

A duality formula between elliptic determinants

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Abstract

We prove a duality formula between two elliptic determinants. We present a proof which is a variant of the Izergin-Korepin method which is a method originally introduced to analyze and compute partition functions of integrable lattice models.

1 Introduction

Elliptic special functions is an active area of research in recent years. Investigating formulas for special determinants and Pfaffians whose matrix elements are given in terms of elliptic functions is one of the most fundamental subjects, and there are developments on the evaluations of the elliptic determinants and Pfaffians in recent years. For example, factorization formulas for various elliptic determinants which are analogues, extensions and variants of the classical (elliptic) Cauchy determinant (Frobenius determinant) formula [1, 2, 3] were found [4, 5, 6, 7, 8, 9, 10]. The Pfaffian analogues of the elliptic Cauchy determinant formula were also found [10, 11, 12, 13], which generalizes the classical Pfaffian formula [14].

Another interesting subject is to find transformation formulas between two elliptic determinants or two different Pfaffians which look totally different at first sight. See [15, 16] for seminal works on this subject. Recently, we found a transformation formula between two elliptic Pfaffians by studying the partition functions of an elliptic integrable model in two ways [17]. A special case of the transformation formula can be proved easily by combining factorization formulas for two elliptic Pfaffians by Rains [10] and Rosengren [12]. However, beyond that special point where no factorized expressions are known, it seems not so easy to prove the transformation formula. A similar situation has already appeared in the work by Rosengren [15], in which he proved a duality between two elliptic determinants which is an elliptic analogue of a duality discovered by Rosengren-Schlosser [16]. A special case can be proved by using the elliptic determinant evaluations by Warnaar [6], but no factorization formulas are known in general case in which giving a proof is not so easy. Rosengren gives

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a proof of his duality formula [15] by using the elliptic Jackson summation formula. See [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] for examples on seminal works and various extensions of the elliptic Jackson summation formulas and corresponding elliptic integral formulas. In general, when we do not have or do not know whether there are factorized expressions, it is not easy to prove the transformation formula between determinants or Pfaffians.

In this paper, we prove another transformation formula for two elliptic determinants which do not seem to have factorized expressions. We prove the following theorem.

Theorem 1.1. *The following identity between two elliptic determinants holds:*

$$\begin{aligned} & \det_N(X_N(z_1, \dots, z_N | w_1, \dots, w_N | h)) \\ &= \frac{[h] \prod_{j=1}^N [h - j/2 + 1]}{[h - N/2] \prod_{j=1}^N [h + (N+1)/2 - j]} \prod_{1 \leq j < k \leq N} \frac{[w_k - w_j + 1/2]}{[w_k - w_j]} \\ & \quad \times \det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)), \end{aligned} \quad (1.1)$$

where $X_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ and $Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ are $N \times N$ matrices whose matrix elements are given by

$$\begin{aligned} & X_N(z_1, \dots, z_N | w_1, \dots, w_N | h)_{jk} \\ &= [h + (j - N)/2 + z_k + w_j] \prod_{\ell=1}^{j-1} [w_\ell + z_k + 1/2] \prod_{\ell=j+1}^N [w_\ell + z_k] \prod_{\ell=1}^N [w_\ell - z_k] \\ & \quad - [h + (j - N)/2 - z_k + w_j] \prod_{\ell=1}^{j-1} [w_\ell - z_k + 1/2] \prod_{\ell=j+1}^N [w_\ell - z_k] \prod_{\ell=1}^N [w_\ell + z_k], \end{aligned} \quad (1.2)$$

$$\begin{aligned} & Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)_{jk} \\ &= [h + (N - 1)/2 + w_k + z_j] \prod_{\substack{\ell=1 \\ \ell \neq j}}^N [z_\ell + w_k] \prod_{\ell=1}^N [z_\ell - w_k] \\ & \quad - [-h - (N - 1)/2 - w_k + z_j] \prod_{\substack{\ell=1 \\ \ell \neq j}}^N [z_\ell - w_k] \prod_{\ell=1}^N [z_\ell + w_k], \end{aligned} \quad (1.3)$$

for $j, k = 1, \dots, N$. Here, $[u]$ is the theta function $[u] = H(\pi i u)$ where $H(u)$ is given by

$$H(u) = 2 \sinh u \prod_{j=1}^{\infty} (1 - 2\mathbf{q}^{2j} \cosh 2u + \mathbf{q}^{4j})(1 - \mathbf{q}^{2j}), \quad (1.4)$$

where \mathbf{q} is the elliptic nome ($0 < \mathbf{q} < 1$).

We prove Theorem 1.1 in this paper. The determinants $\det_N(X_N(z_1, \dots, z_N | w_1, \dots, w_N | h))$ and $\det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h))$ in the Theorem do not seem to factorize, and we present a proof which works for determinants which do not seem to have factorized expressions. The proof is inspired by and can be regarded as a variant of the Izergin-Korepin method [29, 30] in the field of quantum integrable models. The Izergin-Korepin method was

initiated by Korepin [29], which he introduced a way to characterize the domain wall boundary partition functions of the $U_q(\widehat{sl}_2)$ six-vertex model [31, 32, 33, 34, 35, 36] which uniquely define them. Izergin [30] later found a determinant form which satisfies all the properties listed by Korepin, and is now called as the Izergin-Korepin determinant, which today have many applications to other branches of mathematics and further investigations, such as the enumeration of the alternating sign matrices [37, 38, 39, 40, 41, 42] and the thermodynamic limit [43]. The Izergin-Korepin method was also extended to various boundary conditions [39, 44] and other classes of partition functions such as the scalar products [45] and the wavefunctions [46].

The original Izergin-Korepin method is a way to prove identities between partition functions which are functions constructed by the R -matrices of quantum integrable models, and explicit determinants, Pfaffians or symmetric functions. The idea of the Izergin-Korepin method can also be used to prove identities between two functions which look at first sight totally different. The idea of the method do not have to be restricted to the computations of partition functions, and we present one such application in this paper.

This paper is organized as follows. In the next section, we present some properties of the theta functions which will be used in this paper, and check the simplest nontrivial example, i.e., the case $N = 2$. In section 3, we present a proof which is inspired the Izergin-Korepin method. Section 4 is devoted to the conclusion of this paper.

2 Preliminaries and the simplest nontrivial example

In this section, we first list the properties of theta functions used in this paper. One of the most fundamental properties about theta functions is the quasi-periodicities

$$[u + 1] = -[u], \quad (2.1)$$

$$[u - i \log(\mathbf{q})/\pi] = -\mathbf{q}^{-1} \exp(-2\pi i u)[u]. \quad (2.2)$$

Using (2.1) and the fact that $[u]$ is an odd function $[-u] = -[u]$, we get

$$[u + 1/2] = [-u + 1/2], \quad (2.3)$$

which is an important property used in this paper. Another important property is the addition formula for the theta functions

$$[u + x][u - x][v + y][v - y] - [v + x][v - x][u + y][u - y] - [x + y][x - y][u + v][u - v] = 0. \quad (2.4)$$

We will use the above properties repeatedly to check the simplest nontrivial example of Theorem 1.1.

For the proof of Theorem 1.1, besides the above fundamental properties for the theta functions, the following notions and properties about the elliptic polynomials [47, 48] is crucial.

A character is a group homomorphism χ from multiplicative groups $\Gamma = \mathbf{Z} + \tau\mathbf{Z}$ to \mathbf{C}^\times . For each character χ and positive integer n , an n -dimensional space $\Theta_n(\chi)$ is a set of holomorphic functions $\phi(y)$ on \mathbf{C} satisfying the quasiperiodicities

$$\phi(y + 1) = \chi(1)\phi(y), \quad (2.5)$$

$$\phi(y + \tau) = \chi(\tau)e^{-2\pi i n y - \pi i n \tau}\phi(y). \quad (2.6)$$

The elements of the space $\Theta_n(\chi)$ are called elliptic polynomials. The space $\Theta_n(\chi)$ is n -dimensional [47, 48], and the following fact holds for the elliptic polynomials:

Proposition 2.1. [47, 48] *Suppose there are two elliptic polynomials $P(y)$ and $Q(y)$ in $\Theta_n(\chi)$, where $\chi(1) = (-1)^n$ and $\chi(\tau) = (-1)^n e^\alpha$. If these two polynomials are equal at n points y_j , $j = 1, \dots, n$, satisfying $y_j - y_k \notin \Gamma$ and $\sum_{k=1}^N y_k - \alpha \notin \Gamma$, that is, $P(y_j) = Q(y_j)$, then the two polynomials are exactly the same: $P(y) = Q(y)$.*

These properties played important roles for developing methods for elliptic quantum integrable models, such as the separation of the variables method and the Izergin-Korepin method. For example, it was used to analyze and compute the explicit forms of the domain wall boundary partition functions of the Andrews-Baxter-Forrester model [49]. See Refs. [47], [48], [50], and [51] for examples for seminal works of the developments. We use this property in the next section to prove Theorem 1.1.

In the end of this section, let us check Theorem 1.1 by the simplest nontrivial case $N = 2$ by elementary manipulations (the case $N = 1$ is trivial to check). Using (2.3) and (2.4), one can show the following four relations.

$$\begin{aligned}
& [h - 1/2 + z_1 + w_1][w_2 + z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
& - [h - 1/2 + z_2 + w_1][w_2 + z_2][h + z_1 + w_2][w_1 + z_1 + 1/2] \\
& = - [h + 1/2 + z_1 + w_1][w_2 + z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
& + [h + 1/2 + z_2 + w_1][w_2 + z_2][h + z_1 + w_2][w_1 + z_1 + 1/2] \\
& = - [h][w_2 - w_1 + 1/2][h + 1/2 + z_1 + z_2 + w_1 + w_2][z_1 - z_2], \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
& [h - 1/2 + z_1 + w_1][w_2 + z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
& - [h - 1/2 - z_2 + w_1][w_2 - z_2][h + z_1 + w_2][w_1 + z_1 + 1/2] \\
& = - [h + 1/2 + z_1 + w_1][w_2 + z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
& + [h + 1/2 - z_2 + w_1][w_2 - z_2][h + z_1 + w_2][w_1 + z_1 + 1/2] \\
& = - [h][w_2 - w_1 + 1/2][h + 1/2 + z_1 - z_2 + w_1 + w_2][z_1 + z_2], \tag{2.8}
\end{aligned}$$

$$\begin{aligned}
& [h - 1/2 - z_1 + w_1][w_2 - z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
& - [h - 1/2 + z_2 + w_1][w_2 + z_2][h - z_1 + w_2][w_1 - z_1 + 1/2] \\
& = - [h + 1/2 - z_1 + w_1][w_2 - z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
& + [h + 1/2 + z_2 + w_1][w_2 + z_2][h - z_1 + w_2][w_1 - z_1 + 1/2] \\
& = [h][w_2 - w_1 + 1/2][h + 1/2 - z_1 + z_2 + w_1 + w_2][z_1 + z_2], \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
& [h - 1/2 - z_1 + w_1][w_2 - z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
& - [h - 1/2 - z_2 + w_1][w_2 - z_2][h - z_1 + w_2][w_1 - z_1 + 1/2] \\
& = - [h + 1/2 - z_1 + w_1][w_2 - z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
& + [h + 1/2 - z_2 + w_1][w_2 - z_2][h - z_1 + w_2][w_1 - z_1 + 1/2] \\
& = [h][w_2 - w_1 + 1/2][h + 1/2 - z_1 - z_2 + w_1 + w_2][z_1 - z_2]. \tag{2.10}
\end{aligned}$$

Using (2.7), (2.8), (2.9) and (2.10), one can rewrite $\det_2(X_2(z_1, z_2|w_1, w_2|h))$ as

$$\begin{aligned}
& \det_2(X_2(z_1, z_2|w_1, w_2|h)) \\
&= [w_1 - z_1][w_2 - z_1][w_1 - z_2][w_2 - z_2] \\
&\times ([h - 1/2 + z_1 + w_1][w_2 + z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
&- [h - 1/2 + z_2 + w_1][w_2 + z_2][h + z_1 + w_2][w_1 + z_1 + 1/2]) \\
&- [w_1 - z_1][w_2 - z_1][w_1 + z_2][w_2 + z_2] \\
&\times ([h - 1/2 + z_1 + w_1][w_2 + z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
&- [h - 1/2 - z_2 + w_1][w_2 - z_2][h + z_1 + w_2][w_1 + z_1 + 1/2]) \\
&- [w_1 + z_1][w_2 + z_1][w_1 - z_2][w_2 - z_2] \\
&\times ([h - 1/2 - z_1 + w_1][w_2 - z_1][h + z_2 + w_2][w_1 + z_2 + 1/2] \\
&- [h - 1/2 + z_2 + w_1][w_2 + z_2][h - z_1 + w_2][w_1 - z_1 + 1/2]) \\
&+ [w_1 + z_1][w_2 + z_1][w_1 + z_2][w_2 + z_2] \\
&\times ([h - 1/2 - z_1 + w_1][w_2 - z_1][h - z_2 + w_2][w_1 - z_2 + 1/2] \\
&- [h - 1/2 - z_2 + w_1][w_2 - z_2][h - z_1 + w_2][w_1 - z_1 + 1/2]) \\
&= [w_1 - z_1][w_2 - z_1][w_1 - z_2][w_2 - z_2] \\
&\times (-[h][w_2 - w_1 + 1/2][h + 1/2 + z_1 + z_2 + w_1 + w_2][z_1 - z_2]) \\
&- [w_1 - z_1][w_2 - z_1][w_1 + z_2][w_2 + z_2] \\
&\times (-[h][w_2 - w_1 + 1/2][h + 1/2 + z_1 - z_2 + w_1 + w_2][z_1 + z_2]) \\
&- [w_1 + z_1][w_2 + z_1][w_1 - z_2][w_2 - z_2] \\
&\times [h][w_2 - w_1 + 1/2][h + 1/2 - z_1 + z_2 + w_1 + w_2][z_1 + z_2] \\
&+ [w_1 + z_1][w_2 + z_1][w_1 + z_2][w_2 + z_2] \\
&\times [h][w_2 - w_1 + 1/2][h + 1/2 - z_1 - z_2 + w_1 + w_2][z_1 - z_2] \\
&= [h][w_2 - w_1 + 1/2] \\
&\times (-[w_1 - z_1][w_1 - z_2][w_2 - z_1][w_2 - z_2][z_1 - z_2][h + 1/2 + z_1 + z_2 + w_1 + w_2] \\
&+ [w_1 - z_1][w_1 + z_2][w_2 - z_1][w_2 + z_2][z_1 + z_2][h + 1/2 + z_1 - z_2 + w_1 + w_2] \\
&- [w_1 + z_1][w_1 - z_2][w_2 + z_1][w_2 - z_2][z_1 + z_2][h + 1/2 - z_1 + z_2 + w_1 + w_2] \\
&+ [w_1 + z_1][w_1 + z_2][w_2 + z_1][w_2 + z_2][z_1 - z_2][h + 1/2 - z_1 - z_2 + w_1 + w_2]), \tag{2.11}
\end{aligned}$$

which is a simplification of the left hand side of (1.1) for the case $N = 2$.

Let us next examine the right hand side. The right hand side of (1.1) for the case $N = 2$ is $\frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \det_2(Y_2(z_1, z_2|w_1, w_2|h))$. Using the addition formula (2.4), one can show

the following four relations.

$$\begin{aligned}
& [h + 1/2 + w_1 + z_1][z_2 + w_1][h + 1/2 + w_2 + z_2][z_1 + w_2] \\
& - [h + 1/2 + w_1 + z_2][z_1 + w_1][h + 1/2 + w_2 + z_1][z_2 + w_2] \\
& = [h + 1/2][z_1 - z_2][h + 1/2 + z_1 + z_2 + w_1 + w_2][w_1 - w_2],
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
& [h + 1/2 + w_1 + z_1][z_2 - w_1][h + 1/2 + w_2 - z_2][z_1 + w_2] \\
& - [h + 1/2 + w_2 + z_1][z_2 - w_2][h + 1/2 + w_1 - z_2][z_1 + w_1] \\
& = [h + 1/2][z_1 + z_2][h + 1/2 + z_1 - z_2 + w_1 + w_2][w_2 - w_1],
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
& [h + 1/2 + w_1 - z_1][z_2 + w_1][h + 1/2 + w_2 + z_2][z_1 - w_2] \\
& - [h + 1/2 + w_2 - z_1][z_2 + w_2][h + 1/2 + w_1 + z_2][z_1 - w_1] \\
& = [h + 1/2][z_1 + z_2][h + 1/2 - z_1 + z_2 + w_1 + w_2][w_1 - w_2],
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
& [h + 1/2 + w_1 - z_1][z_2 - w_1][h + 1/2 + w_2 - z_2][z_1 - w_2] \\
& - [h + 1/2 + w_2 - z_1][z_2 - w_2][h + 1/2 + w_1 - z_2][z_1 - w_1] \\
& = [h + 1/2][z_2 - z_1][h + 1/2 - z_1 - z_2 + w_1 + w_2][w_1 - w_2].
\end{aligned} \tag{2.15}$$

Using the four relations (2.12), (2.13), (2.14) and (2.15), one can simplify

$$\begin{aligned}
& \frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \det_2(Y_2(z_1, z_2|w_1, w_2|h)) \text{ as} \\
& \frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \det_2(Y_2(z_1, z_2|w_1, w_2|h)) = \frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \\
& \times \{ [z_2 - w_2][z_1 - w_2][z_2 - w_1][z_1 - w_1]([h + 1/2 + w_1 + z_1][z_2 + w_1][h + 1/2 + w_2 + z_2][z_1 + w_2] \\
& - [h + 1/2 + w_1 + z_2][z_1 + w_1][h + 1/2 + w_2 + z_1][z_2 + w_2]) \\
& + [z_1 - w_1][z_1 - w_2][z_2 + w_1][z_2 + w_2]([h + 1/2 + w_1 + z_1][z_2 - w_1][h + 1/2 + w_2 - z_2][z_1 + w_2] \\
& - [h + 1/2 + w_2 + z_1][z_2 - w_2][h + 1/2 + w_1 - z_2][z_1 + w_1]) \\
& + [z_1 + w_1][z_1 + w_2][z_2 - w_1][z_2 - w_2]([h + 1/2 + w_1 - z_1][z_2 + w_1][h + 1/2 + w_2 + z_2][z_1 - w_2] \\
& - [h + 1/2 + w_2 - z_1][z_2 + w_2][h + 1/2 + w_1 + z_2][z_1 - w_1]) \\
& + [z_1 + w_1][z_2 + w_2][z_1 + w_2][z_2 + w_1]([h + 1/2 + w_1 - z_1][z_2 - w_1][h + 1/2 + w_2 - z_2][z_1 - w_2] \\
& - [h + 1/2 + w_2 - z_1][z_2 - w_2][h + 1/2 + w_1 - z_2][z_1 - w_1]) \} \\
& = \frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \\
& \times \{ [z_2 - w_2][z_1 - w_2][z_2 - w_1][z_1 - w_1] \\
& \times [h + 1/2][z_1 - z_2][h + 1/2 + z_1 + z_2 + w_1 + w_2][w_1 - w_2] \\
& + [z_1 - w_1][z_1 - w_2][z_2 + w_1][z_2 + w_2] \\
& \times [h + 1/2][z_1 + z_2][h + 1/2 + z_1 - z_2 + w_1 + w_2][w_2 - w_1] \\
& + [z_1 + w_1][z_1 + w_2][z_2 - w_1][z_2 - w_2] \\
& \times [h + 1/2][z_1 + z_2][h + 1/2 - z_1 + z_2 + w_1 + w_2][w_1 - w_2] \\
& + [z_1 + w_1][z_2 + w_2][z_1 + w_2][z_2 + w_1] \\
& \times [h + 1/2][z_2 - z_1][h + 1/2 - z_1 - z_2 + w_1 + w_2][w_1 - w_2] \}
\end{aligned}$$

$$\begin{aligned}
&= [h][w_2 - w_1 + 1/2] \\
&\times (-[w_1 - z_1][w_1 - z_2][w_2 - z_1][w_2 - z_2][z_1 - z_2][h + 1/2 + z_1 + z_2 + w_1 + w_2] \\
&+ [w_1 - z_1][w_1 + z_2][w_2 - z_1][w_2 + z_2][z_1 + z_2][h + 1/2 + z_1 - z_2 + w_1 + w_2] \\
&- [w_1 + z_1][w_1 - z_2][w_2 + z_1][w_2 - z_2][z_1 + z_2][h + 1/2 - z_1 + z_2 + w_1 + w_2] \\
&+ [w_1 + z_1][w_1 + z_2][w_2 + z_1][w_2 + z_2][z_1 - z_2][h + 1/2 - z_1 - z_2 + w_1 + w_2]). \tag{2.16}
\end{aligned}$$

Since the simplifications (2.11) of $\det_2(X_2(z_1, z_2|w_1, w_2|h))$ and (2.16) of $\frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \det_2(Y_2(z_1, z_2|w_1, w_2|h))$ are the same, one has checked that

$$\det_2(X_2(z_1, z_2|w_1, w_2|h)) = \frac{[h][w_2 - w_1 + 1/2]}{[h + 1/2][w_2 - w_1]} \det_2(Y_2(z_1, z_2|w_1, w_2|h)), \tag{2.17}$$

holds.

3 Proof

In this section, we prove Theorem 1.1. We prove the following equivalent theorem which both hand sides of (1.1) in Theorem 1.1 are multiplied by $\prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]}$.

Theorem 3.1. *The following identity between two elliptic determinants holds:*

$$\begin{aligned}
&\prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \det_N(X_N(z_1, \dots, z_N|w_1, \dots, w_N|h)) \\
&= \frac{[h] \prod_{j=1}^N [h - j/2 + 1]}{[h - N/2] \prod_{j=1}^N [h + (N + 1)/2 - j]} \prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2][w_k - w_j + 1/2]}{[z_j - z_k][w_k - w_j]} \\
&\quad \times \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)). \tag{3.1}
\end{aligned}$$

Proof. Let us denote the left hand side and right hand side of (3.1) as $L_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ and $R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ respectively.

$$\begin{aligned}
&L_N(z_1, \dots, z_N|w_1, \dots, w_N|h) \\
&= \prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \det_N(X_N(z_1, \dots, z_N|w_1, \dots, w_N|h)), \tag{3.2}
\end{aligned}$$

$$\begin{aligned}
&R_N(z_1, \dots, z_N|w_1, \dots, w_N|h) \\
&= \frac{[h] \prod_{j=1}^N [h - j/2 + 1]}{[h - N/2] \prod_{j=1}^N [h + (N + 1)/2 - j]} \prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2][w_k - w_j + 1/2]}{[z_j - z_k][w_k - w_j]} \\
&\quad \times \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)). \tag{3.3}
\end{aligned}$$

To prove Theorem 3.1, we first show the following properties for $L_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$.

Proposition 3.2. *The functions $L_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ satisfy, and are uniquely determined by, the following properties:*

1. The functions $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ are elliptic polynomials in w_N of degree $2N$ with the following quasi-periodicities:

$$L_N(z_1, \dots, z_N | w_1, \dots, w_N + 1 | h) = (-1)^{2N} L_N(z_1, \dots, z_N | w_1, \dots, w_N | h), \quad (3.4)$$

$$\begin{aligned} & L_N(z_1, \dots, z_N | w_1, \dots, w_N - i \log(\mathbf{q})/\pi | h) \\ &= (-\mathbf{q}^{-1})^{2N} \exp(-2\pi i(2Nw_N + h)) L_N(z_1, \dots, z_N | w_1, \dots, w_N | h). \end{aligned} \quad (3.5)$$

2. The following relations hold:

$$\begin{aligned} L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = -z_m} &= [h] [-2z_m] \prod_{\substack{j=1 \\ j \neq m}}^N [z_m - z_j + 1/2] [z_m + z_j] \\ &\times \prod_{j=1}^{N-1} [z_m + w_j + 1/2] [z_m - w_j] L_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h - 1/2), \end{aligned} \quad (3.6)$$

$$\begin{aligned} L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = z_m} &= [h] [-2z_m] \prod_{\substack{j=1 \\ j \neq m}}^N [z_m - z_j + 1/2] [z_m + z_j] \\ &\times \prod_{j=1}^{N-1} [z_m - w_j + 1/2] [z_m + w_j] L_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h + 1/2), \end{aligned} \quad (3.7)$$

for $m = 1, \dots, N$, and \hat{z}_m in $L_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h - 1/2)$ and $L_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h + 1/2)$ means that z_m is removed.

3. The following holds:

$$L_1(z_1 | w_1 | h) = [h + z_1 + w_1][w_1 - z_1] - [h - z_1 + w_1][w_1 + z_1]. \quad (3.8)$$

Proposition 3.2 is a version of the so-called Korepin's Lemma in the field of quantum integrable models [29], which list the properties of a sequence of functions which uniquely define them. Let us explain about the uniqueness. Property 1 together with Proposition 2.1 means that $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is uniquely determined by its evaluation at $2N$ points. The evaluations at $2N$ points are Property 2, which relates $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ at $z_N = \pm w_m$, ($m = 1, \dots, N$) with $L_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h \pm 1/2)$. This means that $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is uniquely determined from $L_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h)$, and Property 3 corresponds to the determination of the initial term of the sequence of functions $\{L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) | N \in \mathbb{N}\}$.

Let us show Properties 1–3 in Proposition 3.2. We first expand (3.2) as

$$\begin{aligned} & L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) \\ &= \prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \sum_{\sigma \in S_N} \sum_{\tau_1, \dots, \tau_N = \pm 1} \text{sgn}(\sigma) (-1)^{|\tau|} \prod_{j=1}^N [h + (j - N)/2 + \tau_{\sigma(j)} z_{\sigma(j)} + w_j] \\ &\times \prod_{j=1}^N \prod_{\ell=1}^{j-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)} + 1/2] \prod_{j=1}^{N-1} \prod_{\ell=j+1}^N [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^N \prod_{\ell=1}^N [w_\ell - \tau_{\sigma(j)} z_{\sigma(j)}], \end{aligned} \quad (3.9)$$

where $|\tau|$ is the number of τ_j 's ($j = 1, \dots, N$) satisfying $\tau_j = -1$. Let us prove Property 1 from the expansion (3.9). One finds that each summand in (3.9) contains the following factors

$$f_{\sigma, \tau}(w_N | z_1, \dots, z_N | h) = [h + \tau_{\sigma(N)} z_{\sigma(N)} + w_N] \prod_{j=1}^{N-1} [w_N + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^N [w_N - \tau_{\sigma(j)} z_{\sigma(j)}], \quad (3.10)$$

from which all the w_N -dependence come. It is easy to calculate the quasi-periodicities for $f_{\sigma, \tau}(w_N | z_1, \dots, z_N | h)$

$$f_{\sigma, \tau}(w_N + 1 | z_1, \dots, z_N | h) = (-1)^{2N} f_{\sigma, \tau}(w_N | z_1, \dots, z_N | h), \quad (3.11)$$

$$\begin{aligned} & f_{\sigma, \tau}(w_N - i \log(\mathbf{q}) / \pi | z_1, \dots, z_N | h) \\ &= (-\mathbf{q}^{-1})^{2N} \exp(-2\pi i(2Nw_N + h)) f_{\sigma, \tau}(w_N | z_1, \dots, z_N | h). \end{aligned} \quad (3.12)$$

The quasi-periodicities do not depend on σ nor τ , from which one finds (3.4) and (3.5).

Now let us show Property 2. First, we note that the determinant of the matrix $X_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ whose matrix elements are given by (1.2) is antisymmetric with respect to $z_j \longleftrightarrow z_k$ ($j \neq k$). The antisymmetry also holds for $\prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]}$, and since $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is a product of $\det_N(X_N(z_1, \dots, z_N | w_1, \dots, w_N | h))$ and $\prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]}$, we find that $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is symmetric with respect to $z_j \longleftrightarrow z_k$ ($j \neq k$). From this symmetry, it is enough to show (3.6) and (3.7) for the case $m = N$

$$\begin{aligned} L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)|_{w_N = -z_N} &= [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \\ &\times \prod_{j=1}^{N-1} [z_N + w_j + 1/2] [z_N - w_j] L_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h - 1/2), \end{aligned} \quad (3.13)$$

$$\begin{aligned} L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)|_{w_N = z_N} &= [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \\ &\times \prod_{j=1}^{N-1} [z_N - w_j + 1/2] [z_N + w_j] L_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2). \end{aligned} \quad (3.14)$$

The other cases (3.6) and (3.7) for $m = 1, \dots, N-1$ follows from (3.13) and (3.14) by using the property that $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is a symmetric function with symmetric variables z_j ($j = 1, \dots, N$).

Let us show (3.13). After the substitution $w_N = -z_N$, one finds that the only the summands satisfying $\sigma(N) = N$, $\tau_N = +1$ in (3.9) survive. Keeping this in mind, one

rewrites $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = -z_N}$ as a sum over $\sigma \in S_{N-1}$ and $\tau_1, \dots, \tau_{N-1}$ as

$$\begin{aligned}
& L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = -z_N} \\
&= \prod_{1 \leq j < k \leq N-1} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} \sum_{\sigma \in S_{N-1}} \sum_{\tau_1, \dots, \tau_{N-1} = \pm 1} \text{sgn}(\sigma) (-1)^{|\tau|} \\
&\times [h] \prod_{j=1}^{N-1} [h - 1/2 + (j - (N-1))/2 + \tau_{\sigma(j)} z_{\sigma(j)} + w_j] \\
&\times \prod_{\ell=1}^{N-1} [w_\ell + z_N + 1/2] \prod_{j=1}^{N-1} \prod_{\ell=1}^{j-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)} + 1/2] \\
&\times \prod_{j=1}^{N-1} [-z_N + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^{N-2} \prod_{\ell=j+1}^{N-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)}] \\
&\times [-2z_N] \prod_{j=1}^{N-1} [-z_N - \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{\ell=1}^{N-1} [w_\ell - z_N] \prod_{j=1}^{N-1} \prod_{\ell=1}^{N-1} [w_\ell - \tau_{\sigma(j)} z_{\sigma(j)}]. \tag{3.15}
\end{aligned}$$

Using

$$\prod_{j=1}^{N-1} [-z_N + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^{N-1} [-z_N - \tau_{\sigma(j)} z_{\sigma(j)}] = \prod_{j=1}^{N-1} [-z_N + z_j] [-z_N - z_j], \tag{3.16}$$

one can further rearrange (3.15) as

$$\begin{aligned}
& L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = -z_N} = [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \\
&\times \prod_{j=1}^{N-1} [z_N + w_j + 1/2] [z_N - w_j] \prod_{1 \leq j < k \leq N-1} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \\
&\times \sum_{\sigma \in S_{N-1}} \sum_{\tau_1, \dots, \tau_{N-1} = \pm 1} \text{sgn}(\sigma) (-1)^{|\tau|} \prod_{j=1}^{N-1} [h - 1/2 + (j - (N-1))/2 + \tau_{\sigma(j)} z_{\sigma(j)} + w_j] \\
&\times \prod_{j=1}^{N-1} \prod_{\ell=1}^{j-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)} + 1/2] \prod_{j=1}^{N-2} \prod_{\ell=j+1}^{N-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^{N-1} \prod_{\ell=1}^{N-1} [w_\ell - \tau_{\sigma(j)} z_{\sigma(j)}] \\
&= [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \prod_{j=1}^{N-1} [z_N + w_j + 1/2] [z_N - w_j] \\
&\times L_N(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h - 1/2), \tag{3.17}
\end{aligned}$$

and we find (3.13) holds.

(3.14) can be shown in a similar way. In this case, one notes that the summands satisfying $\sigma(N) = N$, $\tau_N = -1$ in (3.9) survive after the substitution $w_N = z_N$. Then one rewrites $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = z_N}$ as a sum over $\sigma \in S_{N-1}$ and $\tau_1, \dots, \tau_{N-1}$ in the following

way:

$$\begin{aligned}
& L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = z_N} \\
&= \prod_{1 \leq j < k \leq N-1} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} \sum_{\sigma \in S_{N-1}} \sum_{\tau_1, \dots, \tau_{N-1} = \pm 1} (-1)^{\text{sgn}(\sigma)} (-1)^{|\tau|} \\
&\times [h] (-1)^{N-1} \prod_{j=1}^{N-1} [h + 1/2 + (j - (N-1))/2 + \tau_{\sigma(j)} z_{\sigma(j)} + w_j] \\
&\times \prod_{\ell=1}^{N-1} [w_\ell - z_N + 1/2] \prod_{j=1}^{N-1} \prod_{\ell=1}^{j-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)} + 1/2] \\
&\times \prod_{j=1}^{N-1} [z_N + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^{N-2} \prod_{\ell=j+1}^{N-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)}] \\
&\times [2z_N] \prod_{j=1}^{N-1} [z_N - \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{\ell=1}^{N-1} [w_\ell + z_N] \prod_{j=1}^{N-1} \prod_{\ell=1}^{N-1} [w_\ell - \tau_{\sigma(j)} z_{\sigma(j)}] \\
&= [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \\
&\times \prod_{j=1}^{N-1} [z_N - w_j + 1/2] [z_N + w_j] \prod_{1 \leq j < k \leq N-1} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]} \\
&\times \sum_{\sigma \in S_{N-1}} \sum_{\tau_1, \dots, \tau_{N-1} = \pm 1} \text{sgn}(\sigma) (-1)^{|\tau|} \prod_{j=1}^{N-1} [h + 1/2 + (j - (N-1))/2 + \tau_{\sigma(j)} z_{\sigma(j)} + w_j] \\
&\times \prod_{j=1}^{N-1} \prod_{\ell=1}^{j-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)} + 1/2] \prod_{j=1}^{N-2} \prod_{\ell=j+1}^{N-1} [w_\ell + \tau_{\sigma(j)} z_{\sigma(j)}] \prod_{j=1}^{N-1} \prod_{\ell=1}^{N-1} [w_\ell - \tau_{\sigma(j)} z_{\sigma(j)}] \\
&= [h] [-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2] [z_N + z_j] \prod_{j=1}^{N-1} [z_N - w_j + 1/2] [z_N + w_j] \\
&\times L_N(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2), \tag{3.18}
\end{aligned}$$

hence we have shown (3.14).

The remaining thing to prove is Property 3, which is obvious to see from the definition of $L_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ (3.2).

Next, we show that the functions $R_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ satisfy exactly the same properties in Proposition 3.2.

Proposition 3.3. *The functions $R_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ satisfy, and are uniquely determined by, the following properties:*

1. *The functions $R_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ are elliptic polynomials in w_N of degree $2N$*

with the following quasi-periodicities:

$$R_N(z_1, \dots, z_N | w_1, \dots, w_N + 1 | h) = (-1)^{2N} R_N(z_1, \dots, z_N | w_1, \dots, w_N | h), \quad (3.19)$$

$$\begin{aligned} & R_N(z_1, \dots, z_N | w_1, \dots, w_N - i \log(\mathbf{q}) / \pi | h) \\ &= (-\mathbf{q}^{-1})^{2N} \exp(-2\pi i(2Nw_N + h)) R_N(z_1, \dots, z_N | w_1, \dots, w_N | h). \end{aligned} \quad (3.20)$$

2. The following relations hold:

$$\begin{aligned} R_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = -z_m} &= [h] [-2z_m] \prod_{\substack{j=1 \\ j \neq m}}^N [z_m - z_j + 1/2] [z_m + z_j] \\ &\times \prod_{j=1}^{N-1} [z_m + w_j + 1/2] [z_m - w_j] R_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h - 1/2), \end{aligned} \quad (3.21)$$

$$\begin{aligned} R_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = z_m} &= [h] [-2z_m] \prod_{\substack{j=1 \\ j \neq m}}^N [z_m - z_j + 1/2] [z_m + z_j] \\ &\times \prod_{j=1}^{N-1} [z_m - w_j + 1/2] [z_m + w_j] R_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h + 1/2), \end{aligned} \quad (3.22)$$

for $m = 1, \dots, N$, and \hat{z}_m in $R_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h - 1/2)$ and $R_{N-1}(z_1, \dots, \hat{z}_m, \dots, z_N | w_1, \dots, w_{N-1} | h + 1/2)$ means that z_m is removed.

3. The following holds:

$$R_1(z_1 | w_1 | h) = [h + z_1 + w_1][w_1 - z_1] - [h - z_1 + w_1][w_1 + z_1]. \quad (3.23)$$

Let us show Properties 1–3 of Proposition 3.3. We first introduce the notation

$$\begin{aligned} & c_N(z_1, \dots, z_N | w_1, \dots, w_N | h) \\ &= \frac{[h] \prod_{j=1}^N [h - j/2 + 1]}{[h - N/2] \prod_{j=1}^N [h + (N+1)/2 - j]} \prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2][w_k - w_j + 1/2]}{[z_j - z_k][w_k - w_j]}, \end{aligned} \quad (3.24)$$

and write $R_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ as

$$\begin{aligned} & R_N(z_1, \dots, z_N | w_1, \dots, w_N | h) \\ &= c_N(z_1, \dots, z_N | w_1, \dots, w_N | h) \det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)). \end{aligned} \quad (3.25)$$

First, note that from the explicit form of the matrix elements of $Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ (1.3), the dependence on w_N in $\det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h))$ only comes from the N -th column, and the product of factors which depend on w_N in $c_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ is $\prod_{j=1}^{N-1} \frac{[w_N - w_j + 1/2]}{[w_N - w_j]}$. Then it is easy to check the quasi-periodicities (3.19) and (3.20). Note

also that the factor $\prod_{j=1}^{N-1} \frac{[w_N - w_j + 1/2]}{[w_N - w_j]}$ may lead to singularities at $w_N = w_j$, $j = 1, \dots, N-1$, but actually do not, since one can see from the matrix elements of $Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ (1.3) that the determinant $\det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))$ vanishes when $w_j = w_k$ ($j \neq k$), and in particular has zeroes at $w_N = w_j$, $j = 1, \dots, N-1$. $R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ is holomorphic and thus is an elliptic polynomial of w_N of degree $2N$, and Property 1 is proved.

Next we show Property 2. First, rewriting $\det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))$ as

$$\begin{aligned} \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)) &= \prod_{k=1}^N \prod_{\ell=1}^N [z_\ell + w_k][z_\ell - w_k] \\ &\times \det_N([h + (N-1)/2 + w_k + z_j][z_j + w_k]^{-1} - [-h - (N-1)/2 - w_k + z_j][z_j - w_k]^{-1}), \end{aligned} \quad (3.26)$$

one sees that $\det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))$ is antisymmetric with respect to $z_j \longleftrightarrow z_k$ ($j \neq k$). The factor $\prod_{1 \leq j < k \leq N} \frac{[z_j - z_k + 1/2]}{[z_j - z_k]}$ in $c_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ is also antisymmetric with respect to $z_j \longleftrightarrow z_k$ ($j \neq k$), and we find that $R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ is symmetric with respect to $z_j \longleftrightarrow z_k$ ($j \neq k$). From this symmetry, it is enough to show (3.21) and (3.22) for the case $m = N$

$$\begin{aligned} R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)|_{w_N = -z_N} &= [h][-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2][z_N + z_j] \\ &\times \prod_{j=1}^{N-1} [z_N + w_j + 1/2][z_N - w_j] R_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2), \end{aligned} \quad (3.27)$$

$$\begin{aligned} R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)|_{w_N = z_N} &= [h][-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2][z_N + z_j] \\ &\times \prod_{j=1}^{N-1} [z_N - w_j + 1/2][z_N + w_j] R_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h + 1/2). \end{aligned} \quad (3.28)$$

The other cases (3.21) and (3.22) for $m = 1, \dots, N-1$ can be obtained from (3.27) and (3.28) by using the property that $R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ is a symmetric function with symmetric variables z_j ($j = 1, \dots, N$).

Let us show (3.27). First, we find the following relation for $c_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$

$$\begin{aligned} c_N(z_1, \dots, z_N|w_1, \dots, w_N|h)|_{w_N = -z_N} &= \frac{[h][h + 1/2]}{[h - 1/2][h + N/2 - 1/2]} \\ &\times \prod_{j=1}^{N-1} \frac{[z_N + w_j - 1/2]}{[z_N + w_j]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} c_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2). \end{aligned} \quad (3.29)$$

Next, we analyze the determinant $\det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))$. It can be easily seen from the explicit form of the matrix elements of $Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ (1.3) that among

the matrix elements in the N -th column of $Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$, only the matrix element in N -th row is nonzero after the substitution $w_N = -z_N$. Then one expands the determinant $\det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))|_{w_N=-z_N}$ by its N -th column to get

$$\begin{aligned} & \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))|_{w_N=-z_N} \\ &= [h + (N-1)/2] \prod_{\ell=1}^{N-1} [z_\ell - z_N] \prod_{\ell=1}^N [z_\ell + z_N] \det_{N-1}(\bar{Y}_N(z_1, \dots, z_N|w_1, \dots, w_N|h)), \end{aligned} \quad (3.30)$$

where $\bar{Y}_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ is an $(N-1) \times (N-1)$ matrix which is obtained from $Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)$ by removing the N -th row and the N -th column. Since one can show

$$\begin{aligned} & \bar{Y}_N(z_1, \dots, z_N|w_1, \dots, w_N|h)_{jk} \\ &= Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h)_{jk} \\ &= -[z_N + w_k][z_N - w_k] Y_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2)_{jk}, \end{aligned} \quad (3.31)$$

for $j, k = 1, \dots, N-1$, we can further rewrite (3.30) as

$$\begin{aligned} & \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))|_{w_N=-z_N} \\ &= [h + (N-1)/2] \prod_{\ell=1}^{N-1} [z_\ell - z_N] \prod_{\ell=1}^N [z_\ell + z_N] \\ & \times \prod_{k=1}^{N-1} (-[z_N + w_k][z_N - w_k]) \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2)) \\ &= [2z_N][h + (N-1)/2] \prod_{j=1}^{N-1} [z_j - z_N][z_j + z_N] \prod_{j=1}^{N-1} [z_N + w_j][w_j - z_N] \\ & \times \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2)). \end{aligned} \quad (3.32)$$

Combining (3.29) and (3.32), we get

$$\begin{aligned} & R_N(z_1, \dots, z_N|w_1, \dots, w_N|h)|_{w_N=-z_N} \\ &= c_N(z_1, \dots, z_N|w_1, \dots, w_N|h) \det_N(Y_N(z_1, \dots, z_N|w_1, \dots, w_N|h))|_{w_N=-z_N} \\ &= \frac{[h][h + 1/2]}{[h - 1/2][h + N/2 - 1/2]} \prod_{j=1}^{N-1} \frac{[z_N + w_j - 1/2]}{[z_N + w_j]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} \\ & \times [2z_N][h + (N-1)/2] \prod_{j=1}^{N-1} [z_j - z_N][z_j + z_N] \prod_{j=1}^{N-1} [z_N + w_j][w_j - z_N] \\ & \times c_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2) \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2)) \\ &= [h][-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2][z_N + z_j] \prod_{j=1}^{N-1} [z_N + w_j + 1/2][z_N - w_j] \\ & \times R_{N-1}(z_1, \dots, z_{N-1}|w_1, \dots, w_{N-1}|h - 1/2), \end{aligned} \quad (3.33)$$

which is exactly the relation (3.27).

(3.28) can be proved in a similar way. First, we note the following relation holds

$$\begin{aligned}
c_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N=z_N} &= -\frac{[h - N/2 + 1]}{[h + 1/2]} \\
&\times \prod_{j=1}^{N-1} \frac{[z_N - w_j + 1/2]}{[z_N - w_j]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} c_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2).
\end{aligned} \tag{3.34}$$

Next, one can see that only the matrix element in the N -th row is nonzero among the matrix elements in the N -th column of $Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$, and we find that the expansion of the determinant $\det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)) |_{w_N=z_N}$ by its N -th column gives the following relation

$$\begin{aligned}
&\det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)) |_{w_N=z_N} \\
&= [h + (N - 1)/2] \prod_{\ell=1}^{N-1} [z_\ell - z_N] \prod_{\ell=1}^N [z_\ell + z_N] \det_{N-1}(\bar{Y}_N(z_1, \dots, z_N | w_1, \dots, w_N | h)).
\end{aligned} \tag{3.35}$$

Using the relation

$$\begin{aligned}
&\bar{Y}_N(z_1, \dots, z_N | w_1, \dots, w_N | h)_{jk} \\
&= Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)_{jk} \\
&= [z_N + w_k][z_N - w_k] Y_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2)_{jk},
\end{aligned} \tag{3.36}$$

for $j, k = 1, \dots, N - 1$, we can further rewrite (3.35) as

$$\begin{aligned}
&\det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)) |_{w_N=z_N} \\
&= [h + (N - 1)/2] \prod_{\ell=1}^{N-1} [z_\ell - z_N] \prod_{\ell=1}^N [z_\ell + z_N] \\
&\times \prod_{k=1}^{N-1} [z_N + w_k][z_N - w_k] \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2)) \\
&= [2z_N][h + (N - 1)/2] \prod_{j=1}^{N-1} [z_j - z_N][z_j + z_N] \prod_{j=1}^{N-1} [z_N + w_j][z_N - w_j] \\
&\times \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2)).
\end{aligned} \tag{3.37}$$

Combining (3.34) and (3.37), we get

$$\begin{aligned}
& R_N(z_1, \dots, z_N | w_1, \dots, w_N | h) |_{w_N = z_N} \\
&= c_N(z_1, \dots, z_N | w_1, \dots, w_N | h) \det_N(Y_N(z_1, \dots, z_N | w_1, \dots, w_N | h)) |_{w_N = z_N} \\
&= - \frac{[h - N/2 + 1]}{[h + 1/2]} \prod_{j=1}^{N-1} \frac{[z_N - w_j + 1/2]}{[z_N - w_j]} \prod_{j=1}^{N-1} \frac{[z_j - z_N + 1/2]}{[z_j - z_N]} \\
&\quad \times [2z_N][h + (N-1)/2] \prod_{j=1}^{N-1} [z_j - z_N][z_j + z_N] \prod_{j=1}^{N-1} [z_N + w_j][z_N - w_j] \\
&\quad \times c_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2) \det_{N-1}(Y_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2)) \\
&= [h][-2z_N] \prod_{j=1}^{N-1} [z_N - z_j + 1/2][z_N + z_j] \prod_{j=1}^{N-1} [z_N - w_j + 1/2][z_N + w_j] \\
&\quad \times R_{N-1}(z_1, \dots, z_{N-1} | w_1, \dots, w_{N-1} | h + 1/2), \tag{3.38}
\end{aligned}$$

hence the relation (3.28) is proved. Note that in the last equality, we used the identity $\frac{[h + (N-1)/2][h - N/2 + 1]}{[h + 1/2]} = [h]$ which holds for any integer N .

What remains is to show Property 3, which can be easily seen from the definition of $R_N(z_1, \dots, z_N | w_1, \dots, w_N | h)$ (3.3).

Finally, the two propositions we proved (Propositions 3.2 and 3.3) mean that the sequence of functions $\{L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) | N \in \mathbb{N}\}$ and $\{R_N(z_1, \dots, z_N | w_1, \dots, w_N | h) | N \in \mathbb{N}\}$ are exactly the same, and hence

$$L_N(z_1, \dots, z_N | w_1, \dots, w_N | h) = R_N(z_1, \dots, z_N | w_1, \dots, w_N | h), \tag{3.39}$$

for $N \in \mathbb{N}$. This concludes the proof of Theorem 3.1. □

4 Conclusion

In this paper, we proved a duality between two elliptic determinants. The proof presented in this paper is inspired by and can be regarded as a variant of the Izergin-Korepin method. It is originally a method initiated by Korepin and Izergin [29, 30] to study and find explicit forms of partition functions of quantum integrable models. The key of the Izergin-Korepin method is to list the properties for a sequence of functions which uniquely define them, and one can use this idea to prove identities between determinants which do not seem to have factorized expressions and which look different at first sight.

It seems that there are many other transformation formulas between (elliptic) determinants or Pfaffians which do not seem to have factorized expressions, and it is interesting to discover and prove them. The previous studies on factorized formulas for (elliptic) determinants and Pfaffians [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] may give hints to find them. Another interesting resource for the discovery is partition functions of (elliptic) integrable models. We found a duality between two elliptic Pfaffians as a consequence of analyzing a

variant of the domain wall boundary partition functions (OS boundary) [17]. The partition functions of elliptic integrable models may also give clues to find them. As for the trigonometric $U_q(\widehat{sl_2})$ six-vertex model, Kuperberg [39] uses various variations of the domain wall boundary partition functions to compute various generating functions of the enumeration of alternating sign matrices. We lifted one of his variations from the trigonometric model to the elliptic model in [17] and found a duality between two elliptic Pfaffians. It may also be interesting to lift other variations of the domain wall boundary partition functions to the elliptic model and find transformation formulas.

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