Linear Algebra

A Comprehensive list of all Definitions

Swapnil Dash

Contents

1 Linear System

Definition 1.1. ***General solution of a linear system***

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a ***general solution*** *of the system.*

2 Matrix

Definition 2.1. ***Matrix***

A ***matrix*** *is a rectangular array of numbers. The numbers in the array are called the* ***entries*** *in the matrix.*

Definition 2.2. ***Equality of Matrices***

Two matrices are defined to be ***equal*** *if they have the same size and their corresponding entries are equal.*

Definition 2.3. ***Sum and difference of Matrices***

If A and B are matrices of the same size, then the ***sum*** *$A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the* ***difference*** *$A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A.* ***Matrices of different sizes cannot be added or subtracted.***

In matrix notation, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ have the same size, then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \text{ and } (A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Definition 2.4. ***Scalar Multiple of a Matrix***

If A is any matrix and c is any scalar, then the ***product*** *cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a* ***scalar multiple of A.***

In matrix notation, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, then

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

Definition 2.5. ***Multiplication of Matrices***

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** *AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.*

In matrix notation, if $A = \begin{bmatrix} a_{ij} \end{bmatrix}_{m\times r}$ *and* $B = \begin{bmatrix} b_{ij} \end{bmatrix}_{r\times n}$ *matrix, then,*

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \dots & b_{rj} & \dots & b_{rn} \end{bmatrix}$$

the entry $(AB)_{ij}$ *in row* i *and column* j *of* AB *is given by*

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ir}b_{rj}$$

Definition 2.6. ***Linear combination of Matrices***

If $A_1, A_2, \ldots, A_r$ *are matrices of the same size, and if* $c_1, c_2, \ldots, c_r$ *are scalars, then an expression of the form*

$$c_1A_1 + c_2A_2 + \ldots + c_rA_r$$

is called a ***linear combination*** *of* $A_1, A_2, \ldots, A_n$ *with* ***coefficients*** $c_1, c_2, \ldots, c_n$.

Definition 2.7. ***Transpose of a Matrix***

If A *is any* $m \times n$ *matrix, then the* ***transpose of A****, denoted by* A^T*, is defined to be the* $n \times m$ *matrix that results by interchanging the rows and columns of* A*. In matrix notation*

$$(A^T)_{ij} = (A)_{ji}$$

Definition 2.8. ***Trace of a Matrix***

If A *is a square matrix, then the* ***trace of A****, denoted by* $tr(A)$*, is defined to be the sum of the entries on the main diagonal of* A*.* ***The trace of a matrix is undefined if it is not a square matrix.***

Definition 2.9. ***Inverse of a Matrix***

If A *is a* ***square matrix****, and if a matrix* B *of the same size can be found such that* $AB = BA = I$*, then* A *is said to be* ***invertible*** *(or* ***nonsingular****) and* B *is called an* ***inverse*** *of* A*. If no such matrix* B *can be found, then* A *is said to be* ***singular.***

Definition 2.10. ***Row equivalent***

Matrices A *and* B *are said to be* ***row equivalent*** *if either can be obtained from the other by a sequence of elementary row operations.*

Definition 2.11. ***Elementary Matrix***

*A matrix E is called an **elementary matrix** if it can be obtained from an identity matrix by performing a single elementary row operation.*

Definition 2.12. ***Symmetric***

A square matrix A *is said to be **symmetric** if* $A = A^T$

Definition 2.13. ***Matrix Transformation***

*If f is a function with domain R^n and codomain R^m, then we say that f is a **transformation** from R^n to R^m or that f **maps** from R^n to R^m, which is denoted by*

$$f : R^n \to R^m$$

*In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n*

3 Determinants

Definition 3.1. ***Minor and Cofactor***

*If A is a square matrix, then the **minor of entry** a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i_{th} row and j_{th} column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry** a_{ij}*

Definition 3.2. ***Determinant of a Matrix***

*If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the **determinant of A**, and the sums themselves are called **cofactor expansions of A**. That is,*

$$det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

cofactor expansion along the jth column

Definition 3.3. ***Adjoint of a Matrix***

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{i1} & C_{i2} & \ldots & C_{in} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}$$

*is called the **matrix of cofactors from A**. The transpose of this matrix is called the **adjoint of A** and is denoted by $adj(A)$.*

4 Euclidean Vector Spaces

Definition 4.1. ***Ordered n-tuple***

*If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(v_1, v_2, \ldots, v_n)$. The set of all ordered n-tuples is called **n-space** and is denoted by R^n.*

Definition 4.2. ***Equality of Vectors***

*Vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$ are said to be **equal** if*

$$v_1 = w_1, v_2 = w_2, \ldots, v_n = w_n$$

We indicate this by writing $\mathbf{v} = \mathbf{w}$

Definition 4.3. ***Linear combination of Vectors***

*If $\mathbf{w}$ is a vector in $\mathbb{R}^n$, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ if it can be expressed in the form*

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$$

*where $k_1, k_2, \ldots, k_n$ are scalars. These scalars are called the **coefficients** the linear combination.*

Definition 4.4. ***Norm of a Vector***

*If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$, then the **norm** of $\mathbf{v}$ (also called the **length** of $\mathbf{v}$ or the **magnitude of** $\mathbf{v}$) is denoted by $||\mathbf{v}||$, and is defined by this formula*

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

Definition 4.5. ***Distance between Vectors***

*If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in $\mathbb{R}^n$, then we denote the **distance** between $\mathbf{u}$ and $\mathbf{v}$ by $d(\mathbf{u}, \mathbf{v})$ and define it to be*

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}|| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$