STUDY GROUP

p-ADIC GEOMETRY - FOUR DIFFERENT APPROACHES

SUMMER TERM 2021

The aim of p-adic geometry is, simply put, to give meaning to the notion of "spaces" over non-archimedean fields, or more generally over rings endowed with a topology. To make this more precise, one can compare with complex geometry: Instead of considering varieties of $\mathbb{C}$ as schemes, one can instead see these as complex analytic spaces (or just complex manifolds, if the varieties are smooth) and make use of the euclidean topology. This turns out to be quite fruitful, as not only these two concepts are closely intertwined, but we can make stronger statements about the algebraic nature of these varieties using transcendental methods. Next to Hodge theory, an infamous example is:

Theorem. *There is an equivalence of categories*

$$\left\{ \textit{Elliptic curves over } \mathbb{C} \right\} \longrightarrow \left\{ \textit{Complex tori } \mathbb{C}/\Lambda \right\}.$$

In fact this Theorem was an initial motivation: Is it possible to classify elliptic curves over e.g. $\overline{\mathbb{Q}_p}$?
This led to the notion of rigid-analytic spaces, one of the four approaches that we will encounter.
It is worthwhile to note that the idea of a "p-adic manifold" is far more limited than its real/complex analogue. A core issue is that the topology of non-archimedean fields is too "wild", for example $\mathbb{Q}_p$ is a totally disconnected space. This results in difficulties of building such a theory of p-adic "spaces".

As such, there is also no universally correct approach- each one comes with advantages and obstacles. While for example the theory of rigid-analytic spaces was the initial approach, nowadays it has been largely overtaken by Berkovich and adic spaces. Formal schemes play an important role in any case and are widely used (also beyond the applications we are interested in). The goal of this seminar is to first understand each of the four approaches, but also trying to connect them to each other. After all, each of them is trying to accomplish a similar goal.

Let us add a few remarks about the talks. Sometimes we have given multiple references, although that list is by no means exhaustive. All talks are flexible, especially the more advanced ones. Some of the talks are partly independent of each other and cover for example topics of more geometric or arithmetic flavour.

Talk 1: Introduction. Date
Give some motivation behind p-adic geometry. Sketch what kind of properties one would like to acquire from such a theory by comparison to complex geometry. Outline the different approaches and mention applications like Tate's uniformization of elliptic curves, perfectoid spaces, mirror symmetry via Berkovich spaces, Bruhat-Tits buildings via Berkovich spaces, p-adic modular forms etc.

Talk 2: Rings with topologies. Date
This talk should serve as an introduction to the objects that we will be dealing with. This includes groups and rings with topology, topologies induced by an ideal, but also things like bounded, power-bounded, topologically nilpotent elements. This is covered in detail in [9, Section II.1], but it is by no means necessary to go into all the technicalities. Further digressions might include things like completed tensor product. Complement this with [8, Chapter 5].

Talk 3: Rigid-analytic spaces I. Date
Give basic definitions and examples: Tate algebras and more generally affinoid algebras, see [1, Chapter 3] and [2, Chapter 3.1]. State and explain results like Weierstraß Preparation Theorem, Noether normalization and the maximum modulus principle, e.g. [2, Corollary 9-11, Theorem 15]. Give examples to motivate why we need to work with the maximal spectrum (see [2, p.61-63]).

Talk 4: Rigid-analytic spaces II. Date
The purpose of this talk is to globalize the construction in the previous section. This means first defining the G-topology via admissible opens, see [1, Chapter 5+6] or [2, Chapter 5.1]. Here Tate's acyclicity theorem is an essential building block, this is presented in [1, Chapter 7] and in more detail in [2, Chapter 4.3]. As a consequence one can define global rigid-analytic spaces as in [1, Chapter 8] and [2, Chapter 5.3]. If time permits, also discuss coherent sheaves on it.

Talk 5: Rigid-analytic spaces III. Date
This talk consists of two parts covering further advanced topics. In the first part discuss the analytification functor, e.g. [2, Section 5.4]. In the second part sketch ideas for Tate's uniformization of elliptic curve, e.g. [1, Chapter 9]. Optional: Talk about Drinfeld's upper halfplane [3].

Talk 6: Berkovich spaces I. Date
Introduce seminorms and the associated Berkovich spectrum $M(A)$. Discuss the four kinds of points for the affine line $\mathbb{A}^1_{Berk}$. This is covered in the first few chapters of [4]. For the unit disk example one may also slightly modify [11, Section 11.2]). Also [6, Chapter I] is a great resource for this. After discussing some more fundamental properties, sketch the proof of the non-archimedean version of Picard's theorem as in the last 2 pages of [4].

Talk 7: Berkovich spaces II. Date
Discuss the topology on Berkovich spaces and mention difficulties in gluing these together. Nonetheless one can describe analyifications. Introduce classes of maps, such as closed immersions, seperated and proper maps [5, Section 5]. This can be complemented with some topological properties of Berkovich spaces such as local path-connectedness and topological/Shilov boundary. This is covered again in [5,

Section 5] (see in particular Exercise 4.5.1), but also [4]. If time permits, one can also talk about GAGA theorems.

Talk 8: Berkovich spaces III. Date
This talk is more open and can be adjusted according to the speaker's preference. For example, one could introduce analytic curves, skeletons and some pictures visualising them, see [7, Section 3].

Talk 9: Formal schemes. Date
Give basic definitions and examples [2, Section II.7]. Discuss how formal schemes are related to aforementioned constructions via Raynaud's generic fiber functor. Closely related are also formal blow-ups and formal models. This is covered in [5, Section 3.3] and [2, Section II.8.2]. Here one can finish with the statement that the localisation of the class of morphisms of blow-ups at the category of formal schemes yields rigid-analytic spaces.

Talk 10: Adic spaces I. Date
Give basic definitions and examples [8, Chapter 7 + 8]. Afterwards discuss two examples: Type V points that appear for the adic unit disk (but not in the Berkovich unit disk), e.g. [9, III.5.2] or [11, p. 11]. Then present the example $\mathrm{Spa}(\mathbb{Z}_p[[T]])$ [10, Lecture 4].

Talk 11: Adic spaces II. Date
The purpose of this talk is to define the category of adic spaces, but this comes with a few complications: First address sheafiness issues and sketch how to circumvent these (via restricting to certain classes of Huber rings, e.g. stably uniform or strongly noetherian) as in [8, Section 8.2]. Afterwards give examples of fiber products as in [9, II.3.2] and [10, Lecture 4] illustrating the problems that arise. Here one could also finish with the two constructions of $\mathbb{P}^1$: Gluing affine spaces and gluing unit disks.

Talk 12: Adic spaces III. Date
This talk mostly concerns perfectoid spaces and their tilts [10, Section 6.1+6.2]. Further interesting results are covered in [10, Section 7.4]. Finally, it would be nice to see pro-étale torsors, as they are closely related to perfectoid spaces. As this is rather intricate, it would be enough to cover an example, see [10, Section 8.4].

Talk 13: Comparing spaces I(+II?). Date
This talk (possibly multiple ones) will now link all the different kinds of spaces that were introduced before. This can be centered around Riemann-Zariski spaces (see [1, Section 12] for an overview). Possible connections of interest are for example

- Berkovich spaces, realising the maximal Hausdorff quotients of adic spaces. Carefully stated, this can be upgraded to an equivalence of categories, see [13, Section 8.3]
- Adic spaces, realising the inverse limit of all of its formal models, see [12, Chapter 8]

References

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[2] Siegfried Bosch: *Lectures on Formal and Rigid Geometry* , `https://www.math.purdue.edu/~tongliu/seminar/rigid/Bosch.pdf`

[3] Samit Dasgupta, Jeremy Teitelbaum: *The p-adic upper half plane* ,
`https://www.math.arizona.edu/~swc/aws/2007/DasguptaTeitelbaumNotesMar10.pdf`

[4] Matthew Baker: *Non-Archimedean Geometry*,
`https://people.math.gatech.edu/~mbaker/pdf/BerkCourse.pdf`

[5] Brian Conrad: *Several approaches to non-archimedean geometry*,
`http://math.stanford.edu/~conrad/papers/aws.pdf`

[6] Jérôme Poineau, Daniele Turchetti: *Berkovich curves and Schottky uniformization*,
`https://arxiv.org/pdf/2010.09043.pdf`

[7] Eric Katz, Joseph Rabinoff, David Zureick-Brown: *Diophantine and tropical geometry, and uniformity of rational points on curves* ,
`https://arxiv.org/pdf/1606.09618.pdf`

[8] Torsten Wedhorn: *Adic spaces*,
`https://arxiv.org/pdf/1910.05934.pdf`

[9] Sophie Morel: *Adic spaces*,
`https://web.math.princeton.edu/~smorel/adic_notes.pdf`

[10] Peter Scholze, Jared Weinstein: *Berkeley Lectures on p-adic Geometry*,
`http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf`

[11] Brian Conrad: *Number theory learning seminar, Lecture 11*,
`http://virtualmath1.stanford.edu/~conrad/Perfseminar/Notes/L11.pdf`

[12] Bhargav Bhatt: *Lecture notes for a class on perfectoid spaces*
`http://www-personal.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf`

[13] Roland Huber: *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces*,