

Notes working through Spivak's Calculus

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I'm working through Spivak Calculus. Around the chapter on epsilon-delta limits the details get pretty confusing. I started supplementing with David Galvin's notes, which are considerably more clear but are still confusing. This is surprising because the topic doesn't use anything beyond basic middle school math. Feels like it should be simple! And so I'm attempting to write these notes to properly understand the damned thing.

Spivak's chapters 7 (Three Hard Theorems) and 8 (Least Upper Bounds) are swapped in these notes. Spivak first introduces the Intermediate Value theorem and the Extreme Value theorem as facts, proves their consequences, then introduces completeness and its consequences, and finally proves IVT and EVT. I find it distracting and confusing. I introduce completeness and its consequences first. I then introduce and prove IVT and EVT, and cover their consequences. IMO this approach is much less confusing than Spivak's.

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1 Limits

1.1 A hand-wavy limits definition

I'll first do a hand-wavy definition of limits, and then use it to explain the mechanics of computing limits of functions in practice. Basically, pre-calculus stuff. After that I'll do a proper definition and use it to prove the theorems that make the mechanics work.

A hand-wavy definition: a limit of $f(x)$ at a is the value $f(x)$ approaches close to (but not necessarily at) a .

A slightly less hand-wavy definition: let $f : \mathbf{R} \rightarrow \mathbf{R}$, let $a \in \mathbf{R}$ be some number on the x-axis, and let $l \in \mathbf{R}$ be some number on the y-axis. Then as x gets closer to a , $f(x)$ gets closer to l .

The notation for this whole thing is

$$\lim_{x \rightarrow a} f(x) = l$$

So for example $\lim_{x \rightarrow 5} x^2 = 25$ because the closer x gets to 5, the closer x^2 gets to 25 (we'll prove all this properly soon). Now suppose you have some fancy pants function like this one:

$$\lim_{x \rightarrow 0} \frac{1 - \sqrt{x}}{1 - x} \tag{1}$$

If you plot it, it's easy to see that as x approaches 0, the whole shebang approaches 1. But how do you algebraically evaluate the limit of this thing? Can you just plug 0 into the equation? It seems to work, but once we formally define limits, we'll have to prove somehow that plugging $a = 0$ into x gives us the correct result.

1.2 Limits evaluation mechanics

It turns out that it does in fact work because of a few theorems that make practical evaluation of many limits easy. I'll first state these theorems as facts, and then go back and properly prove them once I introduce the formal definition of limits.

1. **Constants.** $\lim_{x \rightarrow a} c = c$, where $c \in \mathbf{R}$. In other words if the function is a constant, e.g. $f(x) = 5$, then $\lim_{x \rightarrow a} f(x) = 5$ for any a .

2. **Identity.** $\lim_{x \rightarrow a} x = a$. In other words if the function is an identity function $f(x) = x$, then $\lim_{x \rightarrow 6} f(x) = 6$. Meaning we simply plug a into x .
3. **Addition**¹. $\lim_{x \rightarrow a} (f + g)(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$. For example $\lim_{x \rightarrow a} (x + 2) = \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} 2 = a + 2$.
4. **Multiplication.** $\lim_{x \rightarrow a} (f \cdot g)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$. For example $\lim_{x \rightarrow a} 2x = \lim_{x \rightarrow a} 2 \cdot \lim_{x \rightarrow a} x = 2a$.
5. **Reciprocal.** $\lim_{x \rightarrow a} \left(\frac{1}{f}\right)(x) = \frac{1}{\lim_{x \rightarrow a} f(x)}$ when the denominator isn't zero. For example $\lim_{x \rightarrow a} \frac{1}{x} = \frac{1}{\lim_{x \rightarrow a} x} = \frac{1}{a}$ for $a \neq 0$.

To come back to 1, these theorems tells us that

$$\lim_{x \rightarrow 0} \frac{1 - \sqrt{x}}{1 - x} = \frac{\lim_{x \rightarrow 0} 1 - (\lim_{x \rightarrow 0} x)^{\frac{1}{2}}}{\lim_{x \rightarrow 0} 1 - \lim_{x \rightarrow 0} x} = \frac{1 - 0^{\frac{1}{2}}}{1 - 0} = 1$$

Holes

What happens if we try to take a limit as $x \rightarrow 1$ rather than $x \rightarrow 0$?

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x}$$

We can't use the same trick and plug in 1 because we get a nonsensical result $0/0$ as the function isn't defined at 0. If we plot it, we clearly see the limit approaches $1/2$ at 0, but how do we prove this algebraically? The answer is to do some trickery to find a way to cancel out the inconvenient term (in this case $1 - \sqrt{x}$)

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

Why is it ok here to divide by $1 - \sqrt{x}$? Good question! Recall that the limit is defined *close to* a (or *around* a , or as x *approaches* a), but not **at** a . In other words $f(a)$ need not even be defined (as is the case here). This means that as we consider $1 - \sqrt{x}$ at different values of x as it approaches a , the limit never requires us to evaluate the function at $x = a$. So we never have to consider $1 - \sqrt{x}$ as $x = 1$, $1 - \sqrt{x}$ never takes on the value of 0, and it is safe to divide it out.

1.3 Formal limits definition

Now we establish a rigorous definition of limits that formalizes the hand-wavy version above.

¹Spivak's book uses a slightly more verbose definition that assumes the limits of f and g exist near a , see p. 103

Definition: $\lim_{x \rightarrow a} f(x) = L$ when for any $\epsilon \in \mathbf{R}$ there exists $\delta \in \mathbf{R}$ such that for all x , $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. (Also $\epsilon > 0, \delta > 0$.)

Here is what this says. Suppose $\lim_{x \rightarrow a} f(x) = L$. You pick any interval on the y-axis around L . Make it as small (or as large) as you want. I'll produce an interval on the x-axis around a . You can take any number from my interval, plug it into f , and the output will stay within the bounds you specified.

So ϵ specifies the distance away from L along the y-axis, and δ specifies the distance away from a along the x-axis. Take any x within δ of a , plug it into f , and the result is guaranteed to be within ϵ of L . $\lim_{x \rightarrow a} f(x) = L$ just means there exists such δ for any ϵ .

This is quite simple, but the mechanics of the limit definition tend to confuse people. I think it's because absolute value inequalities are unfamiliar. Wtf is $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$?! Let's tease it apart²

Here is the intuitive reading. $0 < |x - a| < \delta$ means the difference between x and a is between 0 and δ . And $|f(x) - L| < \epsilon$ means the difference between $f(x)$ and L is less than ϵ . This is actually all this means, but still, let's look at the inequalities more closely.

First, consider $0 < |x - a| < \delta$. There are two inequalities here. The left side, $0 < |x - a|$ is equivalent to $|x - a| > 0$. But $|x - a|$ is an absolute value, it's **always** true that $|x - a| \geq 0$. So this part of the inequality says $x - a \neq 0$, or $x \neq a$. (Remember, we said the limit is defined *around* a but not *at* a). I don't know why mathematicians say $0 < |x - a|$ instead of $x \neq a$, probably because confusing you brings them pleasure.

The right side is $|x - a| < \delta$. Intuitively this says that along the x-axis the difference between x and a should be less than δ . Put differently, x should be within δ of a . We can rewrite this as $a - \delta < x < a + \delta$.

The second equation, $|f(x) - L| < \epsilon$ should now be easy to understand. Intuitively, along the y-axis $f(x)$ should be within ϵ of L , or put differently $L - \epsilon < f(x) < L + \epsilon$.

Limit uniqueness

Suppose $\lim_{x \rightarrow a} f(x) = L$. It's easy to assume L is the only limit around a , but such a thing needs to be proved. We prove this here. More formally, suppose $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} f(x) = M$. We prove that $L = M$.

Suppose for contradiction $L \neq M$. Assume without loss of generality $L > M$. By limit definition, for all $\epsilon > 0$ there exists a positive $\delta \in \mathbf{R}$ such that $0 <$

²The best answer is to go to Khan academy and do a bunch of absolute value inequalities until they become second nature.

$|x - a| < \delta$ implies

- $|f(x) - L| < \epsilon \implies L - \epsilon < f(x)$
- $|f(x) - M| < \epsilon \implies f(x) < M + \epsilon$

for all x . Thus

$$\begin{aligned} L - \epsilon &< f(x) < M + \epsilon \\ \implies L - \epsilon &< M + \epsilon \\ \implies L - M &< 2\epsilon \end{aligned}$$

The above is true for all ϵ . Now let's narrow our attention and consider a concrete $\epsilon = (L - M)/4$, which we easily find leads to a contradiction³:

$$\begin{aligned} L - M &< 2\epsilon \\ \implies (L - M)/4 &< \epsilon/2 && \text{dividing both sides by 4} \\ \implies \epsilon &< \epsilon/2 && \text{recall we set } \epsilon = (L - M)/4 \end{aligned}$$

We have a contradiction, and so $L = M$ as desired.

Half-Value Continuity Lemma

This lemma will come in handy later, so we may as well prove it now. Suppose $M \neq 0$ and $\lim_{x \rightarrow a} g(x) = M$. We show that there exists some δ such that $0 < |x - a| < \delta$ implies $|g(x)| \geq |M|/2$ for all x .

Intuitively, the lemma states the following: when a function g approaches a nonzero limit M near a point, there exists an interval in which the values of g are closer to M than to zero.

Proof. The claim that $|g(x)| \geq |M|/2$ is equivalent to

$$g(x) \leq -|M|/2 \quad \text{or} \quad g(x) \geq |M|/2$$

There are two possibilities: either $M > 0$ or $M < 0$. Let's consider each possibility separately.

Case 1. Suppose $M > 0$. Then to show $|g(x)| \geq |M|/2$ it is sufficient to show either $g(x) \leq -M/2$ or $g(x) \geq M/2$. We will show $g(x) \geq M/2$. Fix $\epsilon = M/2$.

³note we assumed $L > M$, thus $\epsilon = (L - M)/4 > 0$

By limit definition there is some δ such that $0 < |x - a| < \delta$ implies for all x

$$\begin{aligned}
 |g(x) - M| &< M/2 \\
 \implies -M/2 &< g(x) - M \\
 \implies M/2 &< g(x) && \text{add } M \text{ to both sides} \\
 \implies g(x) &> M/2 && \text{note } \geq \text{ is correct but not tight}
 \end{aligned}$$

Case 2. Suppose $M < 0$. We must show either $g(x) \leq M/2$ or $g(x) \geq -M/2$. We will show $g(x) \leq M/2$. Fix $\epsilon = -M/2$. Then

$$\begin{aligned}
 |g(x) - M| &< -M/2 \\
 \implies g(x) - M &< -M/2 \\
 \implies g(x) &< M/2 && \text{add } M \text{ to both sides;} \\
 &&& \text{note } \leq \text{ is correct but not tight}
 \end{aligned}$$

QED.

1.4 Theorems that make evaluation work

Armed with the formal definition, we can use it to rigorously prove the five theorems useful for evaluating limits (constants, identity, addition, multiplication, reciprocal). Let's do that now.

Constants

Let $f(x) = c$. We prove that $\lim_{x \rightarrow a} f(x) = c$ for all a .

Let $\epsilon > 0$ be given. Pick any positive δ . Then for all x such that $0 < |x - a| < \delta$, $|f(x) - c| = |c - c| = 0 < \epsilon$. QED.

(Note that we can pick any positive $\delta > 0$, e.g. 1, 10, $\frac{1}{10}$.)

Identity

Let $f(x) = x$. We prove that $\lim_{x \rightarrow a} f(x) = a$ for all a .

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all x in $0 < |x - a| < \delta$, $|f(x) - a| = |x - a| < \epsilon$. I.e. we need to find a δ such that $|x - a| < \delta$ implies $|x - a| < \epsilon$. This obviously works for any $\delta \leq \epsilon$. QED.

(Note the many options for δ , e.g. $\delta = \epsilon$, $\delta = \frac{\epsilon}{2}$, etc.)

Addition

Let $f, g \in \mathbf{R} \rightarrow \mathbf{R}$. We prove that

$$\lim_{x \rightarrow a} (f + g)(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

Let $L_f = \lim_{x \rightarrow a} f(x)$ and let $L_g = \lim_{x \rightarrow a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(f + g)(x) - (L_f + L_g)| < \epsilon$$

I.e. we're trying to show $\lim_{x \rightarrow a} (f + g)(x)$ equals to $L_f + L_g$, the sum of the other two limits. Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned} |(f + g)(x) - (L_f + L_g)| &= |f(x) + g(x) - (L_f + L_g)| \\ &= |(f(x) - L_f) + (g(x) - L_g)| \\ &\leq |f(x) - L_f| + |g(x) - L_g| \quad \text{by triangle inequality} \end{aligned}$$

By limit definition there exist positive δ_f, δ_g such that for all x

- $0 < |x - a| < \delta_f$ implies $|f(x) - L_f| < \epsilon/2$
- $0 < |x - a| < \delta_g$ implies $|g(x) - L_g| < \epsilon/2$

Recall that we can make ϵ as small as we like. Here we pick deltas for $\epsilon/2$ because it's convenient to make the equations work, as you will see in a second. For all x bounded by $0 < |x - a| < \min(\delta_f, \delta_g)$ we have

$$|f(x) - L_f| < \epsilon/2 \quad \text{and} \quad |g(x) - L_g| < \epsilon/2$$

Fix $\delta = \min(\delta_f, \delta_g)$. Then for all x bounded by $0 < |x - a| < \delta$ we have

$$\begin{aligned} |(f + g)(x) - (L_f + L_g)| &\leq |f(x) - L_f| + |g(x) - L_g| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

as desired.

Multiplication

Let $f, g \in \mathbf{R} \rightarrow \mathbf{R}$. We prove that

$$\lim_{x \rightarrow a} (fg)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

Let $L_f = \lim_{x \rightarrow a} f(x)$ and let $L_g = \lim_{x \rightarrow a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(fg)(x) - (L_f L_g)| < \epsilon$$

(i.e. we're trying to show $\lim_{x \rightarrow a}(fg)(x)$ equals to $L_f L_g$, the product of the other two limits.) Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}
|(fg)(x) - (L_f L_g)| &= |f(x)g(x) - L_f L_g| \\
&= |f(x)g(x) - L_f g(x) + L_f g(x) - L_f L_g| \\
&= |g(x)(f(x) - L_f) + L_f(g(x) - L_g)| \\
&\leq |g(x)(f(x) - L_f)| + |L_f(g(x) - L_g)| \quad \text{by triangle inequality} \\
&= |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| \quad \text{in general } |ab| = |a||b|
\end{aligned}$$

We now need to show there exists δ such that $0 < |x - a| < \delta$ implies

$$|g(x)||f(x) - L_f| + |L_f||g(x) - L_g| < \epsilon$$

We will do that by finding δ such that

1. $|g(x)||f(x) - L_f| < \epsilon/2$
2. $|L_f||g(x) - L_g| < \epsilon/2$

First, we show $|g(x)||f(x) - L_f| < \epsilon/2$.

By limit definition we can find δ_1 to make $|f(x) - L_f|$ as small as we like. But how small? To make $|g(x)||f(x) - L_f| < \epsilon/2$ we must find a delta such that $|f(x) - L_f| < \epsilon/2g(x)$. But to do that we need to get a bound on $g(x)$. Fortunately we know there exists δ_2 such that $|g(x) - L_g| < 1$ (we pick 1 because we must pick some bound, and 1 is as good as any). Thus $|g(x)| < |L_g| + 1$. And so, we can pick δ_1 such that $|f(x) - L_f| < \epsilon/2(|L_g| + 1)$.

Second, we show $|L_f||g(x) - L_g| < \epsilon/2$.

That is easy. By limit definition there exists a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \epsilon/2|L_f|$ for all x . Actually, we need a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \frac{\epsilon}{2(|L_f|+1)}$ for all x to avoid divide by zero, and of course that exists too.

Fix $\delta = \min(\delta_1, \delta_2, \delta_3)$. Now

$$\begin{aligned}
|(fg)(x) - (L_f L_g)| &\leq |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

as desired.

Reciprocal

Let $\lim_{x \rightarrow a} f(x) = L$. We prove $\lim_{x \rightarrow a} \left(\frac{1}{f}\right)(x) = 1/L$ when $L \neq 0$.

First we show $\frac{1}{f}$ is defined near a . By half-value continuity lemma (see 1.3) there exists δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x)| \geq |L|/2$ where $L \neq 0$. Therefore $f(x) \neq 0$ near a , and thus $\frac{1}{f}$ near a is defined.

Now all we must do is find a delta such that $\left|\frac{1}{f}(x) - \frac{1}{L}\right| < \epsilon$. Let's make the equation more convenient:

$$\begin{aligned} \left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\ &= \left|\frac{L - f(x)}{Lf(x)}\right| \\ &= \frac{|f(x) - L|}{|L||f(x)|} \\ &= \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|} \end{aligned}$$

Above we showed there exists δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x)| \geq |L|/2$. Raising both sides to -1 we get $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$. Continuing the chain of reasoning above we get

$$\begin{aligned} \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|} &\leq \frac{|f(x) - L|}{|L|} \cdot \frac{2}{|L|} \\ &= \frac{2}{|L|^2} |f(x) - L| \end{aligned}$$

(if you're confused about why this inequality works, left-multiply both sides of $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$ by $\frac{|f(x) - L|}{|L|}$.) Thus we must find δ_2 such that

$$\frac{2}{|L|^2} |f(x) - L| < \epsilon$$

That is easy. Since $\lim_{x \rightarrow a} f(x) = L$ we can make $|f(x) - L|$ as small as we like. Dividing both sides by $\frac{2}{|L|^2}$, we must make $|f(x) - L| < \frac{|L|^2 \epsilon}{2}$. Thus we must fix $\delta = \min(\delta_1, \delta_2)$. QED.

1.5 Absence of limits

What does it mean to say L is not a limit of $f(x)$ at a ? It flows out of the definition— there exist some ϵ such that for any δ there exists an x in $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \epsilon$.

A stronger version is to say there is no limit of $f(x)$ at a . To do that we must prove that *any* L is not a limit of $f(x)$ at a .

Example: Absolute value fraction

Consider $f(x) = \frac{x}{|x|}$. It's easy to see that

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

We will show there is no limit of $f(x)$ near 0.

Weak version. First, let's prove a weak version— that $\lim_{x \rightarrow 0} f(x) \neq 0$. That is easy. Pick some reasonably small epsilon, say $\epsilon = \frac{1}{10}$. We must show that for any δ there exists an x in $0 < |x - a| < \delta$ such that $|f(x) - 0| \geq \frac{1}{10}$.

Let's pick some arbitrary x out of our permitted interval, say $x = \delta/2$. Then

$$|f(x) - 0| = |f(\delta/2)| = \left| \frac{\delta/2}{|\delta/2|} \right| = 1 \geq \frac{1}{10}$$

Strong version. Now we prove that $\lim_{x \rightarrow 0} f(x) \neq L$ for *any* L . Sticking with $\epsilon = \frac{1}{10}$ we proceed as follows.

If $L < 0$ take $x = \delta/2$. Then

$$|f(x) - L| = |f(\delta/2) - L| = \left| \frac{\delta/2}{|\delta/2|} - L \right| = |1 - L| > \frac{1}{10}$$

Similarly if $L \geq 0$ take $x = -\delta/2$. Then

$$|f(x) - L| = |f(-\delta/2) - L| = \left| \frac{-\delta/2}{|-\delta/2|} - L \right| = |-1 - L| > \frac{1}{10}$$

Example: Dirichlet function

The *dirichlet* function f is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

We prove $\lim_{x \rightarrow a} f(x)$ does not exist for any a .

Proof. Let $\epsilon = \frac{1}{10}$. Suppose for contradiction there exists L such that $\lim_{x \rightarrow a} f(x) = L$. There are two possibilities: either $L \leq \frac{1}{2}$ or $L > \frac{1}{2}$.

First suppose $L \leq \frac{1}{2}$. Pick any rational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |1 - L| \geq \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Similarly, suppose $L > \frac{1}{2}$. Pick any irrational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |0 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Thus $\lim_{x \rightarrow a} f(x)$ does not exist for any a , as desired.

1.6 Appendix: low-level proofs

While high level theorems allow us to easily compute complicated limits, it's instructive to compute a few limits for complicated functions straight from the definition. We do that here.

Aside: inequalities

We will often need to make an inequality of the following form work out:

$$|n||m| < \epsilon$$

Here ϵ is given to us, we have complete control over the upper bound of $|n|$, and $|m|$ can take on values outside our direct control. Obviously we can't make the inequality work without knowing *something* about $|m|$, so we'll try to find a bound for it in terms of other fixed values, or values we control.

For example, suppose $\lim_{x \rightarrow a} f(x) = L$ and we've discovered that $|m| < 3|a| + 4$. Given that we control $|n|$, how do we bound it in terms of ϵ and $|a|$ in such a way that the inequality $|n||m| < \epsilon$ holds?

Since we control $|n|$ and $(3|a| + 4)$ is fixed, we can find $|n|$ small enough so that $|n|(3|a| + 4) < \epsilon$ holds. Then certainly any inequality whose left side is smaller, e.g. $|n|(3|a| + 3) < \epsilon$, will also hold. And since $|m|$ is always smaller than $3|a| + 4$, it follows $|n||m| < \epsilon$ will hold as well.

All we have left to do is find a bound for $|n|$ such that $|n|(3|a| + 4) < \epsilon$ holds, which is of course easy:

$$|n| < \frac{\epsilon}{3|a| + 4}$$

Having bound $|n|$ in this way, we can verify that $|n|(3|a| + 4) < \epsilon$ holds by multiplying both sides of the above inequality by $3|a| + 4$.

Limits of quadratic functions

We will prove directly from the limits definition that $\lim_{x \rightarrow a} x^2 = a^2$. Let $\epsilon > 0$ be given. We must show there exists δ such that $|x^2 - a^2| < \epsilon$ for all x in $0 < |x - a| < \delta$.

Observe that

$$|x^2 - a^2| = |(x - a)(x + a)| = |x - a||x + a|$$

Thus we must pick δ such that $|x - a||x + a| < \epsilon$. Since $0 < |x - a| < \delta$, picking δ conveniently happens to bound $|x - a|$, letting us make it as small as we want. But to know how small, we need to find an upper bound on $|x + a|$. We can do it as follows.

Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. $1/10$, 10 , etc.) Then since $|x - a| < \delta$:

$$\begin{aligned} |x - a| &< 1 \\ \implies -1 &< x - a < 1 \\ \implies 2a - 1 &< x + a < 2a + 1 \quad \text{add } 2a \text{ to both sides} \end{aligned}$$

We now have a bound on $x + a$, but we need one on $|x + a|$. It's easy to see $|x + a| < \max(|2a - 1|, |2a + 1|)$. By triangle inequality ($|a + b| \leq |a| + |b|$):

$$\begin{aligned} |2a - 1| &\leq |2a| + |-1| = |2a| + 1 \\ |2a + 1| &\leq |2a| + |1| = |2a| + 1 \end{aligned}$$

Thus $|x + a| < |2a| + 1$, provided $|x - a| < 1$. Coming back to our original goal, $|x - a||x + a| < \epsilon$ when

- $|x - a| < 1$ and
- $|x - a| < \frac{\epsilon}{|2a| + 1}$

Putting these together, $\delta = \min(1, \frac{\epsilon}{|2a| + 1})$.

Limits of fractions

We will prove directly from the limits definition that $\lim_{x \rightarrow 2} \frac{3}{x} = \frac{3}{2}$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ for all x in $0 < |x - 2| < \delta$.

Let's manipulate $|\frac{3}{x} - \frac{3}{2}|$ to make it more convenient:

$$\left| \frac{3}{x} - \frac{3}{2} \right| = \left| \frac{6 - 3x}{2x} \right| = \frac{3}{2} \frac{|x - 2|}{|x|}$$

Thus we need to find δ such that

$$\begin{aligned}\frac{3}{2} \frac{|x-2|}{|x|} &< \epsilon \\ \implies \frac{|x-2|}{|x|} &< \frac{2\epsilon}{3}\end{aligned}$$

Conveniently $0 < |x-2| < \delta$ bounds $|x-2|$. But now we need to find a bound for $|x|$. It would be extra convenient if we could show $|x| > 1$. Then we could set $\delta = \frac{2\epsilon}{3}$ (and thus bound $|x-2| < \frac{2\epsilon}{3}$). A denominator greater than 1 would only make the fraction smaller than $\frac{2\epsilon}{3}$, ensuring $\frac{|x-2|}{|x|} < \frac{2\epsilon}{3}$ holds.

We will do exactly that. Pick an arbitrary $\delta = 1$ (we may pick any arbitrary delta, e.g. $1/10$, 10 , etc.) Then since $|x-2| < \delta$

$$\begin{aligned}|x-2| &< 1 \\ \implies -1 &< x-2 < 1 \\ \implies 1 &< x < 3 \\ \implies 1 &< |x| < 3\end{aligned}$$

Yes!! Luckily $\delta = 1$ implies $|x| > 1$! Thus, provided that $|x-2| < 1$ and $|x-2| < \frac{2\epsilon}{3}$, the inequality $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ holds. Putting the two constraints together, we get $\delta = \min(1, \frac{2\epsilon}{3})$.

1.7 Problems

Problem 2

Find the following limits.

Solution

(i)

$$\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

(ii)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \sqrt{1-x^2}}{x} &= \lim_{x \rightarrow 0} \frac{(1 - \sqrt{1-x^2})(1 + \sqrt{1-x^2})}{x(1 + \sqrt{1-x^2})} \\ &= \lim_{x \rightarrow 0} \frac{1 - (1-x^2)}{x(1 + \sqrt{1-x^2})} \\ &= \lim_{x \rightarrow 0} \frac{x^2}{x(1 + \sqrt{1-x^2})} \\ &= \lim_{x \rightarrow 0} \frac{x}{1 + \sqrt{1-x^2}} = 0\end{aligned}$$

(iii)

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{1 - \sqrt{1 - x^2}}{x^2} &= \lim_{x \rightarrow 0} \frac{(1 - \sqrt{1 - x^2})(1 + \sqrt{1 - x^2})}{x^2(1 + \sqrt{1 - x^2})} \\&= \lim_{x \rightarrow 0} \frac{1 - (1 - x^2)}{x^2(1 + \sqrt{1 - x^2})} \\&= \lim_{x \rightarrow 0} \frac{x^2}{x^2(1 + \sqrt{1 - x^2})} \\&= \lim_{x \rightarrow 0} \frac{1}{1 + \sqrt{1 - x^2}} = \frac{1}{2}\end{aligned}$$

Problem 3i, ii

In each of the following cases, find a δ such that $|f(x) - l| < \epsilon$ for all x satisfying $0 < |x - a| < \delta$.

Solution

(i) $f(x) = x^4; l = a^4$

Let $\epsilon > 0$ be given. We must find δ such that $0 < |x - a| < \delta$ implies $|x^4 - a^4| < \epsilon$ for all x . Observe that

$$|x^4 - a^4| = |(x^2 + a^2)(x + a)(x - a)| = |x^2 + a^2||x + a||x - a|$$

We must find a bound on $|x + a|$ and $|x^2 + a^2|$. Start by arbitrarily fixing $|x - a| < 1$. Then

$$\begin{aligned}-1 &< x - a < 1 \\ \implies 2a - 1 &< x + a < 2a + 1 && \text{add } 2a \text{ to both sides}\end{aligned}$$

We now have a bound on $x + a$, but we need one on $|x + a|$. It's easy to see $|x + a| < \max(|2a - 1|, |2a + 1|)$. By triangle inequality ($|a + b| \leq |a| + |b|$):

$$\begin{aligned}|2a - 1| &\leq |2a| + |-1| = |2a| + 1 \\ |2a + 1| &\leq |2a| + |1| = |2a| + 1\end{aligned}$$

Thus $|x + a| < |2a| + 1$, provided $|x - a| < 1$. Similarly, we find a bound for $|x^2 + a^2|$:

$$\begin{aligned}-1 &< x - a < 1 \\ \implies a - 1 &< x < a + 1 \\ \implies (a - 1)^2 &< x^2 < (a + 1)^2 && \text{square each side} \\ \implies (a - 1)^2 + a^2 &< x^2 + a^2 < (a + 1)^2 + a^2\end{aligned}$$

Observe that $x^2 + a^2 = |x^2 + a^2|$, thus $|x^2 + a^2| < (a+1)^2 + a^2$. Thus to make $|x^4 - a^4| < \epsilon$ we must set

$$|x - a| < \frac{\epsilon}{(|2a| + 1)((a+1)^2 + a^2)}$$

provided $|x - a| < 1$. Therefore

$$\delta = \min\left(1, \frac{\epsilon}{(|2a| + 1)(2a^2 + 2a + 1)}\right)$$

(ii) $f(x) = \frac{1}{x}; a = 1, l = 1$

Let $\epsilon > 0$ be given. We must find δ such that $0 < |x - 1| < \delta$ implies $|\frac{1}{x} - 1| < \epsilon$ for all x . Observe that

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1}{x} - \frac{x}{x}\right| = \left|\frac{1-x}{x}\right| = \frac{|x-1|}{|x|}$$

Fix $|x - 1| < \frac{1}{10}$. Then

$$\begin{aligned} -\frac{1}{10} &< x - 1 < \frac{1}{10} \\ \implies \frac{9}{10} &< x < \frac{11}{10} \end{aligned}$$

Thus we must set

$$|x - 1| < \frac{\epsilon}{10}$$

provided $|x - 1| < \frac{1}{10}$. Therefore

$$\delta = \min\left(\frac{1}{10}, \frac{\epsilon}{10}\right)$$

Problem 8

Answer the following.

Solution

(a) If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ do not exist, can $\lim_{x \rightarrow a} [f(x) + g(x)]$ or $\lim_{x \rightarrow a} f(x)g(x)$ exist?

Yes. Consider

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$$

Then $(g + f)(x) = 0$ and $(gf)(x) = -1$, both of which have limits for all a .

(b) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} [f(x) + g(x)]$ exists, must $\lim_{x \rightarrow a} g(x)$ exist?

Yes. Let $\epsilon > 0$ be given. Then there exists δ such that for all x in $0 < |x - a| < \delta$ the following inequalities hold:

$$M - \epsilon/2 < f(x) + g(x) < M + \epsilon/2, \quad L - \epsilon/2 < f(x) < L + \epsilon/2$$

Then:

$$\begin{aligned} M - \epsilon/2 &< f(x) + g(x) < M + \epsilon/2 \\ \implies M - \epsilon/2 - f(x) &< g(x) < M + \epsilon/2 - f(x) \\ \implies M - \epsilon/2 - L - \epsilon/2 &< g(x) < M + \epsilon/2 - L + \epsilon/2 \\ (M - L) - \epsilon &< g(x) < (M - L) + \epsilon \\ -\epsilon &< g(x) - (M - L) < \epsilon \\ |g(x) - (M - L)| &< \epsilon \end{aligned}$$

Therefore $\lim_{x \rightarrow a} g(x) = M - L$ and must exist.

(c) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} g(x)$ does not exist, can $\lim_{x \rightarrow a} [f(x) + g(x)]$ exist?

No. Let $\lim_{x \rightarrow a} f(x) = L$. Since $\lim_{x \rightarrow a} g(x)$ does not exist, there exists ϵ such that $|g(x) - M| \geq \epsilon$ for all M . Suppose for contradiction $\lim_{x \rightarrow a} [f(x) + g(x)] = M$ exists. Then

$$\begin{aligned} M - \epsilon/2 &< f(x) + g(x) < M + \epsilon/2 \\ \implies M - \epsilon/2 - f(x) &< g(x) < M + \epsilon/2 - f(x) \\ \implies M - \epsilon/2 - L - \epsilon/2 &< g(x) < M + \epsilon/2 - L + \epsilon/2 \\ (M - L) - \epsilon &< g(x) < (M - L) + \epsilon \\ -\epsilon &< g(x) - (M - L) < \epsilon \\ |g(x) - (M - L)| &< \epsilon \end{aligned}$$

We have a contradiction, thus $\lim_{x \rightarrow a} [f(x) + g(x)]$ does not exist.

(d) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} f(x)g(x)$ exists, does it follow that $\lim_{x \rightarrow a} g(x)$ exists?

No. Consider $g(x) = 1/x$ which has no limit at 0, and $f(x) = 0$. Then $f(x)g(x) = 0$ which has a limit of 0 as $x \rightarrow 0$.

Problem 13

Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$. Prove that $\lim_{x \rightarrow a} g(x)$ exists, and that $\lim_{x \rightarrow a} g(x) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$. (Draw a picture!)

Solution

Let $L = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$. Let $\epsilon > 0$ be given. We must find δ such that $0 < |x - a| < \delta$ implies $|g(x) - L| < \epsilon$.

By limit definition there exists δ_1 such that for all x in $0 < |x - a| < \delta_1$

$$\begin{aligned} |f(x) - L| &< \epsilon \\ -\epsilon &< f(x) - L < \epsilon \\ L - \epsilon &< f(x) < L + \epsilon \end{aligned}$$

Similarly there exists δ_2 such that for all x in $0 < |x - a| < \delta_2$

$$\begin{aligned} |h(x) - L| &< \epsilon \\ -\epsilon &< h(x) - L < \epsilon \\ L - \epsilon &< h(x) < L + \epsilon \end{aligned}$$

By problem statement $f(x) \leq g(x) \leq h(x)$. Fix $\delta = \min(\delta_1, \delta_2)$. Then

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$

Therefore $L - \epsilon < g(x) < L + \epsilon$ which implies $|g(x) - L| < \epsilon$, as desired.

Problem 15

Evaluate the following limits in terms of the number $\alpha = \lim_{x \rightarrow 0} (\sin x)/x$.

Solution

(i)

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{2 \sin x \cos x}{x} = 2\alpha \cos x = 2\alpha$$

(iv)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin^2 2x}{x^2} &= \lim_{x \rightarrow 0} \frac{(2 \sin x \cos x)^2}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{4 \sin^2 x \cos^2 x}{x^2} \\ &= \lim_{x \rightarrow 0} 4\alpha^2 \cos^2 x \\ &= 4\alpha^2 \end{aligned}$$

(vii)

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{x \sin x(1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\
&= \lim_{x \rightarrow 0} \frac{x \sin x(1 + \cos x)}{\sin^2 x} \\
&= \lim_{x \rightarrow 0} \frac{x(1 + \cos x)}{\sin x} \\
&= \lim_{x \rightarrow 0} \frac{1 + \cos x}{\alpha} = \frac{2}{\alpha}
\end{aligned}$$

(ix)

$$\begin{aligned}
\lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)}{x - 1} &= \lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)(x + 1)}{(x - 1)(x + 1)} \\
&= \lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)(x + 1)}{x^2 - 1}
\end{aligned}$$

Let $u = x^2 - 1$. Observe that as $x \rightarrow 1$, $u \rightarrow 0$. Thus

$$\lim_{x \rightarrow 1} \frac{\sin(x^2 - 1)(x + 1)}{x^2 - 1} = \lim_{u \rightarrow 0} \frac{\sin u}{u} \cdot \lim_{x \rightarrow 1} x + 1 = 2\alpha$$

Problem 19

Prove that if $f(x) = 0$ for irrational x and $f(x) = 1$ for rational x , then $\lim_{x \rightarrow a} f(x)$ does not exist for any a .

Solution

Let $\epsilon = \frac{1}{10}$. We handle two cases. First suppose $L < \frac{1}{2}$. Pick any rational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |1 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

Similarly, suppose $L > \frac{1}{2}$. Pick any irrational x from the interval $0 < |x - a| < \delta$. Then $|f(x) - L| = |0 - L| > \frac{1}{2}$. Thus $|f(x) - L| \geq \frac{1}{10}$.

2 Continuity, Part I

2.1 Definition of continuity

A function f is **continuous** at a when

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Inlining the limits definition, f is continuous at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

We can simplify this definition slightly. Observe that in continuous functions $f(a)$ exists, and at $x = a$ we get $f(x) - f(a) = 0$. Thus we can relax the constraint $0 < |x - a| < \delta$ to $|x - a| < \delta$.

A function f is **continuous on an interval** (a, b) if it's continuous at all $c \in (a, b)$ ⁴.

2.2 Recognizing continuous functions

The following theorems allow us to tell at a glance that large classes of functions are continuous (e.g. polynomials, rational functions, etc.)

Five easy proofs

Constants. Let $f(x) = c$. Then f is continuous at all a because

$$\lim_{x \rightarrow a} f(x) = c = f(a)$$

Identity. Let $f(x) = x$. Then f is continuous at all a because

$$\lim_{x \rightarrow a} f(x) = a = f(a)$$

Addition. Let $f, g \in \mathbf{R} \rightarrow \mathbf{R}$ be continuous at a . Then $f + g$ is continuous at a because

$$\lim_{x \rightarrow a} (f + g)(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = f(a) + g(a) = (f + g)(a)$$

Multiplication. Let $f, g \in \mathbf{R} \rightarrow \mathbf{R}$ be continuous at a . Then $f \cdot g$ is continuous at a because

$$\lim_{x \rightarrow a} (fg)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = f(a) \cdot g(a) = (fg)(a)$$

Reciprocal. Let g be continuous at a . Then $\frac{1}{g}$ is continuous at a where $g(a) \neq 0$ because

$$\lim_{x \rightarrow a} \left(\frac{1}{g} \right)(x) = \frac{1}{\lim_{x \rightarrow a} g(x)} = \frac{1}{g(a)} = \left(\frac{1}{g} \right)(a)$$

Slightly harder proof: composition

Let $f, g \in \mathbf{R} \rightarrow \mathbf{R}$. Let g be continuous at a , and let f be continuous at $g(a)$. Then $f \circ g$ is continuous at a . Put differently, we want to show

$$\lim_{x \rightarrow a} (f \circ g)(x) = (f \circ g)(a)$$

⁴Closed intervals are a tiny bit harder, and I'm keeping them out for brevity.

Unpacking the definitions, let $\epsilon > 0$ be given. We want to show there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$\begin{aligned} |(f \circ g)(x) - (f \circ g)(a)| \\ = |f(g(x)) - f(g(a))| < \epsilon \end{aligned}$$

By problem statement we have two continuities.

First, f is continuous at $g(a)$, i.e. $\lim_{X \rightarrow g(a)} f(X) = f(g(a))$. Thus there exists $\delta' > 0$ such that $|X - g(a)| < \delta'$ implies $|f(X) - f(g(a))| < \epsilon$.

Second, g is continuous at a , i.e. $\lim_{x \rightarrow a} g(x) = g(a)$. Thus there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta'$. Since we can make ϵ be anything, we can set it to δ' .

I.e. there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta'$. Intuitively, $g(x)$ is close to $g(a)$. But by the first continuity, any X close to $g(a)$ implies

$$|f(X) - f(g(a))| < \epsilon$$

Thus $|f(g(x)) - f(g(a))| < \epsilon$, as desired.

2.3 Appendix: Stars over Babylon

Stars over Babylon is a modification of the Dirichlet function (see 1.5).

TODO: come back and study this.

2.4 Problems

Problem 1

For which of the following functions f is there a continuous function F with domain \mathbf{R} such that $F(x) = f(x)$ for all x in the domain of f ?

Solution

(i) $f(x) = \frac{x^2 - 4}{x - 2}$

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{(x - 2)} = x + 2$$

when $x \neq 2$. Thus let $F(x) = x + 2$. It follows $F(x) = f(x)$ for all x in the domain of f , as 2 is not in the domain of f . Further, F is continuous in R as

$$\lim_{x \rightarrow a} x + 2 = a + 2$$

(ii) $f(x) = \frac{|x|}{x}$

There is no such function F . Observe that if we define F such that $F(x) = f(x)$ for all x in the domain of f , the only point we have left to manipulate is 0. It's easy to see that no definition of $F(0)$ will allow F to have a limit at 0, and thus F cannot be continuous.

Problem 5

For each number a , find a function which is continuous at a , but not at any other points.

Solution

As per textbook, the following modification of the Dirichlet function is continuous at 0 but nowhere else:

$$f(x) = \begin{cases} x & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

Shifting horizontally by a creates a function continuous at a but nowhere else:

$$g(x) = \begin{cases} x - a & \text{for rational } x, \\ 0 & \text{for irrational } x. \end{cases}$$

Problem 8

Suppose that f is continuous at a and $f(a) = 0$. Prove that if $\alpha \neq 0$, then $f + \alpha$ is nonzero in some open interval containing a .

Solution

Since f is continuous at a , by continuity theorems (addition of identity and constant functions) $f + \alpha$ is also continuous. Also $(f + \alpha)(a) = \alpha$, and thus $\lim_{x \rightarrow a} (f + \alpha)(x) = \alpha$. We can set $\epsilon = \alpha$, and by limit definition there exists an interval in which $|(f + \alpha)(x) - \alpha| < \alpha$. In other words within the interval $f + \alpha$ is less than α away from α , and thus cannot be zero.

3 Complete ordered fields

3.1 Motivation

The twelve ordered field axioms are sufficient to define limits, continuity, and prove all the theorems in the previous sections. Since the set \mathcal{Q} of rational numbers is an ordered field⁵, rationals have been sufficient for the work we've done so far. However, we are about to start proving slightly more sophisticated theorems about continuous functions, and ordered fields will quickly start breaking our intuitions.

For example, consider the function $f(x) = x^2 - 2$ (a parabola shifted down two units). It's easy to see f is a continuous function, and thus our intuition is that we should be able to draw it without "lifting the tip of the pencil off the sheet

⁵The proof is straightforward, so I'm not including it here.

of paper”. Upon reflection however, it becomes obvious that in the universe limited to ordered fields this is impossible. f intersects the x-axis when $x^2 = 2$, but every high school student knows $\sqrt{2} \notin \mathcal{Q}$ (see 3.5 for proof). Thus there is no $x \in \mathcal{Q}$ such that $f(x) = 0$. And since \mathcal{Q} is an ordered field, it follows ordered fields alone aren’t sufficient to resolve this problem.

The *intermediate value theorem* (see 4.1) formalizes the claim that a continuous function segment that starts below the x-axis and ends above the x-axis intersects the x-axis. But as we can see from the example above, this is not possible to prove with ordered field axioms alone. So before we proceed with further study of continuity, we need one more axiom called *the completeness axiom*, which we introduce in this chapter.

Combined with the twelve ordered field axioms, the completeness axiom forms *complete ordered fields*. These objects are sufficient to proceed with our study of calculus. We will see that rational numbers \mathcal{Q} are not a complete ordered field, whereas real numbers \mathcal{R} are.⁶ Thus from here \mathcal{R} -valued functions will become our primary object of study.

3.2 Least upper bound

Definition: b is an **upper bound** for S if $s \leq b$ for all $s \in S$.

For example:

- Any $b \geq 1$ is an upper bound for $S = \{x : 0 \leq x < 1\}$. E.g. 1, 2, 10 are all upper bounds of S .
- By convention, *every* number is an upper bound for \emptyset .
- The set \mathcal{N} of natural numbers has no natural upper bound. The proof is easy. Suppose $b \in \mathcal{N}$ is an upper bound for \mathcal{N} . But $b + 1 \in \mathcal{N}$, and $b + 1 > b$, which is a contradiction. Thus b isn’t an upper bound for \mathcal{N} .⁷

Definition: x is a **least upper bound** of A , if

1. x is an upper bound of A ,
2. *and* if y is an upper bound of A , then $x \leq y$.

A set can have only one least upper bound. The proof is easy. Suppose x and x' are both least upper bounds of S . Then $x \leq x'$ and $x' \leq x$. Thus $x = x'$. Consequently, we can use a convenient notation $\sup A$ to denote the least upper bound of A .

⁶Proof that \mathcal{R} is a complete ordered field requires construction of \mathcal{R} , which doesn’t happen in Spivak until the last chapters. Thus I will not be delving into that here and ask the reader (i.e., currently myself) to take this on faith.

⁷We need to do a little more work to show \mathcal{N} has no upper bound, natural or not. Be patient! We will prove this by the end of the section.

Obligatory examples:

- Let $S = \{x : 0 \leq x < 1\}$. Then $\sup S = 1$.
- By convention, the empty set \emptyset has no least upper bound.

3.3 Completeness axiom

We are now ready to state the completeness axiom.

Completeness [P13]: If A is a non-empty set of numbers that has an upper bound, then it has a least upper bound.

Claim: rational numbers are not complete.

Proof: Let $C = \{x : x^2 < 2 \text{ and } x \in \mathcal{Q}\}$. Suppose for contradiction rational numbers are complete. Then there exists $b \in \mathcal{Q}$ such that $b = \sup C$. Observe that

- $b^2 \neq 2$ as that would imply $b = \sqrt{2}$ and thus $b \notin \mathcal{Q}$.
- $b^2 < 2$ as there would exist some $x \in C$ such that $b^2 < x^2 < 2$. Thus $b < x$ and b is not the upper bound.

Therefore $b^2 > 2$. But this implies there exists some $x \in \mathcal{Q}$ such that $2 < x^2 < b^2$. Thus x is greater than every element in C , and $x < b$. So b is not the *least* upper bound. We have a contradiction, therefore rational numbers are not complete, as desired.

Claim: completeness cannot be derived from ordered fields.

Proof: \mathcal{Q} is not complete and \mathcal{Q} is an ordered field. Thus completeness is not a property of ordered fields.

Claim: real numbers are complete.

Proof [deferred]: The completeness property can be derived from the construction of real numbers \mathcal{R} , which makes reals a **complete ordered field**. The proof requires we study the actual construction of \mathcal{R} , which Spivak leaves until the last chapters. Thus for the moment the proof will be taken on faith. In any case, it is better to build calculus upon abstract complete ordered fields than upon concrete real numbers.

3.4 Consequences of completeness

\mathcal{N} is not bounded above

We've shown \mathcal{N} has no upper bound in \mathcal{N} . Now we show \mathcal{N} has no upper bound in \mathcal{R} .

Suppose for contradiction \mathcal{N} has an upper bound. Since $\mathcal{N} \neq \emptyset$ then by completeness \mathcal{N} has a least upper bound. Let $\alpha = \sup \mathcal{N}$. Then:

$$\begin{aligned} \alpha &\geq n \text{ for all } n \in \mathcal{N} \\ \implies \alpha &\geq n + 1 \text{ for all } n \in \mathcal{N} && \text{since } n + 1 \in \mathcal{N} \text{ if } n \in \mathcal{N} \\ \implies \alpha - 1 &\geq n \text{ for all } n \in \mathcal{N} \end{aligned}$$

Thus $\alpha - 1$ is *also* an upper bound for \mathcal{N} . This contradicts that $\alpha = \sup \mathcal{N}$. Therefore \mathcal{N} is not bounded above, as desired.

$\sqrt{2}$ exists

We show $\sqrt{2} \in \mathcal{R}$. Let $S = \{y \in \mathcal{R} : y^2 < 2\}$. Obviously S is non-empty and has an upper bound. Thus by completeness property it has a least upper bound. Let $x = \sup S$. Note that $1 \in S$ and 2 is an upper bound of S . Thus $1 \leq x \leq 2$. We show $x^2 = 2$ by showing $x^2 \not< 2$ and $x^2 \not> 2$.

Case 1. Suppose for contradiction $x^2 < 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned} (x + \epsilon)^2 &= x^2 + 2\epsilon x + \epsilon^2 \\ &\leq x^2 + 4\epsilon + \epsilon && \text{since } x < 2 \text{ and } \epsilon < 1 \\ &= x^2 + 5\epsilon < 2 && \text{since } x^2 < 2 \text{ (by supposition), we can pick} \\ &&& \text{a small enough } \epsilon \text{ to make this true} \end{aligned}$$

Thus there exists ϵ such that $(x + \epsilon)^2 < 2$. By definition of S it follows $x + \epsilon \in S$, which contradicts that x is the least upper bound. Therefore $x^2 \not< 2$

Case 2. Suppose for contradiction $x^2 > 2$. Let $0 < \epsilon < 1$ be a small number. Then

$$\begin{aligned} (x - \epsilon)^2 &= x^2 - 2\epsilon x + \epsilon^2 \\ &\geq x^2 - 2\epsilon x && \text{since } \epsilon^2 > 0 \\ &\geq x^2 - 4\epsilon && \text{since } x \leq 2 \\ &> 2 && \text{since } x^2 > 2 \text{ (by supposition), we can pick} \\ &&& \text{a small enough } \epsilon \text{ to make this true} \end{aligned}$$

Thus $(x - \epsilon)^2 > 2$, which by definition of S implies $x - \epsilon > y$ for all $y \in S$. So $x - \epsilon$ is an upper bound of S . We have a contradiction—since $x - \epsilon < x$, it follows x is not a least upper bound. Therefore $x^2 \not> 2$ as desired.

Since $x^2 \not< 2$ and $x^2 \not> 2$, it follows $x^2 = 2$ as desired.

Archimedean property

Handwavy definition: the Archimedean property states that you can fill the universe with tiny grains of sand.

Formal definition: let $\epsilon > 0$ be small and let $r > 0$ be large. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > r$.

Proof: suppose for contradiction the property is false. Then there exist ϵ, r such that for all $n \in \mathcal{N}$, $n\epsilon \leq r$. Therefore $n \leq \frac{r}{\epsilon}$. This implies \mathcal{N} is bounded, which is a contradiction.

A useful special case is when $r = 1$. In this case the Archimedean property can be restated as follows. Let $\epsilon > 0$ be small. Then there exists $n \in \mathcal{N}$ such that $n\epsilon > 1$. Put differently, there exists $n \in \mathcal{N}$ such that $\frac{1}{n} < \epsilon$.

A few more notes on the Archimedean property:

- Obviously the Archimedean property follows from completeness, as shown above.
- The Archimedean property is true in \mathcal{Q} and can be proven without being assumed⁸.
- Completeness does not follow from the Archimedean property. The proof is easy: the Archimedean property holds on \mathcal{Q} , and we know \mathcal{Q} is not complete as shown above.

Density

Let $x, y \in \mathcal{R}$. Then S is a **dense subset** of \mathcal{R} if there is an element of S in (x, y) . Put differently, there is an element of S between any two points in \mathcal{R} .

- Obviously \mathcal{R} is a dense subset of itself.
- Integers are not a dense subset of \mathcal{R} . E.g. there is no integer between 1.1 and 1.9.
- The set of positive numbers $\{x : x \in \mathcal{R}, x > 0\}$ is not a dense subset of \mathcal{R} . E.g. there is no positive number between -2 and -1 .

Claim: the set of rational numbers \mathcal{Q} is dense.

Proof: let $x, y \in \mathcal{R}$ be given. Suppose we can show there exists a rational in (x, y) for $0 \leq x < y$. Then:

- Given $x < y \leq 0$, there is a rational r in $(-y, -x)$. So $-r$ is in (x, y) .
- Given $x < 0 < y$, there is a rational r in $(y, 0)$. So r is of course also in (x, y) .

⁸Excluding the proof here, but it's fairly simple

Thus all we must do is prove there exists a rational in (x, y) for $0 \leq x < y$.

Let $0 \leq x < y$ be given.

Claim: the set of irrational numbers $\mathcal{R} \setminus \mathcal{Q}$ is dense.

Proof: let $x, y \in \mathcal{R}$ be given. By density of the rationals there exists $r \in \mathcal{Q}$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Multiplying each side by $\sqrt{2}$, we get $x < \sqrt{2}r < y$. We know $\sqrt{2}r$ is irrational. Thus there exists an irrational number between any two numbers in \mathcal{R} , and the set of irrationals $\mathcal{R} \setminus \mathcal{Q}$ is dense as desired.

3.5 Appendix: $\sqrt{2}$ is irrational

Suppose $\sqrt{2} \in \mathcal{Q}$. Then there exist $a, b \in \mathcal{N}$ such that $\left(\frac{a}{b}\right)^2 = 2$. Assume a, b have no common divisor (since we can obviously keep simplifying until this is the case). Observe that both a and b cannot be even, otherwise we could simplify further.

Now we have $a^2 = 2b^2$. Thus a^2 is even, a must be even⁹, and there exists $k \in \mathcal{N}$ such that $a = 2k$. Then $a^2 = 4k^2 = 2b^2$ so $2k^2 = b^2$. Thus b^2 is even and so b is even. Since both a and b cannot be even, this is a contradiction. Thus $\sqrt{2} \notin \mathcal{Q}$ as desired.

3.6 Problems

Problem 3a

Let f be a continuous function on $[a, b]$ with $f(a) < 0 < f(b)$. The proof of Theorem 1 showed that there is a smallest x in $[a, b]$ with $f(x) = 0$. Is there necessarily a second smallest x in $[a, b]$ with $f(x) = 0$? Show that there is a largest x in $[a, b]$ with $f(x) = 0$. (Try to give an easy proof by considering a new function closely related to f .)

Solution

Problem 5a

Suppose that $y - x > 1$. Prove that there is an integer k such that $x < k < y$. Hint: let l be the largest integer satisfying $l \leq x$, and consider $l + 1$.

Solution

Problem 5b

Suppose $x < y$. Prove that there is a rational number r such that $x < r < y$. Hint: if $1/n < y - x$, then $ny - nx > 1$. (Query: Why have parts (a) and (b) been postponed until this problem set?)

⁹Even numbers have even squares because $(2k)^2 = 4k^2 = 2 \cdot (2k^2)$

Solution

Problem 5c

Suppose $r < s$ are rational numbers. Prove that there is an irrational number between r and s . Hint: As a start, you know that there is an irrational number between 0 and 1.

Solution

Problem 5d

Suppose that $x < y$. Prove that there is an irrational number between x and y . Hint: It is unnecessary to do any more work; this follows from (b) and (c).

Solution

Problems 6a, b

A set A of real numbers is said to be **dense** if every open interval contains a point of A . For example, Problem 5 shows that the set of rational numbers and the set of irrational numbers are each dense.

(a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A , then $f(x) = 0$ for all x .

(b) Prove that if f and g are continuous and $f(x) = g(x)$ for all x in a dense set A , then $f(x) = g(x)$ for all x .

Solution

Problem 11a

Suppose that a_1, a_2, a_3, \dots is a sequence of positive numbers with $a_{n+1} \leq a_n/2$. Prove that for any $\epsilon > 0$ there is some n with $a_n < \epsilon$.

Solution

4 Continuity, Part II

4.1 Intermediate Value Theorem

4.2 Extreme Value Theorem

4.3 Appendix: IVT and EVT consequences