An analytic view towards semistable reduction

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The goal of this note is to present a self-contained proof of the classical semistable reduction theorem of Deligne-Mumford using analytic tools from non-archimedean geometry. The most natural geometric backdrop for this is the theory of Berkovich curves.

First we give some informal background on semistable reduction and we outline our approach. The second section discusses some notions of non-archimedean geometry. Finally we outline a proof of the semistable reduction theorem, based on the book in preparation of A. Ducros [10]. The approach has its roots in the rigid-analytic proof of Bosch-Lütkebohmert [5]. Most of the material stems from [8], [10] and [16].

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1 Introduction

Throughout, we let k be discretely valued non-Archimedean complete field with ring of integers k^0, residue field $\widetilde{k}$ of characteristic p. We fix an algebraic closure k^a of k. All extensions of k that we will consider will be complete and equipped with an isometric embedding into $\widehat{k^a}$.

1.1 Models and reduction

In what comes we give some background on semistable reduction for curves. A reader with enough motivation can safely skip to section 1.4, where we outline our strategy of proof.

If X is a smooth projective variety over k, we can scale the coefficients of its defining equations in order to make sure they lie in the ring of integers k^0. Doing so, we arrive at a projective scheme $\mathcal{X}$ over k^0, which we say is a **model** of X over the complete discrete valuation ring k^0. We reserve the use of calligraphic letters for schemes over k^0.

The upshot is that such a model can be reduced modulo the maximal ideal of k^0, yielding a connected (but possibly reduced and reducible) variety $\widetilde{X} := \mathcal{X}_{\widetilde{k}}$ over the residue field $\widetilde{k}$, which is a field of simpler arithmetic; the hope is that $\widetilde{X}$ also has simpler geometry (like components of small genus or small multiplicity) when the model is chosen with care. One can view the theory of **reduction** as an attempt to answer which models an arithmetic variety admits. From a more geometric point of view, we study the behaviour of X in a family of varieties (recall that a flat scheme over k^0 consists of a "continuous" family of two fibers, namely its generic fiber X/k and a special fiber $\mathcal{X}_{\widetilde{k}}$ over $\widetilde{k}$.).

If embedded resolution of singularities is available, one disposes of a regular model by repeatedly blowing-up in the special fiber and normalising. If one can find a model that is moreover smooth, one says X has **good reduction**. Many results in arithmetic geometry deal with finding criterions for good reduction or measuring to which extent good reduction fails.

Example 1. An abelian variety has good reduction if and only if the Galois action on the first l-adic cohomology (the dual of the l-adic Tate module) for $l \neq p$ is unramified (Néron-Ogg-Shafarevich and Serre-Tate). For $l = p$ there is a similar crystalline version. Liedtke and collaborators found a similar criterion for K3 surfaces involving the second l-adic cohomology.

An elliptic curve over k has potential good reduction if and only if the valuation of its j-invariant is nonnegative. For genus 2 curves there is a similar criterion in terms of Igusa invariants. For higher genus curves no such simple criterion exists, although Oda has shown that good reduction is equivalent to unramified Galois action on the arithmetic fundamental group (this can be viewed as a nonabelian Néron-Ogg-Shafarevich criterion).

As only few varieties admit good reduction (even after passing to finite extensions of k), it is important to find nice models which mimic some properties of good reduction. For example, one could allow mild singularities in the special fiber.

In what follows, we we always denote C for a nice (projective, smooth, geometrically connected) curve over k of genus $g \geq 1$.

Definition 2. A **model** $\mathcal{C}$ is a proper relative curve[1] over k^0 with generic fiber C/k.

We call a model $\mathcal{C}$ **semistable** if it is regular and the geometric fibers are semistable, i.e. reduced connected (possibly reducible) projective curves with at worst nodal singularities. We say C has (potential) semistable reduction if a semistable model exists (resp. after passing to a finite extension of k). Having semistable reduction is stable under base change.

Remark 3. A semistable model is thus a **normal crossings model**, i.e. it is regular and the special fiber is a normal crossings divisor. Using Lipman's resolution of singularities for arithmetical surfaces, one finds that every curve C admits a normal crossings model, and after blowing down $(-1)-$curves using Castelnuovo's criterion we find a minimal such model. In fact, one can show that if C has semistable reduction, then its minimal normal crossings model is reduced and therefore semistable.

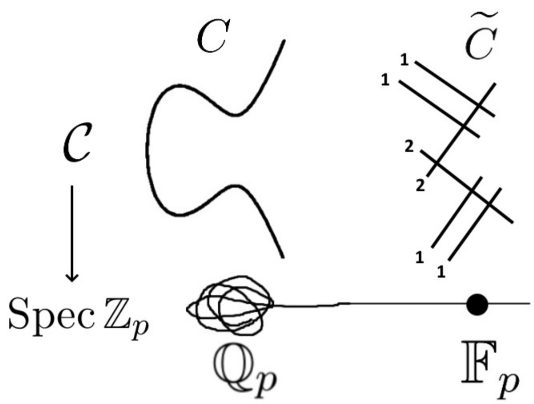

Figure 1: An example of a normal crossings model $\mathcal{C}/\mathbb{Z}_p$ of an elliptic curve $C/\mathbb{Q}_p$, we have indicated the multiplicities of the components of the special fiber $\widetilde{C}$, which are all rational here. The model is in fact minimal because we can no longer blow down (-1)-curves. As the special fiber is not reduced, $C/\mathbb{Q}_p$ is not semistable. However, if $p \neq 2$ and we allow a degree 2 extension of $\mathbb{Q}_p$, the multiplicity 2 components will split into two multiplicity 1 components and we get multiplicative reduction (the special fiber will contain a circle of 4 rational components).

Theorem 4 (Semistable reduction)**.** *Every nice genus ≥ 1 curve C/k is potentially semistable, i.e. admits a semistable model after possibly taking a finite extension of the base field k.*

Example 5. If E/k is an elliptic curve then after a finite extension of k, E will have either multiplicative or good reduction.

[1] i.e. a proper flat k^0-scheme with geometrically connected fibers of pure dimension 1. As C is integral, flatness is equivalent to asking $\mathcal{C}$ is integral. In particular, $\mathcal{C}$ has no irreducible components concentrated in the special fiber.

Note that in any dimension, having a normal crossings model is essentially equivalent to embedded resolution, and the semistable reduction asserts that for curves we can make the special fiber of a normal crossings model reduced after base change. Generalisations of these kinds of statements to all sorts of varieties constitute a very active area of reseach in algebraic geometry.

If the genus is strictly larger than 1, then sometimes a slightly different notion is used, namely that of stable reduction. This is obtained by relaxing the allowed singularities in the special fiber a bit more, by contraction of all the (-2)-curves. Although the resulting model (called the canonical model) is no longer regular, it has ample relative canonical class. Moreover the stable special fiber is a stable curve, i.e. a semistable curve with finite automorphism group. One advantage is that the stable special fiber has a controllable number of components.

from a more classical point of view, the moduli space $\mathcal{M}_g$ of nice (projective, smooth, geometrically connected) curves of genus $g \geq 2$ over k is a connected smooth Deligne-Mumford stack over $\mathbb{Z}$, but it is not proper. The problem is precisely that such curves do not always have potential good reduction. So for geometric applications, like doing intersection theory, one would like to compactify $\mathcal{M}_g$ while still having a geometric interpretion of the boundary. This is accomplished by considering the family of stable curves of genus g, whose functor is representable by a smooth and proper Deligne-Mumford stack $\overline{\mathcal{M}_g}$ containing $\mathcal{M}_g$ as a dense open substack. For this construction it is important that stable curves are stable under base change (that's really where the name comes from), and the stacky valuative criterion of properness for $\overline{\mathcal{M}_g}$ essentially boils down to the (semi)stable reduction theorem.

1.2 A toy case

The following proposition gives a proof of semistable reduction in the case where there exists a normal crossings model where none of the multiplicities are divisible by p (for example, in characteristic zero). A descent argument allows us to suppose that $\tilde{k}$ is seperably closed.

Proposition 6. *Let $\mathcal{C}/k^0$ be a normal crossings model of C/k. Let d be a common multiple of the multiplicities of the components in the special fiber. We suppose that $p \nmid d$.*

Let us denote $k(d) = k[\pi']/(\pi'^d - \pi)$ for the (unique) totally ramified degree d extension of k, and let $k^0(d) = k^0[\pi']/(\pi'^d - \pi)$ be its ring of integers. Finally, denote $\mathcal{C}(d)$ for the normalisation of $\mathcal{C} \times_{k^0} k^0(d)$ over $k^0(d)$. Then $\mathcal{C}(d)$ is a semistable model of C over $k^0(d)$.

Proof. let x be a closed point of the special fiber. Denote A by the completed local ring of x. Then A takes the form

$$A \cong \begin{cases} k^0[[u,v]]/(u^a - \pi) & \text{(1 component through } x) \\ k^0[[u,v]]/(u^a v^b - \pi) & \text{(2 components through } x) \end{cases}$$

where $a, b \geq 1$ are the multiplicities of the components passing through x.

Suppose first that there is one component through x. We seek to normalise

$$A \otimes_{k^0} k^0(d) = R(d)[[u,v]]/(\pi'^d - u^a) = \bigoplus_{\xi \in \mu_a(k^0)} k^0(d)[[u,v]]/(\pi'^{d'} - \xi u)$$

where we denote $d' = d/a$ and $\mu_a(k^0)$ for the a'th roots of unity in k^0, the latter all exists by Hensel's lemma and the fact that $p \nmid a$. The normalisation of $k^0(d)[[u,v]]/(\pi'^{d'} - \xi u)$ is the power series ring $k^0(d)[[v]]$, which is smooth.

Since the normalisation of $A \otimes_{k^0} k^0(d)$ equals the product of the completed local rings of $\mathcal{C}(d)$ for the points lying over x, we see that $\mathcal{C}(d)$ is smooth at the points over x.

The other case is a similar calculation but more lengthy, so we omit the details, see [1]. $\square$

Remark 7. So we see that the real content of the potential semistable reduction theorem lies in the case where p divides some of the multiplicities of the components of the special fiber of a normal crossings model. In case a tame extension of the ground field k suffices for semistable reduction, we say the curve is tame. L. Halle, in his thesis, gives a geometric proof of semistable reduction for tame curves, using resolution of Kato's toric singularities. Saito has given a criterion for a curve to be tame: if a component of the special fiber of the minimal normal crossings model is divisible by p, it needs to be rational, and intersecting exactly 2 other components with prime-to-p multiplicities. So the curve of figure 1 is tame if and only if $p \neq 2$.

1.3 Proofs of the semistable reduction theorem

Here we collect some remarks on the known approaches towards proving the semistable reduction theorem. Each of these has their own technicalities. A detailed overview is given in Abbés's survey [1], although it doesn't mention the non-Archimedean approaches we are interested in.

- Deligne and Mumford gave the first proof in 1969 in their seminal work on the moduli of curves. First they use the fact - which is attributed to Raynaud - that a curve has semistable reduction if and only if its Jacobian has semi-abelian [2] reduction (even if the curve has no rational point). Next, they use Grothendieck's theorem that every abelian variety has potentially semi-abelian reduction, which is proven using the theory of biextensions.

- Artin-Winters gave a more direct proof, which is exhibited in [11]. They use the structure of Picard groups of singular curves to translate the theorem into a complicated combinatorial problem on the possible configurations of special fibers of minimal models.

[2] Recall that every abelian variety A over k admits a Néron model $\mathcal{A}/R$. This is a smooth seperated finite type commutative group scheme whose special fiber $\mathcal{A}_{\tilde{k}}$ encodes the reduction properties of A. In particular, it is proper if and only if A has good reduction. We say A has semi-abelian reduction if the connected component of the identity of $\mathbb{A}_{\tilde{k}}$, denoted $\mathbb{A}_{\tilde{k}}^0$, is a semi-abelian variety, i.e. an extension of an abelian variety by a torus. Chevalley's decomposition theorem on algebraic groups says that in general, $\mathbb{A}_{\tilde{k}}^\circ$ is not semi-abelian, one namely has to deal with a unipotent part too. See the excellent textbook [6] for more on Néron models

- Later Saito found a simple cohomological criterion for a curve to have semistable reduction, namely if the monodromy action on the first l-adic cohomology of the curve is unipotent. By Grothendieck's l-adic monodromy theorem, stating that the aforementioned action is always quasi-unipotent (i.e. unipotent up to a finite index subgroup), potential semistable reduction follows. His arguments combine an elegant use of the complex of l-adic vanishing cycles and a combinatorial argument that is much less involved then in the Artin-Winters proof. Additionally, his methods yields the criterion we mentioned in remark 7 (Stix [12] later reinterpreted this criterion using logarithmic geometry).
- Bosch-Lütkebohmert, Van der Put, Ducros, Temkin and Wewers-Arzdorf have given proofs using non-Archimedean geometry.

1.4 Strategy of the proof

Our approach for the proof of semistable reduction uses the language of Berkovich geometry - for the reader's convenience we recall some notions in section 2. Our plan is roughly as follows:

Step 1. Using a descent argument, we will reduce to a similar statement for k algebraically closed.

Step 2. Consider the Berkovich analytification C^{an} of C, which is a real tree with a compact topology, the so-called **observers topology**, which is coarser than the metric topology. In short, at a branching point we only allow open sets that miss a finite amount of branches. See figure 2 for an example.

Step 3. The points of this infinite tree can be classified according to **four types**.

- Type 1 and 4 points constitute the leafs of the real tree. Type 1 points correspond to points of $C(k)$.
- At type 3 points we have 2 branches.
- At type 2 points, we find an infinite amount of branches, and these branches can be parametrised by a curve over $\widetilde{k}$. The genus of the latter curve is said to be the genus of the type 2 point. Points of type 1,3,4 always have genus 0.

The heart of the proof lies in describing the local branching and topology for each type of point in more detail. We will need tools such as Temkin's graded reduction.

Step 4. By considering all the points of C^{an} that do not admit a neighbourhood that is an open disc, we find a finite subgraph S of C^{an}, called a **skeleton**, so that $C^{an} \setminus S$ consists only of open discs. This implies that C^{an} retracts onto S, so that S really encodes the topology of C^{an}. For example, the genus of the C is the sum of all type 2 point genera and the first Betti number of S.

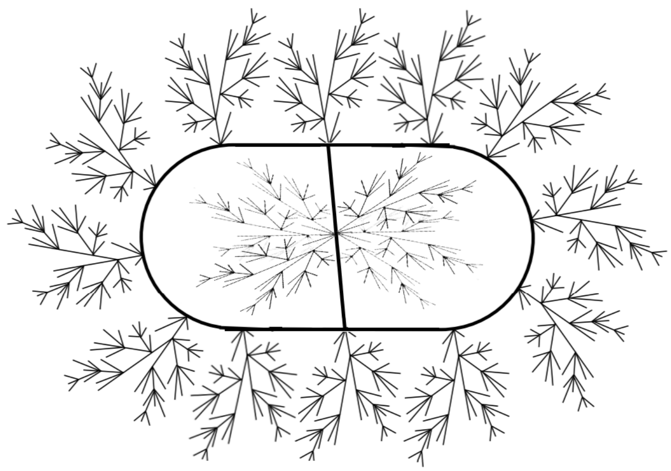

Figure 2: A sketch of what a Berkovich curve looks like. Here the type 2 point in the middle has genus 1 and all other type 2 points have genus 0. The type 2 points are dense in the picture, so we haven't drawn all branches. The skeleton is the fat line, and the total genus of this curve is 3 (we get a contribution of 2 for the two holes of the skeleton).

Figure 3: A closed disc, in the drawing above. It only contains points of genus 0. We get an open disc if we remove the "rooted" branching point.

Another viewpoint for the example of figure 2 is the following: if we remove the middle point of genus 1, and also the two points where the three edges of the skeleton come together, then we get a disjoint union of 4 open annuli[3] and infinitely many open discs. The set of these three type 2 points is also called a **triangulation**. This decomposition of the skeleton into annuli is similar to a pair-of-pants decomposition for a Riemann surface.

Step 5. The skeleton can be used to construct a so-called **semistable formal model** $\mathfrak{C}$, which is a formal scheme over k^0 that has a simple structure. We will introduce these formal models in the coming section.

Step 6. Finally, one can show that the formal model we construct comes from a genuine model over k^0, which is semistable by construction. That concludes our proof.

[3] An open annulus is an open disc minus a closed disc which it contains

2 Analytic Spaces

Here we assume k is an algebraically closed, complete non-trivially valued non-Archimedean field. As before we denote k^0 for the ring of integers and $\widetilde{k}$ for the residue field.

2.1 Basic constructions

Let us recall some standard notions in Berkovich geometry, for a more thorough introduction we refer to [3], [13] and [16]. For a Banach k-algebra A, the spectrum $\mathcal{M}(A)$ is the set of all bounded multiplicative seminorms on A endowed with the weakest topology such that all evaluation maps $x \mapsto |f|_x$ for $f \in A$ are continuous. Any seminorm in $\mathcal{M}(A)$ is automatically bounded by the norm of A, and $\mathcal{M}(A)$ is a nontrivial compact Hausdorff space.

Characters. We can view elements of $\mathcal{M}(A)$ as characters, very much akin to schemes. Note that as a set, $\operatorname{Spec} A$ is the set of all equivalence classes of k-algebra homomorphisms $\phi : A \to L$ for some field L that is an extension of k. Here we say $\phi : A \to L$ and $\phi' : A \to L'$ are equivalent if there exists another such k-algebra homomorphism $\phi'' : A \to L''$ and a diagram

$$\begin{tikzcd}
& L \arrow[dr, hook, dotted] & \\
A \arrow[ur, "\phi"] \arrow[rr, "\phi''"] \arrow[dr, "\phi'"'] & & L'' \\
& L' \arrow[ur, hook, dotted] &
\end{tikzcd}$$

Here $x \in \operatorname{Spec} A$ corresponds with $A \to k(x)$, and we retrieve x by taking the kernel of $A \to k(x)$. If we consider the valued analogy of this, we recover $\mathcal{M}(A)$ as the set of all equivalence classes of Banach k-algebra homomorphisms $\phi : A \to L$ where L is a complete field whose norm extends the one of k. In the equivalence relation one needs to additionally require all complete field extensions to respect the norms. This is proven similarly: $x \in \mathcal{M}(A)$ induces $A \to \mathcal{H}(x)$ where $\mathcal{H}(x)$ is the completion of the fraction field of $A/\ker|.|_x$. Another way to say this is that points of $\mathcal{M}(A)$ correspond to morphisms of affinoid spaces $\mathcal{M}(\mathcal{H}(x)) \to \mathcal{M}(A)$

Affinoids. Recall that if $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$, we write $k\langle r^{-1}T\rangle = k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n\rangle$ for the Tate algebra of power series $\sum a_I T^I$ with $|a_I| r^I \to 0$ as $|I| \to \infty$, it is a Banach k-algebra with respect to the Gauss norm

$$|\sum a_I T^I| = \sup_I |a_I| r^I.$$

An affinoid k-algebra is a Banach k-algebra A admitting an admissible[4] surjection $k\langle r^{-1}T\rangle \to A$. If in addition the radii r_i can be chosen to lie in the value group $|k^\times|$, we say A is **strictly affinoid**. Note that $|k^\times|$ is divisible by our assumption that k is algebraically closed.

Example 8. If $r \notin |k^\times|$, the k-affinoid algebra $K_r = k\langle r^{-1}T, rT^{-1}\rangle = k\langle r^{-1}T, rS\rangle/(ST-1)$ is in fact a field. The point is that any element $\sum_{i\in\mathbb{Z}} a_i T^i \in K_r$ has a unique $i \in \mathbb{Z}$ such that $|\sum_i a_i T^i| = |a_i| r^i$. One can check that if x is the unique type 3 point of $\mathbb{A}_k^{1,an}$ corresponding to the Gauss point of the closed disc $D(0,r) := \mathcal{M}(k\langle r^{-1}T\rangle)$ with centre 0 and radius r, then $K_r \cong \mathcal{H}(x)$.

If $r = (r_1, ..., r_n) \in \mathbb{R}^n_{>0}$ is a tuple of positive reals that are linearly independent over $|k^\times|$, then similarly $K_r = k\langle r^{-1}T, rT^{-1}\rangle$ is a field, and we have $K_r = K_{r_1}\hat{\otimes}_k \dots \hat{\otimes}_k K_{r_n}$. For any affinoid k-algebra A we can find such a tuple r as above so that $A\hat{\otimes}_k K_r$ becomes strictly affinoid over K_r. This allows one to show that many properties that strictly affinoid algebras enjoy (noetherianity, Noether normalisation, nullstellensatz, dimension theory, ...) remain true for general affinoid algebras. However, not everything extends, for example the maximum modulus principle fails in general.

To do sheaf theory, like in rigid geometry we want to identify nice subsets of $\mathcal{M}(A)$ with an associated algebra. If we consider sets on which a given $f \in A$ converges, we arrive at Weierstrass domains. This family of domains is however too coarse to retrieve the topology of $\mathcal{M}(A)$. Therefore we consider Laurent domains, whose associated algebras look like localisations of A. But these are not transitive with respect to inclusion, so we generalise to rational domains. Rational domains enjoy a universal property, which is abstracted as follows.

For an affinoid algebra A, we define an **affinoid domain** $U \subset \mathcal{M}(A)$ similarly as in rigid analytic geometry via a universal property: there exists a bounded morphism of affinoid algebras $A \to A_U$ such that any bounded morphism of affinoid algebras $A \to B$ with $\mathcal{M}(B) \to \mathcal{M}(A)$ factoring through U admits a unique bounded factorisation $A \to A_U \to B$ (recall boundedness is automatic in the strictly affinoid case). It follows that we can identify U with $\mathcal{M}(A_U)$. The Gerritzen-Grauert theorem says that every (strictly) affinoid domain $\mathcal{M}(A)$ is a finite union of (resp. strict) rational domains.

Structure sheaf. In rigid analytic geometry, the G-topology is introduced in order to get rid of annoying covers like the closed unit disc as a union of infinitely many open discs. Such phenomenons no longer occurs in the Berkovich world due to the fact that we have introduced many more points. The G-topology is still there, but no longer essential in understanding the structure sheaf for affinoid spaces.

Suppose A is k-affinoid and $X = \mathcal{M}(A)$, say $V \subset X$ is an **analytic domain** if it is a finite union of affinoid domains. These form the admissable opens for the **G-topology**; the

[4] This means that the Banach norm on A is equivalent to the residue norm coming from the Tate algebra. By [4] 2.8.1 this is equivalent to being continuous, which is automatic in the strictly affinoid case by loc. cit. 6.1.3.

admissable coverings in the latter are given by finite unions of analytic domains. Then we set $\mathcal{O}_X(U) = \lim_{\longleftarrow} A_V$ where V ranges over all analytic domains of U. By Tate acyclicity, $\mathcal{O}_X$ is a G-sheaf, but in fact $\mathcal{O}_X$ is an honest sheaf with respect to the Berkovich topology! To see this, one shows (exercise) that every open cover $\{U_i\}_{i\in I}$ of some open subset $U \subset \mathcal{M}(A)$ (all with respect to the Berkovich topology), and for every analytic domain $V \subset \mathcal{M}(A)$, we can find an open covering $\{V_i\}_{i\in I}$ of V such that $V_i \subset U_i$ for all $i \in I$ and $V_i = \emptyset$ for all but finitely many i.

Good analytic spaces. As with schemes, one can define a good notion of space by defining the category of k-affinoid algebras to be anti-equivalent to the category of k-affinoid algebras, and then glueing these locally ringed spaces.

Definition 9. A **good (strictly) k-analytic space** is defined as a locally ringed space locally isomorphic to the affinoid space $(X = \mathcal{M}(A), \mathcal{O}_X)$, where A is a k-affinoid algebra (resp. strictly k-affinoid algebra). We almost exclusively work with strictly analytic spaces.

An **analytic domain** V in a (strictly) k-analytic space X is a subset of X so that there exists an covering $\{V_i\}_{i\in I}$ of V and an affinoid covering $\{X_i\}_{i\in I}$ of X so that V_i is an affinoid domain in X_i for all i.

We have in fact a fully faithful functor $X \to (X, \mathcal{O}_X)$ from the category of good Berkovich spaces to the category of locally ringed spaces.

Remark 10. There exists a more general notion of a (strictly) analytic space, which we will not explain here. The main point is that unfortunately not all sufficiently nice rigid analytic spaces yield good analytic spaces.

For example, in $\mathbb{A}_k^{2,rig}$ consider the closed unit polydisc minus the open unit polydisc. This is a rigid analytic space whose natural Berkovich analogue

$$\{x \in \mathcal{M}(k\langle T_1, T_2\rangle) \mid |T_1|_x = 1 \text{ or } |T_2|_x = 1\}$$

cannot be a good analytic space as we defined above. This is somewhat subtle and related to the fact that the reduction of this space is $\mathbb{A}^2_{\tilde{k}} \setminus \{(0,0)\}$, which is not affine (see 2.2 and proposition 25 below).

To this end Berkovich defined more general analytic spaces in terms of atlases in his 1993 IHES paper (the one where he constructs an étale cohomology for analytic spaces). In this more general definition affinoid spaces are glued along closed sets via so-called *nets*[5]. Good analytic spaces as we defined above are then recovered as those analytic spaces where all points have an affinoid neighbourhood. Without further mentioning we will always assume our spaces to be good.

[5]the name comes from a notion for metric spaces, not to be confused with the notion in functional analysis. A prototypical example of a net is given by the faces of a polyhedron - where affinoids in this analogy are the maximal faces.

Analytification. One can analytify schemes of finite type over k, similarly to how one obtains complex analytic spaces from schemes over $\mathbb{C}$. If $X = \operatorname{Spec} A$ is affine, we define X^{an} to be the set of multiplicative seminorms on A restricting to the given absolute value on k, endowed with the weakest topology such that all the evaluation maps $X^{an} \to \mathbb{R} : x \mapsto |f|_x$ for $f \in A$ are continuous. Equivalently, one can see that X^{an} is also the set of pairs $(x, |.|)$ with $x \in \operatorname{Spec} A$ and $|.| : k(x) \to \mathbb{R}$ a multiplicative seminorm extending the one on k, such a pair corresponds to the prime $\ker|.|$. The structure sheaf is obtained by taking uniform limits of quotients of regular functions. Globalising this construction one obtains analytifications of arbitrary schemes of finite type over k. We have that X^{an} is a good strictly analytic space that is compact (resp. Hausdorff, arcwise connected) if and only if X is proper (resp. seperated, connected). The topological dimension of X^{an} equals the dimension of X.

This construction also works for trivially valued k, and in fact we recover complex analytification when $k = \mathbb{C}$[6].

Remark 11. One can also analytify qcqs (quasi-compact quasi-seperated) rigid analytic spaces to give (possibly non-good) strictly analytic spaces, and this is an equivalence of categories. It is a rather subtle exercise to give a criterion for a qcqs rigid analytic space to have a good analytification.

Curves. Suppose C/k is a connected smooth projective curve and let C^{an} denote the analytification. The latter is called an **analytic curve**. This is a good strictly analytic space over k. Using Abhyankar's inequality we can classify the points of C^{an} according to four types:

- type 1: $\mathcal{H}(x) = k$
- type 2: $\operatorname{trdeg}_{\widetilde{k}} \widetilde{\mathcal{H}(x)} = 1$ and $|\mathcal{H}(x)^\times| = |k^\times|$
- type 3: $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $\dim_{\mathbb{Q}}(|\mathcal{H}(x)^\times|/|k^\times|) = 1$.
- type 4: $\mathcal{H}(x)$ is a nontrivial immediate[7] extension of k, i.e. $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)^\times| = |k^\times|$.

Note that the type 2 points are exactly those for which $\widetilde{\mathcal{H}(x)}$ is an extension of $\widetilde{k}$ of transcendence degree 1.

Definition 12. At a type 2 point $x \in C^{an}$ we define the **residual curve** $\widetilde{C_x}$ as the unique nonsingular smooth projective curve over $\widetilde{k}$ with function field $\widetilde{\mathcal{H}(x)}$. The genus of $\widetilde{C_x}$ is called the **genus** of x.

Later we show that the closed points of $\widetilde{C_x}$ correspond to the branches emanating from x. We will also reinterprete the residual curve as the so-called germ reduction at x, which is a general construction for analytic spaces defined via Riemann-Zariski spaces.

[6] The nontrivial input here is the Gel'fand-Mazur theorem from functional analysis, which predicts that $\mathcal{H}(x) = \mathbb{C}$ in the construction above

[7] The existence of nontrivial immediate extensions is equivalent to k being spherically incomplete (Kaplansky).

2.2 Affinoid analytic reduction

Spectral seminorm. Recall that every strictly affinoid k-algebra A admits a spectral seminorm $|.|_{\text{sup}}$, defined as $|f|_{\text{sup}} = \sup_{x \in \mathcal{M}(A)} |f|_x$. By the maximum modulus principle, we can change the supremum by a maximum. There is also the following more familiar expression $|f|_{\text{sup}} = \lim_{n\to\infty} |f^n|^{1/n}$, which shows more clearly that $|.|_{sup}$ is the maximal powermultiplicative seminorm. It is a fact that when A is reduced, the $|.|_{\text{sup}}$ is in fact a norm equivalent to the norm of A ([4] 6.2.4).

we denote $A^\circ = \{f \in A | |f|_{\text{sup}} \leq 1\}$ and $A^{\circ\circ} = \{f \in A | |f|_{\text{sup}} < 1\}$. Associated we have the reduced finitely generated $\widetilde{k}$-algebra $\widetilde{A} = A^\circ / A^{\circ\circ}$.

Reduction. Recall that every $x \in \mathcal{M}(A)$ can be identified with a character $A \to \mathcal{H}(x)$. As such a homomorphism of Banach k-algebras is contractive, we get an induced map $A^\circ \to \mathcal{H}(x)^\circ$ which on its turn induces a character $\widetilde{A} \to \widetilde{\mathcal{H}(x)}$. Taking the kernel of this last character yields a well-defined (analytic) *reduction* map $\pi : \mathcal{M}(A) \to \widetilde{\mathcal{M}(A)} := \operatorname{Spec} \widetilde{A}$.

Example 13. Consider the closed unit disc $D(0,1) := \mathcal{M}(k\langle T \rangle)$. Then $\widetilde{k\langle T \rangle} = \widetilde{k}[T]$, and so $\widetilde{D(0,1)} = \mathbb{A}^1_{\widetilde{k}}$. Recall that the Gauss point η of $D(0,1)$ is given by the sup-norm $|\sum a_i T^i|_\eta = \max |a_i|$. One can check that the image of the Gauss point η is the generic point of $\mathbb{A}^1_{\widetilde{k}}$.

If $x \in D(1)$ is distinct from η, there is some $a \in k$ such that $|T - a|_x < 1$, which implies that $\pi(x) = \widetilde{a} \in \widetilde{k}$. This is summarised in the drawing of figure 4.

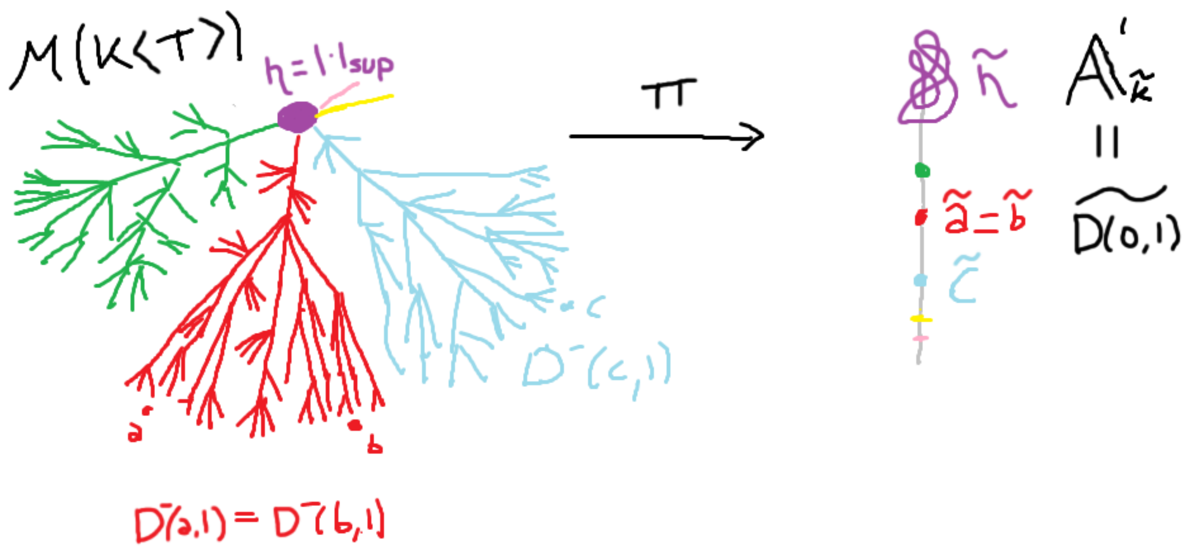

Figure 4: Analytic reduction of the closed unit disc $D(0,1)$

Results of Tate and Berkovich. The example already shows some general behaviour: the reduction map is surjective, and generic points have unique preimages.

Proposition 14. *Suppose A is a strictly k-affinoid algebra, and let $X = \mathcal{M}(A)$.*

(a) The reduction map is anti-continuous, i.e. the inverse of a closed set is open and vice versa.

(b) (Tate) The reduction map $\pi : X \to \widetilde{X}$ is surjective

(c) (Berkovich) for any generic point $\widetilde{x} \in \widetilde{X}$, there exists a unique preimage $x \in X$. If $|A|_{\sup} = |k|$, then $\kappa(\widetilde{x}) \cong \widetilde{\mathcal{H}(x)}$

(d) The union of the preimages of generic points is the **Shilov boundary** *$\Gamma(X)$ of X, defined as the unique minimal set of points of X where all elements in A attain their maximum. It is finite and discrete.*

Proof. (a) This is easy to check by unwinding the definitions. Let $\widetilde{f} \in \widetilde{A}$, and suppose $f \in A^\circ$ is a lift of $\widetilde{f}$. Then

$$\pi^{-1}(V(\widetilde{f})) = \{x \in \mathcal{M}(A) || f(x)| < 1\}.$$

(b) In the case where $A = T_n$ the statement follows from the fact that any map from a polynomial ring over K to $\widetilde{\mathcal{H}(x)}$ lifts uniquely to a map $T_n \to \mathcal{H}(x)$.

For the general case, we have the canonical inclusion $A \to \oplus A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ runs over all the minimal primes of A. Using this we can see it is sufficient to treat the case where A is an integral domain. By Noether normalisation, choose a finite injective map $T_n \to A$. Let K be a normal closure of $\mathrm{Frac}(A)/\mathrm{Frac}(T_n)$, and let G be the Galois group of $K/\mathrm{Frac}(T_n)$. Denote $B = T_n[ga|g \in G]$, which is finite over T_n, therefore canonically affinoid. By going up, we have a diagram

$$\begin{tikzcd}
\mathcal{M}(B) \arrow[r, two heads] \arrow[d, "\pi_{\mathcal{M}(B)}"] & \mathcal{M}(A) \arrow[r, two heads] \arrow[d, "\pi_{\mathcal{M}(A)}"] & \mathcal{M}(T_n) \arrow[d, two heads] \\
\widetilde{\mathcal{M}(B)} \arrow[r, two heads] & \widetilde{\mathcal{M}(A)} \arrow[r, two heads] & \widetilde{\mathcal{M}(T_n)}
\end{tikzcd}$$

We reduce therefore to showing $\pi_{\mathcal{M}(B)}$ surjects. Now G acts on the diagram obtained by removing $\pi_{\mathcal{M}(A)}$, so it suffices to show that G acts transitively on the fibers of $\widetilde{\mathcal{M}(B)} \to \widetilde{\mathcal{M}(T_n)}$.

To do so, suppose y, y' lie above x, and suppose for the sake of contradiction that $\mathfrak{p}_{y'} \not\subset \cup_{g \in G} g^{-1}\mathfrak{p}_y$. Then there exists a $f \in B^\circ$ such that $\widetilde{f} \in \mathfrak{p}_{y'}$ and $g\widetilde{f} \notin \mathfrak{p}_y$ for all

$g \in G$. Suppose that $f^r + a_1 f^{r-1} + ... + a_r = 0$ is an integral equation for f over $Q(T_n)$, then we know by as a_r is a product of conjugates of f that $\widetilde{a}_r \in \mathfrak{p}_{y'} \setminus \mathfrak{p}_y$. Note $a_r \in T_n$. Therefore by $f \in B^\circ$, we see $a_r \in T_n^\circ$, but this yields a contradiction as then $\widetilde{a}_r \in \mathfrak{p}_x = p_y \cap \widetilde{T_n} = p_{y'} \cap \widetilde{T_n}$.

(c) First assume that X is irreducible, i.e. $\widetilde{A}$ is an integral domain. Suppose $\tilde{x}$ is a generic point and let $x \in \pi^{-1}(\widetilde{x})$ (which is non-empty by (b)[8]). Note that for all $f \in A^\circ \setminus A^{\circ\circ}$ we have that $\widetilde{A} \to \widetilde{\mathcal{H}(x)}$ injects, so that by the maximum modulus principle $|f(x)| = 1$.

We claim that in fact for all $f \in A$, we have $|f(x)| = |f|_{sup}$. If $|f|_{sup} = 0$, then $f \in \sqrt{A}$ and so $|f(x)| = 0$. Next, suppose $|f|_{sup} > 0$. Choose a finite map $T_n \to A$, then if $f^n + a_1 f^{n-1} + .. + a_n = 0$ with $a_i \in T_n$, we have

$$|f|_{sup} = \max_i |a_i|_{sup}^{1/i}$$

(the standard Newton polygon lemma). Therefore, there exists an $a \in k^\times$ and an n such that $|f|_{sup}^n = |a|$. Let $g = a^{-1} f^n$. Then we have $|g|_{sup} = 1$. We have $g \in A^\circ \setminus A^{\circ\circ}$, so by $\tilde{x}$ generic we see that $|g(x)| = 1$. This implies $|f|_{sup} = |f(x)|$. This characterises the character $A \to \mathcal{H}(x)$ up to equivalence, hence $\pi^{-1}(\tilde{x}) = \{x\}$.

For the second statement: we have that $\mathrm{Frac}(A)$ is dense in $\mathcal{H}(x)$, therefore elements of $\widetilde{H(x)}$ come from f/g with $|f|_{sup} = |g|_{sup} = |a|$ for some $a \in k$, and $f/g = a^{-1}f/a^{-1}g$ so the canonical embedding $\kappa(\widetilde{x}) \to \widetilde{\mathcal{H}(x)}$ is surjective.

For arbitrary A, we choose some $f \in A^\circ$ such that $\widetilde{f}(\widetilde{x}) \neq 0$ and such that $\tilde{f}$ vanishes on all irreducible components not containing $\tilde{x}$. Let $B = A\langle T \rangle/(fT - 1)$. Then $\mathcal{M}(B)$ is the Laurent domain $X\langle f^{-1} \rangle = \{x \in X ||f(x)| = 1\}$. So $\pi^{-1}(\tilde{x}) \subset X\langle f^{-1} \rangle$, which at the level of the reduction means we can localise and use the previous case. So we are reduced to check that $|B|_{sup} = |A|_{sup}$.

we sketch the argument, which is [4] 7.2.6/2. Denote $\sigma : A \to B$ the canonical surjective map, and let $a \in A$ so that $|\sigma(a)| > 0$. Then $|f^n a| \leq |f^{n+1} a| \leq |\sigma(a)|_{sup}$. As before $\mathrm{Frac}(A)$ is a finite extension of some $\mathrm{Frac}(T_n)$, and the supremum norm is induced by the spectral norm. As before this equals the maximum of finitely many valuations on $\mathrm{Frac}(A)$, and so the sequence $|f^n a|$ is discrete in the reals. Therefore it becomes constant, say $|f^n a| = C$ for n large. Then by the maximum modulus principle, $|f^n a|$ attains its maximum on the affinoid domain $U = \{x \in X ||a(x)| \geq C\}$. If U were disjoint from $\mathcal{M}(B)$, then $|f| < 1$. However then $C \leq |f^n a|_U \leq |f|_U^n |a|_u \to 0$, which is impossible.

(d) for $f \in A^\circ \setminus A^{\circ\circ}$, there exists a generic point $\tilde{x} \in \widetilde{X}$ such that $\tilde{f}(\tilde{x}) \neq 0$; So for the unique point $x \in X$ in the preimage, we have $|f(x)| = 1 = |f|_{sup}$. This implies f attains

[8] actually, the case of (b) where $\tilde{x}$ is generic is much easier to see. Show this as an exercise using compactness of the analytic spectrum and the fact that $\pi^{-1}(\tilde{x}) = \bigcap_{\tilde{f} \notin \mathfrak{p}_{\tilde{x}}} \pi^{-1}(D(\widetilde{f}))$

its maximum on x. for general $f \in A$ with $|f|_{sup} > 0$, we find as before some $a \in k^\times$ so that $|f|_{sup}^n = |a|$, and we reduce to the previous case by considering $a^{-1}f^n$. This proves that the preimages of the generic points are indeed a boundary. Now we show it is minimal.

Suppose that $\tilde{x}$ is a generic point with preimage x with some neighbourhood U. Then $\{x\} = \cap_{\tilde{f}(\tilde{x})\neq 0}\pi^{-1}D(\tilde{f})$, so there exists $f \in A^\circ$ such that $\tilde{f}(\tilde{x}) \neq 0$ and $\{y||f(y)| = 1\} \subset U$. So there exists $\varepsilon > 0$ such that $\{y||f(y)| > 1-\varepsilon\} \subset U$: this implies that the boundary must be minimal, and even finite discrete.

□

For example in example 11, we see that the unit disk has a unique Shilov boundary point, namely its Gauss point.

Remark 15. Part (d) is true more generally for any affinoid algebra. This notion of analytic boundary for affinoid spaces is very particular to the theory of Berkovich, and cannot be found in rigid analytic geometry.

Remark 16. There is an important generalisation of the analytic reduction map due to Temkin, to which we return later. For $r \in \mathbb{R}_{>0}$ let us denote

$$\widetilde{A}_r = \{f \in A||f|_{\sup} \leq r\}/\{f \in A||f|_{\sup} < r\}$$

(so that $\widetilde{A}_1 = \widetilde{A}$). The idea is that more information is contained in the $\mathbb{R}_{>0}$-graded ring

$$\widetilde{\widetilde{A}} := \bigoplus_{r\in\mathbb{R}_{>0}} \widetilde{A}_r.$$

This last ring is a $\widetilde{\widetilde{k}}$-algebra (as $\mathbb{R}_{>0}$-graded rings), where $\widetilde{\widetilde{k}}$ is given the $\mathbb{R}_{>0}$-grading $\widetilde{k}_r = \{x \in k|\ |x| \leq r\}/\{x \in k|\ |x| < r\}$.

For example, if $A = k\langle r^{-1}T\rangle$ for $r \notin |k^\times|$, then $\widetilde{A} = \widetilde{k}$ whereas $\widetilde{\widetilde{A}}$ is quite nontrivial.

If we define $\widetilde{\widetilde{\mathcal{M}(A)}} = \operatorname{Spec}\widetilde{\widetilde{A}}$ as the set of homogeneous primes in $\widetilde{\widetilde{A}}$, Temkin proves that the above results extends to showing that $\mathcal{M}(A) \to \operatorname{Spec}\widetilde{\widetilde{A}}$ is surjective with unique preimages of generic points. To this end one extends results from $\mathbb{Z}$-graded commutative algebra to the $\mathbb{R}_{>0}$-graded setting.

2.3 Discs and annuli

Discs For $r \in \mathbb{R}_{>0}$ we denote $D(r) := \mathcal{M}(k\langle r^{-1}T\rangle)$ for the closed disc of radius r, we also write $D(0, r)$ when we wish to emphasize that the disc can be seen as the affinoid domain $\{x \in \mathbb{A}_k^{1,an}|\ |T|_x \leq r\}$ in $\mathbb{A}_k^{1,an} = \mathcal{M}(k[T])$. We also write $D = D(1)$. Similarly we denote $D^-(r)$ or $D^-(0, r)$ for the open disc of radius r i.e. $D(r)$ with its Shilov boundary removed. If r lies in the value group $|k^\times|$, then $D(r) \cong D$ as analytic spaces. Note that a closed disc deformation retracts onto its Gauss point.

Annuli. Next, for positive reals $r < s$ we denote $A(r,s)$ or $A(0,r,s)$ for the closed annulus $\mathcal{M}\left(k\langle r^{-1}T, s^{-1}S\rangle/(ST-1)\right)$ of radii r, s. Note that $A(r,s) \cong A(r/s, 1)$, for that reason we also write $A(r/s)$.

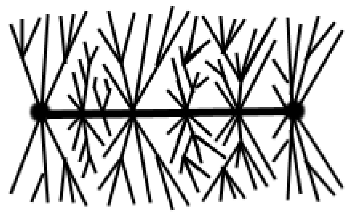

Figure 5: an annulus

One checks that the analytic reduction of $A(r,s)$ is given by $\operatorname{Spec} \widetilde{k}[u,v]/(uv)$, so we have two boundary points. Simarly we have the open annuli $A^{-}(r,s)$, also denoted $A^{-}(r/s)$. By an annulus we mean an open or closed annulus. The central chord of an annulus A, i.e. the path connecting the shilov boundary points, is called the **skeleton** $S(A)$ of the annulus. By the structure theory of $\mathbb{A}_k^{1,an}$ we know that $A \setminus S(A)$ is an infinite disjoint union of open discs. Also any annulus retracts onto its skeleton. To prove this one can restrict attention to an annulus $A(0,r,1) \subset \mathbb{A}_k^{1,an}$ and consider the continuous map $\mathbb{A}_k^{1,an} : x \mapsto |T|_x$. It admits a continuous section

$$\sigma : \mathbb{R}_{\geq 0} \to \mathbb{A}_k^{1,an} : r \mapsto \eta_{0,r},$$

where $\eta_{0,r}$ denotes the Gauss point of the disc $D(0,r)$. With some extra work one shows this is in fact a deformation retraction.

The above works fine for $r = s$ too, but in this case we get a unique boundary point as the reduction is given by $\operatorname{Spec} \widetilde{k}[u,v]/(uv-1)$. We call this the **circle** (or flat annulus) $A(r,r)$. In case $r \notin |k^{\times}|$, we get a point, in the other case we get something isomorphic to $D \setminus D^{-}$ (the closed unit disc minus the open unit disc).

More intrinsically, we can characterise the skeleton of an annulus as follows.

Proposition 17 (Thuillier)**.** *The skeleton of an annulus is the set of all points that do not admit an affinoid neighbourhood isomorphic to an open unit disc.*

Sketch of the proof. By the local structure of $\mathbb{A}_k^{1,an}$ it suffices to show that a point of the skeleton admits no affinoid neighbourhood isomorphic to an open unit disc.

So suppose $r \in (0,1]$ and suppose x is a point on the skeleton of the closed annulus $A(r)(= A(0,r,1)) \subset \mathbb{A}_k^{1,an}$ admitting an affinoid neighbourhood $U \subset A(r)$ isomorphic to an open unit disc. Then $V := \mathbb{A}_k^{1,an} \setminus U$ is connected, since we know what the open discs in $\mathbb{A}_k^{1,an}$ look like. Consider the continuous map σ defined above. We see that V contains $\sigma(0)$ and $\sigma(2)$, therefore it must contain the whole image of σ, and this gives a contradiction as $x \in V \setminus \operatorname{im}(\sigma)$. $\square$

Glueing. Just like schemes, we can glue good analytic spaces along each other. For example, $\mathbb{P}^{1,an}$ is constructed by glueing two copies of the closed unit disc $D(0,1)$ along the circle $A(0,1,1)$, via the map $A(0,1,1) \to A(0,1,1)$ given by $k\langle T, T^{-1}\rangle \to k\langle T, T^{-1}\rangle : T \mapsto T^{-1}$.

We can glue annuli and discs in two ways. First suppose $r \leq s \leq t$ are positive reals. If we glue $A(0,r,s)$ to $A(0,s,t)$ along $A(0,s,s)$ (with the identity map), we get $A(0,r,t)$. Similarly, if $r \leq s$ are positive reals. Glueing $D(0,r)$ with $A(0,r,s)$ along $A(0,r,r)$ we get $D(0,s)$.

Local structure of analytic curves Next, we state the key theorem on the local structure of analytic curves, which will be shown in section 3.3. Suppose C/k is a connected smooth projective curve and let C^{an} denote the analytification.

Definition 18. A **standard set** (or standard neighbourhood) is an open set $U \subset C^{an}$ containing x, such that the connected components of $U \setminus \{x\}$ are given by open discs and finitely many annuli.

In particular, an open disc is a standard neighbourhood of its Gauss point, and an open annulus is a standard neighbourhood of any point on its skeleton.

Theorem 19. *A local basis of neighbourhoods about $x \in C$ is given by*

- *open discs if x is of type 1 or 4;*
- *standard sets if x is of type 2;*
- *annuli if x is of type 3.*

2.4 Formal models and their reductions

★★★★★Under construction.★★★★★

very short summary: For general analytic spaces we need a choice of model to have a similar theory of reduction like for affinoid spaces. This model will look like the formal completion of a k^0-model, where formal completion means we complete the structure sheaf with respect to the ideal sheaf defining the special fiber. Such a formal model has a generic fiber that is an analytic space, and its special fiber is the reduction, a variety over $\widetilde{k}$. The results from 2.2 (reduction map is surjective with unique preimages of generic points) remain true.

Results of Bosch-Lütkebohmert and Berkovich show that for curves, the inverse image of the reduction map of a closed point is an open disc (resp. an open annulus) if and only if the point is smooth (resp. a nodal singularity).

2.5 Formal models vs. algebraic models

★★★★★Under construction. ★★★★★

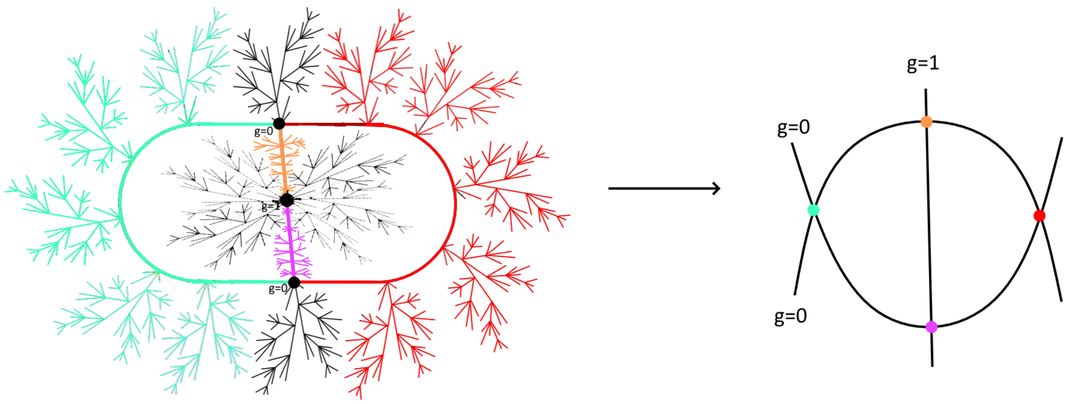

Figure 6: Example of a formal reduction map for the analytification of a genus 3 curve.

very short summary: in nice cases, formal models come from taking formal completion of honest models, by a formal GAGA theorem. So in order to construct a semistable model, it will suffice finding a semistable formal model, which roughly looks locally like

$$\mathrm{Spf}\,(R\langle S,T\rangle/(ST-a))$$

for some $a \in k^0 \setminus \{0\}$.

Here we will also include a proof for why our assumption on k being algebraically closed is sufficient.

2.6 Riemann-Zariski spaces

Definition 20. Recall that a valuation ring R is a domain such that every element $f \in \mathrm{frac}\, R$ satisfies either $f \in R$ or $f^{-1} \in R$ (so it is a local ring). For a field F and a domain A contained in F, the **Riemann-Zariski space associated to** F/A, denoted $\mathbf{P}_F(A)$, is the collection of all valuation rings R with fraction field F that contain A. It is equipped with the weakest topology such that for every $f \in F$, the subset

$$\mathbf{P}_F(A[f]) = \{R \in \mathbf{P}_F(A) |\ f \in R\}$$

is open in $\mathbf{P}_F(A)$. It is easy to see that a basis for the topology is given by the sets $\mathbf{P}_F(A[f_1, \ldots, f_n])$, we call these the affine opens. We write $\mathbf{P}_F(A)^* := \mathbf{P}_F(A) \setminus \{F\}$ (i.e. we exclude the trivial valuation ring F).

Remark 21. Often A is a field. In that case $\mathbf{P}_F(A)$ is a locally ringed space with structure sheaf given by $\mathcal{O}(U) = \bigcap_{R\in U} R$. The stalk at $R \in \mathbf{P}_F(A)$ is just R.

Remark 22. Zariski was the first to consider these spaces in the '40s and called them (confusingly) Riemann spaces. In characteristic zero, he showed these spaces have a smoothness property, which yields local uniformisation, i.e. resolution near a valuation. This allows for a proof of resolution of singularities in dimension ≤ 3 (in characteristic 0[9]).

This is for example how Hartshorne's Algebraic Geometry proves resolution for curves: if F is the function field of a projective integral curve D over a field A, then $\mathbf{P}_F(A)$ is in fact a scheme, given by the unique projective smooth curve birational to D. In dimensions greater than 1, this construction does not yield a scheme in general.

Nagata also used Riemann-Zariski in the proof of his compactification theorem.

Proposition 23. *Let A be a domain contained in a field F. Then the Riemann-Zariski space $X = \mathbf{P}_F(A)$ is qcqs (quasicompact quasiseperated)*

Proof. For a subset $S \subset F^\times$ let us denote $X\{S\} := \{R \in X | \ f \in \mathfrak{m}_R, \text{ for all } f \in S\}$; These are precisely the closed sets of X (indeed, $X \setminus X\{S\}$ is the union of $\mathbf{P}_F(A[f^{-1}])$ for $f \in S$).

Therefore for quasicompactness of X it is sufficient to show that if $X\{S\} = \emptyset$ then already $X\{S'\} = \emptyset$ for some finite subset S' of S.

Now $X\{S\} = \emptyset$ is equivalent to 1 lying in the A-submodule of F generated by the elements of S (if the latter is not the case, use Zorn's lemma to construct an element of $X\{S\}$), and we only need a finite number of elements of S to give an expression of 1 as an A-linear combination.

For quasi-seperatedness one copies the above with X replaced by an affine open of X. □

Graded version. For our purposes, we need to tweak the above Riemann-Zariski spaces slightly, so that they work with the $\mathbb{R}_{>0}$-graded formalism (see remark 16).

For a $\mathbb{R}_{>0}$-graded domain A and a $\mathbb{R}_{>0}$-graded field F, we let $\mathbf{P}_F(A)$ denoted the set of $\mathbb{R}_{>0}$-graded valuation rings R such that $\operatorname{Frac} R = F$ and $A \subset R$ as $\mathbb{R}_{>0}$-graded rings. The analog of proposition 23 still works.

The following lemma helps us computing and comparing these Riemann-Zariski spaces.

Lemma 24. *Suppose $L/K/F$ is a tower of extensions of $\mathbb{R}_{>0}$-graded fields.*

(a) Suppose that $|L^\times|/|K^\times|$ is torsion. Then the canonical map

$$\mathbf{P}_L(K) \to \mathbf{P}_{L_1}(K_1)$$

is a homeomorphism preserving the affine opens.

[9] in positive characteristic, this is also an important ingredient for resolution in dimension ≤ 3 following Abhyankar, Cossart-Piltant

(b) Suppose that $L^\times/K^\times$ is torsion. Then the canonical map

$$\mathbf{P}_L(F) \to \mathbf{P}_K(F)$$

is a homeomorphism preserving the affine opens.

Proof. We start by showing (b). By assumption, if $f \in L$ we have some $n \in \mathbb{Z}_{>0}$ such that $f^n \in K$. For $R \in \mathbf{P}_L(F)$ we have $f^n \in R$ if and only if $f \in R$ and so $\mathbf{P}_L(F) \to \mathbf{P}_K(F)$ is bijective. If $f_1, \ldots, f_e$ lie in L, and n is such that $f_i^n \in K$ for all i, then the image of $\mathbf{P}_L(F[f_1, ..., f_e])$ in $\mathbf{P}_K(F)$ is $\mathbf{P}_L(F[f_1^n, ..., f_e^n])$.

For (a), we can use (b) to reduce to the case where $|L^\times| = |K^\times|$. Then every $f \in L$ can be written as $f = gh$ where $g \in L_1$ and $h \in K^\times$. For $R \in \mathbf{P}_{L_1}(K_1)$ we let R' be the $\mathbb{R}_{>0}$-graded valuation ring consisting of all $f \in L$ such that $g \in R$, one checks this gives a well-defined map $\mathbf{P}_{L_1}(K_1) \to \mathbf{P}_L(K)$ that yields an inverse to the canonical map $\mathbf{P}_L(K) \to \mathbf{P}_{L_1}(K_1)$. For $f_1, ..., f_e \in L$ such that $f_i = g_i h_i$, with $g_i \in L_1, h_i \in K^\times$, we find that the image of $\mathbf{P}_L(K[f_1, \ldots, f_e])$ in $\mathbf{P}_{L_1}(K_1)$ is $\mathbf{P}_{L_1}(K_1[g_1, \ldots, g_n])$. $\square$

Riemann-Zariski spaces and germs of analytic spaces Here we outline how Riemann-Zariski spaces help to understand the local structure of (good) k-analytic spaces. As in remark 16 we will consider $\widetilde{\widetilde{k}}$ as an $\mathbb{R}_{>0}$-graded field via the grading $\tilde{k}_r = \{x \in k | \, |x| \leq r\}/\{x \in k | \, |x| < r\}$. Simarly if X is a k-analytic space we view $\widetilde{\widetilde{\mathcal{H}(x)}}$ as a $\widetilde{\widetilde{k}}$-algebra (as $\mathbb{R}_{>0}$-graded rings).

Suppose $V = \mathcal{M}(A)$ is an affinoid neighbourhood of x in X, and consider the graded reduction $\widetilde{\widetilde{V}}$, this is a $\widetilde{\widetilde{k}}$-variety with a point $\widetilde{\widetilde{x}}$, which is given by a character $\chi : \widetilde{\widetilde{A}} \to \widetilde{\widetilde{\mathcal{H}(x)}}$. To this pointed $\widetilde{\widetilde{k}}$-variety we associate the **birational space**

$$\widetilde{\widetilde{(X,x)}} := \{R \in \mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}(\widetilde{\widetilde{k}}) | \text{ there exists a } g : \operatorname{Spec} R \to \widetilde{\widetilde{V}} \text{ extending } \widetilde{\widetilde{x}} \to \widetilde{\widetilde{V}}\} = \mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}(\chi(\widetilde{\widetilde{A}})),$$

here we define a birational space as a qcqs topological space locally homeomorphic to $\mathbf{P}_K(\widetilde{\widetilde{k}})$ for some $\mathbb{R}_{>0}$-graded field K (morphisms of birational spaces are the obvious ones). In this case $K = \widetilde{\widetilde{\mathcal{H}(x)}}$, and in fact $\widetilde{\widetilde{(X,x)}}$ is an affine open of $\mathbf{P}_K(\widetilde{\widetilde{k}})$[10]. Ones checks the construction is independent of the choice of V, and if V is strictly affinoid, we can just work with $\widetilde{V}$ instead of $\widetilde{\widetilde{V}}$.

Birational spaces were already understood well by Zariski. The category of birational spaces can be constructed by localising the category of pointed $\widetilde{\widetilde{k}}$-varieties at proper maps preserving the residue field at a given point. For general analytic spaces, the upshot is that we can construct, using reduction via formal models, the **germ reduction** functor

$$\textbf{germs of analytic spaces} \to \textbf{birational spaces} : (X,x) \mapsto \widetilde{\widetilde{(X,x)}},$$

[10]This actually true for a general analytic space if and only if X is good at x, i.e. there exists an affinoid neighbourhood V. For general

where the left-hand category is defined as the localisation of pointed k-analytic spaces localised by morphisms that are locally isomorphic about the given point.

Proposition 25 (Temkin [14]). *Germ reduction establishes a bijection between analytic subdomains of (X, x) and birational subspaces of $\widetilde{\widetilde{(X, x)}}$ and preserves intersections, finite unions and inclusions.*

As an application one can show that germ reduction encodes a lot of local properties of analytic spaces (goodness, seperatedness, properness, ...). One can also show that the category of strict k-analytic spaces is a full subcategory of the category of k-analytic spaces.

Remark 26. Another viewpoint is given by adic spaces: the germ reduction is homeomorphic to the closure of a point in the adicification of an analytic space.

3 Analytic curves

Let us denote C^{an} for the analytification of a connected smooth projective curve C over k (we keep the same assumptions on k as in section 2).

3.1 Skeleta and triangulations

Definition 27. Suppose U is an open subset of C^{an}. We define the **analytic skeleton** $S(U)$ of U as the set of all points of U that do not admit an open neighbourhood isomorphic to a unit open disc.

Note that proposition 17 says that for an annulus, the analytic skeleton coincides with what we previously defined to be the skeleton of an annulus.

Definition 28. A **skeleton** is a locally finite closed subgraph S in C^{an} such that each connected component of $C^{an} \setminus S$ is an open disc whose closure is a compact subset meeting S in a unique point.

A **triangulation**[11] T of C^{an} is a finite discrete set of type 2 points such that the complement is a disjoint union of open discs and finitely many annuli.

Below we show two things: assuming theorem 19, we show that the analytic skeleton is indeed a skeleton, and we show that each skeleton gives rise to a triangulation and vice versa. In particular, a triangulation exists.

Remark 29. Note that all skeletons and triangulations, if they exists, necessarily contain the positive genus points, because all points in the unit open disc have genus 0.

Proposition 30. *The analytic skeleton $S(C^{an})$ of C^{an} is a skeleton.*

Proof. Suppose $x \in S(C^{an})$. By theorem 19, x is of type 2 or 3. Suppose V is a standard neighbourhood of x, as in theorem 19.

Suppose A is an annular connected component of $V \setminus \{x\}$, we claim that either $S(C^{an}) \cap A$ is either $S(A)$ or empty. To show the claim, suppose that there exists some $y \in S(A) \setminus S(C^{an})$. Then there exists an open disk $D \subset X$ with $y \in D$. Clearly $x \notin D$. By the fusion lemma 32, we see that $A \cup D$ is an open disc, and must be a connected component of $X \setminus \{x\}$. But this implies $S(C^{an}) \cap A = \emptyset$, and shows the claim.

Let us construct V' from V by adjoining all connected components of $X \setminus \{x\}$ that are open discs. The claim then implies that every connected component of $V' \setminus S(C^{an})$ is an open disc whose closure is compact and has a unique point in $S(C^{an})$.

Now suppose that U is a connected component of $C^{an} \setminus S(C^{an})$, and suppose its topological boundary contains x. Then $U \cap V'$ is both open and closed in $V' \setminus S(C^{an})$, so it is a finite union of connected components of $V' \setminus S(C^{an})$. If U' is one such a connected component, we

[11] At some places in the literature, like in [8], these are also called semistable vertex sets.

see by the above that U' has a unique topological boundary point and is closed in U. Therefore $U' = U$ by U connected, and we are done. $\square$

Proposition 31. *Every triangulation of C^{an} is contained in a skeleton and every skeleton of C^{an} contains a triangulation.*

Proof. Given a triangulation, consider the analytic skeleton $S(C^{an})$. For any point $t \in T \setminus S(C^{an})$, consider the connected component U of t in $C^{an} \setminus S(C^{an})$, and add the unique arc between t and the unique boundary point of U to $S(C^{an})$. One checks this again gives a skeleton.

The other assertion needs more work. Suppose we are given a skeleton S, then we claim that the set T of all points of valency at least 3 in S together with the positive genus points (this is a finite set by lemma 33 below) of C^{an} form a triangulation. Indeed, fix a standard open neighbourhood for every type 2 point in T. For any type 3 point in $S \setminus T$ we fix an annular open neighbourhood A, we necessarily have $S(A) = S \cap A$. This cover has a finite subcover by compactness. Consider a connected component of $C \setminus T$ which is not an open disc. Then by repeated application of the fusion lemma 32 below we find it is an annulus. which concludes the proof. $\square$

Lemma 32 (Fusion lemma)**.** *Suppose U is an open subset of an analytic curve, so that $x \in U$ is of genus 0. Assume that $U \setminus \{x\}$ is a disjoint union of 2 open annuli (resp. 1 open annulus) and infinitely many open discs. Then U is an open annulus (resp. open disc).*

Sketch of the proof. Let us suppose that $U \setminus \{x\}$ is a disjoint union of 2 open annuli A_1, A_2 and open discs. Embed A_1 and A_2 in open discs D_1 and D_2 respectively, and glue $D_1 \sqcup D_2$ with X along $A_1 \sqcup A_2$. The result is a new analytic curve C' with a given point x' of genus 0 such that $C' \setminus \{x'\}$ is a disjoint union of open discs. We claim that C' is in fact $\mathbb{P}^{1,an}$. This suffices because if we remove $D_1 \sqcup D_2$ again, we recover the claim.

★★★★★prove the claim ★★★★★

The other case is dealt with similarly. $\square$

Lemma 33. *An analytic curve has finitely many points of positive genus.*

Proof. ★★★★★prove the lemma ★★★★★ $\square$

What remains for the semistable reduction theorem. In order to show theorem 19, we will study the local topology and branching at the points of analytic curves more closely. This will be done using Riemann-Zariski spaces and Temkin's $\mathbb{R}_{>0}$-graded reduction. Next, we explain how a skeleton - or equivalently a triangulation - gives rise to a semistable formal model, by glueing appropriate formal schemes for the annuli occurring in the decomposition of the complement of a triangulation.

3.2 Branching of curves

Throughout, let C denote the analytification of a connected smooth projective curve over k (which, recall, we assumed to be algebraically closed).

Definition 34. For $x \in C$ we denote $\mathrm{branch}(x)$ for the set of **branches** emanating from x. A branch is defined as an equivalence class of arcs (a subset homeomorphic to a closed interval in $\mathbb{R}$) with a fixed endpoint given by x, where two arcs are equivalent if they share a common arc containing x. For $b \in \mathrm{branch}(x)$ we denote $b(X)$ for its associated connected component of $C \setminus \{x\}$.

For $b \in \mathrm{branch}(x)$ and $\lambda \in \mathbb{R}_{>0}$, we say $|f| \leq \lambda$ **along** b if there exists an arc I in the equivalence class of b such that $|f(z)| \leq \lambda$ for all $z \in Z$.

Suppose $x \in C$ is of type 2 or 3. Let $b \in \mathrm{branch}(x)$. We associate to b an $\mathbb{R}_{>0}$-graded ring

$$\mathcal{O}_b := \bigsqcup_{\lambda \in \mathbb{R}_{>0}} \{\widetilde{f(x)}_\lambda | f \in \mathcal{O}_{X,x} \text{ and } |f| \leq \lambda \text{ along } b\},$$

where $\widetilde{f(x)}_\lambda \in \widetilde{\kappa(x)}_\lambda$.

Proposition 35. *The association $b \mapsto \mathcal{O}_b$ yields a well-defined bijection*

$$\mathrm{branch}(x) \overset{1:1}{\longleftrightarrow} \mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}\left(\widetilde{\widetilde{k}}\right)^*$$

Sketch of proof. Let us denote $P = \mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}\left(\widetilde{\widetilde{k}}\right)^*$. To check that $\mathcal{O}_b \in P$, we use that every $f \in \mathcal{O}_{C,x}$ with $|f(x)| \neq 0$ satisfies either $|f| \leq \lambda$ or $|f| \leq \lambda^{-1}$ along b.

For injectivity, we use Riemann-Roch, to construct an $f \in \mathcal{O}_{X,x}$ such that $|f| > |f(x)|$ along b, and $|f| \leq |f(x)|$ along the other branches. Also in fact, $|f| = |f(x)|$ along almost all branches (add some details ★★★★★)

For surjectivity, suppose $\mathcal{O} \in P$ is a nontrivial $\mathbb{R}_{>0}$-graded valuation ring. Using Riemann-Roch there is a $f \in \mathcal{O}_{X,x}^\times$ such that $\widetilde{f(x)} \in \mathcal{O} \setminus \tilde{k}$. This yields a finite flat morphism of germs $\phi : (X, x) \to (\mathbb{P}^{1,an}, \phi(x))$, and reduces the question to $\mathbb{P}^1$.★★★★★ □

Corollary 36. *A type 3 point has 2 branches, whereas the branches of a type 2 point x are parametrised by the closed points of the residual curve $\widetilde{C_x}$ (see definition 12).*

Proof. For a type 2 point $x \in C$, we know by lemma 24 (a) that

$$\mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}\left(\widetilde{\widetilde{k}}\right)^* = \mathbf{P}_{\widetilde{\mathcal{H}(x)}}\left(\widetilde{k}\right)^*$$

and the latter corresponds, by definition, with the closed points of $\widetilde{C_x}$ (the generic point corresponds with the trivial valuation ring).

Next suppose $x \in C$ is a type 3 points. Then $\mathcal{H}(x)/k$ is a finite extension and $|\mathcal{H}(x)^{\times}|/|k^{\times}|$ is a $\mathbb{Q}$-vector space of dimension 1, let us pick a generator λ. Suppose $\tau \in \widetilde{\widetilde{\mathcal{H}(x)}}$ is an element of degree λ. Denote F for the $\mathbb{R}_{>0}$-graded subfield of $\widetilde{\widetilde{\mathcal{H}(x)}}$ generated by $\tilde{\tilde{k}}$ and τ. Then $F[\tau, \tau^{-1}]$ is also a $\mathbb{R}_{>0}$-graded field, and by lemma 24 (a),(b) we find that

$$\mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}\left(\tilde{\tilde{k}}\right)^{*} = \mathbf{P}_{\widetilde{\widetilde{\mathcal{H}(x)}}}(F)^{*} = \mathbf{P}_{F[\tau,\tau^{-1}]}(F)^{*} = \{F[\tau], F[\tau^{-1}]\}.$$

□

3.3 Local structure of analytic curves

Here our goal is to show theorem 19, i.e. we will identify a fundamental basis of open neighbourhoods for each point of C. The proof will be spread out over the coming paragraphs, according to the type of the point.

type 1 points

★★★★★

type 2 points

Lemma 37. *Let x be a type 2 point. Almost all connected components of $X \setminus \{x\}$ are open discs.*

Proof. Let $\mathcal{C}$ be the residual curve of x, and let f be a rational function on X such that $|f(x)| = 1$ and $\widetilde{f(x)}$ yields a generically étale morphism $\widetilde{f(x)} : \mathcal{C} \to \mathbb{P}^1_{\tilde{k}}$. It induces a finite flat $\varphi : X \to \mathbb{P}^{1,an}_k$. Let Π be the set of connected components V of $X \setminus \{x\}$ such that

1. there is exactly one branch b_v from x emanating to V
2. $\mathcal{C} \to \mathbb{P}^1_{\tilde{k}}$ is unramified at the point corresponding to b_v
3. V contains no preimages of $\varphi(x)$.

by $\mathcal{C} \to \mathbb{P}^1_{\tilde{k}}$ generically étale, almost all connected components of $X \setminus \{x\}$ belong to Π.

There exists a connected component U of $\mathbb{P}^{1,an}_k \setminus \{\varphi(x))\}$ such that V is connected component of $\varphi^{-1}(U)$. ★★★★★In fact, V is isomorphic to U due to the unramifiedness of $\widetilde{f(x)}$ □

type 3 points

Lemma 38 (Gabber). *Let $P_1, P_2 \in C(k)$. Then there exist equivalent effective divisors D_1, D_2 such that D_1 is supported in P_1 and*

$$p \nmid \mathrm{mult}_{P_2}(D_2).$$

Proof. Since $[p]$ is an isogeny on $\mathrm{Jac}(X)$, there is a line bundle $\mathcal{L}$ such that $\mathcal{L}^p = \mathcal{O}(P_1 - P_2)$. By Riemann-Roch there is an effective divisor Δ and $m > 0$ such that

$$\mathcal{L} \otimes \mathcal{O}(mP_2) \cong \mathcal{O}(\Delta).$$

Raising this to the p-th power we get

$$(pm+1)P_1 \sim P_2 + p\Delta.$$

So we can put $D_1 = (pm+1)P_1$ and $D_2 = P_2 + p\Delta$. □

Corollary 39. *Let $x \in X$ be a type 2 or 3 point, and let $b \in \mathrm{branch}(x)$. Then there is a finite étale morphism of germs*

$$\phi : (X, x) \to (\mathbb{P}^{1,an}, \phi(x))$$

such that the degree of ($b \to \phi(b)$) is prime to p.

Proof. Note $\mathrm{branch}(x)$ has at least 2 distinct branches b_1, b_2. Consider rigid points P_1, P_2 in $b_1(X), b_2(X)$ respectively, and apply the lemma to get a rational function f only vanishing in P_1 and all poles in $b_2(X)$ with at least one in P_2.

Now f induces a finite flat morphism $X \to \mathbb{P}^{1,an}$. As $\mathrm{div}(f)$ is not divisible by p, f is generically étale, and so étale at x. Let $U = \phi(b_2(X))$, then $0 \notin U$ and so U is a connected component of $\mathbb{P}^{1,an} \setminus \{\phi(x)\}$. This implies $b_2(X)$ is a connected component of $\phi^{-1}(U)$.

U contains ∞, and counted with multiplicity $\phi_{|}b_2(X)^{-1}(\infty)$ contains a number of points which is prime to p. Now $deg(b \to \phi(b)) = deg(V \to U)$ is prime to p. □

Proposition 40. *Suppose x is a type 3 point of X. Then it admits a basis of open neighbourhoods of annuli.*

Sketch of the proof. x has precisely 2 branches.

let $b \in \mathrm{branch}(x)$ be one of these and let $\phi : (X, x) \to (\mathbb{P}^{1,an}, \phi(x))$ be an étale map of germs of analytic curves, with $d := deg(b \to \phi(b))$ prime to p. after a homothety we can suppose that $\phi(x) = \eta_r$ for some $r \notin |k^\times|$, so that b is the branch for which $|T| < r$.

Because we have a bijection at the level of branches, we get $\deg(\kappa(x)/\kappa(\eta_r)) = d$. So $\kappa(x)/\kappa(\eta_r)$ is a tamely ramified finite extension of Henselian valued fields. As $|\kappa(\eta_r)^\times| = |k^\times| \oplus |T(\eta_r)|^{\mathbb{Z}}$, we have

$$\kappa(x) \cong \kappa(\eta_r)[\zeta]/(\zeta^d - T(\eta_r)^e) \cong k(\eta_r)[\zeta']/(\zeta'^d - T(\eta_r)).$$

so $(X, x) \cong (\mathbb{P}^{1,an}, \eta_{\sqrt[d]{r}})$ as finite étale $(\mathbb{P}^{1,an}, \eta_r)$-germs, where the latter is viewed as such a germ by raising to the d'th power. The result follows from the local description of $\mathbb{P}^{1,an}$. □

type 4 points

★★★★★under construction ★★★★★

3.4 Construction of a semistable formal model

★★★★★under construction ★★★★★

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