Slava A's notes working through Spivak's Calculus

Slava Akhmechet

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I'm working through Spivak Calculus. Around the chapter on epsilon-delta limits the details get pretty confusing. I started supplementing with David Galvin's notes, which are considerabley more clear but are still confusing. This is surprising because the topic doesn't use anything beyond basic middle school math. Feels like it should be simple! And so I'm attempting to write these notes to properly understand the damned thing.

Contents

1 Limits

1.1 A hand-wavy limits definition

I'll first do a hand-wavy definition of limits, and then use it to explain the mechanics of computing limits of functions in practice. Basically, pre-calculus stuff. After that I'll do a proper definition and use it to prove the theorems that make the mechanics work.

A hand-wavy definition: a limit of $f(x)$ at a is the value $f(x)$ approaches close to (but not necessarily at) a.

A slightly less hand-wavy definition: let $f : \mathbf{R} \to \mathbf{R}$, let $a \in \mathbf{R}$ be some number on the x-axis, and let $l \in \mathbf{R}$ be some number on the y-axis. Then as x gets closer to a, $f(x)$ gets closer to l.

The notation for this whole thing is

$$\lim_{x \to a} f(x) = l$$

So for example $\lim_{x\to 5} x^2 = 25$ because the closer x gets to 5, the closer x^2 gets to 25 (we'll prove all this properly soon). Now suppose you have some fancy pants function like this one:

$$\lim_{x \to 0} \frac{1-\sqrt{x}}{1-x} \tag{1}$$

If you plot it, it's easy to see that as x approaches 0, the whole shebang approaches 1. But how do you algebraically evaluate the limit of this thing? Can you just plug 0 into the equation? It seems to work, but once we formally define limits, we'll have to prove somehow that plugging $a = 0$ into x gives us the correct result.

1.2 Limits evaluation mechanics

It turns out that it does in fact work because of a few theorems that make practical evaluation of many limits easy. I'll first state these theorems as facts, and then go back and properly prove them once I introduce the formal definition of limits.

1. **Constants**. $\lim_{x\to a} c = c$, where $c \in \mathbf{R}$. In other words if the function is a constant, e.g. $f(x) = 5$, then $\lim_{x\to a} f(x) = 5$ for any a.

2. **Identity**. $\lim_{x\to a} x = a$. In other words if the function is an identity function $f(x) = x$, then $\lim_{x\to 6} f(x) = 6$. Meaning we simply plug a into x.

3. **Addition**[1]. $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$. For example $\lim_{x\to a}(x+2) = \lim_{x\to a} x + \lim_{x\to a} 2 = a + 2$.

4. **Multiplication**. $\lim_{x\to a}(f \cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. For example $\lim_{x\to a} 2x = \lim_{x\to a} 2 \cdot \lim_{x\to a} x = 2a$.

5. **Reciprocal**. $\lim_{x\to a} \left(\frac{1}{f}\right)(x) = \frac{1}{\lim_{x\to a} f(x)}$ when the denominator isn't zero. For example $\lim_{x\to a} \frac{1}{x} = \frac{1}{\lim_{x\to a} x} = \frac{1}{a}$ for $a \neq 0$.

To come back to 1, these theorems tells us that

$$\lim_{x\to 0} \frac{1-\sqrt{x}}{1-x} = \frac{\lim_{x\to 0} 1 - (\lim_{x\to 0} x)^{\frac{1}{2}}}{\lim_{x\to 0} 1 - \lim_{x\to 0} x} = \frac{1-0^{\frac{1}{2}}}{1-0} = 1$$

[1]Spivak's book uses a slightly more verbose definition that assumes the limits of f and g exist near a, see p. 103

Holes

What happens if we try to take a limit as $x \to 1$ rather than $x \to 0$?

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$$

We can't use the same trick and plug in 1 because we get a nonsensical result 0/0 as the function isn't defined at 0. If we plot it, we clearly see the limit approaches 1/2 at 0, but how do we prove this algebraically? The answer is to do some trickery to find a way to cancel out the inconvenient term (in this case $1-\sqrt{x}$)

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

Why is it ok here to divide by $1-\sqrt{x}$? Good question! Recall that the limit is defined *close to a* (or *around a*, or as *x approaches a*), but not **at** a. In other words $f(a)$ need not even be defined (as is the case here). This means that as we consider $1-\sqrt{x}$ at different values of x as it approaches a, the limit never requires us to evaluate the function at $x = a$. So we never have to consider $1-\sqrt{x}$ as $x = 1$, $1-\sqrt{x}$ never takes on the value of 0, and it is safe to divide it out.

1.3 Formal limits definition

Now we establish a rigorous definition of limits that formalizes the hand-wavy version above.

Definition: $\lim_{x\to a} f(x) = L$ when for any $\epsilon \in \mathbf{R}$ there exists $\delta \in \mathbf{R}$ such that for all x, $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. (Also $\epsilon > 0, \delta > 0$.)

Here is what this says. Suppose $\lim_{x\to a} f(x) = L$. You pick any interval on the y-axis around L. Make it as small (or as large) as you want. I'll produce an interval on the x-axis around a. You can take any number from my interval, plug it into f, and the output will stay within the bounds you specified.

So ϵ specifies the distance away from L along the y-axis, and δ specifies the distance away from a along the x-axis. Take any x within δ of a, plug it into f, and the result is guaranteed to be within ϵ of L. $\lim_{x\to a} f(x) = L$ just means there exists such δ for any ϵ.

This is quite simple, but the mechanics of the limit definition tend to confuse people. I think it's because absolute value inequalities are unfamiliar. Wtf is $0 < |x-a| < \delta$ and $|f(x) - L| < \epsilon$?! Let's tease it apart[2]

[2] The best answer is to go to Khan academy and do a bunch of absolute value inequalities until they become second nature.

Here is the intuitive reading. $0 < |x - a| < \delta$ means the difference between x and a is between 0 and δ. And $|f(x) - L| < \epsilon$ means the difference between $f(x)$ and L is less than ϵ. This is actually all this means, but still, let's look at the inequalities more closely.

First, consider $0 < |x - a| < \delta$. There are two inequalities here. The left side, $0 < |x - a|$ is equivalent to $|x - a| > 0$. But $|x - a|$ is an absolute value, it's **always** true that $|x - a| \geq 0$. So this part of the inequality says $x - a \neq 0$, or $x \neq a$. (Remember, we said the limit is defined *around* a but not *at* a). I don't know why mathematicians say $0 < |x - a|$ instead of $x \neq a$, probably because confusing you brings them pleasure.

The right side is $|x - a| < \delta$. Intuitively this says that along the x-axis the difference between x and a should be less than δ. Put differently, x should be within δ of a. We can rewrite this as $a - \delta < x < a + \delta$.

The second equation, $|f(x) - L| < \epsilon$ should now be easy to understand. Intuitively, along the y-axis $f(x)$ should be within ϵ of L, or put differently $L - \epsilon < f(x) < L + \epsilon$.

Limit uniqueness

Suppose $\lim_{x \to a} f(x) = L$. It's easy to assume L is the only limit around a, but such a thing needs to be proved. We prove this here. More formally, suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. We prove that $L = M$.

Suppose for contradiction $L \neq M$. Assume without loss of generality $L > M$. By limit definition, for all $\epsilon > 0$ there exists a positive $\delta \in \mathbf{R}$ such that $0 < |x - a| < \delta$ implies

- $|f(x) - L| < \epsilon \implies L - \epsilon < f(x)$
- $|f(x) - M| < \epsilon \implies f(x) < M + \epsilon$

for all x. Thus

$$
\begin{aligned}
&L - \epsilon < f(x) < M + \epsilon \\
&\implies L - \epsilon < M + \epsilon \\
&\implies L - M < 2\epsilon
\end{aligned}
$$

The above is true for all ϵ. Now let's narrow our attention and consider a concrete $\epsilon = (L - M)/4$, which we easily find leads to a contradiction[3]:

[3] note we assumed $L > M$, thus $\epsilon = (L - M)/4 > 0$

$$
\begin{aligned}
&L - M < 2\epsilon \\
&\implies (L-M)/4 < \epsilon/2 && \text{dividing both sides by 4} \\
&\implies \epsilon < \epsilon/2 && \text{recall we set } \epsilon = (L-M)/4
\end{aligned}
$$

We have a contradiction, and so $L = M$ as desired.

Half-Value Continuity Lemma

This lemma will come in handy later, so we may as well prove it now. Suppose $M \neq 0$ and $\lim_{x\to a} g(x) = M$. We show that there exists some δ such that $0 < |x - a| < \delta$ implies $|g(x)| \geq |M|/2$ for all x.

Intuitively, the lemma states the following: when a function g approaches a nonzero limit M near a point, there exists an interval in which the values of g are closer to M than to zero.

Proof. The claim that $|g(x)| \geq |M|/2$ is equivalent to

$$g(x) \leq -|M|/2 \quad \text{or} \quad g(x) \geq |M|/2$$

There are two possibilities: either $M > 0$ or $M < 0$. Let's consider each possibility separately.

Case 1. Suppose $M > 0$. Then to show $|g(x)| \geq |M|/2$ it is sufficient to show *either* $g(x) \leq -M/2$ or $g(x) \geq M/2$. We will show $g(x) \geq M/2$. Fix $\epsilon = M/2$. By limit definition there is some δ such that $0 < |x - a| < \delta$ implies for all x

$$
\begin{aligned}
&|g(x) - M| < M/2 \\
&\implies -M/2 < g(x) - M \\
&\implies M/2 < g(x) && \text{add } M \text{ to both sides} \\
&\implies g(x) > M/2 && \text{note } \geq \text{ is correct but not tight}
\end{aligned}
$$

Case 2. Suppose $M < 0$. We must show either $g(x) \leq M/2$ or $g(x) \geq -M/2$. We will show $g(x) \leq M/2$. Fix $\epsilon = -M/2$. Then

$$
\begin{aligned}
&|g(x) - M| < -M/2 \\
&\implies g(x) - M < -M/2 \\
&\implies g(x) < M/2 && \text{add } M \text{ to both sides;} \\
& && \text{note } \leq \text{ is correct but not tight}
\end{aligned}
$$

QED.

1.4 Theorems that make evaluation work

Armed with the formal definition, we can use it to rigorously prove the five theorems useful for evaluating limits (constants, identity, addition, multiplication, reciprocal). Let's do that now.

Constants

Let $f(x) = c$. We prove that $\lim_{x\to a} f(x) = c$ for all a.

Let $\epsilon > 0$ be given. Pick any positive δ. Then for all x such that $0 < |x-a| < \delta$, $|f(x) - c| = |c - c| = 0 < \epsilon$. QED.

(Note that we can pick any positive $\delta > 0$, e.g. $1, 10, \frac{1}{10}$.)

Identity

Let $f(x) = x$. We prove that $\lim_{x\to a} f(x) = a$ for all a.

Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that for all x in $0 < |x - a| < \delta$, $|f(x) - a| = |x - a| < \epsilon$. I.e. we need to find a δ such that $|x - a| < \delta$ implies $|x - a| < \epsilon$. This obviously works for any $\delta \le \epsilon$. QED.

(Note the many options for δ, e.g. $\delta = \epsilon$, $\delta = \frac{\epsilon}{2}$, etc.)

Addition

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x\to a}(f + g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Let $L_f = \lim_{x\to a} f(x)$ and let $L_g = \lim_{x\to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x - a| < \delta$ the following inequality holds:

$$|(f + g)(x) - (L_f + L_g)| < \epsilon$$

I.e. we're trying to show $\lim_{x\to a}(f+g)(x)$ equals to $L_f + L_g$, the sum of the other two limits. Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}
|(f + g)(x) - (L_f + L_g)| &= |f(x) + g(x) - (L_f + L_g)| \\
&= |(f(x) - L_f) + (g(x) - L_g)| \\
&\le |(f(x) - L_f)| + |(g(x) - L_g)| \quad \text{by triangle inequality}
\end{aligned}$$

By limit definition there exist positive δ_f, δ_g such that for all x

- $0 < |x - a| < \delta_f$ implies $|f(x) - L_f| < \epsilon/2$
- $0 < |x - a| < \delta_g$ implies $|g(x) - L_g| < \epsilon/2$

Recall that we can make ϵ as small as we like. Here we pick deltas for $\epsilon/2$ because it's convenient to make the equations work, as you will see in a second. For all x bounded by $0 < |x-a| < \min(\delta_f, \delta_g)$ we have

$$|(f(x) - L_f)| < \epsilon/2 \quad \text{and} \quad |(g(x) - L_g)| < \epsilon/2$$

Fix $\delta = \min(\delta_f, \delta_g)$. Then for all x bounded by $0 < |x-a| < \delta$ we have

$$\begin{aligned}
|(f+g)(x) - (L_f + L_g)| &\leq |(f(x) - L_f)| + |(g(x) - L_g)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

as desired.

Multiplication

Let $f, g \in \mathbf{R} \to \mathbf{R}$. We prove that

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Let $L_f = \lim_{x \to a} f(x)$ and let $L_g = \lim_{x \to a} g(x)$. Let $\epsilon > 0$ be given. We must show there exists $\delta > 0$ such that for all x bounded by $0 < |x-a| < \delta$ the following inequality holds:

$$|(fg)(x) - (L_f L_g)| < \epsilon$$

(i.e. we're trying to show $\lim_{x \to a}(fg)(x)$ equals to $L_f L_g$, the product of the other two limits.) Let's convert the left side of this inequality into a more convenient form:

$$\begin{aligned}
|(fg)(x) - (L_f L_g)| &= |f(x)g(x) - L_f L_g| \\
&= |f(x)g(x) - L_f g(x) + L_f g(x) - L_f L_g| \\
&= |g(x)(f(x) - L_f) + L_f(g(x) - L_g)| \\
&\leq |g(x)(f(x) - L_f)| + |L_f(g(x) - L_g)| && \text{by triangle inequality} \\
&= |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| && \text{in general } |ab| = |a||b|
\end{aligned}$$

We now need to show there exists δ such that $0 < |x-a| < \delta$ implies

$$|g(x)||f(x) - L_f| + |L_f||g(x) - L_g| < \epsilon$$

We will do that by finding δ such that

1. $|g(x)||f(x) - L_f| < \epsilon/2$
2. $|L_f||g(x) - L_g| < \epsilon/2$

First, we show $|g(x)||f(x) - L_f| < \epsilon/2$.

By limit definition we can find δ_1 to make $|f(x) - L_f|$ as small as we like. But how small? To make $|g(x)||f(x) - L_f| < \epsilon/2$ we must find a delta such that $|f(x) - L_f| < \epsilon/2g(x)$. But to do that we need to get a bound on $g(x)$. Fortunately we know there exists δ_2 such that $|g(x) - L_g| < 1$ (we pick 1 because we must pick some bound, and 1 is as good as any). Thus $|g(x)| < |L_g| + 1$. And so, we can pick δ_1 such that $|f(x) - L_f| < \epsilon/2(|L_g| + 1)$.

Second, we show $|L_f||g(x) - L_g| < \epsilon/2$.

That is easy. By limit definition there exists a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \epsilon/2|L_f|$ for all x. Actually, we need a δ_3 such that $0 < |x - a| < \delta_3$ implies $|g(x) - L_g| < \frac{\epsilon}{2(|L_f|+1)}$ for all x to avoid divide by zero, and of course that exists too.

Fix $\delta = \min(\delta_1, \delta_2, \delta_3)$. Now

$$\begin{aligned}|(fg)(x) - (L_f L_g)| &\leq |g(x)||f(x) - L_f| + |L_f||g(x) - L_g| \\ &< e/2 + e/2 = e\end{aligned}$$

as desired.

Reciprocal

Let $\lim_{x\to a} f(x) = L$. We prove $\lim_{x\to a} \left(\frac{1}{f}\right)(x) = 1/L$ when $L \neq 0$.

First we show $\frac{1}{f}$ is defined near a. By half-value continuity lemma (see 1.3) there exists δ_1 such that $0 < |x - a| < \delta_1$ implies $|f(x)| \geq |L|/2$ where $L \neq 0$. Therefore $f(x) \neq 0$ near a, and thus $\frac{1}{f}$ near a is defined.

Now all we must do is find a delta such that $\left|\frac{1}{f}(x) - \frac{1}{L}\right| < \epsilon$. Let's make the equation more convenient:

$$\begin{aligned}\left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\ &= \left|\frac{L - f(x)}{Lf(x)}\right| \\ &= \frac{|f(x) - L|}{|L||f(x)|} \\ &= \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|}\end{aligned}$$

Above we showed there exists δ_1 such that $0 < |x-a| < \delta_1$ implies $|f(x)| \geq |L|/2$. Raising both sides to -1 we get $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$. Continuing the chain of reasoning above we get

$$\begin{aligned}\frac{|f(x)-L|}{|L|} \cdot \frac{1}{|f(x)|} &\leq \frac{|f(x)-L|}{|L|} \cdot \frac{2}{|L|} \\ &= \frac{2}{|L|^2}|f(x)-L|\end{aligned}$$

(if you're confused about why this inequality works, left-multiply both sides of $|\frac{1}{f(x)}| \leq \frac{2}{|L|}$ by $\frac{|f(x)-L|}{|L|}$.) Thus we must find δ_2 such that

$$\frac{2}{|L|^2}|f(x)-L| < \epsilon$$

That is easy. Since $\lim_{x\to a} f(x) = L$ we can make $|f(x)-L|$ as small as we like. Dividing both sides by $\frac{2}{|L|^2}$, we must make $|f(x)-L| < \frac{|L|^2\epsilon}{2}$. Thus we must fix $\delta = \min(\delta_1, \delta_2)$. QED.

1.5 Absence of limits

What does it mean to say L is not a limit of $f(x)$ at a? It flows out of the definition– there exist some ϵ such that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x)-L| \geq \epsilon$.

A stronger version is to say there is no limit of $f(x)$ at a. To do that we must prove that *any* L is not a limit of $f(x)$ at a.

Example: Absolute value fraction

Consider $f(x) = \frac{x}{|x|}$. It's easy to see that

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

We will show there is no limit of $f(x)$ near 0.

Weak version. First, let's prove a weak version first– that $\lim_{x\to 0} f(x) \neq 0$. That is easy. Pick some reasonably small epsilon, say $\epsilon = \frac{1}{10}$. We must show that for any δ there exists an x in $0 < |x-a| < \delta$ such that $|f(x)-0| \geq \frac{1}{10}$.

Let's pick some arbitrary x out of our permitted interval, say $x = \delta/2$. Then

$$|f(x)-0| = |f(\delta/2)| = \left|\frac{\delta/2}{|\delta/2|}\right| = 1 \geq \frac{1}{10}$$

Strong version. Now we prove that $\lim_{x\to 0} f(x) \neq L$ for *any* L. Sticking with $\epsilon = \frac{1}{10}$ we proceed as follows.

If $L < 0$ take $x = \delta/2$. Then

$$|f(x) - L| = |f(\delta/2) - L| = \left|\frac{\delta/2}{|\delta/2|} - L\right| = |1 - L| > \frac{1}{10}$$

Similarly if $L \geq 0$ take $x = -\delta/2$. Then

$$|f(x) - L| = |f(-\delta/2) - L| = \left|\frac{-\delta/2}{|-\delta/2|} - L\right| = |-1 - L| > \frac{1}{10}$$

1.6 Appendix: low-level proofs

While high level theorems allow us to easily compute complicated limits, it's instructive to compute a few limits for complicated functions straight from the definition. We do that here.

Aside: inequalities

We will often need to make an inequality of the following form work out:

$$|n||m| < \epsilon$$

Here ϵ is given to us, we have complete control over the upper bound of $|n|$, and $|m|$ can take on any value outside our control. Obviously we can't make the inequality work without knowing *something* about $|m|$, so we'll try to find a bound for it in terms of $|a|$ (that's the a from $\lim_{x\to a} f(x)$, the value of which we know precisely).

Suppose we've discovered that $|m| < 3|a| + 4$. What does it tell us about $|n||m| < \epsilon$? More precisely, since we control $|n|$, how do we express it in terms of ϵ and $|a|$ in order to make the inequality work?

Well, if the upper bound of $|m|$ is $3|a| + 4$, we can rewrite the inequality as follows

$$|n|(3|a| + 4) < \epsilon$$

The reason for this is that if we can make $|n|(3|a| + 4) < \epsilon$ true, then certainly anything smaller, e.g. $|n|(3|a| + 3) < \epsilon$ will also be true. And $|m|$ is always smaller than $3|a| + 4$. Since $|n|$ is in our control, we can set it as follows

$$|n| < \frac{\epsilon}{3|a| + 4}$$

The above is true (as we can make any bound on $|n|$ true). Multiplying both sides by $3|a| + 4$ it's then easy to see that our desired inequality is also true.

Limits of quadratic functions

We will show from the definition that $\lim_{x \to a} x^2 = a^2$. Let $\epsilon \in \mathbf{R}$ be positive. We must show there exists δ such that $|x^2 - a^2| < \epsilon$ for all x in $0 < |x - a| < \delta$.

Observe that $|x^2 - a^2| = |(x-a)(x+a)| = |x-a||x+a|$. Thus we must pick δ such that $|x-a||x+a| < \epsilon$. We can make $|x-a|$ arbitrarily small by picking δ of our choice– that comes from the limit definition. But how small? For all we know $|x+a|$ could take on any value. There is nothing we can say about the inequality if that's the case. Thus we need to find an upper bound on $|x+a|$. We can do it as follows.

Start by setting $|x-a| < 1$. (This is arbitrary; we may pick 1/10 or 10 just as well). Then

$$\begin{aligned}
&|x-a| < 1 \\
&\implies -1 < x - a < 1 \\
&\implies 2a - 1 < x + a < 2a + 1 \qquad\qquad \text{add } 2a \text{ to both sides}
\end{aligned}$$

We now have a bound on $x + a$, but we need one on $|x + a|$. It's easy to see $|x+a| < \max(|2a-1|, |2a+1|)$. By triangle inequality ($|a+b| \leq |a| + |b|$):

$$\begin{aligned}
|2a-1| &\leq |2a| + |-1| = |2a| + 1 \\
|2a+1| &\leq |2a| + |1| = |2a| + 1
\end{aligned}$$

Thus $|x+a| < |2a| + 1$, provided $|x-a| < 1$. Coming back to our original goal, $|x-a||x+a| < \epsilon$ when

- $|x-a| < 1$ and
- $|x-a| < \frac{\epsilon}{|2a|+1}$

Putting these together, $\delta = \min(1, \frac{\epsilon}{|2a|+1})$.

Limits of fractions

We will show from the definition that $\lim_{x \to 2} \frac{3}{x} = \frac{3}{2}$. Let $\epsilon \in \mathbf{R}$ be positive. We must show there exists δ such that $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ for all x in $0 < |x-2| < \delta$.

Let's manipulate $|\frac{3}{x} - \frac{3}{2}|$ to make it more convenient:

$$\left|\frac{3}{x} - \frac{3}{2}\right| = \left|\frac{6-3x}{2x}\right| = \frac{3}{2}\frac{|x-2|}{|x|}$$

Thus we need to find δ such that

$$\begin{aligned}&\frac{3}{2}\frac{|x-2|}{|x|} < \epsilon \\ &\implies \frac{|x-2|}{|x|} < \frac{2\epsilon}{3}\end{aligned}$$

We need to find a bound for $|x|$. Let's arbitrarily pick delta so that $|x-2| < 1$ (we could as well pick $\frac{1}{10}$ or 10). Then

$$\begin{aligned}&|x-2| < 1 \\ &\implies -1 < x-2 < 1 \\ &\implies 1 < x < 3 \\ &\implies 1 < |x| < 3\end{aligned}$$

Since $|x| > 1$, it take a moment of thought to see that

$$|x-2| < \frac{2\epsilon}{3} \implies \frac{|x-2|}{|x|} < \frac{2\epsilon}{3}$$

(if $|x-2|$ is less than some number, certainly $|x-2|$ divided by something greater than 1 is less than that number.) Coming back to our original goal, $|\frac{3}{x} - \frac{3}{2}| < \epsilon$ provided that $|x-2| < 1$ and $|x-2| < \frac{2\epsilon}{3}$. Putting these together, we get $\delta = \min(1, \frac{2\epsilon}{3})$.

2 Continuity

2.1 Definition of continuity

A function f is **continuous** at a when

$$\lim_{x\to a} f(x) = f(a)$$

Inlining $\epsilon - \delta$ definition, f is continuous at a if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x) - f(a)| < \delta$. Note:

- $|f(x) - f(a)| < \delta$ simply says $f(a)$ is the limit.
- we drop "$0 <$" from $|x-a| < \delta$ because when $x = a$ in continuous functions, $f(x) - f(a) = 0$ and thus we're free to examine the value of $f(a)$.

A function f is **continuous on an interval** (a, b) if it's continuous at all $c \in (a, b)$[4].

[4]Closed intervals are a tiny bit harder, and I'm keeping them out for brevity.

2.2 Recognizing continuous functions

The following theorems allow us to tell at a glance that large classes of functions are continuous (e.g. polynomials, rational functions, etc.)

Five easy proofs

Constants . Let $f(x) = c$. Then f is continuous at all a because $\lim_{x\to a} f(x) = c = f(a)$.

Identity . Let $f(x) = x$. Then f is continuous at all a because $\lim_{x\to a} f(x) = a = f(a)$.

Addition . Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at a. Then $f + g$ is continuous at a because

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a) + g(a) = (f+g)(a)$$

Multiplication . Let $f, g \in \mathbf{R} \to \mathbf{R}$ be continuous at a. Then $f \cdot g$ is continuous at a because

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = f(a)g(a) = (fg)(a)$$

Reciprocal . Let f be continuous at a. Then $\frac{1}{g}$ is continuous at a where $g(a) \neq 0$ because

$$\lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{\lim_{x\to a} g(x)} = \frac{1}{g(a)} = \left(\frac{1}{g}\right)(a)$$

(recall that we proved in the limits chapter that $\frac{1}{g}$ is defined.)

Slightly harder proof: composition

Let $f, g \in \mathbf{R} \to \mathbf{R}$. Let g be continuous at a, and let f be continuous at $g(a)$. Then $f \circ g$ is continuous at a. Put differently, we want to show

$$\lim_{x\to a}(f \circ g)(x) = (f \circ g)(a)$$

Unpacking the definitions, let $\epsilon > 0$ be given. We want to show there exists $\delta > 0$ such that $|x - a| < \delta$ implies

$$\begin{aligned} &|(f \circ g)(x) - (f \circ g)(a)| \\ &\qquad = |f(g(x)) - f(g(a))| < \epsilon \end{aligned}$$

By problem statement we have two continuities.

First, f is continuous at $g(a)$, i.e. $\lim_{y \to g(a)} f(y) = f(g(a))$. Thus there exists $\delta' > 0$ such that $|y - g(a)| < \delta'$ implies $|f(y) - f(g(a))| < \epsilon$.

Second, g is continuous at a, i.e. $\lim_{x \to a} g(x) = g(a)$. Thus there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \epsilon$. Since we can make ϵ be anything, we can set it to δ'.

I.e. there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|g(x) - g(a)| < \delta'$. Which by the first statement implies $|f(g(x)) - f(g(a))| < \epsilon$, as desired.

2.3 Continuity and discontinuity examples