Exercise Solutions for Math 20

Conics (Parabola and Ellipse)

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1.1 Determine the vertex and orientation of the following parabolas.

1.1.a $4y^2 + 4y + x = 2$

$\Rightarrow 4y^2 + 4y = -x + 2$	Isolate y.
$\Rightarrow y^2 + y = -\frac{x}{4} + \frac{2}{4}$	
$\Rightarrow y^2 + y = -\frac{x}{4} + \frac{1}{2}$	
$\Rightarrow y^2 + y + \frac{1}{4} = -\frac{x}{4} + \frac{1}{2} + \frac{1}{4}$	Complete the square.
$\Rightarrow \left(y + \frac{1}{2}\right)^2 = -\frac{x}{4} + \frac{3}{4}$	
$\Rightarrow \left(y + \frac{1}{2}\right)^2 = -\frac{1}{4}(x - 3)$	
$\Rightarrow \left(y + \frac{1}{2}\right)^2 = 4(-\frac{1}{16})(x - 3)$	
$\Rightarrow$ Opening leftward, $(h, k) = (3, -\frac{1}{2})$	Final answer.
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1.1.b $x^2 - 6x - 2y = 7$

$\Rightarrow x^2 - 6x = 2y + 7$	Isolate x.
$\Rightarrow x^2 - 6x + 9 = 2y + 7 + 9$	Complete the square.
$\Rightarrow (x - 3)^2 = 2y + 16$	
$\Rightarrow (x - 3)^2 = 2(y + 8)$	
$\Rightarrow (x - 3)^2 = 4(\frac{1}{2})(y + 8)$	
$\Rightarrow$ Opening upward, $(h, k) = (3, -8)$	Final answer.
	■

1.1.c $2y^2 - 6y - 9x = 0$

$\Rightarrow 2y^2 - 6y = 9x$	Isolate y.
$\Rightarrow y^2 - 3y = \frac{9}{2}x$	
$\Rightarrow y^2 - 3y + \frac{9}{4} = \frac{9}{2}x + \frac{9}{4}$	Complete the square.
$\Rightarrow \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}x + \frac{9}{4}$	
$\Rightarrow \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}(x + \frac{9}{4} \cdot \frac{2}{9})$	
$\Rightarrow \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}(x + \frac{18}{36})$	
$\Rightarrow \left(y - \frac{3}{2}\right)^2 = \frac{9}{2}(x + \frac{1}{2})$	
$\Rightarrow \left(y - \frac{3}{2}\right)^2 = 4(\frac{9}{8})(x + \frac{1}{2})$	
$\Rightarrow$ Opening rightward, $(h, k) = (-\frac{1}{2}, \frac{3}{2})$	Final answer.
	■

1.2 Sketch the graph of the following parabolas.

1.2.a $3y^2 = 8x$

$\Rightarrow y^2 = 4(\frac{2}{3})x$	Rewrite in standard form.
$\Rightarrow V = (0, 0)$	Identify important objects; this is a parabola opening rightward. $V = (h, k)$
$\Rightarrow F = (\frac{2}{3}, 0)$	$F = (h + p, k)$
$\Rightarrow B_1 = (\frac{2}{3}, -\frac{4}{3})$	$B_1 = (h + p, k - 2p)$
$\Rightarrow B_2 = (\frac{2}{3}, \frac{4}{3})$	$B_2 = (h + p, k + 2p)$
$\Rightarrow D \Rightarrow x = -\frac{2}{3}$	$D \Rightarrow x = h - p$
$\Rightarrow$ See Figure 1.	Final answer. Graph the parabola. ■

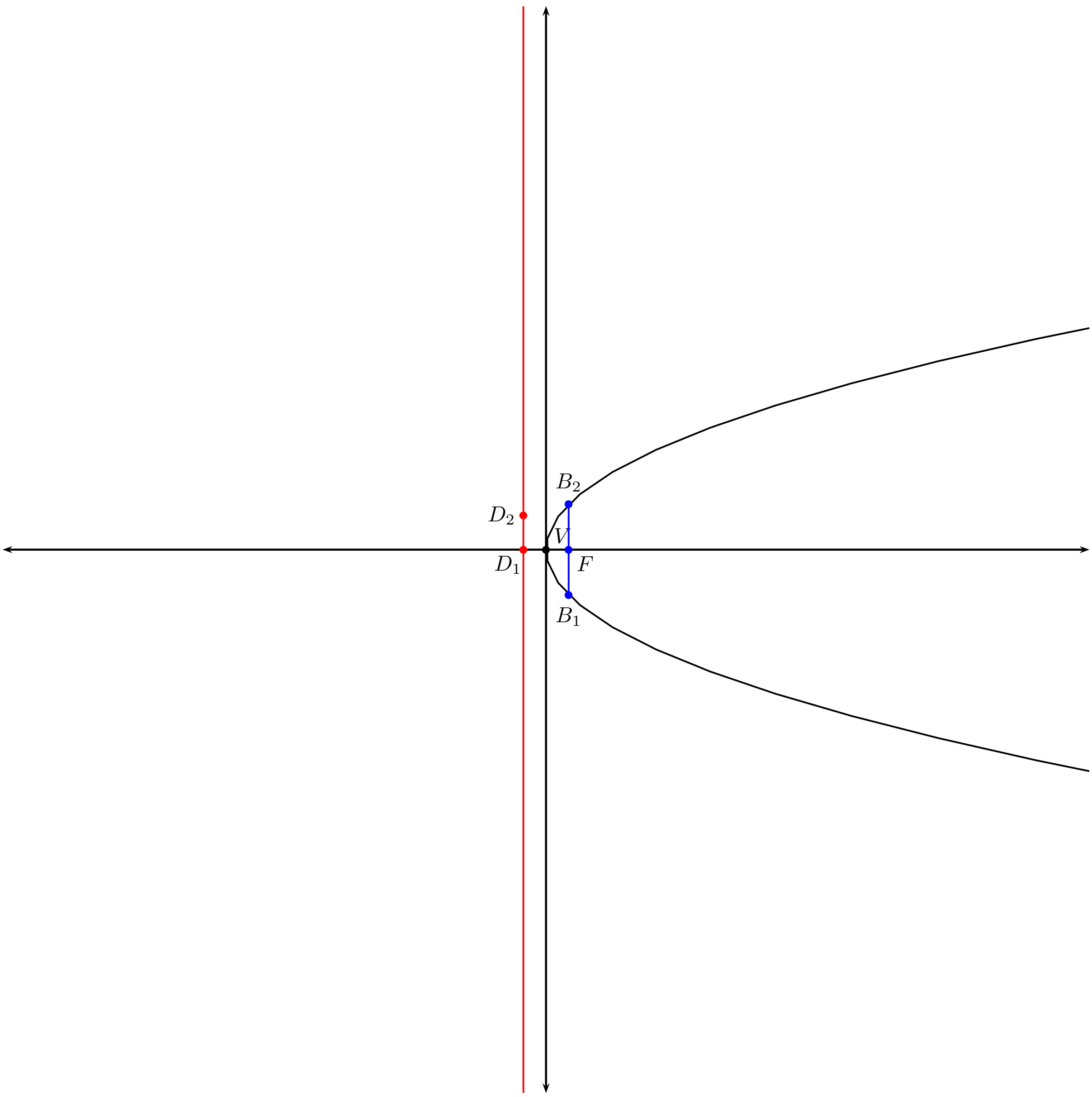

Figure 1. Graph of $y^2 = 4(\frac{2}{3})x$.

1.2.b $x^2 - 8x + 4y = -10$

$\Rightarrow x^2 - 8x = -4y - 10$	Rewrite in standard form.
$\Rightarrow x^2 - 8x + 16 = -4y - 10 + 16$	Complete the square.
$\Rightarrow (x-4)^2 = -4y + 6$	
$\Rightarrow (x-4)^2 = 4(-1)(y - \frac{3}{2})$	
$\Rightarrow V = (4, \frac{3}{2})$	Identify important objects; this is a parabola opening downward. $V = (h, k)$
$\Rightarrow F = (4, \frac{1}{2})$	$F = (h, k + p)$
$\Rightarrow B_1 = (6, \frac{1}{2})$	$B_1 = (h - 2p, k + p)$
$\Rightarrow B_2 = (2, \frac{1}{2})$	$B_2 = (h + 2p, k + p)$
$\Rightarrow D \Rightarrow y = \frac{5}{2}$	$D \Rightarrow y = k - p$
$\Rightarrow$ See Figure 2.	Final answer. Graph the parabola. ■

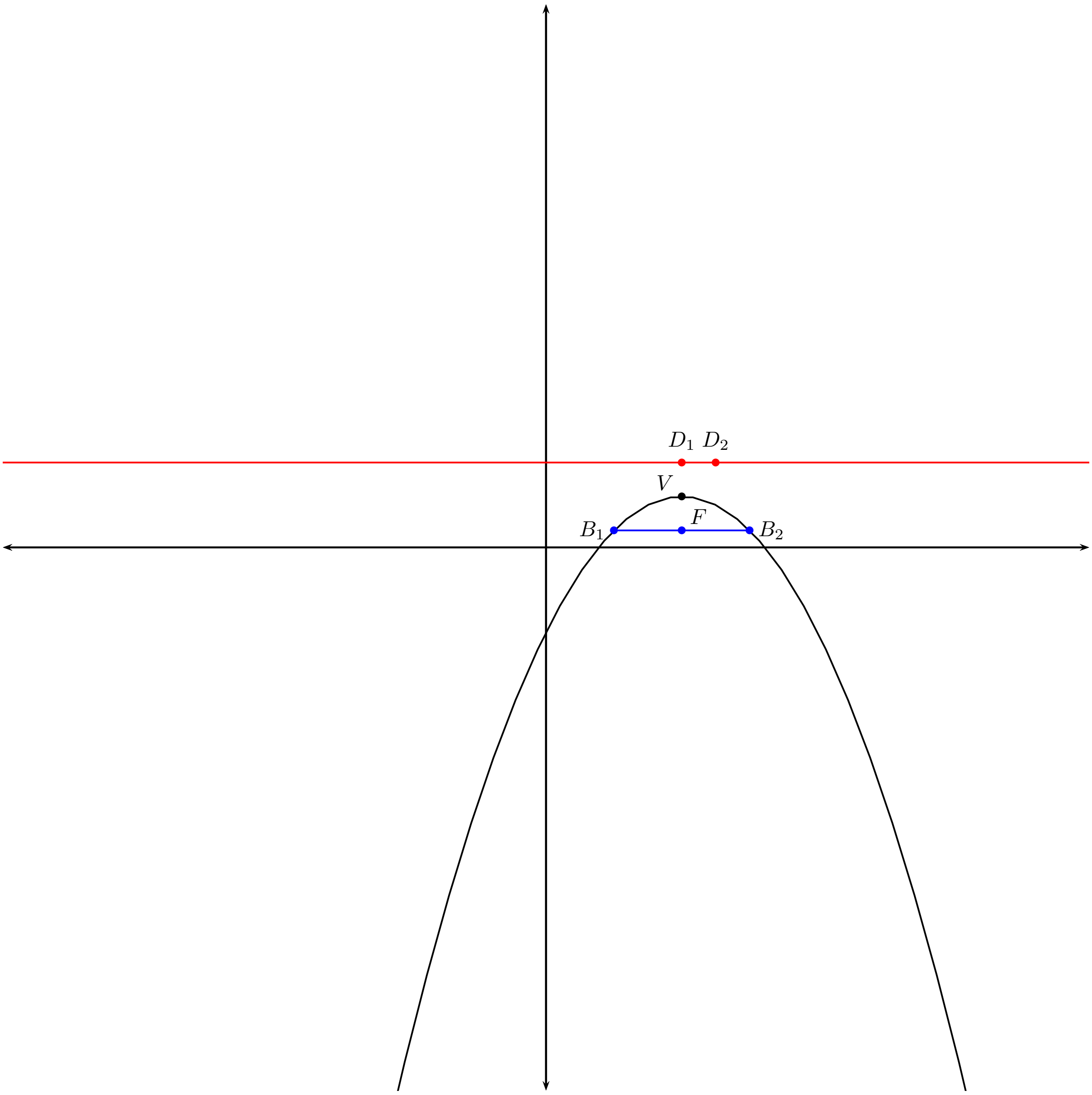

Figure 2. Graph of $(x-4)^2 = 4(-1)(y-\frac{3}{2})$.

1.3 Sketch the graph of $y = -x^2 + 6x - 8$. Label the vertex, x- and y-intercept(s).

$\Rightarrow -x^2 + 6x = y + 8$	Rewrite in standard form.
$\Rightarrow x^2 - 6x = -y - 8$	
$\Rightarrow x^2 - 6x + 9 = -y - 8 + 9$	Complete the square.
$\Rightarrow (x-3)^2 = -y + 1$	
$\Rightarrow (x-3)^2 = 4(-\frac{1}{4})(y-1)$	
$\Rightarrow (x-3)^2 = 4(-\frac{1}{4})(-1)$	Find the x-intercepts.
$\Rightarrow (x-3)^2 = 4(\frac{1}{4})$	
$\Rightarrow (x-3)^2 = 1$	
$\Rightarrow x - 3 = \pm 1$	
$\Rightarrow x = \pm 1 + 3$	
$\Rightarrow x = 1 + 3, x = -1 + 3$	
$\Rightarrow x_i \in \{2, 4\}$	
$\Rightarrow (0-3)^2 = 4(-\frac{1}{4})(y-1)$	Find the y-intercepts.
$\Rightarrow (-3)^2 = 4(-\frac{1}{4})(y-1)$	
$\Rightarrow 9 = -(y-1)$	
$\Rightarrow 9 = -y + 1$	
$\Rightarrow y = 1 - 9$	
$\Rightarrow y_i = -8$	
$\Rightarrow$ See Figure 3.	Final answer. Graph the parabola. ■

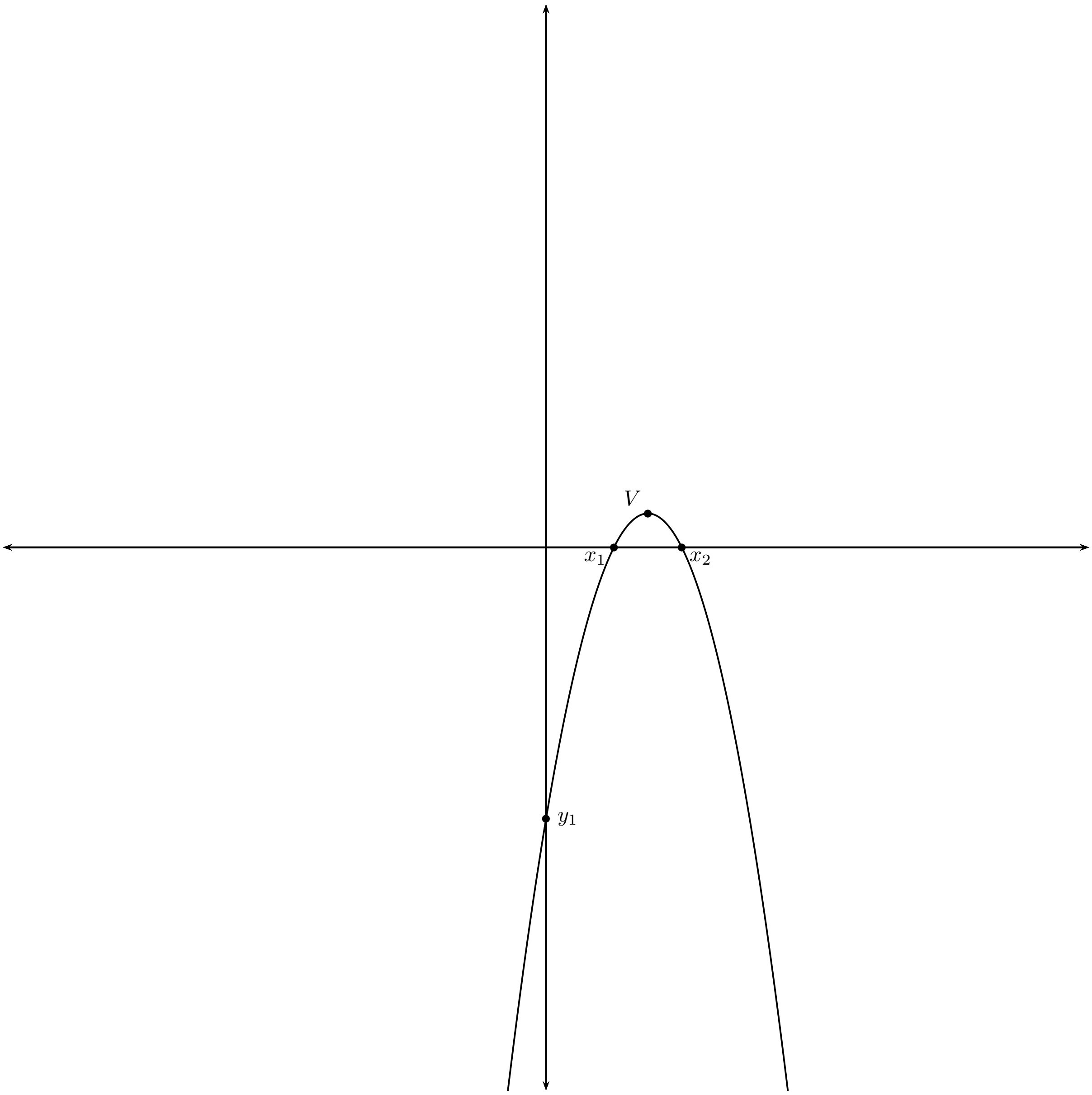

Figure 3. Graph of $(x-3)^2 = 4(-\frac{1}{4})(y-1)$ with x- and y-intercepts.

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2.1 Sketch the graph of the following ellipses.

2.1.a $\frac{x^2}{4} + \frac{(y-1)^2}{9} = 1$

$\Rightarrow \frac{x^2}{2^2} + \frac{(y-1)^2}{3^2} = 1$	Rewrite in standard form.
$\Rightarrow C = (0, 1)$	Identify important objects; this is an ellipse with a vertical major axis. $C = (h, k)$
$\Rightarrow c = \sqrt{5}$	$c = \sqrt{b^2 - a^2}$
$\Rightarrow V_1 = (0, -2)$	$V_1 = (h, k - b)$
$\Rightarrow V_2 = (0, 4)$	$V_2 = (h, k + b)$
$\Rightarrow B_1 = (-2, 1)$	$B_1 = (h - a, k)$
$\Rightarrow B_2 = (2, 1)$	$B_2 = (h + a, k)$
$\Rightarrow F_1 = (0, 1 - \sqrt{5})$	$F_1 = (h, k - c)$
$\Rightarrow F_2 = (0, 1 + \sqrt{5})$	$F_2 = (h, k + c)$
$\Rightarrow$ See Figure 4.	Final answer. Graph the ellipse. ■

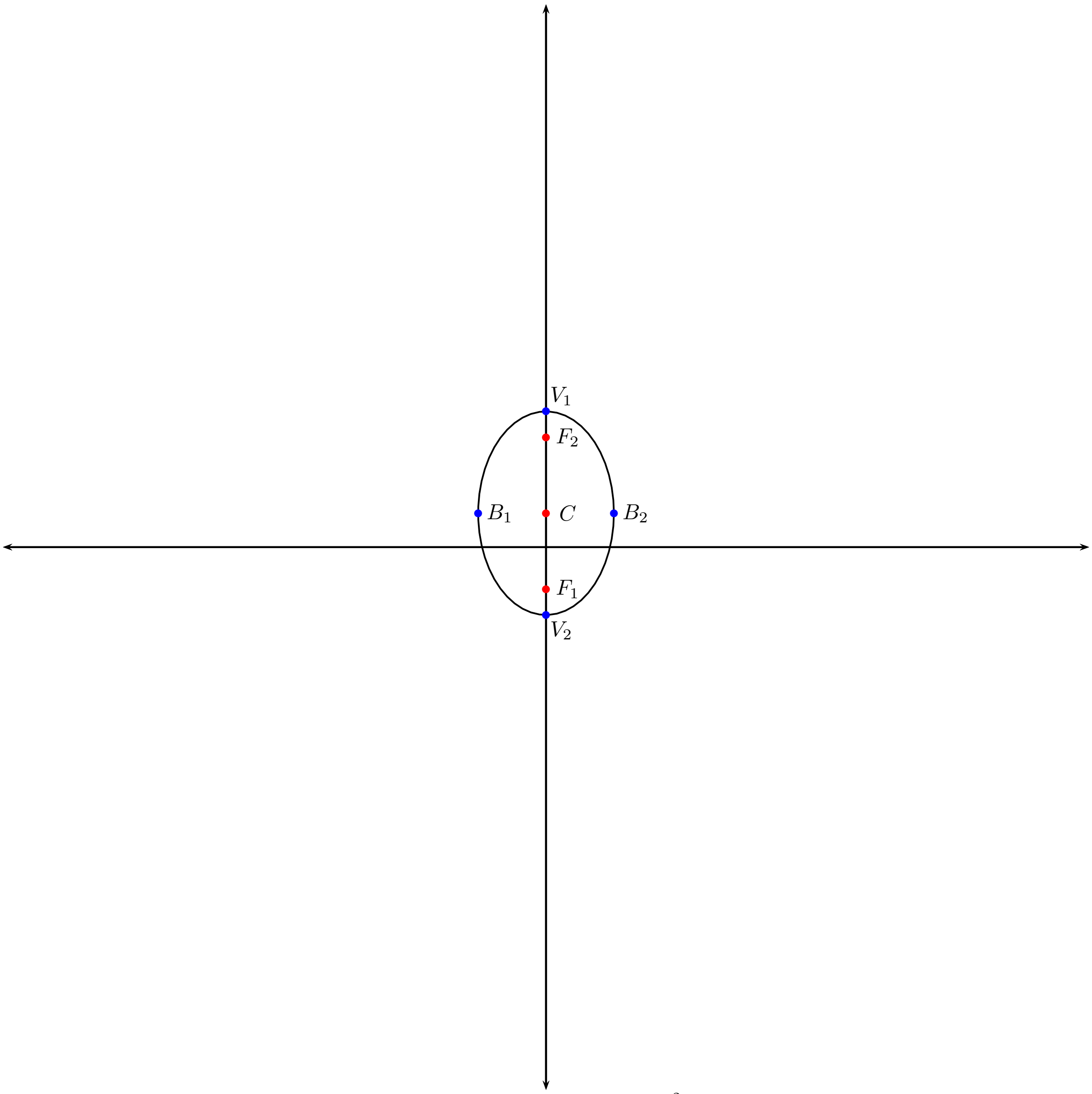

Figure 4. Graph of $\frac{x^2}{2^2} + \frac{(y-1)^2}{3^2} = 1$.

2.1.b $\frac{(x-3)^2}{25} + \frac{y^2+4y+4}{9} = 1$

$\Rightarrow \frac{(x-3)^2}{25} + \frac{(y+2)^2}{9} = 1$	Factor by grouping.
$\Rightarrow \frac{(x-3)^2}{5^2} + \frac{(y+2)^2}{3^2} = 1$	Rewrite in standard form.
$\Rightarrow C = (3, -2)$	Identify important objects; this is an ellipse with a horizontal major axis. $C = (h, k)$
$\Rightarrow c = 4$	$c = \sqrt{a^2 - b^2}$
$\Rightarrow V_1 = (-2, -2)$	$V_1 = (h - a, k)$
$\Rightarrow V_2 = (8, -2)$	$V_2 = (h + a, k)$
$\Rightarrow B_1 = (3, -5)$	$B_1 = (h, k - b)$
$\Rightarrow B_2 = (3, 1)$	$B_2 = (h, k + b)$
$\Rightarrow F_1 = (-1, -2)$	$F_1 = (h - c, k)$
$\Rightarrow F_2 = (7, -2)$	$F_2 = (h + c, k)$
$\Rightarrow$ See Figure 5.	Final answer. Graph the ellipse. ■

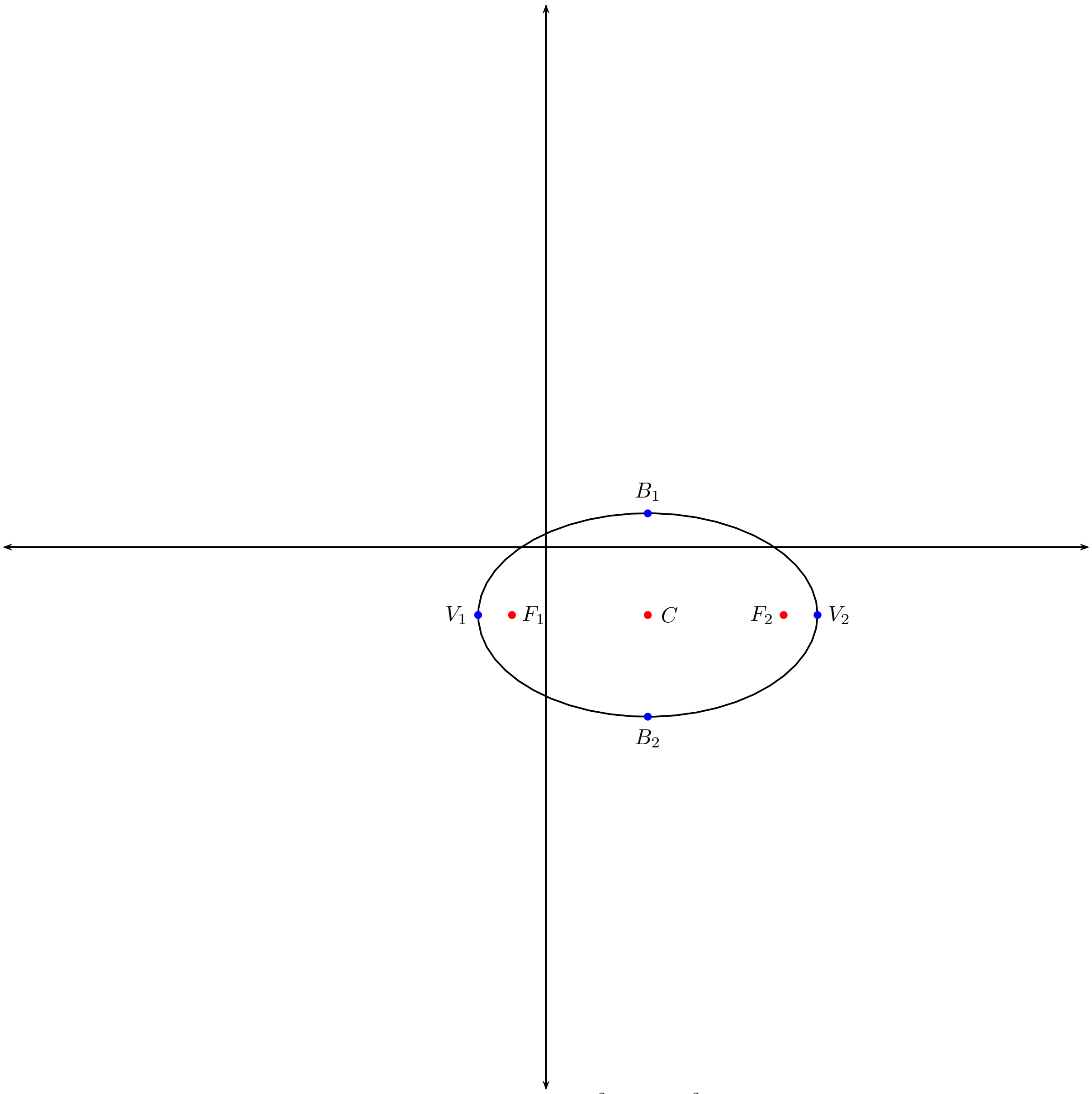

Figure 5. Graph of $\frac{(x-3)^2}{5^2} + \frac{(y+2)^2}{3^2} = 1$.

2.1.c $2x^2 + 3y^2 + 16x - 18y = 13$

$$\begin{aligned}
&\Rightarrow 2x^2 + 16x + 3y^2 - 18y = 13 && \text{Group terms.}\\
&\Rightarrow 2(x^2 + 8x) + 3(y^2 - 6y) = 13 &&\\
&\Rightarrow 2(x^2 + 8x + 16) + 3(y^2 - 6y) = 13 + 2(16) && \text{Complete the square.}\\
&\Rightarrow 2(x^2 + 8x + 16) + 3(y^2 - 6y) = 13 + 32 &&\\
&\Rightarrow 2(x^2 + 8x + 16) + 3(y^2 - 6y) = 45 &&\\
&\Rightarrow 2(x+4)^2 + 3(y^2 - 6y) = 45 &&\\
&\Rightarrow 2(x+4)^2 + 3(y^2 - 6y + 9) = 45 + 3(9) && \text{Complete the square.}\\
&\Rightarrow 2(x+4)^2 + 3(y^2 - 6y + 9) = 45 + 27 &&\\
&\Rightarrow 2(x+4)^2 + 3(y^2 - 6y + 9) = 72 &&\\
&\Rightarrow 2(x+4)^2 + 3(y-3)^2 = 72 &&\\
&\Rightarrow \tfrac{2(x+4)^2}{72} + \tfrac{3(y-3)^2}{72} = 1 &&\\
&\Rightarrow \tfrac{(x+4)^2}{36} + \tfrac{(y-3)^2}{24} = 1 &&\\
&\Rightarrow \tfrac{(x+4)^2}{6^2} + \tfrac{(y-3)^2}{(\sqrt{24})^2} = 1 && \text{Rewrite in standard form.}\\
&\Rightarrow \tfrac{(x+4)^2}{6^2} + \tfrac{(y-3)^2}{(\sqrt{4}\sqrt{6})^2} = 1 &&\\
&\Rightarrow \tfrac{(x+4)^2}{6^2} + \tfrac{(y-3)^2}{(2\sqrt{6})^2} = 1 &&
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow C = (-4, 3) && \text{Identify important objects; this is an ellipse with a horizontal major axis. } C = (h, k)\\
&\Rightarrow c = 2\sqrt{3} && c = \sqrt{a^2 - b^2}\\
&\Rightarrow V_1 = (-10, 3) && V_1 = (h - a, k)\\
&\Rightarrow V_2 = (2, 3) && V_2 = (h + a, k)\\
&\Rightarrow B_1 = (-4, 3 - 2\sqrt{6}) && B_1 = (h, k - b)\\
&\Rightarrow B_2 = (-4, 3 + 2\sqrt{6}) && B_2 = (h, k + b)\\
&\Rightarrow F_1 = (-4 - 2\sqrt{3}, 3) && F_1 = (h - c, k)\\
&\Rightarrow F_2 = (-4 + 2\sqrt{3}, 3) && F_2 = (h + c, k)
\end{aligned}$$

⇒ See Figure 6. Final answer. Graph the ellipse.

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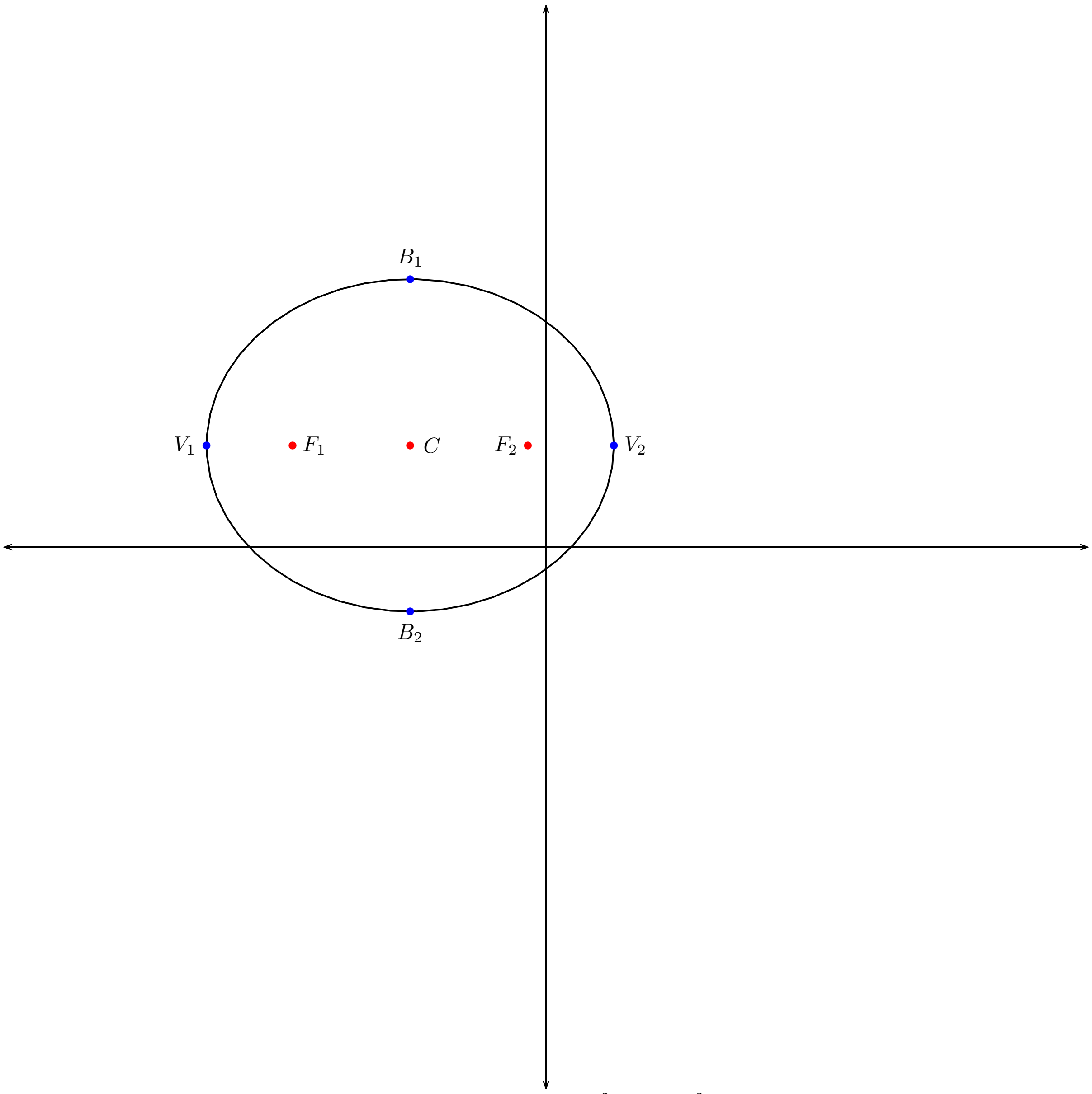

Figure 6. Graph of $\frac{(x+4)^2}{6^2} + \frac{(y-3)^2}{(2\sqrt{6})^2} = 1$

2.2 Find an equation of the parabola that opens downward and whose vertex and focus are the vertices of the ellipse $4(x-2)^2+(y+1)^2=1$

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$\Rightarrow \frac{(x-2)^2}{\frac{1}{4}}+\frac{(y+1)^2}{1}=1$	Rewrite in standard form.
$\Rightarrow \frac{(x-2)^2}{\frac{1}{2}^2}+\frac{(y+1)^2}{1^2}=1$	Since $a<b$, this is an ellipse with a vertical major axis.
$\Rightarrow V_1=(2,-2)$	$V_1=(h,k-b)$
$\Rightarrow V_2=(2,0)$	$V_2=(h,k+b)$
$\Rightarrow V=(2,0), F=(2,-2)$	Derive the vertex and focus. Since this is a parabola opening downward, the lower point is the focus.
$\Rightarrow 0+p=-2$	Derive p; $F=(h,k+p)$
$\Rightarrow p=-2$	
$\Rightarrow (x-2)^2=4(-2)y$	Final answer. Write the parabola equation using (h,k) and p. ■