Exercise Solutions for Math 20

Half-Angle, Product-to-Sum, Sum-to-Product Identities

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1 Given $\sin(\beta) = \frac{3}{5}$ where β lie in the interval $(\frac{\pi}{2}, 2\pi)$, find $\cos\left(\frac{\beta}{2}\right)$

$\Rightarrow \beta \in (\frac{\pi}{2}, \pi)$	Since $\sin(\beta)$ is positive, β must be in $(0, \pi)$. Combining the two intervals, we can see that we are in QII.
$\Rightarrow \cos(\beta) = \pm\sqrt{1 - \sin^2(\beta)}$ $\Rightarrow \cos(\beta) = \pm\sqrt{1 - (\frac{3}{5})^2}$ $\Rightarrow \cos(\beta) = \pm\sqrt{1 - \frac{9}{25}}$ $\Rightarrow \cos(\beta) = \pm\sqrt{\frac{25}{25} - \frac{9}{25}}$ $\Rightarrow \cos(\beta) = \pm\sqrt{\frac{16}{25}}$ $\Rightarrow \cos(\beta) = \pm\frac{4}{5}$ $\Rightarrow \cos(\beta) = -\frac{4}{5}$	Find $\cos(\theta)$ using Pythagoras. Since we are in QII, $\cos(\beta)$ is negative.
$\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1+\cos(\beta)}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1-\frac{4}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{\frac{5}{5}-\frac{4}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{\frac{1}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1}{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{1}{\sqrt{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{\sqrt{10}}{10}$	$\cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$ Rationalize.
$\Rightarrow \cos\left(\frac{\beta}{2}\right) = \frac{\sqrt{10}}{10}$	Final answer. Given $\beta \in (\frac{\pi}{2}, \pi)$, $\frac{\beta}{2} \in (\frac{\pi}{4}, \frac{\pi}{2})$, which is in QI. Therefore $\cos\left(\frac{\beta}{2}\right)$ is positive. ■

2 Express the following as indicated then evaluate if possible.

2.1 $\sin(105°)\cos(15°)$ as a sum.

$\Rightarrow \sin(105°)\cos(15°) = \frac{1}{2}(\sin(105° + 15°) + \sin(105° - 15°))$	$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$
$\Rightarrow \sin(105°)\cos(15°) = \frac{1}{2}(\sin(120°) + \sin(90°))$	
$\Rightarrow \sin(105°)\cos(15°) = \frac{1}{2}(\frac{\sqrt{3}}{2} + 1)$	
$\Rightarrow \sin(105°)\cos(15°) = \frac{\sqrt{3}}{4} + \frac{1}{2}$	
$\Rightarrow \sin(105°)\cos(15°) = \frac{\sqrt{3}}{4} + \frac{2}{4}$	
$\Rightarrow \sin(105°)\cos(15°) = \frac{2+\sqrt{3}}{4}$	Final answer. $\blacksquare$

2.2 $\cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right)$ as a product.

$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\cos\left(\frac{\frac{5\pi}{24} + \frac{\pi}{24}}{2}\right)\cos\left(\frac{\frac{5\pi}{24} - \frac{\pi}{24}}{2}\right)$	$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\cos\left(\frac{\frac{6\pi}{24}}{2}\right)\cos\left(\frac{\frac{4\pi}{24}}{2}\right)$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\cos\left(\frac{\frac{\pi}{4}}{2}\right)\cos\left(\frac{\frac{\pi}{6}}{2}\right)$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}}\sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}}$	$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\sqrt{\frac{2}{4} + \frac{\sqrt{2}}{4}}\sqrt{\frac{2}{4} + \frac{\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2\sqrt{\frac{2+\sqrt{2}}{4}}\sqrt{\frac{2+\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2(\frac{\sqrt{2+\sqrt{2}}}{2})(\frac{\sqrt{2+\sqrt{3}}}{2})$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = \sqrt{2+\sqrt{2}}(\frac{\sqrt{2+\sqrt{3}}}{2})$	Final answer. $\blacksquare$

2.3 $\sin(3x)\cos(2x)$ as a sum.

$\Rightarrow \sin(3x)\cos(2x) = \frac{1}{2}(\sin(3x + 2x) + \sin(3x - 2x))$	$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$
$\Rightarrow \sin(3x)\cos(2x) = \frac{1}{2}(\sin(5x) + \sin(x))$	
$\Rightarrow \sin(3x)\cos(2x) = \frac{\sin(5x)+\sin(x)}{2}$	Final answer. $\blacksquare$

3 Establish the following identities.

3.1 $\frac{\cos(3x)+\cos(x)}{\sin(3x)+\sin(x)} = \frac{\cot(x)-\tan(x)}{2}$

$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)}{\sin(3x)+\sin(x)}$	$\cos(a)+\cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right)}{\sin(3x)+\sin(x)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos(2x)\cos(x)}{\sin(3x)+\sin(x)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos(2x)\cos(x)}{2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)}$	$\sin(a)+\sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos(2x)\cos(x)}{2\sin\left(\frac{4x}{2}\right)\cos\left(\frac{2x}{2}\right)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos(2x)\cos(x)}{2\sin(2x)\cos(x)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2\cos(2x)}{2\sin(2x)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2(\cos^2(x)-\sin^2(x))}{2\sin(2x)}$	$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{2(\cos^2(x)-\sin^2(x))}{2(2\sin(x)\cos(x))}$	$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{\cos^2(x)-\sin^2(x)}{2\sin(x)\cos(x)}$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{1}{2}\left(\frac{\cos^2(x)-\sin^2(x)}{\sin(x)\cos(x)}\right)$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{1}{2}\left(\frac{\cos^2(x)}{\sin(x)\cos(x)} - \frac{\sin^2(x)}{\sin(x)\cos(x)}\right)$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{1}{2}\left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}\right)$	
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{1}{2}\left(\cot(x) - \frac{\sin(x)}{\cos(x)}\right)$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{1}{2}(\cot(x) - \tan(x))$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
$\Rightarrow \frac{\cot(x)-\tan(x)}{2} = \frac{\cot(x)-\tan(x)}{2}$	Final answer.
	$\blacksquare$

3.2 $2\sin^2\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) = \frac{1+\sin(x)}{\csc(x)+\cot(x)}$

$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = 2\sin^2\left(\frac{x}{2}\right) + \frac{\sin(x)}{1+\cos(x)}$	$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = 2\left(\frac{1-\cos(x)}{2}\right) + \frac{\sin(x)}{1+\cos(x)}$	$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = 1 - \cos(x) + \frac{\sin(x)}{1+\cos(x)}$	
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{(1-\cos(x))(1+\cos(x))}{1+\cos(x)} + \frac{\sin(x)}{1+\cos(x)}$	
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{1-\cos^2(x)}{1+\cos(x)} + \frac{\sin(x)}{1+\cos(x)}$	Factor using difference of two squares.
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{\sin^2(x)}{1+\cos(x)} + \frac{\sin(x)}{1+\cos(x)}$	$\sin^2(\theta) + \cos^2(\theta) = 1$
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{\sin^2(x)+\sin(x)}{1+\cos(x)}$	
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{\sin(x)(1+\sin(x))}{1+\cos(x)}$	
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{1+\sin(x)}{\frac{1+\cos(x)}{\sin(x)}}$	
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{1+\sin(x)}{\frac{1}{\sin(x)} + \frac{\cos(x)}{\sin(x)}}$	

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$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{1+\sin(x)}{\csc(x)+\frac{\cos(x)}{\sin(x)}}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\Rightarrow \frac{1+\sin(x)}{\csc(x)+\cot(x)} = \frac{1+\sin(x)}{\csc(x)+\cot(x)}$	Final answer. $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ $\blacksquare$