Convex MINLP formulations

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More details are given in the paper Kronqvist and Misener (2020), and a preprint is available from http://www.optimization-online.org/DB_HTML/2020/08/7957.html

1 Big-M formulation

Using the big-M formulation the problem can be written as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{m} d_k^{i,j} & \\
& d_k^{i,j} \geq p_k^i - p_k^j & \forall k \in D,\ \forall i \in P,\ \forall j \in P^i, \\
& d_k^{i,j} \geq p_k^j - p_k^i & \forall k \in D,\ \forall i \in P,\ \forall j \in P^i, \\
& \sum_{k=1}^{m} \left(p_k^i - c_k^r\right)^2 \leq 1 + M_r(1 - b_{i,r}) & \forall i \in P,\ \forall r \in B, \\
& \sum_{r=1}^{l} b_{i,r} = 1 & \forall i \in P, \\
& \sum_{i=1}^{n} b_{i,r} \leq 1 & \forall r \in B, \\
& p_1^i \leq p_1^{i+1} & \forall i \in P \setminus n, \\
& b_{i,r} \in \{0,1\} & \forall i \in P,\ \forall r \in B, \\
& \mathbf{d}^{i,j} \in [0,10]^m & \forall i \in P,\ \forall j \in P^i, \\
& \mathbf{p}^i \in [0,10]^m & \forall i \in P,
\end{aligned}
\tag{1}
$$

where M_r are sufficiently large constants. For these problems the smallest valid M_r is simply given by

$$
M_r = \max_{i \in B} \left\{ \left(\left\| \mathbf{c}^r - \mathbf{c}^i \right\|_2 + 1 \right)^2 - 1 \right\}. \tag{2}
$$

Here, $\mathbf{c}^i \in R^m$ denotes the center of ball i and c_1^i refers to the first coordinate of the center. Similarly, $\mathbf{p}^i \in R^m$ refers to point i and p_1^i is the first coordinate of the point. To simplify the notation, we use the sets $D = \{1, 2, \ldots, m\}$, $P = \{1, 2, \ldots, n\}$, $P^i = \{i+1, i+2, \ldots, n\}$, and $B = \{1, 2, \ldots, l\}$. The binary variable $b_{i,r}$ selects if point i is assigned to ball r. The ℓ_1-distance is represented by linear constraints and by the auxiliary variables $\mathbf{d}^{i,j} \in R^m$. As before, $d_k^{i,j}$ refers to the k-th component of the vector $\mathbf{d}^{i,j}$. The problem formulation also contains an ordering constraint of the points along the first coordinate to reduce symmetries.

2 Convex hull formulation

Using the convex hull formulation presented by Sawaya and Grossmann (2007), the problem can then be formulated as

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{m} d_k^{i,j} \\
& d_k^{i,j} \geq p_k^i - p_k^j && \forall k \in D,\ \forall i \in P,\ \forall j \in P^i, \\
& d_k^{i,j} \geq p_k^j - p_k^i && \forall k \in D,\ \forall i \in P,\ \forall j \in P^i, \\
& ((1-\epsilon)b_{i,r} + \epsilon) \left(\left\| \frac{\nu_r^{\mathbf{p}^i}}{(1-\epsilon)b_{i,r} + \epsilon} - c^r \right\|_2^2 - 1 \right) \\
& - \epsilon \left(\|c^r\|_2^2 - 1 \right) (1 - b_{i,r}) \leq 0 && \forall i \in P,\ \forall r \in B, \\
& \sum_{r=1}^{l} b_{i,r} = 1 && \forall i \in P, \\
& \sum_{i=1}^{n} b_{i,r} \leq 1 && \forall r \in B, \\
& p_1^i \leq p_1^{i+1} && \forall i \in P \setminus n, \\
& \sum_{r=1}^{l} \nu_r^{\mathbf{p}^i} = \mathbf{p}^i && \forall i \in P \\
& \mathbf{0} \leq \nu_r^{\mathbf{p}^i} \leq \mathbf{10} \cdot b_{i,r} && \forall i \in P, \forall r \in B, \\
& b_{i,r} \in \{0, 1\} && \forall i \in P,\ \forall r \in B, \\
& \mathbf{d}^{i,j} \in [0, 10]^m && \forall i \in P,\ \forall j \in P^i, \\
& \mathbf{p}^i \in [0, 10]^m && \forall i \in P, \\
& \nu_r^{\mathbf{p}^i} \in [0, 10]^m && \forall i \in P,\ \forall r \in B.
\end{aligned}
\tag{3}
$$

Here we are using the same notation as in the big-M formulation, and the variables $\nu_r^{\mathbf{p}^i} \in R^m$ are "copies" of the variables $\mathbf{p}^i$. Here we use $\epsilon = 10^{-9}$ to avoid numerical difficulties.