# Dual Cauchy formula for elliptic Schur functions


Kohei Motegi *

*Faculty of Marine Technology, Tokyo University of Marine Science and Technology, Etchujima 2-1-6, Koto-Ku, Tokyo, 135-8533, Japan*


December 4, 2017


**Abstract**

We present an application of quantum integrability to a derivation of an algebraic identity for elliptic Schur functions. By using the correspondence between the elliptic Schur functions and the wavefunctions of the elliptic Felderhof models and its dual, we derive the dual Cauchy formula for the elliptic Schur functions by comparing two ways of evaluations of the domain wall boundary partition functions.


## 1 Introduction

Recently, the relations between partition functions of integrable lattice models and symmetric functions, and applications to algebraic identities for symmetric functions from the viewpoint of quantum integrability have been extensively investigated.

There exist two types of integrable lattice models [1, 2, 3, 4]. For the case of integrable lattice models having quantum group symmetry of Drinfeld-Jimbo type [5, 6] which are related with the XXX, XXZ and XYZ quantum integrable spin chains and $q$-boson models, what appear as the off-shell Bethe wavefunctions are the Schur, Grothendieck and Hall-Littlewood polynomials and their generalizations. See [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] for examples on various studies of these models.

There is another class of integrable models which are less famous than the former class, but recently revealed by number theorists that it inherits connections with automorphic representation theory and deformation of Weyl character formula (Tokuyama formula) for symmetric functions. See [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45] for examples on the recent developments on the analysis on this type of models as well as related former stories on this subject.

This paper is an application of the correspondence between the symmetric functions and the wavefunctions of the model of this type to a derivation of an algebraic identity. Namely, we derive the dual Cauchy formula for ellptic Schur functions by using quantum


*E-mail: kmoteg0@kaiyodai.ac.jp

integrability. We recently established the exact correspondence between the wavefunctions of the elliptic Felderhof model, Deguchi-Martin (elliptic Perk-Schultz) model and the elliptic Schur functions [44, 45]. This was achieved by extending the Izergin-Korepin anlaysis [46, 47] which was originally a method to anlayze the domain wall boundary partition functions. The correspondence is an elliptic analogue of the ones for the trigonometric free-fermion model obtained by Bump-Brubaker-Friedberg [21] and Bump-McNamara-Nakasuji [29]. In the paper by Bump-McNamara-Nakasuji [29], the dual Cauchy formula for the factorial Schur functions was derived by comparing two ways of evaluations of the domain wall boundary partition functions. In this paper, we apply their idea to the elliptic Felderhof model and derive the dual Cauchy formula for the elliptic Schur functions. We use the factorized form of the explicit form of the domain wall boundary partition functions found by Foda-Wheeler-Zuparic [43]. Together with our results on the correspondence between the wavefunctions, the dual wavefunctions and the elliptic Schur functions, we derive the dual Cauchy formula for the elliptic Schur functions. We remark that the ellipic Schur functions treated in this paper seem to be different from the ones introduced by Schlosser [48] or Noumi [49, 50].

In Section 2, we introduce the (dual) wavefunctions of the elliptic Felderhof model. In Section 3, we introduce two elliptic Schur functions and state the correspondence between the (dual) wavefunctions and the elliptic Schur functions. A detailed proof based on the Izergin-Korepin analysis for the case of the dual wavefunctions are given in Section 4. In Section 5, we derive the dual Cauchy formula for the elliptic Schur functions by two ways of evaluations of the domain wall boundary partition functions.

## 2 Elliptic Felderhof model and wavefunctions

We first introduce elliptic functions and list their properties. The theta functions $H(u)$ is

$$H(u) = 2\mathrm{sinh}u \prod_{n=1}^{\infty} (1 - 2\mathbf{q}^{2n}\cosh(2u) + \mathbf{q}^{4n})(1 - \mathbf{q}^{2n}), \tag{2.1}$$

where $\mathbf{q}$ is the elliptic nome $(0 < \mathbf{q} < 1)$. For the description of the matrix elements of the dynamical $R$-matrix of the elliptic Felderhof model, we introduce the following notation

$$[u] = H(\pi i u). \tag{2.2}$$

The theta function $[u]$ is an odd function $[-u] = -[u]$ and satisfies the quasi-periodicities

$$[u+1] = -[u], \tag{2.3}$$

$$[u - i\log(\mathbf{q})/\pi] = -\mathbf{q}^{-1}\exp(-2\pi i u)[u]. \tag{2.4}$$

We use the following property about the elliptic polynomials [51, 52] presented below.

A character is a group homomorphism $\chi$ from multiplicative groups $\Gamma = \mathbf{Z} + \tau\mathbf{Z}$ to $\mathbf{C}^{\times}$. An $N$-dimensional space $\Theta_N(\chi)$ is defined for each character $\chi$ and positive integer $N$, which consists of holomorphic functions $\phi(y)$ on $\mathbf{C}$ satisfying the quasi-periodicities

$$\phi(y+1) = \chi(1)\phi(y), \tag{2.5}$$

$$\phi(y+\tau) = \chi(\tau)e^{-2\pi i N y - \pi i N \tau}\phi(y). \tag{2.6}$$

The elements of the space $\Theta_N(\chi)$ are called elliptic polynomials. The space $\Theta_N(\chi)$ is $N$-dimensional [51, 52] and the following fact holds for the elliptic polynomials.

**Proposition 2.1.** *[51, 52] Suppose there are two elliptic polynomials $P(y)$ and $Q(y)$ in $\Theta_N(\chi)$, where $\chi(1) = (-1)^N$, $\chi(\tau) = (-1)^N e^{\alpha}$. If those two polynomials are equal $P(y_j) = Q(y_j)$ at $N$ points $y_j$, $j = 1, \ldots, N$ satisfying $y_j - y_k \notin \Gamma$, $\sum_{k=1}^N y_k - \alpha \notin \Gamma$, then the two polynomials are exactly the same $P(y) = Q(y)$.*

This property ensure the uniqueness of the Izergin-Korepin analysis on the wavefunctions of elliptic integrable models. For example, it is used in [51, 53, 54] on the analysis on the domain wall boundary partition functions of the eight-vertex solid-on-solid model [55]). Note that the property above is an elliptic analogue of the following fact for ordinary polynomials: if $P(y)$ and $Q(y)$ are polynomials of degree $N-1$ in $y$, and if these polynomials match at $N$ distinct points, then the two polynomials are exactly the same.

The trigonometric face-type Felderhof model was first introduced by Deguchi-Akutsu [38], and its elliptic extension was constructed by Foda-Wheeler-Zuparic [43]. The dynamical $R$-matrix of the elliptic Felderhof model is given by [43]

$$
\begin{aligned}
&R_{ab}(u,v|p,q|h) \\
&= \begin{pmatrix}
[u-v+p+q] & 0 & 0 & 0 \\
0 & \frac{[h]^{1/2}[h+2p+2q]^{1/2}[u-v+q-p]}{[h+2p]^{1/2}[h+2q]^{1/2}} & \frac{[2p]^{1/2}[2q]^{1/2}[-u+v+q+p+h]}{[h+2p]^{1/2}[h+2q]^{1/2}} & 0 \\
0 & \frac{[2p]^{1/2}[2q]^{1/2}[u-v+q+p+h]}{[h+2p]^{1/2}[h+2q]^{1/2}} & \frac{[h]^{1/2}[h+2p+2q]^{1/2}[u-v-q+p]}{[h+2p]^{1/2}[h+2q]^{1/2}} & 0 \\
0 & 0 & 0 & [-u+v+p+q]
\end{pmatrix},
\end{aligned}
\tag{2.7}
$$

acting on the tensor product $W_a \otimes W_b$ of the complex two-dimensional space $W_a$. The parameters $u$ and $v$ are spectral parameters, and $p$ and $q$ are complex parameters. $h$ is called as the height or dynamical variable. One can think that the space $W_a$ carries the parameters $u$ and $p$, while the parameters $v$ and $q$ are associated with the space $W_b$. See Figure (1) for the graphical representations of the dynamical $R$-matrix (2.7).

We denote the orthonormal basis of $W_a$ and its dual as $\{|0\rangle_a, |1\rangle_a\}$ and $\{{}_a\langle 0|, {}_a\langle 1|\}$, and the matrix elements of the dynamical $R$-matrix as ${}_a\langle\gamma|{}_b\langle\delta|R_{ab}(u,v|p,q|h)|\alpha\rangle_a|\beta\rangle_b = [R(u,v|p,q|h)]^{\gamma\delta}_{\alpha\beta}$. The matrix elements of the dynamical $R$-matrix are explicitly given as

$$
{}_a\langle 0|{}_b\langle 0|R_{ab}(u.v|p,q|h)|0\rangle_a|0\rangle_b = [u-v+p+q], \tag{2.8}
$$

$$
{}_a\langle 0|{}_b\langle 1|R_{ab}(u,v|p,q|h)|0\rangle_a|1\rangle_b = \frac{[h]^{1/2}[h+2p+2q]^{1/2}[u-v+q-p]}{[h+2p]^{1/2}[h+2q]^{1/2}}, \tag{2.9}
$$

$$
{}_a\langle 0|{}_b\langle 1|R_{ab}(u,v|p,q|h)|1\rangle_a|0\rangle_b = \frac{[2p]^{1/2}[2q]^{1/2}[-u+v+q+p+h]}{[h+2p]^{1/2}[h+2q]^{1/2}}, \tag{2.10}
$$

$$
{}_a\langle 1|{}_b\langle 0|R_{ab}(u,v|p,q|h)|0\rangle_a|1\rangle_b = \frac{[2p]^{1/2}[2q]^{1/2}[u-v+q+p+h]}{[h+2p]^{1/2}[h+2q]^{1/2}}, \tag{2.11}
$$

$$
{}_a\langle 1|{}_b\langle 0|R_{ab}(u,v|p,q|h)|1\rangle_a|0\rangle_b = \frac{[h]^{1/2}[h+2p+2q]^{1/2}[u-v-q+p]}{[h+2p]^{1/2}[h+2q]^{1/2}}, \tag{2.12}
$$

$$
{}_a\langle 1|{}_b\langle 1|R_{ab}(u,v|p,q|h)|1\rangle_a|1\rangle_b = [-u+v+p+q]. \tag{2.13}
$$

In statistical physics, $|0\rangle$ or its dual $\langle 0|$ can be regarded as a hole state, while $|1\rangle$ or its dual $\langle 1|$ can be interpretted as a particle state. We thus sometimes use the terms hole states

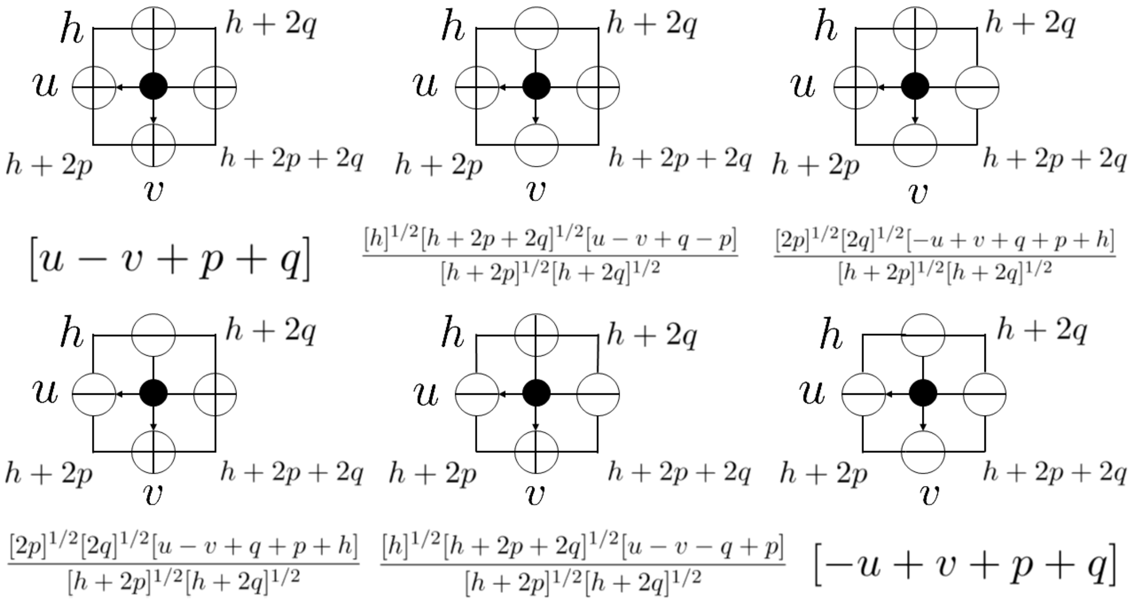


Figure 1: The matrix elements of the elliptic Felderhof model (2.7). The states $|0\rangle$, $\langle 0|$ are represented as $\oplus$, while the states $|1\rangle$, $\langle 1|$ are represented as $\ominus$. This kind of graphical representation for the case of trigonometric vertex models can be found in Bump-Brubaker-Friedberg [21] and Bump-McNamara-Nakasuji [29] for example.

and particle states to describe states constructed from $|0\rangle$, $\langle 0|$, $|1\rangle$ and $\langle 1|$ since they are convenient for the description of the states.

For later convenience, we also define the following Pauli spin operators $\sigma^+$ and $\sigma^-$ as operators acting on the (dual) orthonomal basis as

$$\sigma^+|1\rangle = |0\rangle,\ \sigma^+|0\rangle = 0,\ \langle 0|\sigma^+ = \langle 1|,\ \langle 1|\sigma^+ = 0, \tag{2.14}$$

$$\sigma^-|0\rangle = |1\rangle,\ \sigma^-|1\rangle = 0,\ \langle 1|\sigma^- = \langle 0|,\ \langle 0|\sigma^- = 0. \tag{2.15}$$

The dynamical $R$-matrix (2.7) satisfies the face-type Yang-Baxter relation (Figure 2)

$$\begin{aligned} &R_{ab}(u,v|p,q|h)R_{ac}(u,w|p,r|h+2q)R_{bc}(v,w|q,r|h) \\ =&R_{bc}(v,w|q,r|h+2p)R_{ac}(u,w|p,r|h)R_{ab}(u,v|p,q|h+2r), \end{aligned} \tag{2.16}$$

acting on $W_a \otimes W_b \otimes W_c$.

To construct partition functions of integrable lattice models, we identify one of the complex two-dimensional spaces $W_b$ of the tensor product space $W_a \otimes W_b$ with the quantum space. Let us denote the quantum space by $\mathcal{F}_j$, and define the $L$-operator $L_{aj}(u,v|p,q|h)$ acting on $W_a \otimes \mathcal{F}_j$ as

$$L_{aj}(u,v|p,q|h) = R_{aj}(u,v|p,q|h). \tag{2.17}$$

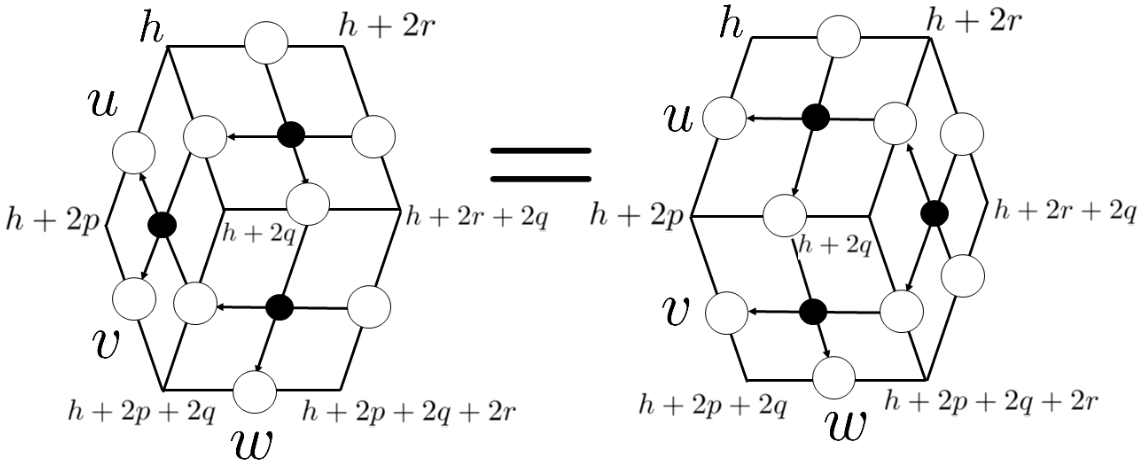


Figure 2: The dynamical Yang-Baxter relation (2.16). The left and right figure represents $R_{ab}(u, v|p, q|h)R_{ac}(u, w|p, r|h + 2q)R_{bc}(v, w|q, r|h)$ and $R_{bc}(v, w|q, r|h + 2p)R_{ac}(u, w|p, r|h)R_{ab}(u, v|p, q|h + 2r)$, respectively.

The next step is to define the monodromy matrix from the $L$-operators. For convenience, one denotes the sum of complex numbers $q_1, q_2, \ldots, q_j$ as $\overline{q_j}$

$$\overline{q_j} := \sum_{k=1}^{j} q_k. \tag{2.18}$$

The monodromy matrix $T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)$ is the product of $L$-operators

$$\begin{aligned} &T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) \\ =&L_{a1}(u, v_1|p, q_1|h)L_{a2}(u, v_2|p, q_2|h + 2\overline{q_1}) \cdots L_{aM}(u, v_M|p, q_M|h + 2\overline{q_{M-1}}), \end{aligned} \tag{2.19}$$

acting on $W_a \otimes \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$ (Figure 3 top).

The $B$-operator is a matrix element of the monodromy matrix (2.19) with respect to the auxiliary space $W_a$

$$B(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h) = {}_a\langle 0|T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|1\rangle_a, \tag{2.20}$$

which acts on $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$ (Figure 3 bottom).

The wavefunctions $W_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h)$ is defined as the matrix elements of the product of the $B$-operators (2.20) as

$$\begin{aligned} &W_{M,N}(u_1, \ldots, u_N|v_1, \ldots, v_M|x_1, \ldots, x_N|h) \\ =&\langle x_1 \cdots x_N|B(u_N|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N-1)p) \\ &\times \cdots \times B(u_2|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2p)B(u_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|\Omega\rangle_M, \end{aligned} \tag{2.21}$$

where $|\Omega\rangle_M := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M$ is the vacuum state in the tensor product of quantum spaces, and $\langle x_1 \cdots x_N|$ are the dual $N$-particle states

$$\langle x_1 \cdots x_N| = ({}_1\langle 0| \otimes \cdots \otimes {}_M\langle 0|) \prod_{j=1}^{N} \sigma_{x_j}^{+} \in \mathcal{F}_1^* \otimes \cdots \otimes \mathcal{F}_M^*, \tag{2.22}$$

which are states labelling the configurations of particles $1 \le x_1 < x_2 < \cdots < x_N \le M$.

We also define the dual wavefunctions $\overline{W_{M,N}}(u_1, \ldots, u_N|v_1, \ldots, v_M|\overline{x_1}, \ldots, \overline{x_N}|h)$ as another type of matrix elements of the $B$-operators as

$$\begin{aligned}
&\overline{W_{M,N}}(u_1, \ldots, u_N|v_1, \ldots, v_M|\overline{x_1}, \ldots, \overline{x_N}|h) \\
=&{}_M\langle\overline{\Omega}|B(u_N|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2(N-1)p) \\
&\times \cdots \times B(u_2|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h + 2p)B(u_1|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)|\overline{x_1} \cdots \overline{x_N}\rangle,
\end{aligned} \tag{2.23}$$

where ${}_M\langle\overline{\Omega}| := {}_1\langle 1| \otimes \cdots \otimes {}_M\langle 1| \in \mathcal{F}_1^* \otimes \cdots \otimes \mathcal{F}_M^*$ is the dual particle-filled state in the tensor product of quantum spaces, and $|\overline{x_1} \cdots \overline{x_N}\rangle$ are the $N$-hole states

$$|\overline{x_1} \cdots \overline{x_N}\rangle = \prod_{j=1}^{N} \sigma_{\overline{x_j}}^{+}(|1\rangle_1 \otimes \cdots \otimes {}_M|1\rangle_M) \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_M, \tag{2.24}$$

which are states labelling the configurations of holes $1 \le \overline{x_1} < \overline{x_2} < \cdots < \overline{x_N} \le M$.

See Figures 4 and 5 for graphical representations of the wavefunctions and the dual wavefunctions.

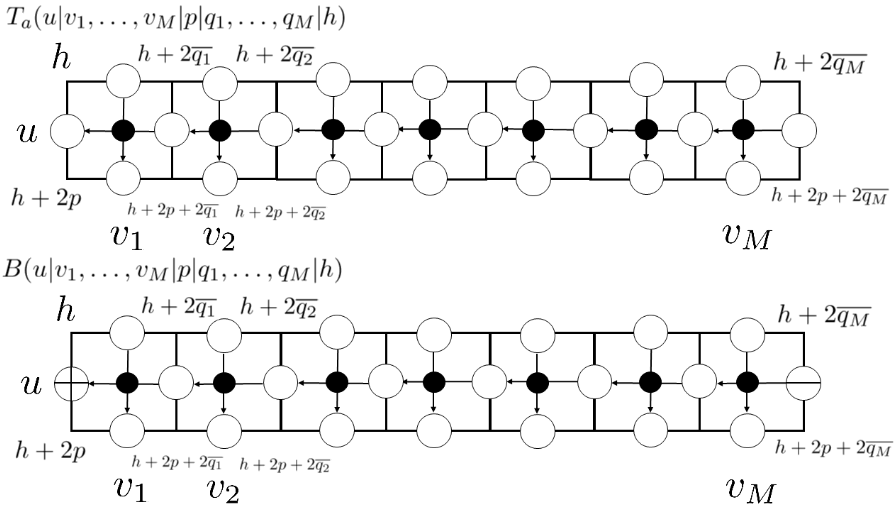


Figure 3: The monodromy matrix $T_a(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)$ (2.19) (top) and the $B$-operator $B(u|v_1, \ldots, v_M|p|q_1, \ldots, q_M|h)$ (2.20) (bottom).

## 3 Wavefunctions and elliptic Schur functions

In this section, we first introduce two elliptic Schur functions.

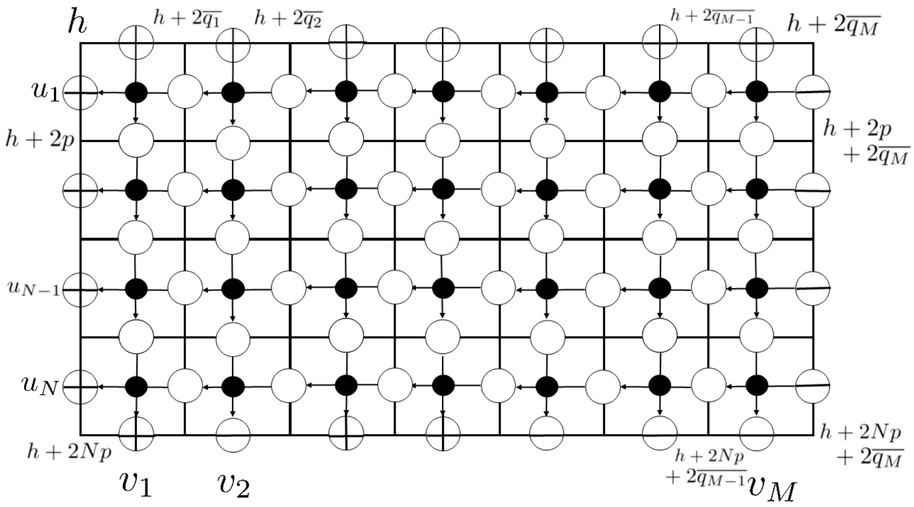


Figure 4: The wavefunctions $W_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$ (2.21). The figure illustrates the case $M=7$, $N=4$, $x_1=2$, $x_2=5$, $x_3=6$, $x_4=7$.

**Definition 3.1.** We define the following elliptic Schur function $S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$ which depends on the symmetric variables $u_1,\dots,u_N$, two sets of complex parameters $v_1,\dots,v_M$ and $q_1,\dots,q_M$, two complex parameters $h$, $p$ and integers $x_1,\dots,x_N$ satisfying $1\le x_1<\cdots<x_N\le M$,

$$\begin{aligned}
&S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)\\
=&\sum_{\sigma\in S_N}\prod_{1\le j<k\le N}\frac{1}{[u_{\sigma(j)}-u_{\sigma(k)}]}\prod_{j=1}^N\prod_{k=x_j+1}^M[u_{\sigma(j)}-v_k-q_k+p]\\
&\times\prod_{j=1}^N\frac{[h+2jp+2\overline{q_M}]^{1/2}[2p]^{1/2}[2q_{x_j}]^{1/2}}{[h+2(j-1)p+2\overline{q_M}]^{1/2}[h+2Np+2\overline{q_{x_j-1}}]^{1/2}[h+2Np+2\overline{q_{x_j}}]^{1/2}}\\
&\times\prod_{j=1}^N[-u_{\sigma(j)}+v_{x_j}+h+(2N-1)p+q_{x_j}+2\overline{q_{x_j-1}}]\prod_{j=1}^N\prod_{k=1}^{x_j-1}[u_{\sigma(j)}-v_k+p+q_k], \qquad (3.1)\\
=&\prod_{1\le j<k\le N}\frac{1}{[u_j-u_k]}\det{}_N(f_{x_j}(u_k|v_1,\dots,v_M)), \qquad (3.2)\\
&f_{x_j}(u|v_1,\dots,v_M)\\
=&\frac{[h+2jp+2\overline{q_M}]^{1/2}[2p]^{1/2}[2q_{x_j}]^{1/2}}{[h+2(j-1)p+2\overline{q_M}]^{1/2}[h+2Np+2\overline{q_{x_j-1}}]^{1/2}[h+2Np+2\overline{q_{x_j}}]^{1/2}}\\
&\times[-u+v_{x_j}+h+(2N-1)p+q_{x_j}+2\overline{q_{x_j-1}}]\prod_{k=1}^{x_j-1}[u-v_k+p+q_k]\prod_{k=x_j+1}^M[u-v_k+p-q_k]. \qquad (3.3)
\end{aligned}$$

Recall that $\overline{q_j}$ is defined as $\overline{q_j}=\sum_{k=1}^j q_k$.

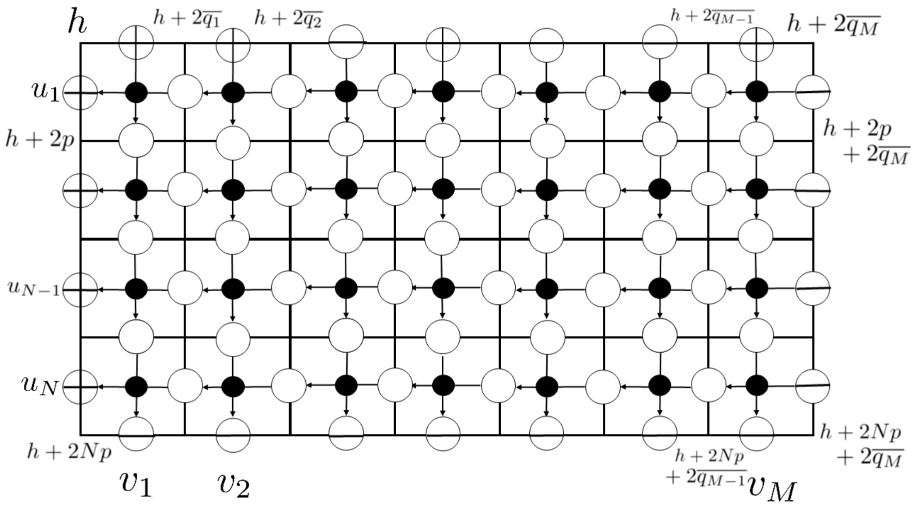


Figure 5: The dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ (2.21). The figure illustrates the case $M=7$, $N=4$, $\overline{x_1}=1$, $\overline{x_2}=2$, $\overline{x_3}=4$, $\overline{x_4}=7$.

We next define another elliptic Schur function $\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ which depends on the symmetric variables $u_1,\dots,u_N$, two sets of complex parameters $v_1,\dots,v_M$ and $q_1,\dots,q_M$, two complex parameters $h$, $p$ and integers $\overline{x_1},\dots,\overline{x_N}$ satisfying $1\le\overline{x_1}<\cdots<\overline{x_N}\le M$,

$$\begin{aligned}
&\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)\\
=&\sum_{\sigma\in S_N}\prod_{1\le j<k\le N}\frac{1}{[u_{\sigma(k)}-u_{\sigma(j)}]}\prod_{j=1}^{N}\prod_{k=\overline{x_j}+1}^{M}[-u_{\sigma(j)}+v_k+q_k+p]\\
&\times\prod_{j=1}^{N}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}}\\
&\times\prod_{j=1}^{N}[-u_{\sigma(j)}+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^{N}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma(j)}-v_k-p+q_k], \qquad (3.4)\\
=&\prod_{1\le j<k\le N}\frac{1}{[u_k-u_j]}\det_N(g_{\overline{x_j}}(u_k|v_1,\dots,v_M)), \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
&g_{\overline{x_j}}(u|v_1,\dots,v_M)\\
=&\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}}\\
&\times[-u+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{k=1}^{\overline{x_j}-1}[u-v_k-p+q_k]\prod_{k=\overline{x_j}+1}^{M}[-u+v_k+p+q_k]. \qquad (3.6)
\end{aligned}$$

Here we define $\overline{q_0}$ as $\overline{q_0}=0$.

The (dual) wavefunctions of the elliptic Felderhof model can be expressed as the product of a one-parameter deformation of the elliptic the Vandermonde determinant $\prod_{1\le j<k\le N}[u_j-$

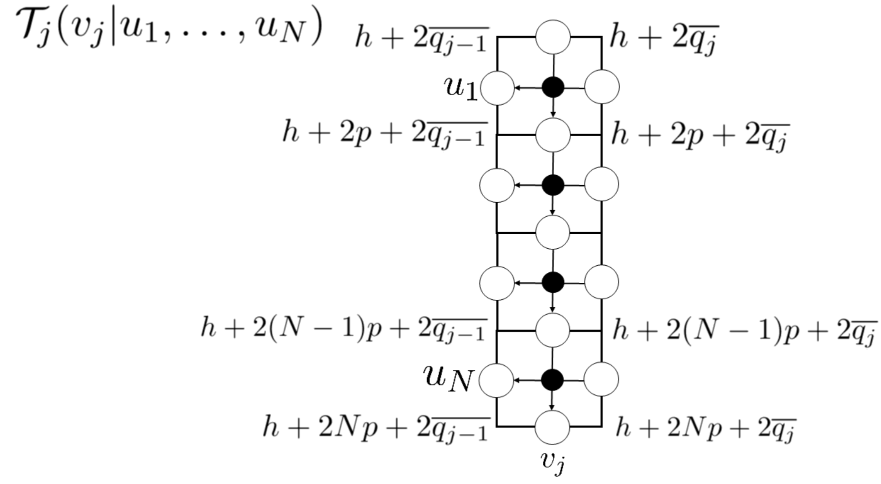


Figure 6: The vertical monodromy matrix $\mathcal{T}_j(v_j|u_1,\dots,u_N)$ (4.5).

$u_k+2p]$ and the elliptic Schur symmetric functions $S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$ ($\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$) defined above.

**Theorem 3.2.** *The wavefunctions of the elliptic Felderhof model* $W_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$ *is explicitly expressed as the product of a one-parameter deformation of an elliptic Vandermonde determinant* $\prod_{1\le j<k\le N}[u_j-u_k+2p]$ *and the elliptic Schur functions* $S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$

$$\begin{aligned}&W_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)\\ =&\prod_{1\le j<k\le N}[u_j-u_k+2p]S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h).\end{aligned}\tag{3.7}$$

*The dual wavefunctions of the elliptic Felderhof model* $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ *is explicitly expressed as the product of a one-parameter deformation of an elliptic Vandermonde determinant* $\prod_{1\le j<k\le N}[u_j-u_k+2p]$ *and the elliptic Schur functions* $\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$

$$\begin{aligned}&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)\\ =&\prod_{1\le j<k\le N}[u_j-u_k+2p]\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h).\end{aligned}\tag{3.8}$$

Theorem 3.2 can be regarded as an elliptic analogue of the correspondence established by Bump-Brubaker-Friedberg [21] and Bump-McNamara-Nakasuji [29] between the wavefunctions of the trigonometric free-fermion model and a one-parameter deformation of the Vandermonde determinant and the (factorial) Schur functions. The relation (3.7) is shown in [45]. In the next section, we give a detailed proof of (3.8).

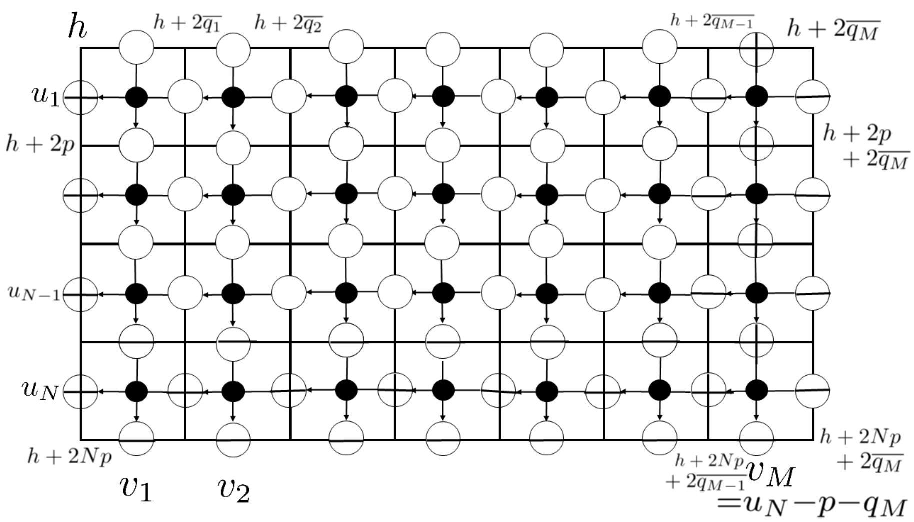


Figure 7: The recursion relation $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$, $\overline{x_N}=M$ evaluated at $v_M=u_N-p-q_M$ (4.2) . The dynamical $R$-matrices of the bottom row and the rightmost column get freezed.

# 4 Details of the proof

In this section, we give a detailed proof of (3.8). We use the idea initiated by Korepin [46], listing the properties of the domain wall boundary partition functions which uniquely characterize it. This characterization lead Izergin [47] to found the determinant representation (Izergin-Korepin determinant) of the domain wall boundary partition functions. We recently extended the Izergin-Korepin analysis to the wavefunctions [44, 45], and we follow this strategy.

**Proposition 4.1.** *The dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ satisfies the following properties.*

*(1) If $\overline{x_N}=M$, the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ is an elliptic polynomial of $v_M$ in $\Theta_N(\chi)$.*

*(2) The dual wavefunctions $\overline{W_{M,N}}(u_\sigma(1),\dots,u_\sigma(N)|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ with the ordering of the spectral parameters permuted $u_{\sigma(1)},\dots,u_{\sigma(N)}$, $\sigma\in S_N$ are related to the original dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ by the following relation*

$$\prod_{\substack{1\le j<k\le N\\ \sigma(j)>\sigma(k)}}[u_{\sigma(j)}-u_{\sigma(k)}+2p]\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$$
$$=\prod_{\substack{1\le j<k\le N\\ \sigma(j)>\sigma(k)}}[u_{\sigma(k)}-u_{\sigma(j)}+2p]\overline{W_{M,N}}(u_{\sigma(1)},\dots,u_{\sigma(N)}|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h). \qquad (4.1)$$

*(3) If $\overline{x_N}=M$, the following recursive relations between the dual wavefunctions hold (Figure*

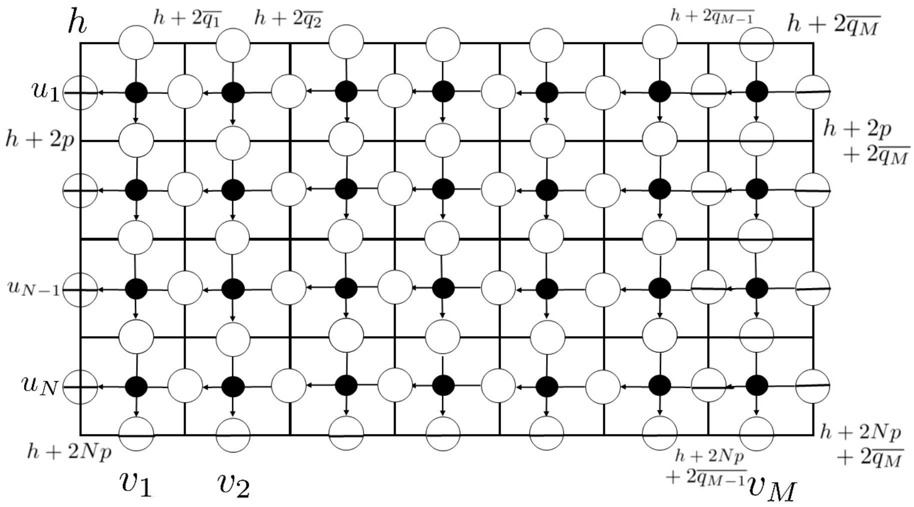


Figure 8: The factorization of $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$, $\overline{x_N}\neq M$ (4.3). The dynamical $R$-matrices of the rightmost column get freezed.

*7):*

$$\begin{aligned}
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)|_{v_M=u_N-p-q_M}\\
=&\frac{[2p]^{1/2}[2q_M]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2(N-1)p]^{1/2}}{[h+2\overline{q_M}]^{1/2}[h+2Np]^{1/2}}\prod_{j=1}^{N-1}[u_j-u_N+2p]\prod_{j=1}^{M-1}[u_N-v_j-p+q_j]\\
&\times\overline{W_{M-1,N-1}}(u_1,\dots,u_{N-1}|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_{N-1}}|h). \qquad (4.2)
\end{aligned}$$

*If $\overline{x_N}\neq M$, the following factorizations hold for the dual wavefunctions (Figure 8):*

$$\begin{aligned}
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)\\
=&\prod_{j=1}^{N}[-u_j+v_M+q_M+p]\overline{W_{M-1,N}}(u_1,\dots,u_N|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_N}|h). \qquad (4.3)
\end{aligned}$$

*(4) The following evaluation holds for the case $N=1$, $\overline{x_1}=M$*

$$\begin{aligned}
&\overline{W_{M,1}}(u|v_1,\dots,v_M|M|h)\\
=&\frac{[h]^{1/2}[2p]^{1/2}[2q_M]^{1/2}[-u+v_M+h+p+q_M+2\overline{q_{M-1}}]}{[h+2p]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2\overline{q_M}]^{1/2}}\prod_{k=1}^{M-1}[u-v_k-p+q_k]. \qquad (4.4)
\end{aligned}$$

*Proof.* To show Property (1) which is a property for the case $\overline{x_N}=M$, we first introduce the vertical transfer matrix (Figure 6)

$$\begin{aligned}
\mathcal{T}_j(v_j|u_1,\dots,u_N)=&L_{a_Nj}(u_N,v_j|p,q_j|h+2\overline{q_{j-1}}+2(N-1)p)\cdots L_{a_1j}(u_1,v_j|p,q_j|h+2\overline{q_{j-1}})\\
&\in\mathrm{End}(W_{a_N}\otimes\cdots\otimes W_{a_1}\otimes\mathcal{F}_j), \qquad (4.5)
\end{aligned}$$

and rewrite the dual wavefunctions as

$$
\begin{aligned}
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\
=&\langle 0|^{\otimes N}{}_M\langle\overline{\Omega}|\mathcal{T}_1(v_1|u_1,\dots,u_N)\cdots\mathcal{T}_M(v_M|u_1,\dots,u_N)|1\rangle^{\otimes N}|\overline{x_1},\dots,\overline{x_{N-1}}\rangle|0\rangle_M, && (4.6)\\
&\langle 0|^{\otimes N}={}_{a_N}\langle 0|\otimes\cdots\otimes{}_{a_1}\langle 0|, && (4.7)\\
&|1\rangle^{\otimes N}=|1\rangle_{a_N}\otimes\cdots\otimes|1\rangle_{a_1}. && (4.8)
\end{aligned}
$$

We use the completeness relation in one hole sector

$$
\begin{aligned}
&\sum_{j=1}^N|1^{N-j}01^{j-1}\rangle\langle 1^{N-j}01^{j-1}|=\mathrm{Id}, && (4.9)\\
&|1^{N-j}01^{j-1}\rangle=|1\rangle_{a_N}\otimes\cdots\otimes|1\rangle_{a_{j+1}}\otimes|0\rangle_{a_j}\otimes|1\rangle_{a_{j-1}}\otimes\cdots\otimes|1\rangle_{a_1}, && (4.10)\\
&\langle 1^{N-j}01^{j-1}|={}_{a_N}\langle 1|\otimes\cdots\otimes{}_{a_{j+1}}\langle 1|\otimes{}_{a_j}\langle 0|\otimes{}_{a_{j-1}}\langle 1|\otimes\cdots\otimes{}_{a_1}\langle 1|, && (4.11)
\end{aligned}
$$

and decompose (4.8) as follows

$$
\begin{aligned}
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\
=&\sum_{j=1}^N\langle 0|^{\otimes N}{}_{M-1}\langle\overline{\Omega}|\mathcal{T}_1(v_1|u_1,\cdots,u_N)\cdots\mathcal{T}_{M-1}(v_{M-1}|u_1,\cdots,u_N)|\overline{x_1},\dots,\overline{x_{N-1}}\rangle|1^{N-j}01^{j-1}\rangle\\
&\times\langle 1^{N-j}01^{j-1}|_M\langle 1|\mathcal{T}_M(v_M|u_1,\cdots,u_N)|1\rangle^{\otimes N}|0\rangle_M. && (4.12)
\end{aligned}
$$

In each summand of the decomposition (4.12), the dependence on $v_M$ comes from $\langle 1^{N-j}01^{j-1}|_M\langle 1|\mathcal{T}_M(v_M|u_1,\cdots,u_N)|1\rangle^{\otimes N}|0\rangle_M=:f_j(v_M)$. The factors $f_j(v_M)$ can be explicitly calculated easily from its graphical description, and we find

$$
\begin{aligned}
f_j(v_M)=&\prod_{k=1}^{j-1}\frac{[h+2\overline{q_{M-1}}+2(k-1)p]^{1/2}[h+2\overline{q_M}+2kp]^{1/2}[u_k-v_M-q_M+p]}{[h+2\overline{q_{M-1}}+2kp]^{1/2}[h+2\overline{q_M}+2(k-1)p]^{1/2}}\\
&\times\frac{[2p]^{1/2}[2q_M]^{1/2}[-u_j+v_M+(2j-1)p+q_M+h+2\overline{q_{M-1}}]}{[h+2\overline{q_{M-1}}+2jp]^{1/2}[h+2\overline{q_M}+2(j-1)p]^{1/2}}\\
&\times\prod_{k=j+1}^N[-u_k+v_M+p+q_M]. && (4.13)
\end{aligned}
$$

From the explicit form (4.13) and recalling the quasi-periodicities of theta functions (2.3) and (2.4), we can calculate the quasi-periodicities of $f_j(v_M)$ as

$$
\begin{aligned}
&f_j(v_M+1)\\
=&(-1)^Nf_j(v_M), && (4.14)\\
&f_j(v_M-i\log(\mathbf{q})/\pi)\\
=&(-\mathbf{q}^{-1})^N\exp\Bigg(-2\pi i\Bigg(Nv_M+h+2\overline{q_{M-1}}+Nq_M+Np-\sum_{j=1}^Nu_j\Bigg)\Bigg)f_j(v_M). && (4.15)
\end{aligned}
$$

We can see that the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h)$ satisfies the same quasi-periodicities with $f_j(v_M)$ as a function of $v_M$ from (4.14), (4.15) and the decomposition (4.12)

$$\begin{aligned}
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M+1|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\
=&(-1)^N\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h), \qquad (4.16)\\
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M-i\mathrm{log}(\mathbf{q})/\pi|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\
=&(-\mathbf{q}^{-1})^N\mathrm{exp}\Bigg(-2\pi i\Bigg(Nv_M+h+2\overline{q_{M-1}}+Nq_M+Np-\sum_{j=1}^N u_j\Bigg)\Bigg)\\
&\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h). \qquad (4.17)
\end{aligned}$$

Comparing with the quasi-periodicities (2.5) and (2.6), one concludes that the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h)$ is an elliptic polynomial of $v_M$ in $\Theta_N(\chi)$ of periods 1 and $\tau=-i\mathrm{log}(\mathbf{q})/\pi$ with the characters $\chi(1)=(-1)^N$ and $\chi(\tau)=(-1)^N\mathrm{exp}\Bigg(-2\pi i\Bigg(h+2\overline{q_{M-1}}+Nq_M+Np-\sum_{j=1}^N u_j\Bigg)\Bigg)$.

Property (2) can be shown in a standard way. The repeated use of the Yang-Baxter relation (2.16) leads to the intertwining relation between the monodromy matrices

$$\begin{aligned}
&R_{ab}(u_{j+1},u_j|p,p|h)T_a(u_{j+1}|v_1,\dots,v_M|p|q_1,\dots,q_M|h+2p)T_b(u_j|v_1,\dots,v_M|p|q_1,\dots,q_M|h)\\
=&T_b(u_j|v_1,\dots,v_M|p|q_1,\dots,q_M|h+2p)T_a(u_{j+1}|v_1,\dots,v_M|p|q_1,\dots,q_M|h)\\
&\times R_{ab}(u_{j+1},u_j|p,p|h+2\overline{q_M}). \qquad (4.18)
\end{aligned}$$

The following commutation relation between the $B$-operators can be obtained as a matrix element of the intertwining relation (4.18)

$$\begin{aligned}
&[u_{j+1}-u_j+2p]B(u_{j+1}|v_1,\dots,v_M|p|q_1,\dots,q_M|h+2p)B(u_j|v_1,\dots,v_M|p|q_1,\dots,q_M|h)\\
=&[u_j-u_{j+1}+2p]B(u_j|v_1,\dots,v_M|p|q_1,\dots,q_M|h+2p)B(u_{j+1}|v_1,\dots,v_M|p|q_1,\dots,q_M|h)
\end{aligned} \qquad (4.19)$$

Using the commutation relations (4.19) and recalling the definition of the dual wavefunctions (2.21), one immediately gets

$$\begin{aligned}
&[u_{j+1}-u_j+2p]\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)\\
=&[u_j-u_{j+1}+2p]\overline{W_{M,N}}(u_1,\dots,u_{j+1},u_j,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h). \qquad (4.20)
\end{aligned}$$

(4.1) can be obtained by using the relation (4.20) repeatedly.

Next we prove Property (3) for the case $\overline{x_N}=M$, i.e. show the recursion relation between the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ (4.2). This can be shown with the help of the graphical representation of the dual wavefunctions (Figure 7). If one sets $v_M=u_N-p-q_M$, one first finds that the dynamical $R$-matrix at the bottom right corner freezes. Next, using the ice-rule of the dynamical $R$-matrix, one finds from the graphical description that all the dynamical $R$-matrices at the rightmost column and the bottom row freeze, and the remaining unfreezed part is nothing but the dual wavefunction $\overline{W_{M-1,N-1}}(u_1,\dots,u_{N-1}|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_{N-1}}|h)$. Thus, we find that the dual wavefunctions of the elliptic model $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ evaluated at $v_M=u_N-p-q_M$ is expressed as the product of the matrix elements of the dynamical $R$-matrices at the rightmost column and the bottom row which can be simplified as

$$\frac{[2p]^{1/2}[2q_M]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2(N-1)p]^{1/2}}{[h+2\overline{q_M}]^{1/2}[h+2Np]^{1/2}}\prod_{j=1}^{N-1}[u_j-u_N+2p]\prod_{j=1}^{M-1}[u_N-v_j-p+q_j],$$

and the smaller one $\overline{W_{M-1,N-1}}(u_1,\dots,u_{N-1}|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_{N-1}}|h)$, which shows (4.2).

Property (3) for the case $\overline{x_N}\neq M$ can also be shown from its graphical representation (Figure 8). One easily finds that all the dynamical $R$-matrices at the rightmost column freeze, whose product of the matrix elements are $\prod_{j=1}^{N}[-u_j+v_M+q_M+p]$. The remaining unfreezed part is exactly the smaller dual wavefunction $\overline{W_{M-1,N}}(u_1,\dots,u_N|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_N}|h)$, hence we find the factorization of the dual wavefunctions (4.3) holds when $\overline{x_N}\neq M$.

Finally, it remains to show Property (4). It is easy to check the explicit expression of $\overline{W_{M,1}}(u|v_1,\dots,v_M|M|h)$ from its graphical representation to get (4.4). □

The Izergin-Korepin analysis consists of two steps. The first step is to construct Proposition 4.1 (Korepin's lemma) and prove it which uniquely defines the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$. The next step is to find the explicit forms of the functions which satisfy all the properties in Proposition 4.1. This is given in the next proposition. The proof of the proposition also concludes the proof of (3.8).

**Proposition 4.2.** *The product of a deformed elliptic Vandermonde determinant and the elliptic Schur functions* $\prod_{1\le j<k\le N}[u_j-u_k+2p]\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ *satisfy all the properties listed in Proposition 4.1, which the dual wavefunctions of the elliptic Felderhof model* $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ *must satisfy.*

*Proof.* We denote $\prod_{1\le j<k\le N}[u_j-u_k+2p]\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ by $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ which is explicitly

$$\begin{aligned}
&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)\\
&=\prod_{1\le j<k\le N}[u_j-u_k+2p]\sum_{\sigma\in S_N}\prod_{1\le j<k\le N}\frac{1}{[u_{\sigma(k)}-u_{\sigma(j)}]}\prod_{j=1}^{N}\prod_{k=\overline{x_j}+1}^{M}[-u_{\sigma(j)}+v_k+q_k+p]\\
&\quad\times\prod_{j=1}^{N}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}}\\
&\quad\times\prod_{j=1}^{N}[-u_{\sigma(j)}+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^{N}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma(j)}-v_k-p+q_k].
\end{aligned}\tag{4.21}$$

We show that the function $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ satisfies Properties (1), (2), (3) and (4) in Proposition 4.1.

We show Property (1). First, note that when $\overline{x_N}=M$, the factors depending on $v_M$ in each summand in $G_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)$ are

$$[-u_{\sigma(N)}+v_M+h+p+q_M+2\overline{q_{M-1}}]\prod_{j=1}^{N-1}[-u_{\sigma(j)}+v_M+q_M+p]:=f_\sigma(v_M). \tag{4.22}$$

Using (2.3) and (2.4), we can calculate the quasi-periodicities of the function $f_\sigma(v_M)$

$$\begin{aligned}&f_\sigma(v_M+1)\\=&(-1)^N f_\sigma(v_M),\end{aligned} \tag{4.23}$$

$$\begin{aligned}&f_\sigma(v_M-i\log(\mathbf{q})/\pi)\\=&(-\mathbf{q}^{-1})^N\exp\Bigg(-2\pi i\Bigg(Nv_M+h+2\overline{q_{M-1}}+Nq_M+Np-\sum_{j=1}^N u_j\Bigg)\Bigg)f_\sigma(v_M).\end{aligned} \tag{4.24}$$

The quasi-periodicities (4.23) and (4.24) are independent of the permutation $\sigma$, from which one finds the same quasi-periodicities hold for the elliptic function $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$

$$\begin{aligned}&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M+1|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\=&(-1)^N\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h),\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M-i\log(\mathbf{q})/\pi|\overline{x_1},\dots,\overline{x_{N-1}},M|h)\\=&(-\mathbf{q}^{-1})^N\exp\Bigg(-2\pi i\Bigg(Nv_M+h+2\overline{q_{M-1}}+Nq_M+Np-\sum_{j=1}^N u_j\Bigg)\Bigg)\\&\times\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h).\end{aligned} \tag{4.26}$$

The quasi-periodicities (4.25) and (4.26) for $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_{N-1}},M|h)$ are exactly the same with the ones for the dual wavefunctions (4.16) and (4.17), hence Property (1) is proved.

Property (2) can be easily proved by noting that the elliptic symmetric functions $\overline{S_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ which constructs a part of the function $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ is symmetric with respect to $u_1,\dots,u_N$

$$\begin{aligned}&S_{M,N}(u_1,\dots,u_N|v_1,\dots,v_M|x_1,\dots,x_N|h)\\=&S_{M,N}(u_{\sigma(1)},\dots,u_{\sigma(N)}|v_1,\dots,v_M|x_1,\dots,x_N|h),\end{aligned} \tag{4.27}$$

since it is a symmetric function with symmetric variables $u_1,\dots,u_N$, and the other factor $\prod_{1\le j<k\le N}[u_j-u_k+2p]$ which constructs $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ satisfies

$$\begin{aligned}&\prod_{\substack{1\le j<k\le N\\ \sigma(j)>\sigma(k)}}[u_{\sigma(j)}-u_{\sigma(k)}+2p]\prod_{1\le j<k\le N}[u_j-u_k+2p]\\=&\prod_{\substack{1\le j<k\le N\\ \sigma(j)>\sigma(k)}}[u_{\sigma(k)}-u_{\sigma(j)}+2p]\prod_{1\le j<k\le N}[u_{\sigma(j)}-u_{\sigma(k)}+2p].\end{aligned} \tag{4.28}$$

Next we show Property (3). We first treat the case $x_N = M$. The factors $\prod_{j=1}^{N-1}[-u_{\sigma(j)} + v_M + q_M + p]$ in each summand in $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ means that we only have to deal with the summands satisfying $\sigma(N) = N$ in (3.1) which survive after the substitution $v_M = u_N - p - q_M$. Then we find that $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ evaluated at $v_M = u_N - p - q_M$ can be expressed using the symmetric group $S_{N-1}$ where every $\sigma' \in S_{N-1}$ satisfies $\{\sigma'(1),\cdots,\sigma'(N-1)\} = \{1,\cdots,N-1\}$ as follows:

$$\begin{aligned}
&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)|_{v_M=u_N-p-q_M} \\
&= \prod_{1\le j<k\le N-1}[u_j - u_k + 2p]\prod_{j=1}^{N-1}[u_j - u_N + 2p] \\
&\times \sum_{\sigma'\in S_{N-1}}\prod_{1\le j<k\le N-1}\frac{1}{[u_{\sigma'(k)} - u_{\sigma'(j)}]}\prod_{j=1}^{N-1}\frac{1}{[u_N - u_{\sigma'(j)}]} \\
&\times \prod_{j=1}^{N-1}\prod_{k=\overline{x_j}+1}^{M-1}[-u_{\sigma'(j)} + v_k + q_k + p]\prod_{j=1}^{N-1}[-u_{\sigma'(j)} + u_N] \\
&\times \frac{[h+2(N-1)p]^{1/2}[2p]^{1/2}[2q_M]^{1/2}}{[h+2Np]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2\overline{q_M}]^{1/2}}\prod_{j=1}^{N-1}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}} \\
&\times [h+2\overline{q_{M-1}}]\prod_{j=1}^{N-1}[-u_{\sigma'(j)} + v_{\overline{x_j}} + h + p + q_{\overline{x_j}} + 2\overline{q_{\overline{x_j}-1}}] \\
&\times \prod_{j=1}^{N-1}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma'(j)} - v_k - p + q_k]\prod_{k=1}^{M-1}[u_N - v_k - p + q_k].
\end{aligned} \tag{4.29}$$

After appropriately cancelling and rearraning the expression above, we find that (4.29) can be rewritten as

$$\begin{aligned}
&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)|_{v_M=u_N-p-q_M} \\
&= \frac{[2p]^{1/2}[2q_M]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2(N-1)p]^{1/2}}{[h+2\overline{q_M}]^{1/2}[h+2Np]^{1/2}}\prod_{j=1}^{N-1}[u_j - u_N + 2p]\prod_{j=1}^{M-1}[u_N - v_j - p + q_j] \\
&\times \prod_{1\le j<k\le N-1}[u_j - u_k + 2p]\sum_{\sigma'\in S_{N-1}}\prod_{1\le j<k\le N-1}\frac{1}{[u_{\sigma'(k)} - u_{\sigma'(j)}]} \\
&\times \prod_{j=1}^{N-1}\prod_{k=\overline{x_j}+1}^{M-1}[-u_{\sigma'(j)} + v_k + q_k + p]\prod_{j=1}^{N-1}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}} \\
&\times \prod_{j=1}^{N-1}[-u_{\sigma'(j)} + v_{\overline{x_j}} + h + p + q_{\overline{x_j}} + 2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^{N-1}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma'(j)} - v_k - p + q_k].
\end{aligned} \tag{4.30}$$

Since

$$\prod_{1\le j<k\le N-1}[u_j-u_k+2p]\sum_{\sigma'\in S_{N-1}}\prod_{1\le j<k\le N-1}\frac{1}{[u_{\sigma'(k)}-u_{\sigma'(j)}]}$$
$$\times\prod_{j=1}^{N-1}\prod_{k=\overline{x_j}+1}^{M-1}[-u_{\sigma'(j)}+v_k+q_k+p]\prod_{j=1}^{N-1}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}}$$
$$\times\prod_{j=1}^{N-1}[-u_{\sigma'(j)}+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^{N-1}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma'(j)}-v_k-p+q_k]$$
$$=\overline{G_{M-1,N-1}}(u_1,\dots,u_{N-1}|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_{N-1}}|h),\tag{4.31}$$

one finds that (4.30) is the following relation between the elliptic function $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$

$$\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)|_{v_M=u_N-p-q_M}$$
$$=\frac{[2p]^{1/2}[2q_M]^{1/2}[h+2\overline{q_{M-1}}]^{1/2}[h+2(N-1)p]^{1/2}}{[h+2\overline{q_M}]^{1/2}[h+2Np]^{1/2}}\prod_{j=1}^{N-1}[u_j-u_N+2p]\prod_{j=1}^{M-1}[u_N-v_j-p+q_j]$$
$$\times\overline{G_{M-1,N-1}}(u_1,\dots,u_{N-1}|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_{N-1}}|h),\tag{4.32}$$

which is exactly the same with the one (4.2) for the dual wavefunctions of the elliptic Felderhof model $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$, hence we have proved Property (3) for the case $\overline{x_N}=M$.

Next, we show Property (3) for the case $\overline{x_N}\neq M$, which is much simpler to show. We rewrite $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ as

$$\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$$
$$=\prod_{1\le j<k\le N}[u_j-u_k+2p]\sum_{\sigma\in S_N}\prod_{1\le j<k\le N}\frac{1}{[u_{\sigma(k)}-u_{\sigma(j)}]}\prod_{j=1}^{N}\prod_{k=\overline{x_j}+1}^{M-1}[-u_{\sigma(j)}+v_k+q_k+p]$$
$$\times\prod_{j=1}^{N}[-u_{\sigma(j)}+v_M+q_M+p]\prod_{j=1}^{N}\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}}$$
$$\times\prod_{j=1}^{N}[-u_{\sigma(j)}+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^{N}\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma(j)}-v_k-p+q_k],\tag{4.33}$$

and use the identity

$$\prod_{j=1}^{N}[-u_{\sigma(j)}+v_M+q_M+p]=\prod_{j=1}^{N}[-u_j+v_M+q_M+p],\tag{4.34}$$

to take this factor out of the sum in (4.33) to get

$$
\begin{aligned}
&\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h) \\
=&\prod_{j=1}^N[-u_j+v_M+q_M+p]\prod_{1\le j<k\le N}[u_j-u_k+2p]\sum_{\sigma\in S_N}\prod_{1\le j<k\le N}\frac{1}{[u_{\sigma(k)}-u_{\sigma(j)}]} \\
&\times\prod_{j=1}^N\prod_{k=\overline{x_j}+1}^{M-1}[-u_{\sigma(j)}+v_k+q_k+p]\prod_{j=1}^N\frac{[h+2(j-1)p]^{1/2}[2p]^{1/2}[2q_{\overline{x_j}}]^{1/2}}{[h+2jp]^{1/2}[h+2\overline{q_{\overline{x_j}-1}}]^{1/2}[h+2\overline{q_{\overline{x_j}}}]^{1/2}} \\
&\times\prod_{j=1}^N[-u_{\sigma(j)}+v_{\overline{x_j}}+h+p+q_{\overline{x_j}}+2\overline{q_{\overline{x_j}-1}}]\prod_{j=1}^N\prod_{k=1}^{\overline{x_j}-1}[u_{\sigma(j)}-v_k-p+q_k] \\
=&\prod_{j=1}^N[-u_j+v_M+q_M+p]\overline{G_{M-1,N}}(u_1,\dots,u_N|v_1,\dots,v_{M-1}|\overline{x_1},\dots,\overline{x_N}|h),
\end{aligned}
\tag{4.35}
$$

which is exactly the same factorization (4.3) with the dual wavefunctions of the elliptic Felderhof model $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ for the case $\overline{x_N}\neq M$.

What remains to show is that $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ satisfies Property (4), which is easy to check from the definition of the elliptic function (4.21). □

Since we have shown Proposition 4.2, we find that the elliptic function $\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$ is nothing but the explicit form of the dual wavefunctions $\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)$

$$
\overline{W_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h)=\overline{G_{M,N}}(u_1,\dots,u_N|v_1,\dots,v_M|\overline{x_1},\dots,\overline{x_N}|h),
\tag{4.36}
$$

hence (3.8) in Theorem 3.2 is proved.

# 5 Dual Cauchy formula for elliiptic Schur functions

Theorem 3.2 gives the correspondence between the wavefunctions and the dual wavefunctions of the elliptic Felderhof model and the elliptic Schur functions. We combine Theorem 3.2 and the factorized form of the domain wall boundary partition functions found by Foda-Wheeler-Zuparic [43] to derive the dual Cauchy formula for the elliptic Schur functions, following the idea of Bump-McNamara-Nakasuji [29] in which way they derived the dual Cauchy formula for the factorial Schur functions.

We derive the identity by two ways of evaluations of the domain wall boundary partition functions [46]. The domain wall boundary partition functions $Z_N(u_1,\dots,u_N|v_1,\dots,v_N|h)$ can be regarded as a special case of the wavefunctions or the dual wavefunctions

$$
\begin{aligned}
Z_N(u_1,\dots,u_N|v_1,\dots,v_N|h)&=W_{N,N}(u_1,\dots,u_N|v_1,\dots,v_N|1,\dots,N|h) &(5.1)\\
&=\overline{W_{N,N}}(u_1,\dots,u_N|v_1,\dots,v_N|1,\dots,N|h). &(5.2)
\end{aligned}
$$

A factorized expression for the domain wall boundary partition functions of the elliptic Felderhof model was found in [43].

**Theorem 5.1.** *(Foda-Wheeler-Zuparic [43]) The domain wall boundary partition functions of the elliptic Felderhof model $Z_N(u_1,\dots,u_N|v_1,\dots,v_N|h)$ has the following factorized form*

$$Z_N(u_1,\dots,u_N|v_1,\dots,v_N|h) = \frac{[h+\sum_{j=1}^N v_j - \sum_{j=1}^N u_j + Np + \sum_{j=1}^N q_j]}{[h+2\sum_{j=1}^N q_j]^{1/2}[h+2Np]^{1/2}} \times \prod_{j=1}^N [2p]^{1/2}[2q_j]^{1/2} \prod_{1\le j<k\le N} [u_j-u_k+2p][v_k-v_j+q_j+q_k]. \tag{5.3}$$

We now apply the idea of Bump-McNamara-Nakasuji [29] to the elliptic Felderhof model to derive the following dual Cauchy formula for the elliptic Schur functions by combining Theorem 3.2 and Theorem 5.1.

**Theorem 5.2.** *We have the following dual Cauchy formula for the elliptic Schur functions*

$$\begin{aligned}\sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}} & S_{n+m,n}(u_1,\dots,u_n|v_1,\dots,v_{n+m}|x_1,\dots,x_n|h) \\ & \times \overline{S_{n+m,m}}(w_1,\dots,w_m|v_1,\dots,v_{n+m}|\overline{x_1},\dots,\overline{x_m}|h+2np) \\ & = \frac{[h+\sum_{j=1}^{n+m} v_j - \sum_{j=1}^n u_j - \sum_{j=1}^m w_j + (n+m)p + \sum_{j=1}^{n+m} q_j]}{[h+2\sum_{j=1}^{n+m} q_j]^{1/2}[h+2(n+m)p]^{1/2}} \\ & \quad \times \prod_{j=1}^{n+m}[2p]^{1/2}[2q_j]^{1/2} \prod_{1\le j<k\le n+m}[v_k-v_j+q_j+q_k]\prod_{j=1}^n\prod_{k=1}^m[u_j-w_k+2p].\end{aligned} \tag{5.4}$$

*Here, the sum* $\sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}}$ *means that we take the sum over $x=\{x_1,\dots,x_n\}$ $(1\le x_1< x_2<\cdots<x_n\le n+m)$ and $\overline{x}=\{\overline{x_1}\dots\overline{x_m}\}$ $(1\le\overline{x_1}<\overline{x_2}<\cdots<\overline{x_m}\le n+m)$ which forms a disjoint union of $\{1,2,\dots,n+m\}$, $x\sqcup\overline{x}=\{1,2\dots,n+m\}$.*

*Proof.* We evaluate the domain wall boundary paritition functions $Z_{n+m}(u_1,\dots,u_n,w_1,\dots,w_m|v_1,\dots,v_{n+m}|h)$ by inserting the completeness relation

$$\sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}} |\overline{x_1}\cdots\overline{x_m}\rangle\langle x_1\cdots x_n| = \mathrm{Id}, \tag{5.5}$$

and using the correspondence between the wavefunctions, dual wavefunctions and the elliptic

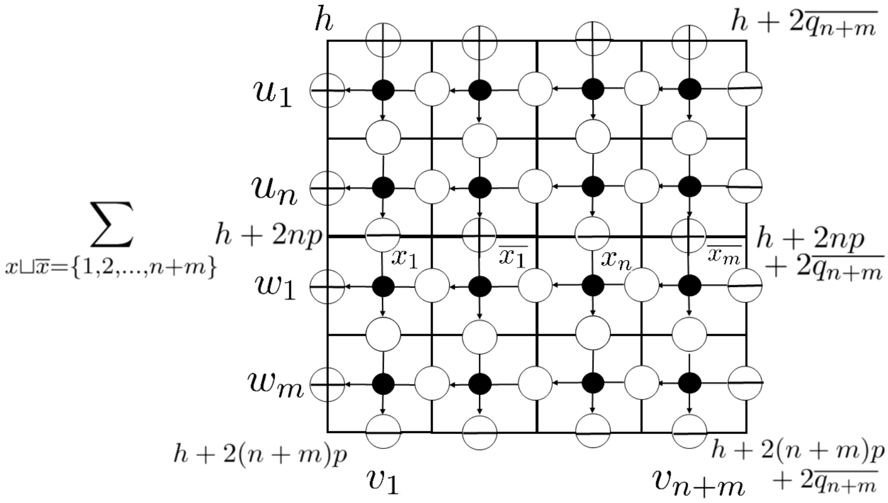


Figure 9: The decomposition of the domain wall boundary partition functions into the wavefunctions and the dual wavefunctions (5.6).

Schur functions (3.7), (3.8) in Theorem 3.2 as

$$
\begin{aligned}
& Z_{n+m}(u_1,\dots,u_n,w_1,\dots,w_m|v_1,\dots,v_{n+m}|h) \\
=& {}_{n+m}\langle\overline{\Omega}|B(w_m|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2(n+m-1)p) \\
& \times\cdots\times B(w_1|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2np) \\
& \times B(u_n|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2(n-1)p) \\
& \times\cdots\times B(u_1|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h)|\Omega\rangle_{n+m}, \\
=& \sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}} {}_{n+m}\langle\overline{\Omega}|B(w_m|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2(n+m-1)p) \\
& \times\cdots\times B(w_1|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2np)|\overline{x_1}\cdots\overline{x_m}\rangle \\
& \times\langle x_1\cdots x_n|B(u_n|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h+2(n-1)p) \\
& \times\cdots\times B(u_1|v_1,\dots,v_{n+m}|p|q_1,\dots,q_M|h)|\Omega\rangle_{n+m}, \\
=& \sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}} W_{n+m,n}(u_1,\dots,u_n|v_1,\dots,v_{n+m}|x_1,\dots,x_n|h) \\
& \times\overline{W_{n+m,m}}(w_1,\dots,w_m|v_1,\dots,v_{n+m}|\overline{x_1},\dots,\overline{x_m}|h+2np) \\
=& \sum_{x\sqcup\overline{x}=\{1,2,\dots,n+m\}} \prod_{1\le j<k\le n}[u_j-u_k+2p]S_{n+m,n}(u_1,\dots,u_n|v_1,\dots,v_{n+m}|x_1,\dots,x_n|h) \\
& \times\prod_{1\le j<k\le m}[w_j-w_k+2p]\overline{S_{n+m,m}}(w_1,\dots,w_m|v_1,\dots,v_{n+m}|\overline{x_1},\dots,\overline{x_m}|h+2np).
\end{aligned}
\tag{5.6}
$$

See Figure 9 for a graphical description of the decomposition (5.6). The dual Cauchy formula for the elliptic Schur functions (5.4) follows from comparing (5.6) with the factorized form

$$\begin{aligned}
&Z_{n+m}(u_1,\dots,u_n,w_1,\dots,w_m|v_1,\dots,v_{n+m}|h)\\
=&\frac{[h+\sum_{j=1}^{n+m}v_j-\sum_{j=1}^{n}u_j-\sum_{j=1}^{m}w_j+(n+m)p+\sum_{j=1}^{n+m}q_j]}{[h+2\sum_{j=1}^{n+m}q_j]^{1/2}[h+2(n+m)p]^{1/2}}\\
&\times\prod_{j=1}^{n+m}[2p]^{1/2}[2q_j]^{1/2}\prod_{1\le j<k\le n+m}[v_k-v_j+q_j+q_k]\prod_{j=1}^{n}\prod_{k=1}^{m}[u_j-w_k+2p]\\
&\times\prod_{1\le j<k\le n}[u_j-u_k+2p]\prod_{1\le j<k\le m}[w_j-w_k+2p],
\end{aligned}\tag{5.7}$$

which is a specialization of Theorem 5.2.

□

## Acknowledgements

This work was partially supported by grant-in-Aid for Scientific Research (C) No. 16K05468.

## References

[1] Bethe, H. *Z. Phys.*, 71, 205 (1931)

[2] Faddeev, L.D. Sklyanin, E.K, and Takhtajan, L.A. *Theor. Math. Phys.* 40, 194 (1979)

[3] Baxter, R.J. *Exactly Solved Models in Statistical Mechanics* (Academic Press, London) (1982)

[4] Korepin, V.E. Bogoliubov, N.M. and Izergin, A.G. *Quantum Inverse Scattering Method and Correlation functions* (Cambridge University Press, Cambridge) (1993)

[5] Drinfeld, V. *Sov. Math. Dokl.* 32, 254 (1985)

[6] Jimbo, M. *Lett. Math. Phys.* 10, 63 (1985)

[7] Bogoliubov, N.M. *J. Phys. A: Math. Gen.* 38, 9415 (2005)

[8] Betea, D., Wheeler, M. and Zinn-Justin, P. *J. Alg. Comb.* 42, 555 (2015)

[9] Betea, D. and Wheeler, M. *J. Comb. Th. Ser. A* 137, 126 (2016)

[10] Wheeler, M. and Zinn-Justin, P. *Adv. Math.* 299, 543 (2016)

[11] van Diejen, J.F. and Emsiz, E. *Commun. Math. Phys.* 350, 1017 (2017)

[12] Motegi, K. and Sakai, K. *J. Phys. A: Math. Theor.* 46, 355201 (2013)

[13] Motegi, K. *J. Math. Phys.* 58, 091703 (2017)

[14] Motegi, K. and Sakai, K. *J. Phys. A: Math. Theor.* 47, 445202 (2014)

[15] Korff, C. and Stroppel, C. *Adv. Math.* 225, 200 (2010)

[16] Korff, C. *Lett. Math. Phys.* 104, 771 (2014)

[17] Borodin, A. *Adv. in Math.* 306, 973 (2017)

[18] Borodin, A. and Petrov, L. *Sel. Math. New Ser.* 1 (2016)

[19] Takeyama, Y. *J. Phys. A* 47, 465203 (2014)

[20] Takeyama, Y. arXiv:1606.00578

[21] Brubaker, B. Bump, D. and Friedberg, S. *Commun. Math. Phys.* 308, 281 (2011)

[22] Tokuyama, T. *J. Math. Soc. Japan* 40, 671 (1988)

[23] Okada, S. *J. Algebraic Comb.* 2, 155 (1993)

[24] Hamel, A. and King, R.C. *J. Algebraic Comb.* 16, 269 (2002)

[25] Hamel, A. and King, R.C. *J. Algebraic Comb.* 21, 395 (2005)

[26] Ivanov, D. *Symplectic Ice.* in: Multiple Dirichlet series, $L$-functions and automorphic forms, vol 300 of Progr. Math. Birkhäuser/Springer, New York, 205-222 (2012)

[27] Brubaker, B. Bump, D. Chinta, G. and Gunnells, P.E. *Metaplectic Whittaker Functions and Crystals of Type B.* in: Multiple Dirichlet series, $L$-functions and automorphic forms, vol 300 of Progr. Math. Birkhäuser/Springer, New York, 93-118 (2012)

[28] Tabony, S.J. *Deformations of characters, metaplectic Whittaker functions and the Yang-Baxter equation,* PhD. Thesis, Massachusetts Institute of Technology, USA (2011)

[29] Bump, D, McNamara, P. and Nakasuji, M. *Comm. Math. Univ. St. Pauli* 63, 23 (2014)

[30] Hamel, A.M. and King, R.C. *Elect. J. Comb.* 22, 2, P2.42 (2015)

[31] Brubaker, B. and Schultz, A. *J. Alg. Comb.* 42, 917 (2015)

[32] Brubaker, B. Buciumas, V. and Bump, D. arXiv:1604.02206

[33] Motegi, K. *Lett. Math. Phys.* 107, 1235 (2017)

[34] Perk, J-H-H. and Schultz, C.L. *Phys. Lett. A* 84, 407 (1981)

[35] Yamane, H. *Publ. Res. Inst. Math. Sci.* 35, 321 (1999)

[36] Felderhof, B. *Physica* 65, 421 (1973)

[37] Murakami, J. *Infinite analysis. Adv. Ser. Math. Phys.* 16B, 765 (1991)

[38] Deguchi, T. and Akutsu, Y. *J. Phys. Soc. Jpn.* 62, 19 (1993)

[39] Zhao, S-Y. and Zhang, Y-Z. *J. Math. Phys.* 48, 023504 (2007)

[40] Foda, O. Caradoc, A.D. Wheeler, M. and Zuparic, M.L. *J. Stat. Mech.* 0703, P03010 (2007)

[41] Zhao, S-Y. Yang, W-L. and Zhang, Y-Z. *J. Math. Phys.* 47, 013302 (2006)

[42] Deguchi, T. and Martin, P. *Int. J. Mod. Phys. A* 7, Suppl. 1A, 165 (1992)

[43] Foda, O. Wheeler, M. and Zuparic, M. *J. Stat. Mech.* P02001 (2008)

[44] Motegi, K. arXiv:1704.07035, Progress of Theoretical and Experimental Physics.

[45] Motegi, K. Symmetric functions and wavefunctions of the six-vertex model and elliptic Felderhof model, preprint.

[46] Korepin, V.E. *Commun. Math. Phys.* 86, 391 (1982)

[47] Izergin, A. *Sov. Phys. Dokl.* 32, 878 (1987)

[48] Schlosser, M. *J. Comb. Th. A.* 114, 505 (2007)

[49] Noumi, M. http://www.lorentzcenter.nl/lc/web/2010/423/presentations/Noumi.pdf

[50] Noumi, M. http://www.lorentzcenter.nl/lc/web/2013/541/presentations/Noumi.pdf

[51] Pakuliak, S. Rubtsov, V. and Silantyev, A. *J. Phys. A:Math. Theor.* 41, 295204 (2008)

[52] Felder, G. and Schorr, A. *J. Phys. A: Math.Gen.* 32, 8001 (1999)

[53] Rosengren, H. *Adv. Appl. Math.* 43, 137 (2009)

[54] Yang, W-L. and Zhang, Y-Z. *J. Math. Phys.* 50, 083518 (2009)

[55] Andrews, G.E. Baxter, R.J. and Forrester, P.J. *J. Stat. Phys.* 35, 193 (1984)