STUDY GROUP

EMERTON-GEE STACKS AND APPLICATIONS

SUMMER TERM 2022

Moduli spaces of (local) Galois representations initially came up in the form of Mazur's rings [8] governing the deformation theory of mod p Galois representations. The role of his Galois deformation rings quickly became of central importance in the Langlands program and, in particular, the close study of their geometry proved to be a fruitful area of research. Despite the success of the theory, it is clearly limited by the fact that we first need to fix a mod p Galois representation $\bar{\rho}$ to be able to move in families of Galois representations. More precisely, it is plausible to wish for a global geometric object $\mathcal{X}_d$ parameterizing d-dimensional (continuous) Galois representations $G_K \to \mathrm{GL}_d(A)$ of a p-adic field K with A a topological $\mathbf{Z}_p$-algebra such that the deformation theory of its $\overline{\mathbf{F}}_p$-points recovers Mazur's deformation rings.

Such an object has been constructed by Carl Wang-Erickson in [9] in the form of a formal algebraic stack. However, the families described by his stacks are in some sense still quite limited. In particular, they decompose into disjoint unions of moduli spaces of Galois representations whose residual Galois representation has fixed semisimplification.

Instead, [1] takes a different approach by redefining mod p Galois representations as Fontaine's (φ, Γ)-modules and considering families of (φ, Γ)-modules. As the main theorem of [1] explains, this stack is again a formal algebraic stack with nice geometric properties and dimension theory. Moreover, it contains Wang-Erickson's stack as a substack and their deformation theory at $\overline{\mathbf{F}}_p$-valued points coincides! However, over the stack of Emerton and Gee we see more interesting families appear. Moreover, (φ, Γ)-modules provide a natural ground field to connect this stack to p-adic Hodge theory, in particular, to Kisin's (potentially) semistable/crystalline deformation rings. Furthermore, as the proof of Colmez's p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ is based on the use of (φ, Γ)-modules, it is natural to expect the Emerton-Gee stack to be the right space to formulate a p-adic local Langlands in families.

The goal of this study group is to get a feeling for the construction and geometry of the Emerton-Gee stack, see it in action in form of some applications, and to learn about its (conjectural) role in the Langlands program. For this the obvious main reference is the paper of Emerton and Gee [1]. However, since the foundations of their theory are often on the technical side, when discussing the construction of their stacks we will try to focus on developing an intuition for the results appearing in [1] rather than discussing the actual arguments which are mostly beyond the scope of this study group. Therefore, for the first 6 talks we will probably closely follow the very well-written survey article [2] on the subject. For the last two talks we are planning to learn about the speculations on the role of the Emerton-Gee stack in the (p-adic local) Langlands program.

Talk 1: Overview/motivation. Bence Hevesi - 12/05
We motivate the problem of globalizing Mazur's local deformation rings. We further hint on the obstacles and difficulties arising in the $\ell = p$ setup as opposed to the case of $\ell \neq p$ and how it leads to considering families of (φ, Γ)-modules. Finally, we give a brief overview of the plan for the rest of the study group.

Talk 2: Formal algebraic stacks for the working mathematician. Name - 19/05
The purpose of this talk is to go from schemes and stacks to their formal and ind-analagoues. You may briefly recall what a(n algebraic) stack is (as a category fibered in groupoids) and then proceed to define Ind-algebraic stacks see [1, Appendix A], in particular A.5-A.13. Then we are in the position to talk about versal rings (again see [1, Appendix A]) and scheme-theoretic images. It would be good to at least mention [3, Theorem 1.1.1] and explain briefly the presence of certain assumptions in the statement. Use the rich variety of examples in [2, Section 4.3] to illustrate the technical definitions mentioned before.

Talk 3: Preliminary definitions and the geometry of stacks of Breuil-Kisin and étale φ-modules. Name - 26/05
The first goal of this talk is to introduce the notion of Breuil-Kisin, φ and (φ, Γ)-modules and explain their relations to Galois representations. For simplicity, the speaker is advised to assume that the base field is unramified over $\mathbf{Q}_p$. Finally, as a preparation for the next talk, sketch the proof of [2, Corollary 4.2.3] about the geometry of the stack of étale φ-modules. The main reference for this talk is [2, Lecture 3, Lecture 4]. For more details see [1, Section 2, Section 3.1] and [4].

Talk 4: Stacks of (φ, Γ)-modules and their crystalline/semistable substacks. Name - 31/05 (?)
In this talk we finally introduce the main objects of interest of the study group. First introduce the stack of (φ, Γ)-modules and state [2, Proposition 5.3.9]. Say a few words on how to deduce it from [2, Corollary 4.2.3]. The rest of the talk is concerned with introducing the crystalline and semistable closed substacks of the Emerton-Gee stack $\mathcal{X}_d$ using Breuil-Kisin-Fargues G_K-modules (admitting all descents). After defining these substacks, state [1, Theorem 4.8.12, Theorem 4.8.14] and explain as much of their proof as time permits. The main reference for the necessary definitions and sketch of arguments is [2, Section 3.4, Lecture 5, Lecture 6]. For more details one can consult the references therein (e.g., [1, Section 3, Section 4, Appendix F]).

Talk 5: The Herr complex and the geometry of $\mathcal{X}_d$. Name - 10/06
This talk is on the proof of most parts of the main theorem [1, Theorem 1.2.1]. More precisely, the goal is to sketch the proof [1, Theorem 5.5.12]. It would be nice and useful to see some of the arguments involved in action on concrete examples. For reference see [1, Section 5] and [2, Lecture 7].

Talk 6: Existence of crystalline lifts. Name - 17/06
This talk is the first on the applications of the Emerton-Gee stack and its geometry. Explain the proof of the main result of [1] on the existence of (potentially diagonalizable) crystalline lifts (see Theorem 6.4.4 in *loc.cit.*). It would be nice to point out the crucial extra feature of the global object $\mathcal{X}_d$ which was missing in previous attempts

(e.g. [5]) to prove such a result. Again, the speaker is encouraged to demonstrate the arguments say in the case of $d = 2$. Deduce the last remaining bit of the proof of [1, Theorem 1.2.1] (see *loc.cit.* Theorem 6.5.1). If time permits, sketch the application of the existence of potentially diagonalizable crystalline lifts to globalization of local Galois representations [1, Corollary 6.4.7] (see the appendix of [6]).

Talk 7-8: Emerton-Gee stacks and the Langlands program. Name - 23/06 + 05/07
For the last two talks the plan is less concrete. The idea would be to elaborate on [2, Lecture 10]. More precisely, we would like to focus on Section 10.3 and 10.4 of *loc.cit.* on investigating the relationship of the Emerton-Gee stack with local Langlands. For these talks we hope to have access at some point to a draft of [7].

References

[1] Matthew Emerton, Toby Gee: *Moduli stacks of étale* (φ, Γ)*-modules and the existence of crystalline lifts*,
`https://www.ma.imperial.ac.uk/~tsg/Index_files/moduli.pdf`

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`https://www.ma.imperial.ac.uk/~tsg/Index_files/bonn-lectures-survey-article.pdf`

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`https://www.ma.imperial.ac.uk/~tsg/Index_files/proper.pdf`

[4] George Pappas, Michael Rapoport: Φ*-modules and coefficient spaces, Mosc. Math. J. 9 (2009), no. 3, 625-663, back matter*

[5] Toby Gee, Florian Herzig, Tong Liu, and David Savitt, Potentially crystalline lifts of certain prescribed types, Doc. Math. 22 (2017), 397-422.

[6] Matthew Emerton and Toby Gee, A geometric perspective on the Breuil-Mézard conjecture, J. Inst. Math. Jussieu 13 (2014), no. 1, 183 223.

[7] Matthew Emerton, Toby Gee and Eugen Hellmann, p-adic moduli stacks of Galois representations, and the p-adic Langlands program, 2022

[8] Barry Mazur, Deforming Galois representations, Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer, New York, 1989, pp. 385-437.

[9] Carl Wang-Erickson, Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings, Math. Ann. 371 (2018), no. 3-4, 1615-1681.