Notes working through Spivak's Calculus

Slava Akhmechet

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I'm working through Spivak Calculus. Around the chapter on epsilon-delta limits the details get pretty confusing. I started supplementing with David Galvin's notes, which are considerabley more clear but are still confusing. This is surprising because the topic doesn't use anything beyond basic middle school math. Feels like it should be simple! And so I'm attempting to write these notes to properly understand the damned thing.

Spivak's chapters 7 (Three Hard Theorems) and 8 (Least Upper Bounds) are swapped in these notes. Spivak first introduces the Intermediate Value theorem and the Extreme Value theorem as facts, proves their consequences, then introduces completeness and its consequences, and finally proves IVT and EVT. I find it distracting and confusing. I introduce completeness and its consequences first. I then introduce and prove IVT and EVT, and cover their consequences. IMO this approach is much less confusing than Spivak's.

Contents

1 Limits, Part I (The Blessed Path)

1.1 Precalculus

I'll first do a hand-wavy definition of limits, and then use it to explain the mechanics of computing limits of functions in practice. Basically, pre-calculus stuff. After that I'll do a proper definition and use it to prove the theorems that make the mechanics work.

A hand-wavy definition: a limit of $f(x)$ at a is the value $f(x)$ approaches close to (but not necessarily at) a.

A slightly less hand-wavy definition: let $f : \mathbf{R} \to \mathbf{R}$, let $a \in \mathbf{R}$ be some number on the x-axis, and let $l \in \mathbf{R}$ be some number on the y-axis. Then as x gets closer to a, $f(x)$ gets closer to l.

The notation for this whole thing is

$$\lim_{x \to a} f(x) = l$$

So for example $\lim_{x\to 5} x^2 = 25$ because the closer x gets to 5, the closer x^2 gets to 25 (we'll prove all this properly soon). Now suppose you have some fancy pants function like this one:

$$\lim_{x \to 0} \frac{1-\sqrt{x}}{1-x} \tag{1}$$

If you plot it, it's easy to see that as x approaches 0, the whole shebang approaches 1. But how do you algebraically evaluate the limit of this thing? Can you just plug 0 into the equation? It seems to work, but once we formally define limits, we'll have to prove somehow that plugging $a = 0$ into x gives us the correct result.

Limits evaluation mechanics

It turns out that it does in fact work because of a few theorems that make practical evaluation of many limits easy. I'll first state these theorems as facts, and then go back and properly prove them once I introduce the formal definition of limits.

1. **Constants**. $\lim_{x\to a} c = c$, where $c \in \mathbf{R}$. In other words if the function is a constant, e.g. $f(x) = 5$, then $\lim_{x\to a} f(x) = 5$ for any a.

2. **Identity**. $\lim_{x\to a} x = a$. In other words if the function is an identity function $f(x) = x$, then $\lim_{x\to 6} f(x) = 6$. Meaning we simply plug a into x.

3. **Addition**[1]. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. For example $\lim_{x\to a}(x+2) = \lim_{x\to a} x + \lim_{x\to a} 2 = a+2$.

4. **Multiplication**. $\lim_{x\to a}(f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. For example $\lim_{x\to a} 2x = \lim_{x\to a} 2 \cdot \lim_{x\to a} x = 2a$.

5. **Reciprocal**. $\lim_{x\to a} \left(\frac{1}{f}\right)(x) = \frac{1}{\lim_{x\to a} f(x)}$ when the denominator isn't zero. For example $\lim_{x\to a} \frac{1}{x} = \frac{1}{\lim_{x\to a} x} = \frac{1}{a}$ for $a \neq 0$.

To come back to 1, these theorems tells us that

$$\lim_{x\to 0} \frac{1-\sqrt{x}}{1-x} = \frac{\lim_{x\to 0} 1 - (\lim_{x\to 0} x)^{\frac{1}{2}}}{\lim_{x\to 0} 1 - \lim_{x\to 0} x} = \frac{1-0^{\frac{1}{2}}}{1-0} = 1$$

Holes

What happens if we try to take a limit as $x \to 1$ rather than $x \to 0$?

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$$

We can't use the same trick and plug in 1 because we get a nonsensical result 0/0 as the function isn't defined at 0. If we plot it, we clearly see the limit approaches 1/2 at 0, but how do we prove this algebraically? The answer is to do some trickery to find a way to cancel out the inconvenient term (in this case $1-\sqrt{x}$)

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

Why is it ok here to divide by $1-\sqrt{x}$? Good question! Recall that the limit is defined *close to* a (or *around* a, or as x *approaches* a), but not **at** a. In other words $f(a)$ need not even be defined (as is the case here). This means that as we consider $1-\sqrt{x}$ at different values of x as it approaches a, the limit never requires us to evaluate the function at $x = a$. So we never have to consider $1-\sqrt{x}$ as $x = 1$, $1-\sqrt{x}$ never takes on the value of 0, and it is safe to divide it out.

1.2 Formal limits definition

Now we establish a rigorous definition of limits that formalizes the hand-wavy version above.

[1]Spivak's book uses a slightly more verbose definition that assumes the limits of f and g exist near a, see p. 103

Definition: $\lim_{x\to a} f(x) = L$ when for any $\epsilon \in \mathbf{R}$ there exists $\delta \in \mathbf{R}$ such that for all x, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. (Also $\epsilon > 0, \delta > 0$.)

Here is what this says. Suppose $\lim_{x\to a} f(x) = L$. You pick any interval on the y-axis around L. Make it as small (or as large) as you want. I'll produce an interval on the x-axis around a. You can take any number from my interval, plug it into f, and the output will stay within the bounds you specified.

So ϵ specifies the distance away from L along the y-axis, and δ specifies the distance away from a along the x-axis. Take any x within δ of a, plug it into f, and the result is guaranteed to be within ϵ of L. $\lim_{x\to a} f(x) = L$ just means there exists such δ for any ϵ.

This is quite simple, but the mechanics of the limit definition tend to confuse people. I think it's because absolute value inequalities are unfamiliar. Wtf is $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$?! Let's tease it apart[2]

Here is the intuitive reading. $0 < |x - a| < \delta$ means the difference between x and a is between 0 and δ. And $|f(x) - L| < \epsilon$ means the difference between $f(x)$ and L is less than ϵ. This is actually all this means, but still, let's look at the inequalities more closely.

First, consider $0 < |x - a| < \delta$. There are two inequalities here. The left side, $0 < |x - a|$ is equivalent to $|x - a| > 0$. But $|x - a|$ is an absolute value, it's **always** true that $|x - a| \geq 0$. So this part of the inequality says $x - a \neq 0$, or $x \neq a$. (Remember, we said the limit is defined *around* a but not *at* a). I don't know why mathematicians say $0 < |x - a|$ instead of $x \neq a$, probably because confusing you brings them pleasure.

The right side is $|x - a| < \delta$. Intuitively this says that along the x-axis the difference between x and a should be less than δ. Put differently, x should be within δ of a. We can rewrite this as $a - \delta < x < a + \delta$.

The second equation, $|f(x) - L| < \epsilon$ should now be easy to understand. Intuitively, along the y-axis $f(x)$ should be within ϵ of L, or put differently $L - \epsilon < f(x) < L + \epsilon$.

Limit uniqueness

Suppose $\lim_{x\to a} f(x) = L$. It's easy to assume L is the only limit around a, but such a thing needs to be proved. We prove this here. More formally, suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$. We prove that $L = M$.

Suppose for contradiction $L \neq M$. Assume without loss of generality $L > M$. By limit definition, for all $\epsilon > 0$ there exists a positive $\delta \in \mathbf{R}$ such that $0 <$

[2] The best answer is to go to Khan academy and do a bunch of absolute value inequalities until they become second nature.

$|x - a| < \delta$ implies

- $|f(x) - L| < \epsilon \implies L - \epsilon < f(x)$
- $|f(x) - M| < \epsilon \implies f(x) < M + \epsilon$

for all x. Thus

$$\begin{aligned} & L - \epsilon < f(x) < M + \epsilon \\ \implies & L - \epsilon < M + \epsilon \\ \implies & L - M < 2\epsilon \end{aligned}$$

The above is true for all ϵ. Now let's narrow our attention and consider a concrete $\epsilon = (L - M)/4$, which we easily find leads to a contradiction[3]:

$$\begin{aligned} & L - M < 2\epsilon & \\ \implies & (L - M)/4 < \epsilon/2 & \text{dividing both sides by 4} \\ \implies & \epsilon < \epsilon/2 & \text{recall we set } \epsilon = (L - M)/4 \end{aligned}$$

We have a contradiction, and so $L = M$ as desired.

Half-Value Neighborhood Lemma

This lemma will come in handy later, so we may as well prove it now. Suppose $M \neq 0$ and $\lim_{x \to a} g(x) = M$. We show that there exists some δ such that $0 < |x - a| < \delta$ implies $|g(x)| \geq |M|/2$ for all x.

Intuitively, the lemma states the following: when a function g approaches a nonzero limit M near a point, there exists an interval in which the values of g are closer to M than to zero.

Proof. The claim that $|g(x)| \geq |M|/2$ is equivalent to

$$g(x) \leq -|M|/2 \quad \text{or} \quad g(x) \geq |M|/2$$

There are two possibilities: either $M > 0$ or $M < 0$. Let's consider each possibility separately.

Case 1. Suppose $M > 0$. Then to show $|g(x)| \geq |M|/2$ it is sufficient to show *either* $g(x) \leq -M/2$ or $g(x) \geq M/2$. We will show $g(x) \geq M/2$. Fix $\epsilon = M/2$.

[3] note we assumed $L > M$, thus $\epsilon = (L - M)/4 > 0$

By limit definition there is some δ such that $0 < |x - a| < \delta$ implies for all x

$$\begin{aligned}
&|g(x) - M| < M/2 \\
&\implies -M/2 < g(x) - M \\
&\implies M/2 < g(x) && \text{add } M \text{ to both sides} \\
&\implies g(x) > M/2 && \text{note } \geq \text{ is correct but not tight}
\end{aligned}$$

Case 2. Suppose $M < 0$. We must show either $g(x) \leq M/2$ or $g(x) \geq -M/2$. We will show $g(x) \leq M/2$. Fix $\epsilon = -M/2$. Then

$$\begin{aligned}
&|g(x) - M| < -M/2 \\
&\implies g(x) - M < -M/2 \\
&\implies g(x) < M/2 && \text{add } M \text{ to both sides;} \\
& && \text{note } \leq \text{ is correct but not tight}
\end{aligned}$$

QED.

1.3 Theorems that make evaluation work

Armed with the formal definition, we can use it to rigorously prove the five theorems useful for evaluating limits (constants, identity, addition, multiplication, reciprocal). Let's do that now.

Constants

Let $f(x) = c$. We prove that $\lim_{x \to a} f(x) = c$ for all a.

Let $\epsilon > 0$ be given. Pick any positive δ. Then for all x such that $0 < |x - a| < \delta$, $|f(x) - c| = |c - c| = 0 < \epsilon$. QED.

(Note that we can pick any positive $\delta > 0$, e.g. $1, 10, \frac{1}{10}$.)

Identity

Let $f(x) = x$. We prove that $\lim_{x \to a} f(x) = a$ for all a.

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all x in $0 < |x - a| < \delta$, $|f(x) - a| = |x - a| < \epsilon$. I.e. we need to find a δ such that $|x - a| < \delta$ implies $|x - a| < \epsilon$. This obviously works for any $\delta \leq \epsilon$. QED.

(Note the many options for δ, e.g. $\delta = \epsilon$, $\delta = \frac{\epsilon}{2}$, etc.)

Addition

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Let $L_f = \lim_{x\to a} f(x)$ and let $L_g = \lim_{x\to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(f+g)(x) - (L_f + L_g)| < \epsilon$$

I.e. we're trying to show $\lim_{x\to a}(f+g)(x)$ equals to $L_f + L_g$, the sum of the other two limits. Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}
|(f+g)(x) - (L_f + L_g)| &= |f(x) + g(x) - (L_f + L_g)| \\
&= |(f(x) - L_f) + (g(x) - L_g)| \\
&\leq |(f(x) - L_f)| + |(g(x) - L_g)| \quad \text{by triangle inequality}
\end{aligned}$$

By limit definition there exist positive δ_f, δ_g such that for all x

- $0 < |x - a| < \delta_f$ implies $|f(x) - L_f| < \epsilon/2$
- $0 < |x - a| < \delta_g$ implies $|g(x) - L_g| < \epsilon/2$

Recall that we can make ϵ as small as we like. Here we pick deltas for $\epsilon/2$ because it's convenient to make the equations work, as you will see in a second. For all x bounded by $0 < |x - a| < \min(\delta_f, \delta_g)$ we have

$$|(f(x) - L_f)| < \epsilon/2 \quad \text{and} \quad |(g(x) - L_g)| < \epsilon/2$$

Fix $\delta = \min(\delta_f, \delta_g)$. Then for all x bounded by $0 < |x - a| < \delta$ we have

$$\begin{aligned}
|(f+g)(x) - (L_f + L_g)| &\leq |(f(x) - L_f)| + |(g(x) - L_g)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

as desired.

Multiplication

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

Let $L_f = \lim_{x\to a} f(x)$ and let $L_g = \lim_{x\to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(fg)(x) - (L_f L_g)| < \epsilon$$

(i.e. we're trying to show $\lim_{x\to a}(fg)(x)$ equals to $L_f L_g$, the product of the other two limits.) Let's convert the left side of this inequality into a more convenient form:

$$
\begin{aligned}
|(fg)(x) - (L_f L_g)| &= |f(x)g(x) - L_f L_g| \\
&= |f(x)g(x) - L_f g(x) + L_f g(x) - L_f L_g| \\
&= |g(x)(f(x) - L_f) + L_f(g(x) - L_g)| \\
&\le |g(x)(f(x) - L_f)| + |L_f(g(x) - L_g)| && \text{by triangle inequality} \\
&= |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| && \text{in general } |ab| = |a||b|
\end{aligned}
$$

We now need to show there exists δ such that $0 < |x - a| < \delta$ implies

$$|g(x)||f(x) - L_f| + |L_f||g(x) - L_g| < \epsilon$$

We will do that by finding δ such that

1. $|g(x)||f(x) - L_f| < \epsilon/2$
2. $|L_f||g(x) - L_g| < \epsilon/2$

First, we show $|g(x)||f(x) - L_f| < \epsilon/2$.

By limit definition we can find δ_1 to make $|f(x) - L_f|$ as small as we like. But how small? To make $|g(x)||f(x) - L_f| < \epsilon/2$ we must find a delta such that $|f(x) - L_f| < \epsilon/2g(x)$. But to do that we need to get a bound on $g(x)$. Fortunately we know there exists δ_2 such that $|g(x) - L_g| < 1$ (we pick 1 because we must pick some bound, and 1 is as good as any). Thus $|g(x)| < |L_g| + 1$. And so, we can pick δ_1 such that $|f(x) - L_f| < \epsilon/2(|L_g| + 1)$.

Second, we show $|L_f||g(x) - L_g| < \epsilon/2$.

That is easy. By limit definition there exists a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \epsilon/2|L_f|$ for all x. Actually, we need a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \frac{\epsilon}{2(|L_f|+1)}$ for all x to avoid divide by zero, and of course that exists too.

Fix $\delta = \min(\delta_1, \delta_2, \delta_3)$. Now

$$
\begin{aligned}
|(fg)(x) - (L_f L_g)| &\le |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| \\
&< e/2 + e/2 = e
\end{aligned}
$$

as desired.

Reciprocal

Let $\lim_{x\to a} f(x) = L$. We prove $\lim_{x\to a}\left(\frac{1}{f}\right)(x) = 1/L$ when $L \neq 0$.

First we show $\frac{1}{f}$ is defined near a. By half-value neighborhood lemma (see 1.2) there exists δ_1 such that $0 < |x-a| < \delta_1$ implies $|f(x)| \geq |L|/2$ where $L \neq 0$. Therefore $f(x) \neq 0$ near a, and thus $\frac{1}{f}$ near a is defined.

Now all we must do is find a delta such that $\left|\frac{1}{f}(x) - \frac{1}{L}\right| < \epsilon$. Let's make the equation more convenient:

$$\begin{aligned}\left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\ &= \left|\frac{L-f(x)}{Lf(x)}\right| \\ &= \frac{|f(x)-L|}{|L||f(x)|} \\ &= \frac{|f(x)-L|}{|L|}\cdot\frac{1}{|f(x)|}\end{aligned}$$

Above we showed there exists δ_1 such that $0 < |x-a| < \delta_1$ implies $|f(x)| \geq |L|/2$. Raising both sides to -1 we get $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$. Continuing the chain of reasoning above we get

$$\begin{aligned}\frac{|f(x)-L|}{|L|}\cdot\frac{1}{|f(x)|} &\leq \frac{|f(x)-L|}{|L|}\cdot\frac{2}{|L|} \\ &= \frac{2}{|L|^2}|f(x)-L|\end{aligned}$$

(if you're confused about why this inequality works, left-multiply both sides of $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$ by $\frac{|f(x)-L|}{|L|}$.) Thus we must find δ_2 such that

$$\frac{2}{|L|^2}|f(x)-L| < \epsilon$$

That is easy. Since $\lim_{x\to a} f(x) = L$ we can make $|f(x)-L|$ as small as we like. Dividing both sides by $\frac{2}{|L|^2}$, we must make $|f(x)-L| < \frac{|L|^2\epsilon}{2}$. Thus we must fix $\delta = \min(\delta_1, \delta_2)$. QED.

1.4 Appendix: low-level proofs

While high level theorems allow us to easily compute complicated limits, it's instructive to compute a few limits for complicated functions straight from the definition. We do that here.

Aside: inequalities

We will often need to make an inequality of the following form work out:

$$|n||m| < \epsilon$$

Here ϵ is given to us, we have complete control over the upper bound of $|n|$, and $|m|$ can take on values outside our direct control. Obviously we can't make the inequality work without knowing *something* about $|m|$, so we'll try to find a bound for it in terms of other fixed values, or values we control.

For example, suppose $\lim_{x\to a} f(x) = L$ and we've discovered that $|m| < 3|a|+4$. Given that we control $|n|$, how do we bound it in terms of ϵ and $|a|$ in such a way that the inequality $|n||m| < \epsilon$ holds?

Since we control $|n|$ and $(3|a| + 4)$ is fixed, we can find $|n|$ small enough so that $|n|(3|a| + 4) < \epsilon$ holds. Then certainly any inequality whose left side is smaller, e.g. $|n|(3|a| + 3) < \epsilon$, will also hold. And since $|m|$ is always smaller than $3|a| + 4$, it follows $|n||m| < \epsilon$ will hold as well.

All we have left to do is find a bound for $|n|$ such that $|n|(3|a| + 4) < \epsilon$ holds, which is of course easy:

$$|n| < \frac{\epsilon}{3|a| + 4}$$

Having bound $|n|$ in this way, we can verify that $|n|(3|a| + 4) < \epsilon$ holds by multiplying both sides of the above inequality by $3|a| + 4$.

Limits of quadratic functions

We will prove directly from the limits definition that $\lim_{x\to a} x^2 = a^2$. Let $\epsilon > 0$ be given. We must show there exists δ such that $|x^2 - a^2| < \epsilon$ for all x in $0 < |x - a| < \delta$.

Observe that

$$|x^2 - a^2| = |(x - a)(x + a)| = |x - a||x + a|$$

Thus we must pick δ such that $|x - a||x + a| < \epsilon$. Since $0 < |x - a| < \delta$, picking δ conveniently happens to bound $|x - a|$, letting us make it as small as we want. But to know how small, we need to find an upper bound on $|x + a|$. We can do it as follows.

Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. 1/10, 10, etc.) Then since $|x - a| < \delta$:

$$\begin{aligned}
&|x - a| < 1 \\
&\implies -1 < x - a < 1 \\
&\implies 2a - 1 < x + a < 2a + 1 && \text{add } 2a \text{ to both sides}
\end{aligned}$$

We now have a bound on $x + a$, but we need one on $|x + a|$. It's easy to see $|x + a| < \max(|2a - 1|, |2a + 1|)$. By triangle inequality ($|a + b| \leq |a| + |b|$):

$$|2a - 1| \leq |2a| + |-1| = |2a| + 1$$
$$|2a + 1| \leq |2a| + |1| = |2a| + 1$$

Thus $|x + a| < |2a| + 1$, provided $|x - a| < 1$. Coming back to our original goal, $|x - a||x + a| < \epsilon$ when

- $|x - a| < 1$ and
- $|x - a| < \frac{\epsilon}{|2a|+1}$

Putting these together, $\delta = \min(1, \frac{\epsilon}{|2a|+1})$.

Limits of fractions

We will prove directly from the limits definition that $\lim_{x\to 2} \frac{3}{x} = \frac{3}{2}$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ for all x in $0 < |x - 2| < \delta$.

Let's manipulate $|\frac{3}{x} - \frac{3}{2}|$ to make it more convenient:

$$\left|\frac{3}{x} - \frac{3}{2}\right| = \left|\frac{6 - 3x}{2x}\right| = \frac{3}{2}\frac{|x - 2|}{|x|}$$

Thus we need to find δ such that

$$\frac{3}{2}\frac{|x - 2|}{|x|} < \epsilon$$
$$\implies \frac{|x - 2|}{|x|} < \frac{2\epsilon}{3}$$

Conveniently $0 < |x - 2| < \delta$ bounds $|x - 2|$. But now we need to find a bound for $|x|$. It would be extra convenient if we could show $|x| > 1$. Then we could set $\delta = \frac{2\epsilon}{3}$ (and thus bound $|x - 2| < \frac{2\epsilon}{3}$). A denominator greater than 1 would only make the fraction smaller than $\frac{2\epsilon}{3}$, ensuring $\frac{|x-2|}{|x|} < \frac{2\epsilon}{3}$ holds.

We will do exactly that. Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. 1/10, 10, etc.) Then since $|x - 2| < \delta$

$$|x - 2| < 1$$
$$\implies -1 < x - 2 < 1$$
$$\implies 1 < x < 3$$
$$\implies 1 < |x| < 3$$

Yes!! Luckily $\delta = 1$ implies $|x| > 1$! Thus, provided that $|x - 2| < 1$ and $|x - 2| < \frac{2\epsilon}{3}$, the inequality $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ holds. Putting the two constraints together, we get $\delta = \min(1, \frac{2\epsilon}{3})$.

2 Limits, Part II (Edge Cases)

2.1 Absence of limits

What does it mean to say L is not a limit of $f(x)$ at a? It flows out of the definition– there exist some ϵ such that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x) - L| \geq \epsilon$.

A stronger version is to say there is no limit of $f(x)$ at a. To do that we must prove that *any* L is not a limit of $f(x)$ at a.

Example: Absolute value fraction

Consider $f(x) = \frac{x}{|x|}$. It's easy to see that

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

We will show there is no limit of $f(x)$ near 0.

Weak version. First, let's prove a weak version– that $\lim_{x\to 0} f(x) \neq 0$. That is easy. Pick some reasonably small epsilon, say $\epsilon = \frac{1}{10}$. We must show that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x) - 0| \geq \frac{1}{10}$.

Let's pick some arbitrary x out of our permitted interval, say $x = \delta/2$. Then

$$|f(x) - 0| = |f(\delta/2)| = \left|\frac{\delta/2}{|\delta/2|}\right| = 1 \geq \frac{1}{10}$$

Strong version. Now we prove that $\lim_{x\to 0} f(x) \neq L$ for *any* L. Sticking with $\epsilon = \frac{1}{10}$ we proceed as follows.

If $L < 0$ take $x = \delta/2$. Then

$$|f(x) - L| = |f(\delta/2) - L| = \left|\frac{\delta/2}{|\delta/2|} - L\right| = |1 - L| > \frac{1}{10}$$

Similarly if $L \geq 0$ take $x = -\delta/2$. Then

$$|f(x) - L| = |f(-\delta/2) - L| = \left|\frac{-\delta/2}{|-\delta/2|} - L\right| = |-1 - L| > \frac{1}{10}$$

Example: Dirichlet function

The *dirichlet* function f is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

We prove $\lim_{x\to a} f(x)$ does not exist for any a.

Proof. Let $\epsilon = \frac{1}{10}$. Suppose for contradiction there exists L such that $\lim_{x \to a} f(x) = L$. There are two possibilities: either $L \leq \frac{1}{2}$ or $L > \frac{1}{2}$.

First suppose $L \leq \frac{1}{2}$. Pick any rational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |1 - L| \geq \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Similarly, suppose $L > \frac{1}{2}$. Pick any irrational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |0 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Thus $\lim_{x \to a} f(x)$ does not exist for any a, as desired.

2.2 One-sided limits

TODO

2.3 Infinities

TODO

3 Continuity, Part I

3.1 Definition of continuity

A function f is **continuous** at a when

$$\lim_{x \to a} f(x) = f(a)$$

Inlining the limits definition, f is continuous at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

We can simplify this definition slightly. Observe that in continuous functions $f(a)$ exists, and at $x = a$ we get $f(x) - f(a) = 0$. Thus we can relax the constraint $0 < |x - a| < \delta$ to $|x - a| < \delta$.

A function f is **continuous on an interval** (a, b) if it's continuous at all $c \in (a, b)$[4].

Nonzero Neighborhood Lemma

Armed with these definitions we can extend the half-value neighborhood lemma (see 1.2) in a useful way. The *nonzero neighborhood lemma* will come in handy when we prove the intermediate value theorem (see 5.1), so we may as well prove the lemma now.

Suppose f is continuous at a, and $f(a) \neq 0$. Then there exists $\delta > 0$ such that:

[4] Closed intervals are a tiny bit harder, and I'm keeping them out for brevity.

1. if $f(a) < 0$ then $f(x) < 0$ for all x in $|x - a| < \delta$.
2. if $f(a) > 0$ then $f(x) > 0$ for all x in $|x - a| < \delta$.

Intuitively the lemma states that there is some interval around a on which $f(x) \neq 0$ and has the same sign as $f(a)$.

Proof. The proof follows trivially from the half-value neighborhood lemma.

3.2 Recognizing continuous functions

The following theorems allow us to tell at a glance that large classes of functions are continuous (e.g. polynomials, rational functions, etc.)

Five easy proofs

Constants. Let $f(x) = c$. Then f is continuous at all a because

$$\lim_{x \to a} f(x) = c = f(a)$$

Identity. Let $f(x) = x$. Then f is continuous at all a because

$$\lim_{x \to a} f(x) = a = f(a)$$

Addition. Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at a. Then $f + g$ is continuous at a because

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

Multiplication. Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at a. Then $f \cdot g$ is continuous at a because

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = f(a) \cdot g(a) = (fg)(a)$$

Reciprocal. Let g be continuous at a. Then $\frac{1}{g}$ is continuous at a where $g(a) \neq 0$ because

$$\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{\lim_{x \to a} g(x)} = \frac{1}{g(a)} = \left(\frac{1}{g}\right)(a)$$

Slightly harder proof: composition

Let $f, g \in \mathbf{R} \to \mathbf{R}$. Let g be continuous at a, and let f be continuous at $g(a)$. Then $f \circ g$ is continuous at a. Put differently, we want to show

$$\lim_{x \to a} (f \circ g)(x) = (f \circ g)(a)$$

Unpacking the definitions, let $\epsilon > 0$ be given. We want to show there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$\begin{aligned} &|(f \circ g)(x) - (f \circ g)(a)| \\ &\qquad = |f(g(x)) - f(g(a))| < \epsilon \end{aligned}$$

By problem statement we have two continuities.

First, f is continuous at $g(a)$, i.e. $\lim_{X \to g(a)} f(X) = f(g(a))$. Thus there exists $\delta' > 0$ such that $|X - g(a)| < \delta'$ implies $|f(X) - f(g(a))| < \epsilon$.

Second, g is continuous at a, i.e. $\lim_{x \to a} g(x) = g(a)$. Thus there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \epsilon$. Since we can make ϵ be anything, we can set it to δ'.

I.e. there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta'$. Intuitively, $g(x)$ is close to $g(a)$. But by the first continuity, any X close to $g(a)$ implies

$$|f(X) - f(g(a))| < \epsilon$$

Thus $|f(g(x)) - f(g(a))| < \epsilon$, as desired.

3.3 Appendix: Stars over Babylon

Stars over Babylon is a modification of the Dirichlet function (see 2.1).

TODO: come back and study this.

4 Complete ordered fields

4.1 Motivation

The twelve ordered field axioms are sufficient to define limits, continuity, and prove all the theorems in the previous sections. Since the set $\mathcal{Q}$ of rational numbers is an ordered field[5], rationals have been sufficient for the work we've done so far. However, we are about to start proving slightly more sophisticated theorems about continous functions, and ordered fields will quickly start breaking our intuitions.

For example, consider the function $f(x) = x^2 - 2$ (a parabola shifted down two units). It's easy to see f is a continuous function, and thus our intuition is that we should be able to draw it without "lifting the tip of the pencil off the sheet of paper". Upon reflection however, it becomes obvious that in the universe limited to ordered fields this is impossible. f intersects the x-axis when $x^2 = 2$, but every high school student knows $\sqrt{2} \notin \mathcal{Q}$ (see 4.1 for proof). Thus there is

[5]The proof is straightforward, so I'm not including it here.

no $x \in \mathcal{Q}$ such that $f(x) = 0$. And since $\mathcal{Q}$ is an ordered field, it follows ordered fields alone aren't sufficient to resolve this problem.

The *intermediate value theorem* (see 5.1) formalizes the claim that a continuous function segment that starts below the x-axis and ends above the x-axis intersects the x-axis. But as we can see from the example above, this is not possible to prove with ordered field axioms alone. So before we proceed with further study of continuity, we need one more axiom called *the completeness axiom*, which we introduce in this chapter.

Combined with the twelve ordered field axioms, the completeness axiom forms *complete ordered fields*. These objects are sufficient to proceed with our study of calculus. We will see that rational numbers $\mathcal{Q}$ are not a complete ordered field, whereas real numbers $\mathcal{R}$ are.[6] Thus from here $\mathcal{R}$-valued functions will become our primary object of study.

Aside: sqrt(2) is irrational

Suppose $\sqrt{2} \in \mathcal{Q}$. Then there exist $a, b \in \mathcal{N}$ such that $\left(\frac{a}{b}\right)^2 = 2$. Assume a, b have no common divisor (since we can obviously keep simplifying until this is the case). Observe that both a and b cannot be even, otherwise we could simplify further.

Now we have $a^2 = 2b^2$. Thus a^2 is even, a must be even[7], and there exists $k \in \mathcal{N}$ such that $a = 2k$. Then $a^2 = 4k^2 = 2b^2$ so $2k^2 = b^2$. Thus b^2 is even and so b is even. Since both a and b cannot be even, this is a contradiction. Thus $\sqrt{2} \notin \mathcal{Q}$ as desired.

4.2 Least Upper Bound

Definition: b is an **upper bound** for S if $s \leq b$ for all $s \in S$.

For example:

- Any $b \geq 1$ is an upper bound for $S = \{x : 0 \leq x < 1\}$. E.g. $1, 2, 10$ are all upper bounds of S.
- By convention, *every* number is an upper bound for $\emptyset$.
- The set $\mathcal{N}$ of natural numbers has no natural upper bound. The proof is easy. Suppose $b \in \mathcal{N}$ is an upper bound for $\mathcal{N}$. But $b + 1 \in \mathcal{N}$, and $b + 1 > b$, which is a contradiction. Thus b isn't an upper bound for $\mathcal{N}$.[8]

[6] Proof that $\mathcal{R}$ is a complete ordered field requires construction of $\mathcal{R}$, which doesn't happen in Spivak until the last chapters. Thus I will not be delving into that here and ask the reader (i.e., currently myself) to take this on faith.

[7] Even numbers have even squares because $(2k)^2 = 4k^2 = 2 \cdot (2k^2)$

[8] We need to do a little more work to show $\mathcal{N}$ has no upper bound, natural or not. Be patient! We will prove this by the end of the section.

Definition: x is a **least upper bound** of A, if

1. x is an upper bound of A,
2. *and* if y is an upper bound of A, then $x \leq y$.

A set can have only one least upper bound. The proof is easy. Suppose x and x' are both least upper bounds of S. Then $x \leq x'$ and $x' \leq x$. Thus $x = x'$. Consequently, we can use a convenient notation $\sup A$ to denote the least upper bound of A.

Obligatory examples:

- Let $S = \{x : 0 \leq x < 1\}$. Then $\sup S = 1$.
- By convention, the empty set $\emptyset$ has no least upper bound.

4.3 Completeness axiom

We are now ready to state the completeness axiom.

Completeness [P13]: If A is a non-empty set of numbers that has an upper bound, then it has a least upper bound.

Claim: rational numbers are not complete.
Proof: Let $C = \{x : x^2 < 2 \text{ and } x \in \mathcal{Q}\}$. Suppose for contradiction rational numbers are complete. Then there exists $b \in \mathcal{Q}$ such that $b = \sup C$. Observe that

- $b^2 \neq 2$ as that would imply $b = \sqrt{2}$ and thus $b \notin \mathcal{Q}$.
- $b^2 \not< 2$ as there would exist some $x \in C$ such that $b^2 < x^2 < 2$. Thus $b < x$ and b is not the upper bound.

Therefore $b^2 > 2$. But this implies there exists some $x \in \mathcal{Q}$ such that $2 < x^2 < b^2$. Thus x is greater than every element in C, and $x < b$. So b is not the *least* upper bound. We have a contradiction, therefore rational numbers are not complete, as desired.

Claim: completeness cannot be derived from ordered fields.
Proof: $\mathcal{Q}$ is not complete and $\mathcal{Q}$ is an ordered field. Thus completeness is not a property of ordered fields.

Claim: real numbers are complete.
Proof [deferred]: The completeness property can be derived from the construction of real numbers $\mathcal{R}$, which makes reals a **complete ordered field**. The proof requires we study the actual construction of $\mathcal{R}$, which Spivak leaves until the last chapters. Thus for the moment the proof will be taken on faith. In any case, it is better to build calculus upon abstract complete ordered fields than upon concrete real numbers.

4.4 Consequences of completeness

$\mathcal{N}$ is not bounded above

We've shown $\mathcal{N}$ has no upper bound in $\mathcal{N}$. Now we show $\mathcal{N}$ has no upper bound in $\mathcal{R}$.

Suppose for contradiction $\mathcal{N}$ has an upper bound. Since $\mathcal{N} \neq \emptyset$ then by completeness $\mathcal{N}$ has a least upper bound. Let $\alpha = \sup \mathcal{N}$. Then:

$$\begin{aligned} &\alpha \geq n \text{ for all } n \in \mathcal{N} \\ \implies &\alpha \geq n+1 \text{ for all } n \in \mathcal{N} && \text{since } n+1 \in \mathcal{N} \text{ if } n \in \mathcal{N} \\ \implies &\alpha - 1 \geq n \text{ for all } n \in \mathcal{N} \end{aligned}$$

Thus $\alpha - 1$ is *also* an upper bound for $\mathcal{N}$. This contradicts that $\alpha = \sup \mathcal{N}$. Therefore $\mathcal{N}$ is not bounded above, as desired.

$\sqrt{2}$ exists

We show $\sqrt{2} \in \mathcal{R}$. Let $S = \{y \in \mathcal{R} : y^2 < 2\}$. Obviously S is non-empty and has an upper bound. Thus by completeness property it has a least upper bound. Let $x = \sup S$. Note that $1 \in S$ and 2 is an upper bound of S. Thus $1 \leq x \leq 2$. We show $x^2 = 2$ by showing $x^2 \not< 2$ and $x^2 \not> 2$.

Case 1. Suppose for contradiction $x^2 < 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned} (x+\epsilon)^2 =& x^2 + 2\epsilon x + \epsilon^2 \\ &\leq x^2 + 4\epsilon + \epsilon && \text{since } x < 2 \text{ and } \epsilon < 1 \\ &= x^2 + 5\epsilon < 2 && \text{since } x^2 < 2 \text{ (by supposition), we can pick} \\ &&& \text{a small enough } \epsilon \text{ to make this true} \end{aligned}$$

Thus there exists ϵ such that $(x+\epsilon)^2 < 2$. By definition of S it follows $x+\epsilon \in S$, which contradics that x is the least upper bound. Therefore $x^2 \not< 2$

Case 2. Suppose for contradiction $x^2 > 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned} (x-\epsilon)^2 =& x^2 - 2\epsilon x + \epsilon^2 \\ &\geq x^2 - 2\epsilon x && \text{since } \epsilon^2 > 0 \\ &\geq x^2 - 4\epsilon && \text{since } x \leq 2 \\ &> 2 && \text{since } x^2 > 2 \text{ (by supposition), we can pick} \\ &&& \text{a small enough } \epsilon \text{ to make this true} \end{aligned}$$

Thus $(x-\epsilon)^2 > 2$, which by definition of S implies $x-\epsilon > y$ for all $y \in S$. So $x-\epsilon$ is an upper bound of S. We have a contradiction– since $x-\epsilon < x$, it follows x is not a least upper bound. Therefore $x^2 \not> 2$ as desired.

Since $x^2 \not< 2$ and $x^2 \not> 2$, it follows $x^2 = 2$ as desired.

Archimedean property

Handwavy definition: the Archimedean property states that you can fill the universe with tiny grains of sand.

Formal defition: let $\epsilon > 0$ be small and let $r > 0$ be large. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > r$.

Proof: suppose for contradiction the property is false. Then there exist ϵ, r such that for all $n \in \mathcal{N}$, $n\epsilon \leq r$. Therefore $n \leq \frac{r}{\epsilon}$. This implies $\mathcal{N}$ is bounded, which a contradiction.

A useful special case is when $r = 1$. In this case the Archimedean property can be restated as follows. Let $\epsilon > 0$ be small. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > 1$. Put differently, there exists $n \in \mathcal{N}$ such that $\frac{1}{n} < \epsilon$.

A few more notes on the Archimedean property:

- Obviously the Archimedean property follows from completeness, as shown above.
- The Archimedean property is true in $\mathcal{Q}$ and can be proven without being assumed[9].
- Completeness does not follow from the Archimedean property. The proof is easy: the Archimedean property holds on $\mathcal{Q}$, and we know $\mathcal{Q}$ is not complete as shown above.

Density

Let $x, y \in \mathcal{R}$. Then S is a **dense subset** of $\mathcal{R}$ if there is an element of S in (x, y). Put differently, there is an element of S between any two points in $\mathcal{R}$.

- Obviously $\mathcal{R}$ is a dense subset of itself.
- Integers are not a dense subset of $\mathcal{R}$. E.g. there is no integer between 1.1 and 1.9.
- The set of positive numbers $\{x : x \in \mathcal{R}, x > 0\}$ is not a dense subset of $\mathcal{R}$. E.g. there is no positive number between -2 and -1.

[9]Excluding the proof here, but it's fairly simple

Claim: the set of rational numbers $\mathcal{Q}$ is dense.
Proof: let $x, y \in \mathcal{R}$ be given. Suppose we can show there exists a rational in (x, y) for $0 \leq x < y$. Then:

- Given $x < y \leq 0$, there is a rational r in $(-y, -x)$. So $-r$ is in (x, y).
- Given $x < 0 < y$, there is a rational r in $(0, y)$. So r is of course also in (x, y).

Thus all we must do is prove there exists a rational in (x, y) for $0 \leq x < y$.

Let $0 \leq x < y$ be given. By the Archimedean property there exists $n \in \mathcal{N}$ such that $\frac{1}{n} < y - x$. Because (a) $\mathcal{N}$ is unbounded and (b) $\mathcal{N}$ is well-ordered, there exists the least integer $m \in \mathcal{N}$ such that $m \geq ny$.

First, observe that

$$\begin{aligned} & m - 1 < ny && \text{or } m \text{ wouldn't be the } \textit{least} \text{ integer } m \geq ny \\ \implies & \frac{m-1}{n} < y \end{aligned}$$

Second, suppose for contradiction $\frac{m-1}{n} \leq x$. Then

$$\begin{aligned} & \frac{m-1}{n} \leq x \\ \implies & \frac{m}{n} - \frac{1}{n} \leq x \\ \implies & -\frac{1}{n} \leq x - \frac{m}{n} \\ \implies & \frac{1}{n} \geq \frac{m}{n} - x \\ \implies & \frac{1}{n} \geq y - x && \text{recall } m \geq ny\text{, thus } \frac{m}{n} \geq y \end{aligned}$$

This is a contradiction, thus $\frac{m-1}{n} > x$.

Therefore $\frac{m-1}{n} \in (x, y)$ as desired.

Claim: the set of irrational numbers $\mathcal{R} \setminus \mathcal{Q}$ is dense.
Proof: let $x, y \in \mathcal{R}$ be given. By density of the rationals there exists $r \in \mathcal{Q}$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Multiplying each side by $\sqrt{2}$, we get $x < \sqrt{2}r < y$. We know $\sqrt{2}r$ is irrational. Thus there exists an irrational number between any two numbers in $\mathcal{R}$, and the set of irrationals $\mathcal{R} \setminus \mathcal{Q}$ is dense as desired.

5 Continuity, Part II

5.1 Intermediate Value Theorem

Theorem: if f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there exists $x \in [a, b]$ such that $f(x) = 0$.

Or intuitively, if $f(a)$ is below zero and $f(b)$ is above zero, f must cross the x-axis somewhere.

Proof: intuitively, we will locate the smallest number x on the x-axis where $f(x)$ first crosses from negative to positive, and show that $f(x)$ must be zero.

First, we define a set A that contains all inputs to f before f crosses from negative to positive for the first time:

$$A = \{x : a \leq x \leq b, \text{and } f \text{ is negative on the interval } [a, x]\}$$

We know $A \neq \emptyset$ since $a \in A$, and b is an upper bound of A. Thus A has a least upper bound α such that $a \leq \alpha \leq b$. By nonzero neighborhood lemma (see 3.1) we know there is some interval around a on which f is negative, and some interval around b on which f is positive. Thus we can further refine the bound on α to $a < \alpha < b$.

We now show $f(\alpha) = 0$ by eliminating the possibilities $f(\alpha) < 0$ and $f(\alpha) > 0$.

Case 1. Suppose for contradiction $f(\alpha) < 0$. By nonzero neighborhood lemma there exists $\delta > 0$ such $|x - \alpha| < \delta$ implies $f(x) < 0$ for all x. But that means numbers in $(\alpha - \delta, \alpha + \delta)$ are in A. E.g. $(\alpha + \delta/2) \in A$. Since $\alpha + \delta/2 > \alpha$, α is not an upper bound of A, and is thus not the least upper bound.

Case 2. Suppose for contradiction $f(\alpha) > 0$. By nonzero neighborhood lemma there exists $\delta > 0$ such $|x - \alpha| < \delta$ implies $f(x) > 0$ for all x. But that means numbers in $(\alpha - \delta, \alpha + \delta)$ are *not* in A, and there exist many upper bounds of A less than α. E.g. $\alpha - \delta/2$ is an upper bound of A, and since $\alpha - \delta/2 < \alpha$, α is not the *least* upper bound.

Both cases lead to contradiction, therefore $f(\alpha) = 0$. QED.

IVT generalization

The intermediate value theorem is usually presented in a more general way. If f is continuous on $[a, b]$ and $f(a) < c < f(b)$ or $f(a) > c > f(b)$ then there is some x in $[a, b]$ such that $f(x) = c$.

Intuitively, f takes on any value between $f(a)$ and $f(b)$ at some point in the interval $[a, b]$.

Proof. This trivially follows from the the theorem as initially stated. There are two cases:

Case 1: $f(a) < c < f(b)$. Let $g = f - c$. Then g is continuous and $g(a) < 0 < g(b)$. Thus there is some x in $[a, b]$ such that $g(x) = 0$. But that means $f(x) = c$.

Case 2: $f(a) > c > f(b)$. Observe that $-f$ is continuous on $[a, b]$ and $-f(a) < -c < -f(b)$. By case 1 there is some x in $[a, b]$ such that $-f(x) = -c$, which means $f(x) = c$.

QED.

5.2 Extreme Value Theorem

We will build up to the *extreme value theorem* by proving three progressively more important claims (the last one being the theorem itself):

1. If f is continuous at a then there is some interval around a on which f is bounded above.
2. If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$.
3. Finally, if f is continuous on $[a, b]$, f attains its maximum on $[a, b]$.

To see why we need the extreme value theorem, consider $f = \frac{1}{x}$. f is discontinuous at 0 and approaches infinity. Thus f does not attain a maximum value on the interval $[0, 1]$.

***

Claim 1 (bounded neighborhood lemma): if f is continuous at a, then there is $\delta > 0$ such that f is bounded above on the interval $(a - \delta, a + \delta)$.

Proof: The proof is trivial. Inlining the definition of continuity, for any $\epsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ for all x. Thus $f(a) + \epsilon$ is the upper bound on f within $(a - \delta, a + \delta)$, as desired.

(Note that we can pick any ϵ to concretize the proof, for example $\epsilon = 1$.)

***

Claim 2 (boundedness theorem): if f is continuous on $[a, b]$, then f is bounded above on $[a, b]$. I.e. there is some numbers N such that $f(x) \leq N$ for all x in $[a, b]$.

Intuitively, the claim means the graph of f lies below some line.

Proof: intuitively, we will try to find the smallest number x on the x-axis where $f(x)$ becomes unbounded above, and discover that there is no such number in $[a, b]$.

First, we define a set A that contains all inputs to f before f stops being bounded above:

$$A = \{x : a \leq x \leq b, \text{ and } f \text{ is bounded above on } [a, x]\}$$

By bounded neighborhood lemma f is bounded above in the neighborhood of a[10]. Thus we know $A \neq \emptyset$ because $a \in A$. Further, b is an upper bound of A. Thus A has a least upper bound.

Let $\alpha = \sup A$. To prove the boundedness theorem we must prove two claims:

1. $\alpha = b$, i.e. f does not ever stop being bounded above before b.
2. $(\alpha = b) \in A$, as $\sup A$ is not necessarily a member of A.

First, we prove $\alpha = b$. Suppose for contradiction $\alpha < b$. By bounded neighborhood lemma there is some $\delta > 0$ such that f is bounded above in $(\alpha - \delta, \alpha + \delta)$. But that means there are many upper bounds greater than α, for example $\alpha + \delta/2$. Thus α is not the *least* upper bound. We have a contradiction, and so $\alpha = b$.

Second, we prove $(\alpha = b) \in A$. By bounded neighborhood lemma there is some $\delta > 0$ such that f is bounded above in $(b - \delta, b]$. Pick any x_0 such that $b - \delta < x_0 < b$. Then:

- $x_0 < b = \alpha$. Since α is the least upper bound it follows $x_0 \in A$. Thus f is bounded above on $[a, x_0]$.
- f is bounded above on $[x_0, b]$.

Since f is bounded above on $[a, x_0]$ and on $[x_0, b]$, it follows f is bounded above on $[a, b]$ as desired. QED.

***

Claim 3: (extreme value theorem): If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$.

Intuitively, the theorem states that f achieves its maximum.

Proof: Let A be the set of f's outputs on $[a, b]$:

$$A = \{f(x) : x \text{ in } [a, b]\}$$

[10] We are being sloppy here as we actually need a left-sided and right-sided version of the bounded neighborhood lemma. I am papering over this for now, but will need to fix at some point by giving proper one sided proofs

Since $[a, b]$ isn't empty, $A \neq \emptyset$. By boundedness theorem, f is bounded on $[a, b]$, and so A has an upper bound. Thus A has a least upper bound. Let $\alpha = \sup A$. By definition $\alpha \geq f(x)$ for x in $[a, b]$. Thus it suffices to show $\alpha \in A$ (i.e. $\alpha = f(y)$ for some y in $[a, b]$).

Let's consider a function g[11]:

$$g = \frac{1}{\alpha - f(x)}, \quad x \text{ in } \ [a, b]$$

Suppose for contradiction $\alpha \notin A$. Then the denominator is never zero and g is continuous. Therefore:

$$\begin{aligned}
\frac{1}{\alpha - f(x)} &< M && \text{by boundedness theorem} \\
& && \text{for some bound } M \\
\implies \alpha - f(x) &> \frac{1}{M} && \text{take reciprocal} \\
\implies -f(x) &> \frac{1}{M} - \alpha && \\
\implies f(x) &< \alpha - \frac{1}{M} && \text{times } -1
\end{aligned}$$

But this contradicts that α is the *least* upper bound. Thus $\alpha \in A$ as desired. QED.

EVT generalization

Claim 2 above proves a continuous f is bounded above. The *boundedness theorem* is usually presented slightly more generally, i.e. it proves f is bounded above *and* below. This is easy to derive from claim 2.

Claim 2a (boundedness theorem): if f is continuous on $[a, b]$, then f is bounded *below* on $[a, b]$. I.e. there is some numbers N such that $f(x) \geq N$ for all x in $[a, b]$.

Proof: observe that $-f$ is continuous on $[a, b]$. By claim 2 there exists a number M such that $-f(x) \leq M$ for all x in $[a, b]$. But that means $f(x) \geq -M$ for all x in $[a, b]$. QED.

***

Similarly, claim 3 above proves a continuous f attains its maximum. The extreme value theorem is usually presented slightly more generally, i.e. it states a continuous f attains both its maximum and its minimum. Again, this is easy to derive from claim 3.

[11] g is a bit of a rabbit pulled out of a magic hat, but to quote a great British statesman, them's the breaks

Cliam 3a (extreme value theorem): If f is continuous on $[a, b]$, then there is some number y in $[a, b]$ such that $f(y) \leq f(x)$ for all x in $[a, b]$.

Proof: Observe that $-f$ is continous on $[a, b]$. By claim 3 there is some y in $[a, b]$ such that $-f(y) \geq -f(x)$ for all x in $[a, b]$. But that means that $f(y) \leq f(x)$ for all x in $[a, b]$. QED.

5.3 Appendix: IVT and EVT consequences

Claim 1a: Every positive number has a square root. I.e. if $\alpha > 0$, then there is some number x such that $x^2 = \alpha$.

Proof: TBD.

***

Claim 1b: Every positive number has an nth root. I.e. if $\alpha > 0$, then there is some number x such that $x^n = \alpha$.

Proof: TBD.

***

Claim 1c: Let n be odd. Then every number has an nth root. I.e. there is some number x such that $x^n = \alpha$ for all α.

Proof: TBD.

***

Claim 2: If n is odd, then any equation of the form

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$$

has a root.

Proof: TBD.

***

Claim 3: If n is even and $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then there is a number y such that $f(y) \leq f(x)$ for all x.

Proof: TBD.

***

Claim 4: Consider the equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_0 = c$$

and suppose n is even. Then there is a number m such that the equation has a solution for $c \geq m$ and has no solution for $c < m$.

Proof: TBD.