Exercise Solutions for Math 20

Equations in Quadratic Form and with Radicals and Absolute Values

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1 Solve for x.

1.1 $\sqrt{2x+3} - \sqrt{x-2} = \sqrt{x+1}$

$\Rightarrow (\sqrt{2x+3} - \sqrt{x-2})^2 = x+1$	Square both sides.
$\Rightarrow 2x+3-2\sqrt{2x+3}\sqrt{x-2}+x-2 = x+1$	
$\Rightarrow 2x+3+x-2-x-1 = 2\sqrt{2x+3}\sqrt{x-2}$	
$\Rightarrow 2x = 2\sqrt{2x+3}\sqrt{x-2}$	
$\Rightarrow x = \sqrt{2x+3}\sqrt{x-2}$	
$\Rightarrow x^2 = (2x+3)(x-2)$	Square both sides.
$\Rightarrow x^2 = 2x^2-4x+3x-6$	
$\Rightarrow x^2 = 2x^2-x-6$	
$\Rightarrow 2x^2-x^2-x-6 = 0$	
$\Rightarrow x^2-x-6 = 0$	
$\Rightarrow (x-3)(x+2) = 0$	Factor by grouping.
$\Rightarrow x \in \{-2, 3\}$	
$\Rightarrow \sqrt{2(-2)+3} - \sqrt{-2-2} = \sqrt{-2+1}$	Verify $x=-2$.
$\Rightarrow \sqrt{-4+3} - \sqrt{-2-2} = \sqrt{-2+1}$	
$\Rightarrow \sqrt{-1} - \sqrt{-4} = \sqrt{-1}$	
$\Rightarrow i - 2i = i$	
$\Rightarrow -i = i$	
$\Rightarrow x \neq -2$	
$\Rightarrow \sqrt{2(3)+3} - \sqrt{3-2} = \sqrt{3+1}$	Verify $x=3$.
$\Rightarrow \sqrt{6+3} - \sqrt{3-2} = \sqrt{3+1}$	
$\Rightarrow \sqrt{9} - \sqrt{1} = \sqrt{4}$	
$\Rightarrow 3-1 = 2$	
$\Rightarrow 2 = 2$	
$\Rightarrow x = 3$	Final answer.
	$\blacksquare$

1.2 $1 = x + \sqrt{2x-3}$

$\Rightarrow 1-x = \sqrt{2x-3}$	Isolate the root.
$\Rightarrow (1-x)^2 = 2x-3$	Square both sides.
$\Rightarrow 1-2x+x^2 = 2x-3$	
$\Rightarrow 1-2x+x^2-2x+3 = 0$	
$\Rightarrow x^2-4x+4 = 0$	
$\Rightarrow (x-2)^2$	Factor by grouping.
$\Rightarrow x = 2$	
$\Rightarrow 1 = 2+\sqrt{2(2)-3}$	Verify $x=2$
$\Rightarrow 1 = 2+\sqrt{4-3}$	

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$\Rightarrow 1 = 2 + \sqrt{1}$	
$\Rightarrow 1 = 2 + 1$	
$\Rightarrow 1 = 3$	
$\Rightarrow x \neq 2$	
$\Rightarrow x \in \emptyset$	Final answer.
	$\blacksquare$

1.3 $\left\lvert \frac{3x-4}{2x+3} \right\rvert = 1$

$\Rightarrow \frac{3x-4}{2x+3} = -1$	$\lvert a\rvert = b \Rightarrow a = \pm b$. Solve for $a = -b$.
$\Rightarrow \frac{3x-4}{2x+3} = -\frac{2x+3}{2x+3}$	
$\Rightarrow 3x - 4 = -(2x+3)$	Eliminate denominator. $x = -\frac{3}{2}$ is an undefined point.
$\Rightarrow 3x - 4 = -2x - 3$	
$\Rightarrow 3x + 2x = -3 + 4$	
$\Rightarrow 5x = 1$	
$\Rightarrow x = \frac{1}{5}$	
$\Rightarrow \frac{3x-4}{2x+3} = 1$	$\lvert a\rvert = b \Rightarrow a = \pm b$. Solve for $a = +b$.
$\Rightarrow \frac{3x-4}{2x+3} = \frac{2x+3}{2x+3}$	
$\Rightarrow 3x - 4 = 2x + 3$	Eliminate denominator. $x = -\frac{3}{2}$ is an undefined point.
$\Rightarrow 3x - 2x = 3 + 4$	
$\Rightarrow x = 7$	
$\Rightarrow x \in \{\frac{1}{5}, 7\}$	Final answer.
	$\blacksquare$

1.4 $-7(\frac{1}{x} - 1) = 4 - 2(\frac{1}{x} - 1)^2$

$\Rightarrow -7t = 4 - 2t^2$	$t = (\frac{1}{x} - 1)$. $x = 0$ is an undefined point.
$\Rightarrow 2t^2 - 7t - 4 = 0$	
$\Rightarrow 2t^2 - 8t + t - 4 = 0$	Factor by grouping.
$\Rightarrow 2t(t-4) + 1(t-4) = 0$	
$\Rightarrow (2t+1)(t-4) = 0$	
$\Rightarrow (2t+1)(t-4) = 0$	
$\Rightarrow t \in \{-\frac{1}{2}, 4\}$	
$\Rightarrow \frac{1}{x} - 1 = -\frac{1}{2}$	Solve for x using $t = -\frac{1}{2}$.
$\Rightarrow \frac{1}{x} = -\frac{1}{2} + 1$	
$\Rightarrow \frac{1}{x} = \frac{1}{2}$	
$\Rightarrow x = 2$	
$\Rightarrow \frac{1}{x} - 1 = 4$	Solve for x using $t = 4$.
$\Rightarrow \frac{1}{x} = 4 + 1$	
$\Rightarrow \frac{1}{x} = 5$	

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$\Rightarrow x = \frac{1}{5}$	
$\Rightarrow x \in \{\frac{1}{5}, 2\}$	Final answer. ■

1.5 $x^2(x^2-1) - 9(x^2-1) = 0$

$\Rightarrow (x^2-9)(x^2-1) = 0$	Factor by grouping.
$\Rightarrow x^2 - 9 = 0$	Solve for x.
$\Rightarrow (x-3)(x+3) = 0$	Factor using difference of two squares.
$\Rightarrow x \in \{-3, 3\}$	
$\Rightarrow x^2 - 1 = 0$	Solve for x.
$\Rightarrow (x-1)(x+1) = 0$	Factor using difference of two squares.
$\Rightarrow x \in \{-1, 1\}$	
$\Rightarrow x \in \{-3, -1, 1, 3\}$	Final answer. ■

1.6 $2(x^2+x+1) + \sqrt{x^2+x+1} - 3 = 0$

$\Rightarrow 2t + \sqrt{t} - 3 = 0$	$t = x^2 + x + 1$.
$\Rightarrow 2t - 3 = \sqrt{t}$	Isolate the root.
$\Rightarrow (2t-3)^2 = t$	Square both sides.
$\Rightarrow 4t^2 - 12t + 9 = t$	
$\Rightarrow 4t^2 - 13t + 9 = 0$	
$\Rightarrow 4t^2 - 4t - 9t + 9 = 0$	Factor by grouping.
$\Rightarrow 4t(t-1) - 9(t-1) = 0$	
$\Rightarrow (4t-9)(t-1) = 0$	
$\Rightarrow t \in \{1, \frac{9}{4}\}$	
$\Rightarrow x^2 + x + 1 = 1$	Solve for x using $t = 1$.
$\Rightarrow x^2 + x = 0$	
$\Rightarrow x(x+1) = 0$	
$\Rightarrow x \in \{-1, 0\}$	
$\Rightarrow x^2 + x + 1 = \frac{9}{4}$	Solve for x using $t = \frac{9}{4}$.
$\Rightarrow x^2 + x + 1 - \frac{9}{4} = 0$	
$\Rightarrow x^2 + x + \frac{4}{4} - \frac{9}{4} = 0$	
$\Rightarrow x^2 + x - \frac{5}{4} = 0$	
$\Rightarrow 4x^2 + 4x - 5 = 0$	
$\Rightarrow \frac{-4 \pm \sqrt{4^2 - 4(4)(-5)}}{2(4)}$	Use the quadratic formula.
$\Rightarrow \frac{-4 \pm \sqrt{16 - 4(4)(-5)}}{2(4)}$	
$\Rightarrow \frac{-4 \pm \sqrt{16+80}}{8}$	
$\Rightarrow \frac{-4 \pm \sqrt{96}}{8}$	

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$\Rightarrow \frac{-4 \pm \sqrt{16}\sqrt{6}}{8}$ $\Rightarrow \frac{-4 \pm 4\sqrt{6}}{8}$ $\Rightarrow \frac{-1 \pm \sqrt{6}}{2}$ $\Rightarrow x \in \{\frac{-1+\sqrt{6}}{2}, \frac{-1-\sqrt{6}}{2}\}$	
$\Rightarrow 2((-1)^2 - 1 + 1) + \sqrt{(-1)^2 - 1 + 1} - 3 = 0$ $\Rightarrow 2(1 - 1 + 1) + \sqrt{1 - 1 + 1} - 3 = 0$ $\Rightarrow 2(1) + \sqrt{1} - 3 = 0$ $\Rightarrow 2 + 1 - 3 = 0$ $\Rightarrow 0 = 0$ $\Rightarrow x = -1$	Verify $x = -1$.
$\Rightarrow 2(0^2 - 0 + 1) + \sqrt{0^2 - 0 + 1} - 3 = 0$ $\Rightarrow 2(1) + \sqrt{1} - 3 = 0$ $\Rightarrow 2 + 1 - 3 = 0$ $\Rightarrow 0 = 0$ $\Rightarrow x = 0$	Verify $x = 0$.
$\Rightarrow x \in \{-1, 0\}$	Final answer. A quadratic equation can have at most two solutions. ■