Spivak Calculus/Math 113H homework

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Only the first four chapters are here. I started taking notes at chapter 5, so the homework problems are inlined in the notes.

Contents

1 Basic Properties of Numbers

Problem 1v

Prove $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$

Solution

$$\begin{aligned}
&(x-y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}) \\
&= x(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}) - y(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}) \\
&= (x^n + x^{n-1}y + \ldots + x^2y^{m-2} + xy^{n-1}) - (x^{n-1}y + x^{n-2}y^2 + \ldots + xy^{n-1} + y^n) \\
&= x^n + (x^{n-1}y + \ldots + x^2y^{m-2} + xy^{n-1} - x^{n-1}y + x^{n-2}y^2 + \ldots + xy^{n-1}) + y^n \\
&= x^n - y^n
\end{aligned}$$

Problem 1vi

Prove $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Solution

$$\begin{aligned}
(x + y)(x^2 - xy + y^2) &= x(x^2 - xy + y^2) + y(x^2 - xy + y^2) \\
&= (x^3 - x^2y + xy^2) + (x^2y - xy^2 + y^3) \\
&= x^3 + (-x^2y + xy^2 + x^2y - xy^2) + y^3 \\
&= x^3 + y^3
\end{aligned}$$

Problem 5iii

Prove that if $a < b$ and $c > d$ then $a - c < b - d$.

Solution

Observe that $a < b \implies b - a > 0$ and $c > d \implies c - d > 0$. Then

$$\begin{aligned}
(b - d) - (a - c) &= b - d - a + c \\
&= (b - a) + (c - d) > 0
\end{aligned}$$

Thus $a - c < b - d$ as desired.

Problem 5vii

Prove that if $0 < a < 1$ then $a^2 < a$.

Solution

Observe that $a - a^2 = a(1 - a)$. Further $a > 0$ and $1 - a > 0$, thus $a(1 - a) > 0$. Therefore $a - a^2 > 0$ and thus $a^2 < a$ as desired.

Problem 5viii

Prove that if $0 \le a < b$ and $0 \le c < d$, then $ac < bd$.

Solution

Suppose $c = 0$. Then $ac = 0$. Since $b > 0$ and $d > 0$, $bd > 0$, and thus $ac < bd$.

Alternatively, suppose $c > 0$. Since $a < b$ and $c > 0$, it follows $ac < bc$. Similarly since $c < d$ and $b > 0$ it follows $bc < bd$. So, $ac < bc, bc < bd$, therefore $ac < bd$ as desired.

Problem 11

Find all numbers x for which

Solution

(i) $|x-3| = 8$

- $x - 3 = 8 \implies x = 11$
- $x - 3 = -8 \implies x = -5$

(ii) $|x-3| < 8$

$$\begin{aligned} -8 < x - 3 < 8 \\ -5 < x < 11 \end{aligned}$$

(iii) $|x+4| < 2$

$$\begin{aligned} -2 < x + 4 < 2 \\ -6 < x < -2 \end{aligned}$$

(iv) $|x-1| + |x-2| > 1$
Checking the intervals $(-\infty, 1), (1, 2), (2, \infty)$, the two intervals that work are $x < 1$ and $x > 2$.

(v) $|x-1| + |x+1| < 2$
Checking the intervals $(-\infty, 1), (-1, 1), (1, \infty)$, there is no solution that satisfies the equation.

(vi) $|x-1| + |x+1| < 1$
Since there is no solution for (v), there is certainly no solution for (vi) as it's a stricter inequality.

(vii) $|x-1| \cdot |x+1| = 0$
$x = 1, x = -1$

(viii) $|x-1| \cdot |x+2| = 3$
There are three cases:

- $x < -2$: The equation becomes $(1-x)(-x-2) = 3$.
- $-2 < x < 1$: The equation becomes $(1-x)(x+2) = 3$
- $x > 1$ The equation becomes $(x-1)(x+2) = 3$.

The first and last equations simplify to $x^2+x-5=0$ and have two obvious roots (based on quadratic formula). The middle equation has no solutions. Plotting confirms this.

Problem 13

Prove that

$$\max(x,y) = \frac{x+y+|y-x|}{2}$$

$$\min(x,y) = \frac{x+y-|y-x|}{2}$$

Derive a formula for $\max(x,y,z)$ and $\min(x,y,z)$.

Solution

First we prove $\max(x,y) = \frac{x+y+|y-x|}{2}$. Suppose $x > y$, i.e. $\max(x,y) = x$. Then $|y-x| = x-y$. Therefore

$$\frac{x+y+|y-x|}{2} = \frac{x+y+x-y}{2} = x$$

as desired. Conversely suppose $x < y$, i.e. $\max(x,y) = y$. Then $|y-x| = y-x$. Therefore

$$\frac{x+y+|y-x|}{2} = \frac{x+y+y-x}{2} = y$$

as desired.

Similarly we prove $\min(x,y) = \frac{x+y-|y-x|}{2}$. Suppose $x > y$, i.e. $\min(x,y) = y$. Then $|y-x| = x-y$. Therefore

$$\frac{x+y-|y-x|}{2} = \frac{x+y-x+y}{2} = y$$

as desired. Conversely suppose $x < y$, i.e. $\min(x,y) = x$. Then $|y-x| = y-x$. Therefore

$$\frac{x+y-|y-x|}{2} = \frac{x+y-y+x}{2} = x$$

as desired.

It's easy to see that if $x = y$, both formulas compute the correct result

$$\min(x,y) = \max(x,y) = x = y$$

We now derive $\max(x, y, z)$:

$$\begin{aligned}
\max(x,y,z) &= \max(x, \max(y,z)) \\
&= \frac{x + \max(y,z) + |\max(y,z) - x|}{2} \\
&= \frac{x + \frac{y+z+|z-y|}{2} + |\frac{y+z+|z-y|}{2} - x|}{2} \\
&= \frac{\frac{2x+y+z+|z-y|}{2} + |\frac{y+z+|z-y|-2x}{2}|}{2} \\
&= \frac{2x + y + z + |z-y| + |y + z + |z-y| - 2x|}{4}
\end{aligned}$$

Similarly

$$\begin{aligned}
\min(x,y,z) &= \min(x, \min(y,z)) \\
&= \frac{x + \frac{y+z-|z-y|}{2} - |\frac{y+z-|z-y|}{2} - x|}{2} \\
&= \frac{\frac{2x+y+z-|z-y|}{2} - |\frac{y+z-|z-y|-2x}{2}|}{2} \\
&= \frac{2x + y + z - |z-y| - |y + z - |z-y| - 2x|}{4}
\end{aligned}$$

2 Numbers of various sorts

Problem 1i

Prove the following formula by induction

$$1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

Observe that the base case $n = 1$ holds:

$$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$$

Suppose the equation holds for some integer k. Adding $(k+1)^2$ to both sides we get:

$$\begin{aligned}
1^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k^2+k)(2k+1)}{6} + k^2 + 2k + 1 \\
&= \frac{2k^3 + 3k^2 + k}{6} + \frac{6k^2 + 12k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

Consider the case $k+1$:

$$\begin{aligned}
1^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k^2+3k+2)(2k+3)}{6} \\
&= \frac{2k^3 + 3k^2 + 6k^2 + 9k + 4k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}$$

Thus the formula holds for any positive integer n as desired.

Problem 1ii

Prove the following formula by induction

$$1^3 + \ldots + n^3 = (1 + \ldots + n)^2$$

Solution

Observe that the base case $n = 1$ holds:

$$1^3 = 1^2$$

Suppose the equation holds for some integer k. Recall that $1 + \ldots + k = \frac{k(k+1)}{2}$.

Adding $(k+1)^3$ to both sides we get:

$$\begin{aligned}
1^3 + \ldots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k^2+k)^2}{4} + k^3 + 3k^2 + 3k + 1 \\
&= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}
\end{aligned}$$

Now consider the case $k+1$:

$$\begin{aligned}
1^3 + \ldots + k^3 + (k+1)^3 &= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= \frac{(k^2+3k+2)^2}{4} \\
&= \frac{k^4 + 3k^3 + 2k^2 + 3k^3 + 9k^2 + 6k + 2k^2 + 6k + 4}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}
\end{aligned}$$

Thus the formula holds for any positive integer n as desired.

Problem 5

(a) Prove by induction on n that

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

if $r \neq 1$.
Observe that the base case $n = 1$ holds:

$$1 + r = \frac{1 - r^2}{1 - r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$$

Suppose the equation holds for some integer k. Adding r^{k+1} to both sides we get:

$$\begin{aligned}
1 + r + r^2 + \ldots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\
&= \frac{1 - r^{k+1} + (1-r)r^{k+1}}{1 - r} \\
&= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\
&= \frac{1 - r^{k+2}}{1 - r}
\end{aligned}$$

Thus the equation holds as desired.

(b) Derive this result by setting $S = 1 + r + \ldots + r^n$, multiplying this equation by r, and solving the two equations for S.

$$
\begin{aligned}
&rS = r + r^2 + \ldots + r^{n+1} \\
&\implies S - rS = (1 + r + \ldots + r^n) - (r + r^2 + \ldots + r^{n+1}) \\
&\implies S(1-r) = 1 - r^{n+1} \\
&\implies S = \frac{1 - r^{n+1}}{1-r}
\end{aligned}
$$

Solution

Problem 10

Prove the principle of mathematical induction from the well-ordering principle.

Solution

Let P be a property indexed on natural numbers such that: P_1 is true, and P_{k+1} is true if P_k is true for $k \in \mathbf{N}$. We must show P_n is true for all $n \in \mathbf{N}$.

Let S be a set of natural numbers for which P is false, i.e. $S = \{j : P_j \text{ is false}\}$. Suppose for contradiction S is not empty. Then by well ordering principle there exists $m \in S$ such that m is the smallest element in S.

Since $m \in S$, P_m is false. Further, since m is the smallest element of S, P_{m-1} is true (if it weren't, $m-1$ would be in S and m wouldn't be the smallest element). But P_{m-1} being true implies by induction P_m is true. We have a contradiction. Therefore S is empty, and P_n is true for all $n \in \mathbf{N}$ as desired.

Problem 10-modified

Prove the well-ordering principle from the principle of mathematical induction. [*Adding for my own understanding.*]

Solution

Let S be a set of natural numbers with no least element. We show that S must be empty (and thus every non-empty set of natural numbers has a least element).

Let B be the set of natural numbers n such that $1 \ldots n \notin S$. We will show by induction that all $n \in \mathbf{N}$ are in B (and thus S is empty).

First, $1 \in B$ (if 1 were in S, then S would have 1 as its smallest member). Thus the base case holds.

Suppose $k \in B$ (and thus $1 \ldots k \notin S$). Consider $k+1$. Since $1 \ldots k \notin S$, then $k+1 \notin S$ (otherwise $k+1$ would be the smallest member of S). Therefore $k+1 \in B$. Thus by induction $n \in B$ for all $n \in N$, and S is empty as desired.

Problem 14a

Prove that $\sqrt{2} + \sqrt{6}$ is irrational.

Solution

Suppose $\sqrt{2} + \sqrt{6}$ is rational. Then its square $8 + 4\sqrt{3}$ must be rational as well. Recall that $\mathcal{Q}$ is closed under addition, thus $8 + 4\sqrt{3} + (-8) = 4\sqrt{3}$ is rational. Similarly, $\mathcal{Q}$ is closed under multiplication, thus $4\sqrt{3} \cdot \frac{1}{4} = \sqrt{3}$ is rational. We have a contradiction, and thus $\sqrt{2} + \sqrt{6}$ is irrational as desired.

3 Functions

Problem 3ii

Find the domain of the function defined by the following formula

$$f(x) = \sqrt{1 - \sqrt{1 - x^2}}$$

Solution

There are two constraints: $x^2 \leq 1$ and $\sqrt{1 - x^2} \leq 1$. Taking a square root of the first constraint we get $x \leq 1$ and $x \geq -1$. With the second constraint:

$$\begin{aligned} &\sqrt{1 - x^2} \leq 1 \\ &\implies 1 - x^2 \leq 1 \\ &\implies x^2 \geq 0 \end{aligned}$$

Observe that $x^2 \geq 0$ is true for all x. Thus the domain of f is $[-1, 1]$.

Problem 9a

If A is any set of real numbers, define a function C_A as follows:

$$C_A(x) = \begin{cases} 1, x \in A \\ 2, x \notin A \end{cases}$$

Find expressions for $C_{A \cap B}$ and $C_{A \cup B}$ and $C_{\mathbf{R}-A}$, in terms of C_A and C_B.

Solution

$$C_{A\cap B} = C_A \cdot C_B$$
$$C_{A\cup B}(x) = C_A + C_B - C_A \cdot C_B$$
$$C_{\mathbf{R}-A} = 1 - C_A$$

Problem 12a,b,c

A function f is **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$. For example, f is even if $f(x) = x^2$ or $f(x) = |x|$ or $f(x) = \cos x$, while f is odd if $f(x) = x$ or $f(x) = \sin x$.

Solution

(a) Determine whether $f + g$ is even, odd, or not not necessarily either, in the four cases obtained by choosing f even or odd and g even or odd.

Suppose both f, g are even. Then $f + g$ is even:

$$(f+g)(x) = f(-x) + g(-x) = (f+g)(-x)$$

Suppose both f, g are odd. Then $f + g$ is odd:

$$(f+g)(x) = -f(-x) - g(-x) = -(f+g)(-x)$$

Suppose f is even, g is odd. Then $f + g$ is neither:

$$(f+g)(x) = f(-x) - g(-x) = (f-g)(-x)$$

Suppose f is odd, g is even. Then $f + g$ is neither:

$$(f+g)(x) = -f(-x) + g(-x) = (g-f)(-x)$$

(b) Do the same for $f \cdot g$.

Suppose both f, g are even. Then $f \cdot g$ is even:

$$(f \cdot g)(x) = f(-x) \cdot g(-x) = (f \cdot g)(-x)$$

Suppose both f, g are odd. Then $f \cdot g$ is even:

$$(f \cdot g)(x) = -f(-x) \cdot -g(-x) = (f \cdot g)(-x)$$

Suppose f is even, g is odd. Then $f \cdot g$ is odd:

$$(f \cdot g)(x) = f(-x) \cdot -g(-x) = -(f \cdot g)(-x)$$

Suppose f is odd, g is even. Then $f \cdot g$ is odd:

$$(f \cdot g)(x) = -f(-x) \cdot g(-x) = -(f \cdot g)(-x)$$

(c) Do the same for $f \circ g$.

Suppose both f, g are even. Then $f \circ g$ is even:

$$(f \circ g)(x) = f(g(x)) = f(g(-x)) = (f \circ g)(-x)$$

Suppose both f, g are odd. Then $f \circ g$ is odd:

$$(f \circ g)(x) = f(g(x)) = -f(g(-x)) = (-f \circ g)(-x) = -(f \circ g)(-x)$$

Suppose f is even, g is odd. Then $f \circ g$ is even:

$$(f \circ g)(x) = f(g(x)) = f(-g(-x)) = f(g(-x)) = (f \circ g)(-x)$$

Suppose f is odd, g is even. Then $f \circ g$ is even:

$$(f \circ g)(x) = f(g(x)) = f(g(-x)) = (f \circ g)(-x)$$

4 Graphs

Problem 1iv

Indicate on a straight line the set of all x satisfying the following condition. Name the set, using the notation for intervals.

$$|x^2 - 1| < \frac{1}{2}$$

Solution

$$\begin{aligned}
&-\frac{1}{2} < x^2 - 1 < \frac{1}{2} \\
&\implies \frac{1}{2} < x^2 < \frac{3}{2} \\
&\implies x \in ((-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)) \cap (-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}) \\
&\implies (-\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}})
\end{aligned}$$

Problem 4

Draw the set of all points (x, y) satisfying the following conditions:

Solution

(i) $|x| + |y| = 1$
(ii) $|x| - |y| = 1$

I nailed the answers to these in my notebook (I used a red pen on top of the page so I can find the page easily by looking at the closed notebook). There is a way to plot in latex but it sounds like a pain to learn. I'll do it later if it keeps coming up.

Problem 8a

Prove that the graphs of the functions

$$f(x) = mx + b$$
$$g(x) = nx + c$$

are perpendicular if $mn = -1$ by computing the squares of the lengths of the sides of the triangle in Figure 29. (Why is this special case, where the lines intersect at the origin, as good as the general case?)

Solution

Figure 29 assumes $b = 0, c = 0$, fixes $x = 1$, and draws two orthogonal lines from the origin– one to $(1, m)$, the other to $(1, n)$. To pull in a little linear algebra

$$\langle (1, m), (1, n) \rangle = 1 + mn$$

thus $\langle (1, m), (1, n) \rangle = 0$ when $mn = -1$ as desired. Since we didn't cover orthogonality in the book, another way to approach this problem is to recall the Pythagorean theorem. Observe that the hypotenuse is equal to $m+(-n) = m-n$ and the squares of the sides are equal to $1 + m^2$ and $1 + n^2$. Thus:

$$\begin{aligned}
&(m - n)^2 = (1 + m^2) + (1 + n^2) \\
&\implies m^2 - 2mn + n^2 = 2 + m^2 + n^2 \\
&\implies -2mn = 2 \\
&\implies mn = -1
\end{aligned}$$

Suppose the two lines intersect at a point (x', y') that isn't the origin. Observe that translation of the lines such that $(x', y') = (0, 0)$ (i.e. translation to the origin) doesn't change the slope of the lines, and thus doesn't change the angles. Therefore the special case here applies to the more general case in which the lines don't intersect at the origin.

Problem 8b

Prove that the two straight lines consisting of all (x, y) satisfying the conditions

$$Ax + By + C = 0$$
$$A'x + B'y + C' = 0$$

are perpendicular if and only if $AA' + BB' = 0$.

Solution

We can define two functions f, g that represent each line as follows:

$$f(x) = -\frac{A}{B}x - C$$
$$g(x) = -\frac{A'}{B'} - C'$$

From 8a we know f and g are perpendicular when

$$\frac{A}{B} \cdot \frac{A'}{B'} = -1$$
$$\implies AA' = -1BB'$$
$$\implies AA' + BB' = 0$$

as desired.

Problem 10

Sketch the graphs of the following functions, plotting enough points to get a good idea of the general appearance. (Part of the problem is to make a reasonable decision how many is "enough"; the queries posed below are meant to show that a little thought will often be more valuable than hundreds of individual points.)

Solution

(i) $f(x) = x + \frac{1}{x}$. (What happens for x near 0, and for large x? Where does the graph lie in relation to the graph of the identity function? Why does it suffice to consider only positive x at first?)

I started with three points, $x = 1, x = 100, x = 0.01$. It's obvious that in the first quadrant $f(1) = 2$. From there it stays above the diagonal/identity function $x = y$ and gets closer and closer to it the larger x gets. Similarly, as x gets smaller and smaller, f approaches the y axis but never crosses it. The function is smooth around $f(1)$, which I unfortunately didn't nail. It's mirror image is in the third quadrant.

(ii) $f(x) = x - \frac{1}{x}$

Here $f(1) = 0$. As x gets larger, $f(x)$ approaches the diagonal, but always stays below it. As x gets smaller, $f(x)$ approaches $-\infty$ on the y axis. It's mirror image is on the 2nd and 3rd quadrants when x is negative.

Problem 14

Describe the graph g in terms of the graph of f.

Solution

(vii) $g(x) = |f(x)|$
When $f(x) \geq 0, g(x) = f(x)$. When $f(x) < 0, g(x) = -f(x)$, i.e. any part of $f(x)$ under the x axis gets drawn as a mirror image above the x axis.

(viii) $g(x) = \max(f, 0)$
Above the x axis $g(x) = f(x)$. Below the x axis $g(x) = 0$, i.e. it gets squashed as a line onto the x axis.

(ix) $g(x) = \min(f, 0)$
Below the x axis $g(x) = f(x)$. Above the x axis $g(x) = 0$ (i.e. above the x axis everything gets squashed as a line onto the x axis).

(x) $g(x) = \max(f, 1)$
Similar to viii, but instead of x axis everything below $y = 1$ line gets squashed as a line onto $y = 1$. I.e. nothing gets drawn below that line.

5 Limits

Problem 2

Find the following limits.

Solution

(i)

$$\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

(ii)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x} &= \lim_{x\to 0}\frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{1-(1-x^2)}{x(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{x^2}{x(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{x}{1+\sqrt{1-x^2}} = 0
\end{aligned}$$

(iii)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{x^2} &= \lim_{x\to 0}\frac{(1-\sqrt{1-x^2})(1+\sqrt{1-x^2})}{x^2(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{1-(1-x^2)}{x^2(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{x^2}{x^2(1+\sqrt{1-x^2})}\\
&= \lim_{x\to 0}\frac{1}{1+\sqrt{1-x^2}} = \frac{1}{2}
\end{aligned}$$

Problem 3i, ii

In each of the following cases, find a δ such that $|f(x)-l| < \epsilon$ for all x satisfying $0 < |x-a| < \delta$.

Solution

(i) $f(x) = x^4; l = a^4$

Let $\epsilon > 0$ be given. We must find δ such that $0 < |x-a| < \delta$ implies $|x^4-a^4| < \epsilon$ for all x. Observe that

$$|x^4 - a^4| = |(x^2+a^2)(x+a)(x-a)| = |x^2+a^2||x+a||x-a|$$

We must find a bound on $|x+a|$ and $|x^2+a^2|$. Start by arbitrarily fixing $|x-a| < 1$. Then

$$\begin{aligned}
&-1 < x-a < 1\\
&\implies 2a-1 < x+a < 2a+1 \qquad\qquad \text{add } 2a \text{ to both sides}
\end{aligned}$$

We now have a bound on $x+a$, but we need one on $|x+a|$. It's easy to see $|x+a| < \max(|2a-1|, |2a+1|)$. By triangle inequality ($|a+b| \le |a|+|b|$):

$$\begin{aligned}
|2a-1| &\le |2a| + |-1| = |2a|+1\\
|2a+1| &\le |2a| + |1| = |2a|+1
\end{aligned}$$

Thus $|x+a| < |2a|+1$, provided $|x-a| < 1$. Similarly, we find a bound for $|x^2+a^2|$:

$$\begin{aligned}
&-1 < x-a < 1 \\
&\implies a-1 < x < a+1 \\
&\implies (a-1)^2 < x^2 < (a+1)^2 && \text{square each side} \\
&\implies (a-1)^2+a^2 < x^2+a^2 < (a+1)^2+a^2
\end{aligned}$$

Observe that $x^2+a^2 = |x^2+a^2|$, thus $|x^2+a^2| < (a+1)^2+a^2$. Thus to make $|x^4-a^4| < \epsilon$ we must set

$$|x-a| < \frac{\epsilon}{(|2a|+1)((a+1)^2+a^2)}$$

provided $|x-a| < 1$. Therefore

$$\delta = \min(1, \frac{\epsilon}{(|2a|+1)(2a^2+2a+1)})$$

(ii) $f(x) = \frac{1}{x}; a = 1, l = 1$

Let $\epsilon > 0$ be given. We must find δ such that $0 < |x-1| < \delta$ implies $|\frac{1}{x}-1| < \epsilon$ for all x. Observe that

$$\left|\frac{1}{x}-1\right| = \left|\frac{1}{x}-\frac{x}{x}\right| = \left|\frac{1-x}{x}\right| = \frac{|x-1|}{|x|}$$

Fix $|x-1| < \frac{1}{10}$. Then

$$\begin{aligned}
-\frac{1}{10} &< x-1 < \frac{1}{10} \\
\implies \frac{9}{10} &< x < \frac{11}{10}
\end{aligned}$$

Thus we must set

$$|x-1| < \frac{\epsilon}{10}$$

provided $|x-1| < \frac{1}{10}$. Therefore

$$\delta = \min(\frac{1}{10}, \frac{\epsilon}{10})$$

Problem 8

Answer the following.

Solution

(a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a}[f(x)+g(x)]$ or $\lim_{x\to a} f(x)g(x)$ exist?

Yes. Consider

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$$

Then $(g+f)(x) = 0$ and $(gf)(x) = -1$, both of which have limits for all a.

(b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}[f(x)+g(x)]$ exists, must $\lim_{x\to a} g(x)$ exist?

Yes. Let $\epsilon > 0$ be given. Then there exists δ such that for all x in $0 < |x-a| < \delta$ the following inequalities hold:

$$M - \epsilon/2 < f(x) + g(x) < M + \epsilon/2, \quad L - \epsilon/2 < f(x) < L + \epsilon/2$$

Then:

$$\begin{aligned}
&M - \epsilon/2 < f(x) + g(x) < M + \epsilon/2 \\
&\implies M - \epsilon/2 - f(x) < g(x) < M + \epsilon/2 - f(x) \\
&\implies M - \epsilon/2 - L - \epsilon/2 < g(x) < M + \epsilon/2 - L + \epsilon/2 \\
&(M - L) - \epsilon < g(x) < (M - L) + \epsilon \\
&-\epsilon < g(x) - (M - L) < \epsilon \\
&|g(x) - (M - L)| < \epsilon
\end{aligned}$$

Therefore $\lim_{x\to a} g(x) = M - L$ and must exist.

(c) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}[f(x)+g(x)]$ exist?

No. Let $\lim_{x\to a} f(x) = L$. Since $\lim_{x\to a} g(x)$ does not exist, there exists ϵ such that $|g(x) - M| \geq \epsilon$ for all M. Suppose for contradiction $\lim_{x\to a}[f(x)+g(x)] = M$ exists. Then

$$\begin{aligned}
&M - \epsilon/2 < f(x) + g(x) < M + \epsilon/2 \\
&\implies M - \epsilon/2 - f(x) < g(x) < M + \epsilon/2 - f(x) \\
&\implies M - \epsilon/2 - L - \epsilon/2 < g(x) < M + \epsilon/2 - L + \epsilon/2 \\
&(M - L) - \epsilon < g(x) < (M - L) + \epsilon \\
&-\epsilon < g(x) - (M - L) < \epsilon \\
&|g(x) - (M - L)| < \epsilon
\end{aligned}$$

We have a contradiction, thus $\lim_{x\to a}[f(x)+g(x)]$ does not exist.

(d) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x)g(x)$ exists, does it follow that $\lim_{x\to a} g(x)$ exists?

No. Consider $g(x) = 1/x$ which has no limit at 0, and $f(x) = 0$. Then $f(x)g(x) = 0$ which has a limit of 0 as $x \to 0$.

Problem 13

Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Prove that $\lim_{x\to a} g(x)$ exists, and that $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. (Draw a picture!)

Solution

Let $L = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Let $\epsilon > 0$ be given. We must find δ such that $0 < |x-a| < \delta$ implies $|g(x) - L| < \epsilon$.

By limit definition there exists δ_1 such that for all x in $0 < |x-a| < \delta_1$

$$\begin{aligned}
&|f(x) - L| < \epsilon \\
&-\epsilon < f(x) - L < \epsilon \\
&L - \epsilon < f(x) < L + \epsilon
\end{aligned}$$

Similarly there exists δ_2 such that for all x in $0 < |x-a| < \delta_2$

$$\begin{aligned}
&|h(x) - L| < \epsilon \\
&-\epsilon < h(x) - L < \epsilon \\
&L - \epsilon < h(x) < L + \epsilon
\end{aligned}$$

By problem statement $f(x) \leq g(x) \leq h(x)$. Fix $\delta = \min(\delta_1, \delta_2)$. Then

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

Therefore $L - \epsilon < g(x) < L + \epsilon$ which implies $|g(x) - L| < \epsilon$, as desired.

Problem 15

Evaluate the following limits in terms of the number $\alpha = \lim_{x\to 0}(\sin x)/x$.

Solution

(i)

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{x\to 0}\frac{2\sin x\cos x}{x} = 2\alpha\cos x = 2\alpha$$

(iv)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 2x}{x^2} &= \lim_{x\to 0}\frac{(2\sin x\cos x)^2}{x^2}\\ &= \lim_{x\to 0}\frac{4\sin^2 x\cos^2 x}{x^2}\\ &= \lim_{x\to 0} 4\alpha^2\cos^2 x\\ &= 4\alpha^2\end{aligned}$$

(vii)

$$\begin{aligned}\lim_{x\to 0}\frac{x\sin x}{1-\cos x} &= \lim_{x\to 0}\frac{x\sin x(1+\cos x)}{(1-\cos x)(1+\cos x)}\\ &= \lim_{x\to 0}\frac{x\sin x(1+\cos x)}{\sin^2 x}\\ &= \lim_{x\to 0}\frac{x(1+\cos x)}{\sin x}\\ &= \lim_{x\to 0}\frac{1+\cos x}{\alpha} = \frac{2}{\alpha}\end{aligned}$$

(ix)

$$\begin{aligned}\lim_{x\to 1}\frac{\sin(x^2-1)}{x-1} &= \lim_{x\to 1}\frac{\sin(x^2-1)(x+1)}{(x-1)(x+1)}\\ &= \lim_{x\to 1}\frac{\sin(x^2-1)(x+1)}{x^2-1}\end{aligned}$$

Let $u = x^2 - 1$. Observe that as $x \to 1, u \to 0$. Thus

$$\lim_{x\to 1}\frac{\sin(x^2-1)(x+1)}{x^2-1} = \lim_{u\to 0}\frac{\sin u}{u}\cdot\lim_{x\to 1} x+1 = 2\alpha$$

Problem 19

Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x, then $\lim_{x\to a} f(x)$ does not exist for any a.

Solution

Let $\epsilon = \frac{1}{10}$. We handle two cases. First suppose $L < \frac{1}{2}$. Pick any rational x from the interval $0 < |x-a| < \delta$. Then $|f(x) - L| = |1 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Similarly, suppose $L > \frac{1}{2}$. Pick any irrational x from the interval $0 < |x-a| < \delta$. Then $|f(x) - L| = |0 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

6 Continuity

Problem 1

For which of the following functions f is there a continuous function F with domain $\mathbf{R}$ such that $F(x) = f(x)$ for all x in the domain of f?

Solution

(i) $f(x) = \frac{x^2-4}{x-2}$

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{(x-2)} = x+2$$

when $x \neq 2$. Thus let $F(x) = x+2$. It follows $F(x) = f(x)$ for all x in the domain of f, as 2 is not in the domain of f. Further, F is continuous in R as

$$\lim_{x \to a} x + 2 = a + 2$$

(ii) $f(x) = \frac{|x|}{x}$
There is no such function F. Observe that if we define F such that $F(x) = f(x)$ for all x in the domain of f, the only point we have left to manipulate is 0. It's easy to see that no definition of $F(0)$ will allow F to have a limit at 0, and thus F cannot be continuous.

Problem 5

For each number a, find a function which is continuous at a, but not at any other points.

Solution

As per textbook, the following modification of the Dirichlet function is continuous at 0 but nowhere else:

$$f(x) = \begin{cases} x & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

Shifting horizontally by a creates a function continous at a but nowhere else:

$$g(x) = \begin{cases} x - a & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

Problem 8

Suppose that f is continuous at a and $f(a) = 0$. Prove that if $\alpha \neq 0$, then $f + \alpha$ is nonzero in some open interval containing a.

Solution

Since f is continous at a, by continuity theorems (addition of identity and constant functions) $f + \alpha$ is also continuous. Also $(f + \alpha)(a) = \alpha$, and thus $\lim_{x \to a}(f + \alpha)(x) = \alpha$. We can set $\epsilon = \alpha$, and by limit definition there exists an interval in which $|(f + \alpha)(x) - \alpha| < \alpha$. In another words within the interval $f + \alpha$ is less than α away from α, and thus cannot be zero.

7 Three Hard Theorems

Problem 1

Decide which functions are bounded above or below on the given interval, and which take on their min and max values. (Note, f may have these properties even if it isn't continuous or the interval isn't closed.)

Solution

(i) $f(x) = x^2$ on $(-1, 1)$
Bounded above and below, and takes on min value. Note f doesn't take on the maximum value in $(-1, 1)$. I.e. there is no y in $(-1, 1)$ such that $f(y) \geq f(x)$ for all x in $(-1, 1)$.

(ii) $f(x) = x^3$ on $(-1, 1)$
Bounded above and below.

(iii) $f(x) = x^2$ on $\mathbf{R}$
Bounded below and takes on min value.

(iv) $f(x) = x^2$ on $[0, \infty)$
Bounded below and takes on min value.

(v) $f(x) = \begin{cases} x^2, x \leq a \\ a + 2, x > a \end{cases}$ on $(-a - 1, a + 1)$.
TODO

(vi) $f(x) = \begin{cases} x^2, x < a \\ a+2, x \geq a \end{cases}$ on $[-a-1, a+1]$.

TODO

(vii) $f(x) = \begin{cases} 0, x \text{ is irrational} \\ 1/q \quad x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

Bounded below by 0 and achieves it. Bounded above by 1 and achieves it.

(viii) $f(x) = \begin{cases} 1, x \text{ is irrational} \\ 1/q \quad x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

Bounded below by 0 but doesn't achieve it. Bounded above by 1 and achieves it.

(ix) $f(x) = \begin{cases} 1, x \text{ is irrational} \\ -1/q \quad x = p/q \text{ in lowest terms} \end{cases}$ on $[0, 1]$.

Bounded above by 1 and achieves it. Bounded below by -1 and achieves it.

(x) $f(x) = \begin{cases} x, x \text{ is rational} \\ 0 \quad x \text{ is irrational} \end{cases}$ on $[0, a]$.

Bounded below by 0 and achieves it. Achieves maximum value if a is rational, otherwise does not achieve it.

Problem 3

Prove that there is some number x such that

Solution

(i) $x^{179} + \frac{163}{1+x^2+\sin^2 x} = 119$

Consider $f(x) = x^{179} + \frac{163}{1+x^2+\sin^2 x} - 119$. Observe that $f(-1) < 0$ and $f(0) > 0$. Thus by intermediate value theorem there exists y in $[-1, 0]$ such that $f(y) = 0$.

(ii) $\sin x = x - 1$

Consider $f(x) = \sin x - x + 1$. Observe that $f(-5) > 0$ and $f(5) < 0$. Thus by intermediate value theorem there exists y in $[-5, 5]$ such that $f(y) = 0$.

Problem 5

Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f?

Solution

$f(x) = c, c \in \mathcal{Q}$. For suppose this isn't the case. Then there exist x, y such that $f(x) = a$ and $f(y) = b$, and $a \neq b$. Let d be irrational and $a < d < b$. By intermediate value theorem there exists z in $[x, y]$ such that $f(z) = d$. But f is always rational, so we have a contradiction.

Problem 17

Suppose that f is a continuous function with $f(x) > 0$ for all x, and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. (Draw a picture.) Prove that there is some number y such that $f(y) \geq f(x)$ for all x.

Solution

Let $a = f(0)$. By limit definition:

- There exists $N_1 < 0$ such that $f(x) < a$ for all x in $(-\infty, N_1)$.
- There exists $N_2 > 0$ such that $f(x) < a$ for all x in (N_2, ∞).

By extreme value theorem there is some y in $[N_1, N_2]$ such that $f(y) \geq f(x)$ for all x in $[N_1, N_2]$. Observe that $f(y) \geq a$, otherwise $f(y)$ wouldn't be a maximum. Thus $f(y) \geq f(x)$ for all $x \in \mathcal{R}$, as desired.

8 Least upper bounds

Problem 3a

Let f be a continous function on $[a, b]$ with $f(a) < 0 < f(b)$. The proof of Theorem 1 showed that there is a smallest x in $[a, b]$ with $f(x) = 0$. Is there necessarily a second smallest x in $[a, b]$ with $f(x) = 0$? Show that there is a largest x in $[a, b]$ with $f(x) = 0$. (Try to give an easy proof by considering a new function closely related to f.)

Solution

There isn't necessarily a second smallest x in $[a, b]$ with $f(x) = 0$. Consider $f(x) = x$ on $[-1, 1]$. The smallest and only x where $f(x) = 0$ is 0.

We want to find a function g that varies from b to a the way f varies from a to b. In particular, we want:

- g(a)=f(b)
- g(b)=f(a)

Observe that $g(x) = f(a + b - x)$ is such a function. By intermediate value theorem there is a smallest x in $[a, b]$ with $g(x) = 0$. But that's the largest x in $[a, b]$ with $f(x) = 0$.

Problem 5a

Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. Hint: let l be the largest integer satisfying $l \leq x$, and consider $l + 1$.

Solution

Lemma: there exists the largest integer l such that $l \leq x$.
Proof: since $\mathcal{N}$ is unbounded, there exists $n \in \mathcal{N}$ such that $-n < x < n$. There is a finite number of integers between $-n$ and x. Pick the largest.

Let l be the largest integer such that $l \leq x$. Then

- $l + 1 > x$, otherwise l wouldn't be the *largest* integer such that $l \leq x$.
- $l + 1 \leq x + 1$. But $y > x + 1$, thus $l + 1 < y$.

Therefore $x < l + 1 < y$ as desired.

Problem 5b

Suppose $x < y$. Prove that there is a rational number r such that $x < r < y$. Hint: if $1/n < y - x$, then $ny - nx > 1$. (Query: Why have parts (a) and (b) been postponed until this problem set?)

Solution

In 5a we've proven there exists an integer l such that $nx < l < ny$. Dividing each side by n we get $x < \frac{l}{n} < y$, as desired.

For 5a we need the unboundedness of $\mathcal{N}$ and in 5b we need the Archimedean property for the existence of n such that $1/n < y - x$. Both depend on least upper bound.

Problem 5c

Suppose $r < s$ are rational numbers. Prove that there is an irrational number between r and s. Hint: As a start, you know that there is an irrational number between 0 and 1.

Solution

Let $x = r + \frac{\sqrt{2}(s-r)}{2}$. Obviously x is irrational and $x > r$. Also $r + (s - r)/2 < s$, and multiplying $(s - r)/2$ by $0 < \sqrt{2} < 1$ will only make it smaller. Thus $x < s$.

Problem 5d

Suppose that $x < y$. Prove that there is an irrational number between x and y. Hint: It is unnecessary to do any more work; this follows from (b) and (c).

Solution

By 5b there exists a rational $x < r < y$, and by 5b again there exists a rational $r < s < y$. Then by 5c there exists an irrational m such that $r < m < s$.

Problems 6a, b

A set A of real numbers is said to be **dense** if every open interval contains a point of A. For example, Problem 5 shows that the set of rational numbers and the set of irrational numbers are each dense.

Solution

(a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A, then $f(x) = 0$ for all x.

Suppose for contradiction there exists $a \in \mathcal{R}$ such that $f(a) \neq 0$. Fix $\epsilon = |f(a)|/2$. By continuity definition, there exists $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Since A is dense, there exists $x \in (a-\delta, a+\delta)$ such that $x \in A$, and thus $f(x) = 0$. But that would imply $|0 - f(a)| = |f(a)| < |f(a)|/2$, which is a contradiction. Thus $f(x) = 0$ for all x as desired.

(b) Prove that if f and g are continous and $f(x) = g(x)$ for all x in a dense set A, then $f(x) = g(x)$ for all x.

Since f, g are continuous, so is $f - g$. We can thus apply 6a to $f - g$, and we are done.

Problem 11a

Suppose that $a_1, a_2, a_3, \ldots$ is a sequence of positive numbers with $a_{n+1} \leq a_n/2$. Prove that for any $\epsilon > 0$ there is some n with $a_n < \epsilon$.

Solution

Observe that

$$
\begin{aligned}
&a_2 \leq \frac{a_1}{2} \\
&a_3 \leq \frac{a_2}{2} \implies a_3 \leq \frac{a_1}{4} = \frac{a_1}{2^2} \\
&a_4 \leq \frac{a_3}{2} \implies a_4 \leq \frac{a_1}{8} = \frac{a_1}{2^3} \\
&\ldots \\
&a_n \leq \frac{a_1}{2^{n-1}}
\end{aligned}
$$

Thus to find $a_n < \epsilon$, we must choose n such that:

$$\begin{aligned} &\frac{a_1}{2^{n-1}} < \epsilon \\ \implies &2^{n-1} > \frac{a_1}{\epsilon} \end{aligned}$$