Exercise Solutions for Math 20

Some Types of Functions, Operations

Nile Jocson <novoseiversia@gmail.com>

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1.1

Given

$$g(x) = \begin{cases} x+4, & \text{if } x < -2 \\ |x|, & \text{if } -1 < x < 1 \\ 2, & \text{if } x > 3 \end{cases}$$

Sketch its graph and label its x- and y- intercepts.

$\Rightarrow 0 = x + 4$	Find the x-intercepts of $y = x + 4$
$\Rightarrow x_i = -4$	
$\Rightarrow 0 = \|x\|$	Find the x-intercepts of $y = \|x\|$
$\Rightarrow x_i = 0$	
$\Rightarrow y = \|0\|$	Find the y-intercepts of $y = \|x\|$
$\Rightarrow y_i = 0$	
$\Rightarrow$ See Figure 1.	Final answer. Graph the system.
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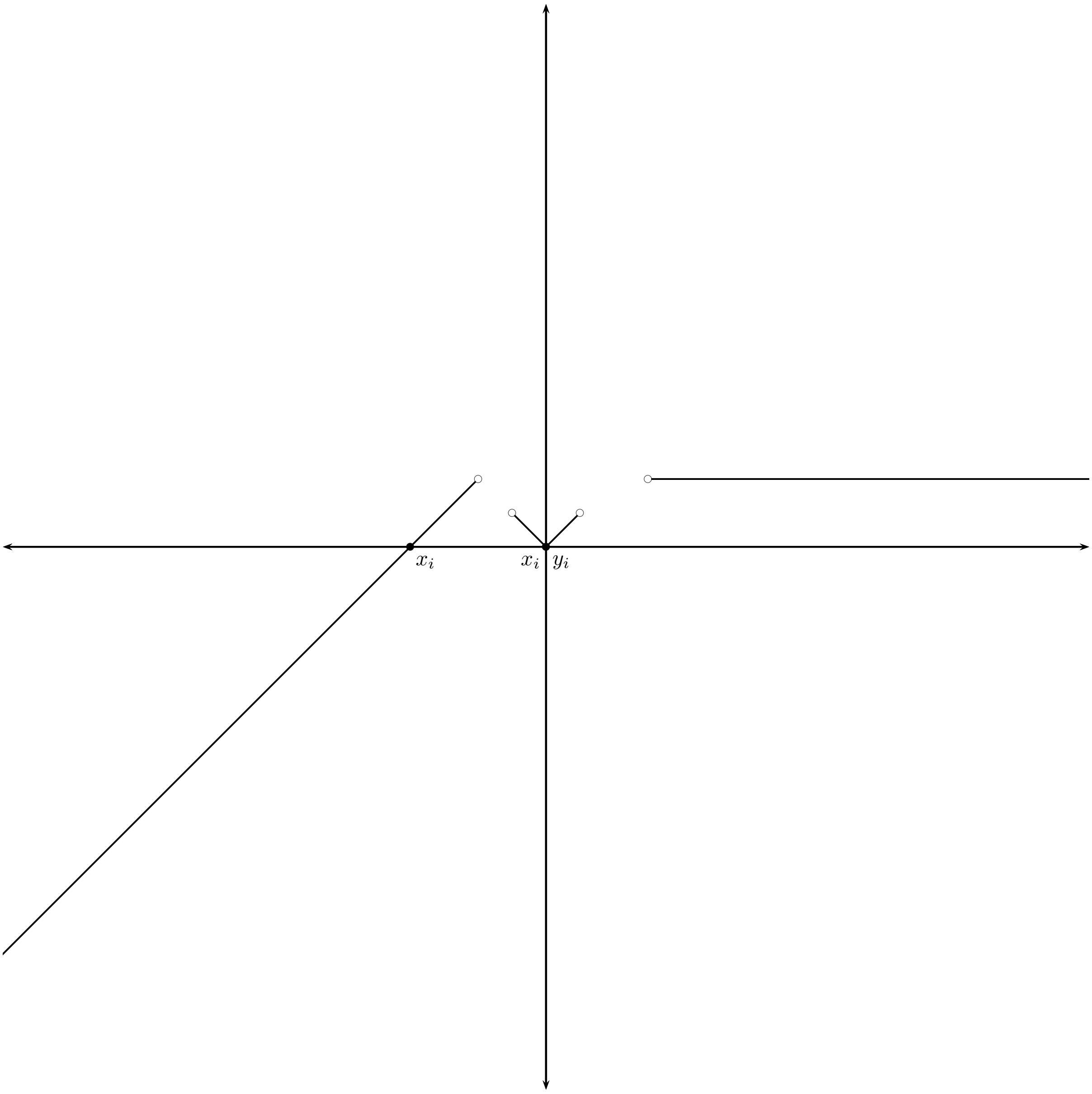

Figure 1.

1.2

Let f, g, and h be the functions defined from the expressions below.

$$f(x) = \frac{x+2}{x-3}$$

$$g(x) = \frac{x+3}{x+2}$$

$$h(x) = \sqrt{3x-1}$$

Determine $(f-g)(x)$, $(\frac{f}{g})(x)$, $(fg)(x)$, $(f \circ g)(x)$, $(h \circ f)(x)$, $(g \circ h)(x)$ and obtain their respective domains.

$\Rightarrow (f-g)(x) = \frac{x+2}{x-3} - \frac{x+3}{x+2}$	Find $(f-g)(x)$.
$\Rightarrow (f-g)(x) = \frac{(x+2)(x+2)}{(x-3)(x+2)} - \frac{(x-3)(x+3)}{(x-3)(x+2)}$	
$\Rightarrow (f-g)(x) = \frac{(x+2)(x+2)-(x-3)(x+3)}{(x-3)(x+2)}$	
$\Rightarrow (f-g)(x) = \frac{x^2+4x+4-(x^2-9)}{(x-3)(x+2)}$	
$\Rightarrow (f-g)(x) = \frac{x^2+4x+4-x^2+9}{(x-3)(x+2)}$	
$\Rightarrow (f-g)(x) = \frac{4x+13}{(x-3)(x+2)}$	
$\Rightarrow \mathrm{dom}(f-g) = \mathbb{R} \setminus \{-2, 3\}$	
$\Rightarrow (\frac{f}{g})(x) = \frac{\frac{x+2}{x-3}}{\frac{x+3}{x+2}}$	Find $(\frac{f}{g})(x)$.
$\Rightarrow (\frac{f}{g})(x) = \frac{x+2}{x-3} \cdot \frac{x+2}{x+3}$	
$\Rightarrow (\frac{f}{g})(x) = \frac{(x+2)^2}{(x-3)(x+3)}$	
$\Rightarrow \mathrm{dom}(\frac{f}{g}) = \mathbb{R} \setminus \{-3, -2, 3\}$	
$\Rightarrow (fg)(x) = \frac{x+2}{x-3} \cdot \frac{x+3}{x+2}$	Find $(fg)(x)$.
$\Rightarrow (fg)(x) = \frac{(x+2)(x+3)}{(x-3)(x+2)}$	
$\Rightarrow (fg)(x) = \frac{x+3}{x-3}$	
$\Rightarrow \mathrm{dom}(fg) = \mathbb{R} \setminus \{-2, 3\}$	
$\Rightarrow (f \circ g)(x) = \frac{\frac{x+3}{x+2}+2}{\frac{x+3}{x+2}-3}$	Find $(f \circ g)(x)$.
$\Rightarrow (f \circ g)(x) = \frac{\frac{x+3}{x+2}+\frac{2x+4}{x+2}}{\frac{x+3}{x+2}-\frac{3x+6}{x+2}}$	
$\Rightarrow (f \circ g)(x) = \frac{\frac{x+3+(2x+4)}{x+2}}{\frac{x+3-(3x+6)}{x+2}}$	
$\Rightarrow (f \circ g)(x) = \frac{\frac{x+3+2x+4}{x+2}}{\frac{x+3-3x-6}{x+2}}$	
$\Rightarrow (f \circ g)(x) = \frac{\frac{3x+7}{x+2}}{\frac{-2x-3}{x+2}}$	
$\Rightarrow (f \circ g)(x) = \frac{3x+7}{x+2} \cdot \frac{x+2}{-2x-3}$	
$\Rightarrow (f \circ g)(x) = \frac{3x+7}{-2x-3}$	
$\Rightarrow \mathrm{dom}(f \circ g) = \mathbb{R} \setminus \{-2, -\frac{3}{2}\}$	
$\Rightarrow (h \circ f)(x) = \sqrt{3(\frac{x+2}{x-3}) - 1}$	Find $(h \circ f)(x)$.
$\Rightarrow (h \circ f)(x) = \sqrt{\frac{3x+6}{x-3} - 1}$	
$\Rightarrow (h \circ f)(x) = \sqrt{\frac{3x+6}{x-3} - \frac{x-3}{x-3}}$	

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$\Rightarrow (h \circ f)(x) = \sqrt{\frac{3x+6-(x-3)}{x-3}}$

$\Rightarrow (h \circ f)(x) = \sqrt{\frac{3x+6-x+3}{x-3}}$

$\Rightarrow (h \circ f)(x) = \sqrt{\frac{2x+9}{x-3}}$

$\Rightarrow \frac{2x+9}{x-3} \geq 0$ Solve for the domain. Note that $x = 3$ is undefined.

Create a table of signs.

	$-\frac{9}{2}$	3	
$2x+9$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$\frac{2x+9}{x-3}$	$+$	$-$	$+$

$\Rightarrow \text{dom}(h \circ f) = (-\infty, -\frac{9}{2}] \cup (3, +\infty)$

$\Rightarrow (g \circ h)(x) = \frac{\sqrt{3x-1}+3}{\sqrt{3x-1}+2}$ Find $(g \circ h)(x)$.

$\Rightarrow (g \circ h)(x) = \frac{\sqrt{3x-1}+3}{\sqrt{3x-1}+2} \cdot \frac{\sqrt{3x-1}-2}{\sqrt{3x-1}-2}$ Rationalize.

$\Rightarrow (g \circ h)(x) = \frac{(\sqrt{3x-1}+3)(\sqrt{3x-1}-2)}{3x-1-4}$

$\Rightarrow (g \circ h)(x) = \frac{(\sqrt{3x-1}+3)(\sqrt{3x-1}-2)}{3x-5}$

$\Rightarrow 3x - 1 \geq 0$ Solve for the domain. Note that $x = \frac{5}{3}$ is undefined.

$\Rightarrow 3x \geq 1$

$\Rightarrow x \geq \frac{1}{3}$

$\Rightarrow \text{dom}(g \circ h) = [\frac{1}{3}, +\infty) \setminus \{\frac{5}{3}\}$

$\Rightarrow (f - g)(x) = \frac{4x+13}{(x-3)(x+2)}$ Final answer.

$\Rightarrow \text{dom}(f - g) = \mathbb{R} \setminus \{-2, 3\}$

$\Rightarrow (\frac{f}{g})(x) = \frac{(x+2)^2}{(x-3)(x+3)}$

$\Rightarrow \text{dom}(\frac{f}{g}) = \mathbb{R} \setminus \{-3, -2, 3\}$

$\Rightarrow (fg)(x) = \frac{x+3}{x-3}$

$\Rightarrow \text{dom}(fg) = \mathbb{R} \setminus \{-2, 3\}$

$\Rightarrow (f \circ g)(x) = \frac{3x+7}{-2x-3}$

$\Rightarrow \text{dom}(f \circ g) = \mathbb{R} \setminus \{-2, -\frac{3}{2}\}$

$\Rightarrow (h \circ f)(x) = \sqrt{\frac{2x+9}{x-3}}$

$\Rightarrow \text{dom}(h \circ f) = (-\infty, -\frac{9}{2}] \cup (3, +\infty)$

$\Rightarrow (g \circ h)(x) = \frac{(\sqrt{3x-1}+3)(\sqrt{3x-1}-2)}{3x-5}$

$\Rightarrow \text{dom}(g \circ h) = [\frac{1}{3}, +\infty) \setminus \{\frac{5}{3}\}$

$\blacksquare$

1.3

A ball is thrown upward from the roof of a building that is 30 meters high. If it is known that the position of the ball with respect to the ground after x seconds is $h(x) = -5x^2 + 20x + 30$ meters, determine the maximum height reached by the ball, and how long it takes before the ball hits the ground.

— Let:	
☐ M	Maximum height.
☐ t	Time to hit the ground.
$\Rightarrow -5x^2 + 20x + 30 = y$	The trajectory of a ball is most likely a parabola. Rewrite $h(x)$ in standard form and in terms of y.
$\Rightarrow -5x^2 + 20x = y - 30$	
$\Rightarrow -5(x^2 - 4x) = y - 30$	
$\Rightarrow -5(x^2 - 4x + 4) = y - 30 - 5(4)$	
$\Rightarrow -5(x^2 - 4x + 4) = y - 30 - 20$	
$\Rightarrow -5(x^2 - 4x + 4) = y - 50$	
$\Rightarrow -5(x-2)^2 = y - 50$	
$\Rightarrow (x-2)^2 = (-\frac{1}{5})y - 50$	
$\Rightarrow (x-2)^2 = 4(-\frac{1}{20})y - 50$	
$\Rightarrow -5x^2 + 20x + 30 = 0$	The x-intercept of the function is the time is takes for the ball to hit the ground.
$\Rightarrow x = \frac{-20 \pm \sqrt{(-20)^2 - 4(-5)(30)}}{2(-5)}$	Use the quadratic formula.
$\Rightarrow x = \frac{-20 \pm \sqrt{400+600}}{-10}$	
$\Rightarrow x = \frac{-20 \pm \sqrt{1000}}{-10}$	
$\Rightarrow x = \frac{-20 \pm \sqrt{100}\sqrt{10}}{-10}$	
$\Rightarrow x = \frac{-20 \pm 10\sqrt{10}}{-10}$	
$\Rightarrow x = 2 \pm \sqrt{10}$	
$\Rightarrow x = 2 + \sqrt{10}$	We only care about the cases where $x \geq 0$ since we can't reverse time.
$\Rightarrow M = 10$ meters	Final answer. The maximum height is the y-coordinate of the vertex of the parabola.
$\Rightarrow t = 2 + \sqrt{10}$ seconds	
$\Rightarrow$ See Figure 2 for a visualization.	
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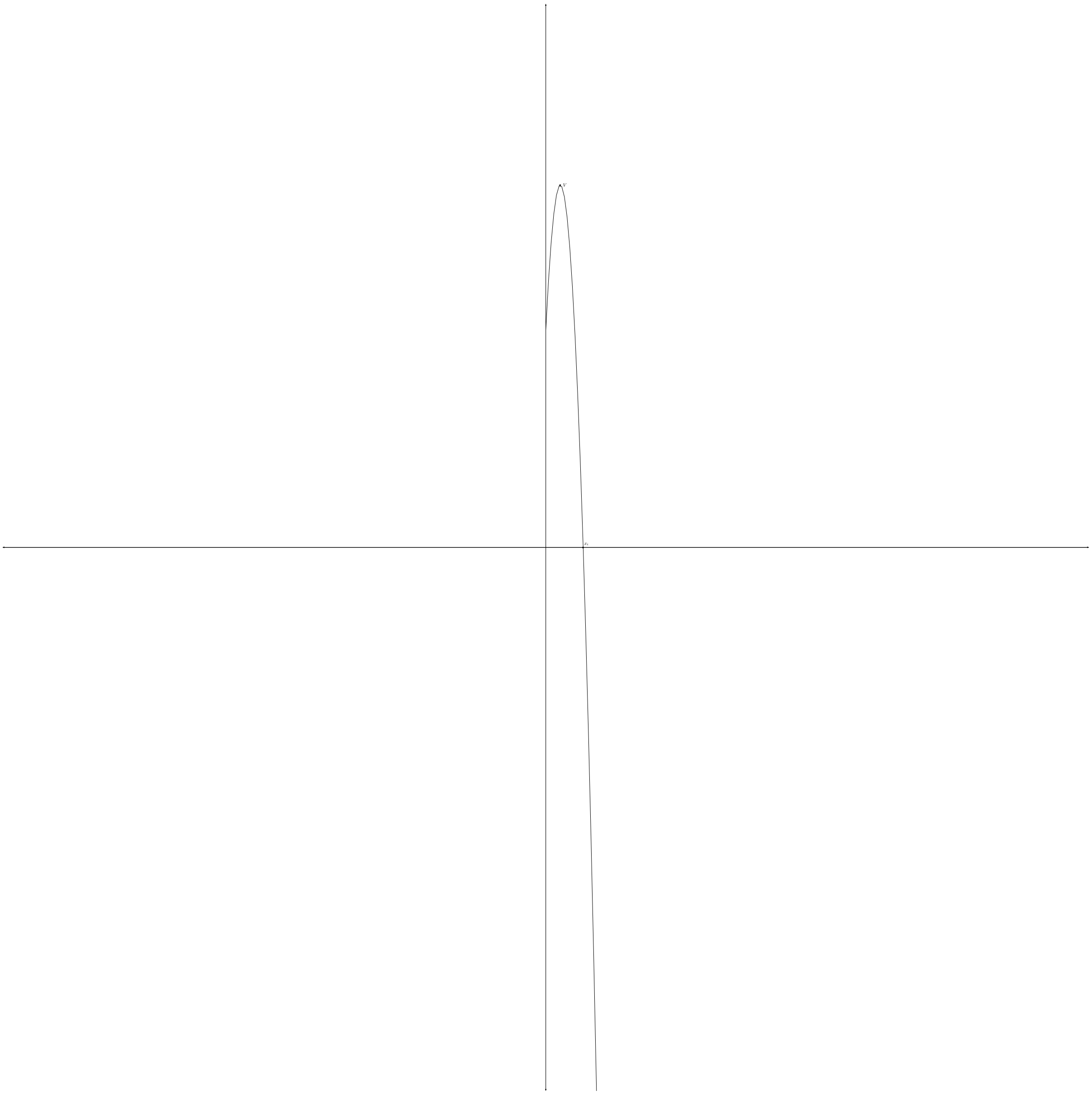

Figure 2. Zoom in to see labels.