Problem 46

In this problem, we used the fourth order Taylor method:

$$y(x) \approx y(x_0) + y'h + \frac{1}{2!}y''h^2 + \frac{1}{3!}y^{(3)}h^3 + \frac{1}{4!}y^{(4)}h^4.$$

$y^{(n)}$ up to $n = 4$ are computed as below.

- $y' = f(x, y) = x - y^2$,
- $y'' = 1 - 2y(x - y^2)$,
- $y^{(3)} = -2\left(y + (x - y^2)(x - 3y^2)\right)$,
- $y^{(4)} = -2\left(3x - 6y^2 + 4y(x - y^2)(3y^2 - 2)\right)$.

As we compare the numerical results in the graphs, the approximations with $h = .5$ fails in this problem as x approaches to 10, while the approximations seem to converge to the correct values in the cases of $h \leq .25$.
We add the global error estimate (6.10.24) in the textbook, then confirm that the error estimate decreases as h increases as shown at the bottom in the rightmost columns on the tables($1.7 \times 10^{-1} \to 1.5 \times 10^{-3}$ as $h : .25 \to .125$).