

Exercise Solutions for Math 20

Sum, Difference, Cofunction, Double Measure Identities

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1 Given $\sin(\beta) = \frac{3}{5}$ where β lie in the interval $(\frac{\pi}{2}, 2\pi)$, find $\cos(\frac{\beta}{2})$

$\Rightarrow \beta \in (\frac{\pi}{2}, \pi)$	Since $\sin(\beta)$ is positive, β must be in $(0, \pi)$. Combining the two intervals, we can see that we are in QII.
$\Rightarrow \cos(\beta) = \pm\sqrt{1 - \sin^2(\beta)}$ $\Rightarrow \cos(\beta) = \pm\sqrt{1 - (\frac{3}{5})^2}$ $\Rightarrow \cos(\beta) = \pm\sqrt{1 - \frac{9}{25}}$ $\Rightarrow \cos(\beta) = \pm\sqrt{\frac{25}{25} - \frac{9}{25}}$ $\Rightarrow \cos(\beta) = \pm\sqrt{\frac{16}{25}}$ $\Rightarrow \cos(\beta) = \pm\frac{4}{5}$ $\Rightarrow \cos(\beta) = -\frac{4}{5}$	Find $\cos(\theta)$ using Pythagoras. Since we are in QII, $\cos(\beta)$ is negative.
$\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1+\cos(\beta)}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1-\frac{4}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{\frac{5}{5}-\frac{4}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{\frac{1}{5}}{2}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1}{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{1}{\sqrt{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$ $\Rightarrow \cos\left(\frac{\beta}{2}\right) = \pm\frac{\sqrt{10}}{10}$	$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$ Rationalize.
$\Rightarrow \cos\left(\frac{\beta}{2}\right) = \frac{\sqrt{10}}{10}$	Final answer. Given $\beta \in (\frac{\pi}{2}, \pi)$, $\frac{\beta}{2} \in (\frac{\pi}{4}, \frac{\pi}{2})$, which is in QI. Therefore $\cos\left(\frac{\beta}{2}\right)$ is positive. ■

2 Express the following as indicated then evaluate if possible.

2.1 $\sin(105^\circ) \cos(15^\circ)$ as a sum.

$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{1}{2}(\sin(105^\circ + 15^\circ) + \sin(105^\circ - 15^\circ))$	$\sin(a) \cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$
$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{1}{2}(\sin(120^\circ) + \sin(90^\circ))$	
$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)$	
$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{\sqrt{3}}{4} + \frac{1}{2}$	
$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{\sqrt{3}}{4} + \frac{2}{4}$	
$\Rightarrow \sin(105^\circ) \cos(15^\circ) = \frac{2+\sqrt{3}}{4}$	Final answer. ■

2.2 $\cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right)$ as a product.

$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \cos\left(\frac{\frac{5\pi}{24} + \frac{\pi}{24}}{2}\right) \cos\left(\frac{\frac{5\pi}{24} - \frac{\pi}{24}}{2}\right)$	$\cos(a) + \cos(b) = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \cos\left(\frac{\frac{6\pi}{24}}{2}\right) \cos\left(\frac{\frac{4\pi}{24}}{2}\right)$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \cos\left(\frac{\frac{\pi}{4}}{2}\right) \cos\left(\frac{\frac{\pi}{6}}{2}\right)$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} \sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}}$	$\cos\left(\frac{\theta}{2}\right) = \pm \sqrt{\frac{1+\cos(\theta)}{2}}$
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}} \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \sqrt{\frac{2}{4} + \frac{\sqrt{2}}{4}} \sqrt{\frac{2}{4} + \frac{\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \sqrt{\frac{2+\sqrt{2}}{4}} \sqrt{\frac{2+\sqrt{3}}{4}}$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = 2 \left(\frac{\sqrt{2+\sqrt{2}}}{2}\right) \left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)$	
$\Rightarrow \cos\left(\frac{5\pi}{24}\right) + \cos\left(\frac{\pi}{24}\right) = \sqrt{2 + \sqrt{2}} \left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)$	Final answer. ■

2.3 $\sin(3x) \cos(2x)$ as a sum.

$\Rightarrow \sin(3x) \cos(2x) = \frac{1}{2}(\sin(3x + 2x) + \sin(3x - 2x))$	$\sin(a) \cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$
$\Rightarrow \sin(3x) \cos(2x) = \frac{1}{2}(\sin(5x) + \sin(x))$	
$\Rightarrow \sin(3x) \cos(2x) = \frac{\sin(5x) + \sin(x)}{2}$	Final answer. ■