Problem 6

In this problem, the error bound of $\max_{x\in[a,b]}|y(x)-y_h(x)|$ from above can be written as $\frac{1}{L}(\exp((b-a)L)-1)\frac{h}{2}\|y''\|_\infty$ by the theorem 6.3 in the textbook, where

- $\|y''\|_\infty = \max_{x\in[a,b]}|y''(x)|$,
- $(a,b)=(0,4)$,
- $L=1$ is the Lipschitz constant of this function,
- h is the step size for the Euler method.

In the numerical results, when we focus on the rightmost and the second right columns on the tables, we notice that all the value of $|y(x)-y_h(x)|$ are less than the bounds from above, which agrees with the claim of the theorem 6.3.

The bound is proportional to h because other parameters and values are constants. Hence, if h halves, then the error bound from above does so. We confirmed this conjecture is true in the numerical results. Indeed, the value of $|y(x)-y_h(x)|$ at the bottom on each table becomes one half of it approximately as h halves ($.87218 \to .43737 \to .219$ as $h : .25 \to .125 \to .0625$).