ and $\alpha \in (\frac{\pi}{2}, 2\pi)$, we are in QIV. Therefore, $O < 0$ and $A > 0$.
$\Rightarrow O = -\sqrt{5^2 - 3^2}$	From $H = \sqrt{A^2 + O^2}$, we can derive $O = \pm\sqrt{H^2 - A^2}$. Remember that in this case, $A > 0$.
$\Rightarrow O = -\sqrt{25 - 9}$	
$\Rightarrow O = -\sqrt{16}$	
$\Rightarrow O = -4$	
$\Rightarrow \sin(\alpha) = -\frac{4}{5}$	$\sin(\theta) = \frac{O}{H}$
$\Rightarrow \sin(2\alpha) = 2(-\frac{4}{5})(\frac{3}{5})$	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\Rightarrow \sin(2\alpha) = 2(-\frac{12}{25})$	
$\Rightarrow \sin(2\alpha) = -\frac{24}{25}$	Final answer.
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3.2 $\sin(3\alpha)$

$\Rightarrow \cos(2\alpha) = (\frac{3}{5})^2 - (-\frac{4}{5})^2$	$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
$\Rightarrow \cos(2\alpha) = \frac{9}{25} - \frac{16}{25}$	
$\Rightarrow \cos(2\alpha) = -\frac{7}{25}$	
$\Rightarrow \sin(3\alpha) = \sin(\alpha + 2\alpha)$	
$\Rightarrow \sin(\alpha + 2\alpha) = (-\frac{4}{5})(-\frac{7}{25}) + (\frac{3}{5})(-\frac{24}{25})$	$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$
$\Rightarrow \sin(\alpha + 2\alpha) = \frac{28}{125} - \frac{72}{125}$	
$\Rightarrow \sin(\alpha + 2\alpha) = \frac{44}{125}$	
$\Rightarrow \sin(3\alpha) = \frac{44}{125}$	Final answer.
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4 Establish the following identities.

4.1 $\frac{\sin(\alpha-\beta)}{\cos(\alpha)\cos(\beta)} = \tan(\alpha) - \tan(\beta)$

$$\Rightarrow \tan(\alpha) - \tan(\beta) = \frac{\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)} \qquad \sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\Rightarrow \tan(\alpha) - \tan(\beta) = \frac{\sin(\alpha)\cos(\beta)}{\cos(\alpha)\cos(\beta)} - \frac{\cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta)}$$

$$\Rightarrow \tan(\alpha) - \tan(\beta) = \frac{\sin(\alpha)}{\cos(\alpha)} - \frac{\sin(\beta)}{\cos(\beta)}$$

$$\Rightarrow \tan(\alpha) - \tan(\beta) = \tan(\alpha) - \tan(\beta)$$

Final answer. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

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4.2 $\csc(x+y) = \frac{\csc(x)\csc(y)}{\csc(x)\cos(x)+\csc(y)\cos(y)}$

$$\Rightarrow \csc(x+y) = \frac{(\frac{1}{\sin(x)})(\frac{1}{\sin(y)})}{(\frac{1}{\sin(x)})\cos(x)+(\frac{1}{\sin(y)})\cos(y)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\Rightarrow \csc(x+y) = \frac{\frac{1}{\sin(x)\sin(y)}}{\frac{\cos(x)}{\sin(x)}+\frac{\cos(y)}{\sin(y)}}$$

$$\Rightarrow \csc(x+y) = \frac{\frac{1}{\sin(x)\sin(y)}}{\frac{\cos(x)\sin(y)}{\sin(x)\sin(y)}+\frac{sin(x)\cos(y)}{\sin(x)\sin(y)}}$$

$$\Rightarrow \csc(x+y) = \frac{\frac{1}{\sin(x)\sin(y)}}{\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\sin(x)\sin(y)}}$$

$$\Rightarrow \csc(x+y) = \frac{1}{\sin(x)\cos(y)+\cos(x)\sin(y)}$$

$$\Rightarrow \csc(x+y) = \frac{1}{\sin(x+y)} \qquad \sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\Rightarrow \csc(x+y) = \csc(x+y)$$

Final answer. $\csc(\theta) = \frac{1}{\sin(\theta)}$

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4.3 $\frac{\sin(2\theta)+\sin(\theta)}{\cos(2\theta)+\cos(\theta)+1} = \tan(\theta)$

$$\Rightarrow \tan(\theta) = \frac{2\sin(\theta)\cos(\theta)+\sin(\theta)}{\cos(2\theta)+\cos(\theta)+1} \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\Rightarrow \tan(\theta) = \frac{2\sin(\theta)\cos(\theta)+\sin(\theta)}{2\cos^2(\theta)-1+\cos(\theta)+1} \qquad \cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\Rightarrow \tan(\theta) = \frac{2\sin(\theta)\cos(\theta)+\sin(\theta)}{2\cos^2(\theta)+\cos(\theta)}$$

$$\Rightarrow \tan(\theta) = \frac{\sin(\theta)(2\cos(\theta)+1)}{\cos(\theta)(2\cos(\theta)+1)}$$

$$\Rightarrow \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\Rightarrow \tan(\theta) = \tan(\theta)$$

Final answer. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

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4.4 $\sec(2x) = \frac{\tan(x)+\cot(x)}{\cot(x)-\tan(x)}$

$$\Rightarrow \sec(2x) = \frac{\frac{\sin(x)}{\cos(x)}+\cot(x)}{\cot(x)-\frac{\sin(x)}{\cos(x)}} \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

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$\Rightarrow \sec(2x) = \frac{\frac{\sin(x)}{\cos(x)}+\frac{\cos(x)}{\sin(x)}}{\frac{\cos(x)}{\sin(x)}-\frac{\sin(x)}{\cos(x)}}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\Rightarrow \sec(2x) = \frac{\frac{\sin^2(x)}{\sin(x)\cos(x)}+\frac{\cos^2(x)}{\sin(x)\cos(x)}}{\frac{\cos^2(x)}{\sin(x)\cos(x)}-\frac{\sin^2(x)}{\sin(x)\cos(x)}}$	
$\Rightarrow \sec(2x) = \frac{\frac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)}}{\frac{\cos^2(x)-\sin^2(x)}{\sin(x)\cos(x)}}$	
$\Rightarrow \sec(2x) = \frac{\sin^2(x)+\cos^2(x)}{\cos^2(x)-\sin^2(x)}$	
$\Rightarrow \sec(2x) = \frac{1}{\cos^2(x)-\sin^2(x)}$	$\sin^2(\theta)+\cos^2(\theta) = 1$
$\Rightarrow \sec(2x) = \frac{1}{\cos(2x)}$	$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta)$
$\Rightarrow \sec(2x) = \sec(2x)$	Final answer. $\sec(\theta) = \frac{1}{\cos(\theta)}$
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4.5 $\csc(2\beta) - \cot(2\beta) = \tan(\beta)$

$\Rightarrow \tan(\beta) = \frac{1}{\sin(2\beta)} - \cot(2\beta)$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\Rightarrow \tan(\beta) = \frac{1}{\sin(2\beta)} - \frac{\cos(2\beta)}{\sin(2\beta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\Rightarrow \tan(\beta) = \frac{1-\cos(2\beta)}{\sin(2\beta)}$	
$\Rightarrow \tan(\beta) = \frac{1-(\cos^2(\beta)-\sin^2(\beta))}{\sin(2\beta)}$	$\cos(2\theta) = \cos^2(\theta)-\sin^2(\theta)$
$\Rightarrow \tan(\beta) = \frac{1-\cos^2(\beta)+\sin^2(\beta)}{\sin(2\beta)}$	
$\Rightarrow \tan(\beta) = \frac{1-\cos^2(\beta)+\sin^2(\beta)}{2\sin(\beta)\cos(\beta)}$	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\Rightarrow \tan(\beta) = \frac{\sin^2(\beta)+\sin^2(\beta)}{2\sin(\beta)\cos(\beta)}$	$\sin^2(\theta)+\cos^2(\theta) = 1$
$\Rightarrow \tan(\beta) = \frac{2\sin^2(\beta)}{2\sin(\beta)\cos(\beta)}$	
$\Rightarrow \tan(\beta) = \frac{\sin(\beta)}{\cos(\beta)}$	
$\Rightarrow \tan(\beta) = \tan(\beta)$	Final answer. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
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