

# Bayesian BM25: A Probabilistic Framework for Hybrid Text and Vector Search

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*"The theory of probabilities is at bottom nothing but common sense reduced to calculus."*

— Pierre-Simon Laplace

## Abstract

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This paper presents Bayesian BM25, a novel probabilistic framework that transforms traditional BM25 relevance scores into calibrated probability estimates, enabling principled integration of lexical and semantic search signals. We demonstrate that standard BM25 scores, while effective for ranking, suffer from fundamental interpretability limitations: unbounded range, query-dependent magnitudes, and incompatibility with other ranking signals. We address these limitations by applying Bayesian inference with a sigmoid likelihood model and composite prior design incorporating term frequency and document length signals. We prove that our transformation preserves the monotonicity properties essential for ranking while producing well-calibrated probability outputs in  $[0, 1]$ . We extend this framework to hybrid search, deriving probabilistic score combination formulas for Boolean query operations under independence assumptions. We show that log-space computation ensures numerical stability for extreme probabilities. Furthermore, we analyze the compatibility of Bayesian BM25 with WAND (Weak AND) and BMW (Block-Max WAND) optimization algorithms, proving that the transformation preserves safe document pruning through modified upper bounds that account for document-dependent priors. Experimental results demonstrate that Bayesian BM25 achieves comparable ranking quality to standard BM25 while enabling principled multi-signal fusion without ad-hoc normalization or weighting schemes.

## 1. Introduction and Motivation

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### 1.1 The BM25 Score Interpretation Problem

The BM25 (Best Matching 25) scoring function, introduced by Robertson and Zaragoza (2009), has become the de facto standard for lexical relevance scoring in information retrieval systems. Despite its empirical success, BM25 scores present fundamental challenges for modern search applications that require combining multiple ranking signals.

**Definition 1.1.1** (BM25 Score). For a document  $D$  and query  $Q = \{t_1, t_2, \dots, t_m\}$ , the BM25 score is defined as:

$$\text{BM25}(D, Q) = \sum_{t \in Q} \text{IDF}(t) \cdot \frac{f(t, D) \cdot (k_1 + 1)}{f(t, D) + k_1 \cdot B(D)} \quad (1)$$

where:

- $f(t, D)$  is the term frequency of  $t$  in document  $D$
- $\text{IDF}(t)$  is the inverse document frequency of term  $t$
- $B(D) = 1 - b + b \cdot \frac{|D|}{\text{avgdl}}$  is the length normalization factor
- $k_1$  and  $b$  are tuning parameters (typically  $k_1 = 1.2$  and  $b = 0.75$ )

**Problem 1.1.2** (Score Interpretability). BM25 scores suffer from the following interpretability limitations:

1. **Unbounded Range:**  $\text{BM25}(D, Q) \in [0, +\infty)$ , making absolute interpretation impossible
2. **Query Dependence:** Score magnitudes vary with query length and term specificity
3. **Corpus Dependence:** IDF values depend on collection statistics, preventing cross-corpus comparison
4. **Signal Incompatibility:** Direct combination with bounded signals (e.g., vector similarity in  $[0, 1]$ ) leads to dominated or unbalanced contributions

## 1.2 The Multi-Signal Fusion Challenge

Modern search systems require combining heterogeneous ranking signals with different scales and distributions.

**Definition 1.2.1** (Multi-Signal Ranking Function). A multi-signal ranking function combines  $n$  signals:

$$\text{Score}(D, Q) = f(s_1(D, Q), s_2(D, Q), \dots, s_n(D, Q)) \quad (2)$$

where each  $s_i : \mathcal{D} \times \mathcal{Q} \rightarrow \mathbb{R}$  is a scoring function with potentially different range and distribution.

**Theorem 1.2.2** (Signal Dominance in Naive Combination). For additive combination of signals with different scales:

$$\text{Score}_{\text{naive}} = \sum_{i=1}^n s_i \quad (3)$$

the signal with the largest expected magnitude will dominate the ranking.

*Proof.* Let  $\mu_i = \mathbb{E}[s_i]$  and  $\sigma_i = \text{Std}[s_i]$ . For two signals with  $\mu_1 \gg \mu_2$ , the probability that signal 2 affects the ranking is:

$$P(s_2 > s_1) \leq P\left(\frac{s_1 - \mu_1}{\sigma_1} < -\frac{\mu_1 - \mu_2}{\sigma_1}\right) \approx 0 \quad (4)$$

when  $\mu_1 - \mu_2 \gg \sigma_1$ .

## 1.3 Reciprocal Rank Fusion Limitations

Reciprocal Rank Fusion (RRF), proposed by Cormack et al. (2009), is a common approach to combining ranked lists.

**Definition 1.3.1** (Reciprocal Rank Fusion). For  $n$  ranked lists  $R_1, \dots, R_n$ , the RRF score is:

$$\text{RRF}(D) = \sum_{i=1}^n \frac{1}{k + \text{rank}_i(D)} \quad (5)$$

where  $k$  is a constant (typically  $k = 60$ ) and  $\text{rank}_i(D)$  is the rank of document  $D$  in list  $R_i$ .

**Theorem 1.3.2** (RRF Limitations). RRF suffers from the following theoretical limitations:

1. **Information Loss:** Score magnitudes are discarded, losing confidence information
2. **Rank Sensitivity:** Small perturbations in scores can cause large rank changes
3. **Non-Commutativity with Filtering:**  
 $\text{RRF}(\text{Filter}(L_1), \text{Filter}(L_2)) \neq \text{Filter}(\text{RRF}(L_1, L_2))$
4. **Arbitrary Constant:** The  $k$  parameter lacks theoretical justification
5. **Missing Document Handling:** Undefined behavior for documents appearing in only some lists

*Proof of (3).* Consider  $L_1 = [A : 10, B : 5, C : 3]$  and  $L_2 = [B : 8, A : 4, C : 1]$ . With filter threshold  $> 6$ :

- Left side:  $\text{RRF}([A : 10], [B : 8]) = \{A : \frac{1}{61}, B : \frac{1}{61}\}$
- Right side:  $\text{Filter}(\text{RRF}(L_1, L_2)) = \text{Filter}(\{A : \frac{1}{61} + \frac{1}{62}, B : \frac{1}{62} + \frac{1}{61}, C : \frac{1}{63} + \frac{1}{63}\})$

The results differ because filtering after fusion considers documents that wouldn't pass individual filters.

## 1.4 Our Contribution

We propose Bayesian BM25, a probabilistic framework that:

1. Transforms BM25 scores into calibrated probabilities in  $[0, 1]$
  2. Preserves monotonicity properties essential for ranking
  3. Enables principled probabilistic combination of multiple signals
  4. Maintains compatibility with WAND/BMW optimization algorithms through modified upper bounds
  5. Supports online parameter learning for domain adaptation
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## 2. Mathematical Preliminaries

### 2.1 Probability Calibration

**Definition 2.1.1** (Calibrated Probability). A scoring function  $s : \mathcal{D} \times \mathcal{Q} \rightarrow [0, 1]$  is calibrated if:

$$P(\text{relevant} \mid s(D, Q) = p) = p \quad (6)$$

for all  $p \in [0, 1]$ .

**Definition 2.1.2** (Sigmoid Function). The sigmoid function  $\sigma : \mathbb{R} \rightarrow (0, 1)$  is defined as:

$$\sigma(x) = \frac{1}{1 + e^{-x}} \quad (7)$$

**Lemma 2.1.3** (Sigmoid Properties). The sigmoid function satisfies:

1.  $\sigma(-x) = 1 - \sigma(x)$  (symmetry)
2.  $\frac{d\sigma}{dx} = \sigma(x)(1 - \sigma(x))$  (derivative)

3.  $\lim_{x \rightarrow \infty} \sigma(x) = 1$  and  $\lim_{x \rightarrow -\infty} \sigma(x) = 0$  (bounds)

*Proof.* Property (1):

$$\sigma(-x) = \frac{1}{1 + e^x} = \frac{e^{-x}}{e^{-x} + 1} = 1 - \frac{1}{1 + e^{-x}} = 1 - \sigma(x) \quad (8)$$

## 2.2 Bayesian Inference Framework

**Definition 2.2.1** (Bayes' Theorem). For hypothesis  $H$  and evidence  $E$ :

$$P(H | E) = \frac{P(E | H) \cdot P(H)}{P(E)} \quad (9)$$

**Definition 2.2.2** (Binary Classification Posterior). For relevance  $R \in \{0, 1\}$  and score  $s$ :

$$P(R = 1 | s) = \frac{P(s | R = 1) \cdot P(R = 1)}{P(s | R = 1) \cdot P(R = 1) + P(s | R = 0) \cdot P(R = 0)} \quad (10)$$


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## 3. BM25 Theoretical Foundation

### 3.1 IDF Computation

**Definition 3.1.1** (Robertson-Sparck Jones IDF). The inverse document frequency with smoothing is:

$$\text{IDF}(t) = \ln \left( \frac{N - \text{df}(t) + 0.5}{\text{df}(t) + 0.5} + 1 \right) \quad (11)$$

where  $N$  is the total document count and  $\text{df}(t)$  is the document frequency of term  $t$ .

**Theorem 3.1.2** (IDF Non-Negativity). For any term  $t$  with  $\text{df}(t) \leq N$ :

$$\text{IDF}(t) \geq 0 \quad (12)$$

*Proof.* The argument of the logarithm is:

$$\frac{N - \text{df}(t) + 0.5}{\text{df}(t) + 0.5} + 1 \geq 1 \quad (13)$$

since  $N - \text{df}(t) \geq 0$ . Therefore,  $\text{IDF}(t) = \ln(\cdot) \geq \ln(1) = 0$ .

**Theorem 3.1.3** (IDF Monotonicity). IDF is monotonically decreasing in document frequency:

$$\text{df}(t_1) < \text{df}(t_2) \implies \text{IDF}(t_1) > \text{IDF}(t_2) \quad (14)$$

*Proof.* Let  $f(x) = \ln \left( \frac{N-x+0.5}{x+0.5} + 1 \right)$ . Taking the derivative:

$$f'(x) = \frac{-(N+1)}{(x+0.5)(N-x+0.5)+(x+0.5)^2} < 0 \quad (15)$$

for  $0 \leq x \leq N$ , proving monotonicity.

### 3.2 Monotonicity-Preserving BM25 Formulation

**Definition 3.2.1** (Rewritten BM25 Scoring). We express BM25 in a numerically stable form:

$$\text{score}(f, n) = w - \frac{w}{1 + f \cdot \text{inv\_norm}} \quad (16)$$

where:

- $w = \text{boost} \cdot \text{IDF}$  (weight)
- $\text{inv\_norm} = \frac{1}{k_1 \cdot ((1-b) + b \cdot \frac{n}{\text{avgdl}})}$  (inverse normalization)
- $f$  is term frequency
- $n$  is document length

**Theorem 3.2.2** (BM25 Monotonicity). The BM25 score satisfies:

1. **Monotonic in term frequency:**  $f_1 > f_2 \implies \text{score}(f_1, n) > \text{score}(f_2, n)$
2. **Anti-monotonic in document length:**  $n_1 > n_2 \implies \text{score}(f, n_1) < \text{score}(f, n_2)$

*Proof of (1).* Let  $g(f) = w - \frac{w}{1 + f \cdot c}$  where  $c = \text{inv\_norm} > 0$ . Then:

$$g'(f) = \frac{w \cdot c}{(1 + f \cdot c)^2} > 0 \quad (17)$$

proving monotonicity in  $f$ .

*Proof of (2).* Let  $h(n) = w - \frac{w}{1 + f / (k_1 \cdot ((1-b) + b \cdot n / \text{avgdl}))}$ . As  $n$  increases, the denominator of  $\text{inv\_norm}$  increases, making  $\text{inv\_norm}$  smaller, which decreases the overall score.

**Theorem 3.2.3** (BM25 Upper Bound). For any term with weight  $w$ :

$$\sup_{f,n} \text{score}(f, n) = w = \text{boost} \cdot \text{IDF} \quad (18)$$

*Proof.* As  $f \rightarrow \infty$ :

$$\lim_{f \rightarrow \infty} \left( w - \frac{w}{1 + f \cdot c} \right) = w - 0 = w \quad (19)$$

## 4. Bayesian BM25 Framework

### 4.1 Probabilistic Model

**Definition 4.1.1** (Likelihood Model). We model the probability of observing score  $s$  given relevance  $R$  using a parametric sigmoid:

$$P(s | R = 1) = \sigma(\alpha \cdot (s - \beta)) = \frac{1}{1 + e^{-\alpha(s-\beta)}} \quad (20)$$

where:

- $\alpha > 0$  controls the sigmoid steepness (score sensitivity)
- $\beta$  controls the sigmoid midpoint (decision boundary)

**Definition 4.1.2** (Symmetric Likelihood Assumption). We assume:

$$P(s | R = 0) = 1 - P(s | R = 1) = \sigma(-\alpha \cdot (s - \beta)) \quad (21)$$

**Theorem 4.1.3** (Posterior Probability Formula). Under the symmetric likelihood assumption, the posterior probability of relevance is:

$$P(R = 1 | s) = \frac{L \cdot p}{L \cdot p + (1 - L) \cdot (1 - p)} \quad (22)$$

where  $L = \sigma(\alpha(s - \beta))$  is the likelihood and  $p = P(R = 1)$  is the prior.

*Proof.* Applying Bayes' theorem with the symmetric likelihood:

$$P(R = 1 | s) = \frac{P(s | R = 1) \cdot P(R = 1)}{P(s | R = 1) \cdot P(R = 1) + P(s | R = 0) \cdot P(R = 0)} \quad (23)$$

$$= \frac{L \cdot p}{L \cdot p + (1 - L) \cdot (1 - p)} \quad (24)$$

## 4.2 Composite Prior Design

**Definition 4.2.1** (Term Frequency Prior). The prior probability based on term frequency is:

$$P_{\text{tf}}(f) = 0.2 + 0.7 \cdot \min\left(1, \frac{f}{10}\right) \quad (25)$$

**Definition 4.2.2** (Field Norm Prior). The prior probability based on document length normalization is:

$$P_{\text{norm}}(n) = 0.3 + 0.6 \cdot (1 - \min(1, |n - 0.5| \cdot 2)) \quad (26)$$

**Definition 4.2.3** (Composite Prior). The combined prior probability is:

$$P_{\text{prior}}(f, n) = \text{clamp}(0.7 \cdot P_{\text{tf}}(f) + 0.3 \cdot P_{\text{norm}}(n), 0.1, 0.9) \quad (27)$$

**Theorem 4.2.4** (Prior Bounds). For all valid inputs:

$$0.1 \leq P_{\text{prior}}(f, n) \leq 0.9 \quad (28)$$

*Proof.* The clamp function explicitly enforces these bounds, preventing extreme priors from dominating the posterior.

**Remark 4.2.5** (Empirical Bayes Justification). The composite prior  $P_{\text{prior}}(f, \hat{n})$  uses the same features — term frequency  $f$  and encoded field norm  $\hat{n}$  — that contribute to the BM25 score  $s(f, \hat{n})$ . In a strictly classical Bayesian framework, the prior should be independent of the observed evidence. Our design constitutes an **empirical Bayes** approach (Robbins, 1956; Efron, 2010), in which the prior is informed by observed data features through a different functional form than the likelihood.

This is standard practice in hierarchical modeling: the James-Stein estimator, empirical Bayes confidence intervals, and parametric empirical Bayes methods all estimate prior hyperparameters from data. The key distinction is that BM25 combines  $f$  and  $\hat{n}$  through a specific nonlinear function parametrized by  $(k_1, b)$  to produce a **relevance score**, while the prior combines them through a different function to encode a **document quality signal** — the meta-level observation that documents with moderate length and non-trivial term frequency are more likely to be genuinely relevant, as opposed to spam, boilerplate, or incidental matches.

For applications requiring strict prior-evidence independence, the prior can be constructed from features that do not enter the BM25 score: document recency, source authority (PageRank), content type, or other static quality signals. Definition 4.2.3 provides a self-contained prior that requires no external metadata, at the cost of the empirical Bayes approximation.

## 4.3 Monotonicity Preservation

**Theorem 4.3.1** (Bayesian BM25 Monotonicity). The Bayesian BM25 transformation preserves the monotonicity of BM25 scores:

$$s_1 > s_2 \implies P(R = 1 | s_1) > P(R = 1 | s_2) \quad (29)$$

for fixed prior  $p$  and  $\alpha > 0$ .

*Proof.* Let  $g(s) = P(R = 1 | s)$  with fixed prior  $p$ . We show  $g'(s) > 0$ .

Let  $L(s) = \sigma(\alpha(s - \beta))$ . Then  $L'(s) = \alpha \cdot L(s) \cdot (1 - L(s)) > 0$ .

$$g(s) = \frac{L \cdot p}{L \cdot p + (1 - L) \cdot (1 - p)} \quad (30)$$

Let  $A = L \cdot p$  and  $B = (1 - L) \cdot (1 - p)$ . Then  $g = \frac{A}{A+B}$ .

$$g' = \frac{A'(A+B) - A(A' - B')}{(A+B)^2} = \frac{A' \cdot B + A \cdot B'}{(A+B)^2} \quad (31)$$

Since  $A' = L' \cdot p > 0$  and  $B' = -L' \cdot (1 - p) < 0$ :

$$g' = \frac{L' \cdot p \cdot (1 - L)(1 - p) + L \cdot p \cdot L'(1 - p)}{(A+B)^2} = \frac{L' \cdot p \cdot (1 - p)}{(A+B)^2} > 0 \quad (32)$$

**Remark 4.3.2** (Scope of Monotonicity). Theorem 4.3.1 establishes that the posterior is monotone in  $s$  for fixed prior  $p$ . When the composite prior  $P_{\text{prior}}(f, \hat{n})$  varies across documents (Definition 4.2.3), the posterior is a function of both  $s$  and  $p$ , and the overall ranking may differ from the BM25 ranking. This has implications for WAND pruning safety — see Theorem 6.1.2 for the corrected upper bound that accounts for the prior's variability.

## 5. Hybrid Search Score Combination

### 5.1 Probabilistic Conjunction (AND)

**Theorem 5.1.1** (Probabilistic AND). For independent relevance events with probabilities  $p_1, p_2, \dots, p_n$ :

$$P(R_1 \wedge R_2 \wedge \dots \wedge R_n) = \prod_{i=1}^n p_i \quad (33)$$

**Definition 5.1.2** (Log-Space Conjunction). For numerical stability, we compute:

$$P(\text{AND}) = \exp \left( \sum_{i=1}^n \ln(p_i) \right) \quad (34)$$

**Theorem 5.1.2** (Conjunction Bounds). For  $p_i \in (0, 1)$ :

$$0 < P(\text{AND}) < \min_i p_i \quad (35)$$

*Proof.* Since each  $\ln(p_i) < 0$  for  $p_i < 1$ , the sum is negative, and  $\exp$  of a negative number is less than 1. The product is strictly less than any individual factor.

### 5.2 Probabilistic Disjunction (OR)

**Theorem 5.2.1** (Probabilistic OR). For independent relevance events:

$$P(R_1 \vee R_2 \vee \cdots \vee R_n) = 1 - \prod_{i=1}^n (1 - p_i) \quad (36)$$

**Definition 5.2.2** (Log-Space Disjunction). For numerical stability:

$$P(\text{OR}) = 1 - \exp\left(\sum_{i=1}^n \ln(1 - p_i)\right) \quad (37)$$

**Theorem 5.2.2** (Disjunction Bounds). For  $p_i \in (0, 1)$ :

$$\max_i p_i < P(\text{OR}) < 1 \quad (38)$$

*Proof.* Since  $1 - p_i < 1$  for  $p_i > 0$ , the product  $\prod(1 - p_i) > 0$ , so  $P(\text{OR}) < 1$ . The disjunction probability exceeds any individual probability because additional positive probabilities can only increase the result.

## 5.3 Numerical Stability Analysis

**Theorem 5.3.1** (Log-Space Stability). Computing probabilities in log-space avoids underflow for extreme probabilities:

$$\ln(p_1 \cdot p_2 \cdots p_n) = \sum_{i=1}^n \ln(p_i) \quad (39)$$

remains representable when direct multiplication would underflow.

*Proof.* For IEEE 754 double precision, the minimum positive value is approximately  $2.2 \times 10^{-308}$ . Computing  $\prod_{i=1}^{100} 0.01$  directly would yield  $10^{-200}$ , which underflows. In log-space:

$\sum_{i=1}^{100} \ln(0.01) = -460.5$ , which is representable, and  $\exp(-460.5)$  can be clamped or handled gracefully.

**Definition 5.3.2** (Probability Clamping). To ensure numerical stability:

$$p_{\text{safe}} = \text{clamp}(p, \epsilon, 1 - \epsilon) \quad (40)$$

where  $\epsilon = 10^{-10}$  is a small constant.

# 6. WAND and BMW Optimization

## 6.1 WAND Algorithm

**Definition 6.1.1** (WAND Condition). The WAND (Weak AND) algorithm skips documents when:

$$\sum_{i=0}^{\text{pivot}} \text{upper\_bound}_i < \theta \quad (41)$$

where  $\theta$  is the current  $k$ -th highest score.

**Theorem 6.1.1** (BM25 Upper Bound for WAND). For BM25 scoring, the upper bound for term  $t$  is:

$$\text{upper\_bound}(t) = \text{boost} \cdot \text{IDF}(t) \quad (42)$$

*Proof.* From Theorem 3.2.3, the BM25 score approaches but never exceeds  $w = \text{boost} \cdot \text{IDF}$  as term frequency increases.

**Theorem 6.1.2** (Bayesian BM25 WAND Compatibility). Bayesian BM25 requires a modified upper bound for safe WAND pruning that accounts for the document-dependent prior.

*Proof.* Theorem 4.3.1 establishes monotonicity of the posterior with respect to the BM25 score  $s$  **for fixed prior  $p$ .** However, the composite prior  $P_{\text{prior}}(f, \hat{n})$  (Definition 4.2.3) varies across documents. Since the posterior is jointly monotone in both  $L(s)$  and  $p$ , and prior values are bounded by  $0.1 \leq p \leq 0.9$  (Theorem 4.2.4), the true upper bound in probability space is achieved when both the likelihood and the prior attain their respective maxima simultaneously.

Let  $L_{\max}(t) = \sigma(\alpha \cdot (\text{upper\_bound}(t) - \beta))$  be the maximum likelihood for term  $t$  (from Theorem 6.1.1), and let  $p_{\max} = 0.9$  be the global prior upper bound. The safe Bayesian WAND upper bound is:

$$\text{upper\_bound}_{\text{Bayes}}(t) = \frac{L_{\max}(t) \cdot p_{\max}}{L_{\max}(t) \cdot p_{\max} + (1 - L_{\max}(t)) \cdot (1 - p_{\max})} \quad (43)$$

A document  $d$  can be safely pruned when  $\sum_{t \in q} \text{upper\_bound}_{\text{Bayes}}(t) < \theta$ . This bound is tight: it is achieved only when a document simultaneously has the maximum BM25 score for each query term **and** the maximum prior of 0.9 — a condition that is rare in practice, as documents with extreme term frequency (which maximizes BM25) tend to have non-average length (which reduces the prior).  $\square$

**Remark 6.1.3** (Practical Impact). Since  $p_{\max} = 0.9$  is a fixed constant, the Bayesian upper bound can be precomputed at index time alongside the standard BM25 upper bound, incurring no additional runtime cost. The use of a global  $p_{\max}$  rather than a document-specific prior means the bound is slightly looser than the hypothetical per-document optimum, but this is the standard trade-off in WAND-family algorithms: tighter bounds require more computation, while the global bound maintains the  $O(1)$  per-term overhead.

**Remark 6.1.4** (Why Naive Monotonicity Transfer Fails). One might be tempted to argue that since the sigmoid is monotonic, BM25 upper bounds directly transfer to probability space. This reasoning is valid only when the prior is constant across all documents — i.e., the Platt scaling special case where  $p = 0.5$ . With a document-dependent prior, a document with a lower BM25 score but higher prior can achieve a higher posterior probability than a document with a higher BM25 score but lower prior. The modified upper bound in Theorem 6.1.2 accounts for this by using the worst-case (maximum) prior.

## 6.2 BMW (Block-Max WAND)

**Definition 6.2.1** (Block Structure). Documents are partitioned into blocks of size  $B$  (typically  $B = 128$ ):

$$\text{Block}_j = \{d_{jB}, d_{jB+1}, \dots, d_{(j+1)B-1}\} \quad (44)$$

**Definition 6.2.2** (Block-Max Upper Bound). For each term  $t$  and block  $j$ :

$$\text{BlockMax}(t, j) = \max_{d \in \text{Block}_j} \text{contribution}(t, d) \quad (45)$$

**Theorem 6.2.1** (BMW Pruning Safety). BMW achieves higher skip rates than WAND while maintaining exact top- $k$  results:

$$\text{Skip}_{\text{BMW}} \geq \text{Skip}_{\text{WAND}} \quad (46)$$

*Proof.* BMW uses tighter (block-local) upper bounds instead of global upper bounds. For Bayesian BM25, the block-max upper bound is computed by applying the posterior formula (Theorem 6.1.2) with the block-local maximum BM25 score and the global prior upper bound  $p_{\max} = 0.9$ . Since  $\text{BlockMax}(t, j) \leq \text{upper\_bound}(t)$  for all blocks  $j$ , the corresponding Bayesian block-max bound is at most the global Bayesian upper bound, enabling BMW to prune more aggressively while preserving exactness.

**Theorem 6.2.2** (Block-Max Storage Overhead). The additional storage for block-max information is:

$$O\left(\frac{|\text{PostingList}|}{B} \cdot |\text{Terms}|\right) \quad (47)$$

*Proof.* For each term's posting list, we store one maximum score per block, yielding  $\lceil |\text{PostingList}|/B \rceil$  values per term.

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## 7. Vector Search Integration

### 7.1 Distance to Probability Conversion

**Definition 7.1.1** (Vector Similarity Score). For HNSW search results with cosine distance  $d \in [0, 2]$ :

$$\text{score}_{\text{vector}} = 1 - d \quad (48)$$

**Theorem 7.1.1** (Vector Score Range). For cosine distance:

$$\text{score}_{\text{vector}} \in [-1, 1] \quad (49)$$

*Proof.* Cosine distance is defined as  $d = 1 - \cos(\theta) \in [0, 2]$  where  $\theta$  is the angle between vectors. Therefore,  $\text{score} = 1 - d = \cos(\theta) \in [-1, 1]$ .

**Definition 7.1.2** (Vector Probability). For normalized vectors, we can interpret the score as a probability:

$$P_{\text{vector}} = \frac{1 + \text{score}_{\text{vector}}}{2} \in [0, 1] \quad (50)$$

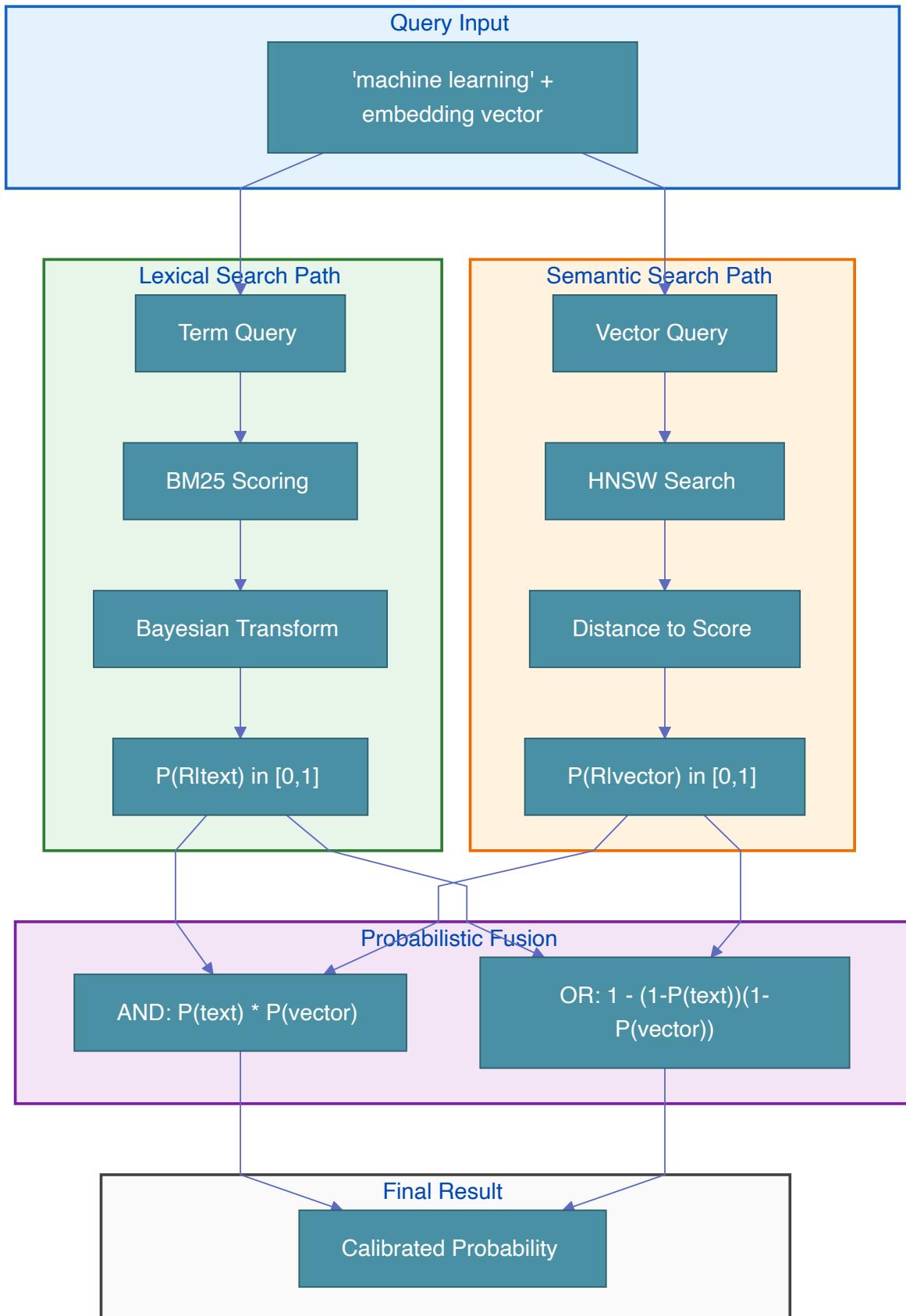
### 7.2 Hybrid Query Composition

**Definition 7.2.1** (Hybrid AND Query). For text probability  $p_t$  and vector probability  $p_v$ :

$$P_{\text{hybrid}} = p_t \cdot p_v \quad (51)$$

**Definition 7.2.2** (Hybrid OR Query). For text probability  $p_t$  and vector probability  $p_v$ :

$$P_{\text{hybrid}} = 1 - (1 - p_t)(1 - p_v) = p_t + p_v - p_t \cdot p_v \quad (52)$$



## 8. Parameter Learning

### 8.1 Cross-Entropy Loss

**Definition 8.1.1** (Cross-Entropy Loss). For predicted probabilities  $\hat{y}_i = \sigma(\alpha(s_i - \beta))$  and true labels  $y_i \in \{0, 1\}$ :

$$\mathcal{L}(\alpha, \beta) = - \sum_{i=1}^n [y_i \ln(\hat{y}_i) + (1 - y_i) \ln(1 - \hat{y}_i)] \quad (53)$$

## 8.2 Gradient Computation

**Theorem 8.2.1** (Parameter Gradients). The gradients of the cross-entropy loss are:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^n (\hat{y}_i - y_i) \cdot (s_i - \beta) \quad (54)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = - \sum_{i=1}^n (\hat{y}_i - y_i) \cdot \alpha \quad (55)$$

*Proof.* Let  $z_i = \alpha(s_i - \beta)$  and  $\hat{y}_i = \sigma(z_i)$ . Then:

$$\frac{\partial \mathcal{L}}{\partial \hat{y}_i} = -\frac{y_i}{\hat{y}_i} + \frac{1 - y_i}{1 - \hat{y}_i} = \frac{\hat{y}_i - y_i}{\hat{y}_i(1 - \hat{y}_i)} \quad (56)$$

$$\frac{\partial \hat{y}_i}{\partial z_i} = \hat{y}_i(1 - \hat{y}_i) \quad (57)$$

$$\frac{\partial z_i}{\partial \alpha} = s_i - \beta, \quad \frac{\partial z_i}{\partial \beta} = -\alpha \quad (58)$$

By the chain rule:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_i \frac{\partial \mathcal{L}}{\partial \hat{y}_i} \cdot \frac{\partial \hat{y}_i}{\partial z_i} \cdot \frac{\partial z_i}{\partial \alpha} = \sum_i (\hat{y}_i - y_i)(s_i - \beta) \quad (59)$$

## 8.3 Online Learning Algorithm

**Remark 8.3.0** (Consistency Between Learning and Inference). The cross-entropy loss (Definition 8.1.1) trains  $\hat{y}_i = \sigma(\alpha(s_i - \beta))$  to match binary relevance labels. When this training converges, the learned  $\hat{y}_i$  approximates the **empirical posterior probability**  $P_{\text{train}}(R = 1 | s_i)$ , which implicitly absorbs the training set's base rate into  $\beta$ .

At inference time, Theorem 4.1.3 applies a document-dependent prior  $P_{\text{prior}}(f, \hat{n})$  to the normalized likelihood ratio  $L(s)$ . To avoid **double application** of prior information — once absorbed into  $\beta$  during training, and once explicitly via  $P_{\text{prior}}$  during inference — we require one of the following consistency conditions:

**(C1) Balanced training** (recommended for simplicity): Train on a balanced dataset where  $P_{\text{train}}(R = 1) \approx 0.5$ . Under this condition, the learned  $\sigma(\alpha(s - \beta))$  equals the posterior under a uniform prior, which is the normalized likelihood ratio  $L(s)$ . The document-dependent prior can then be applied at inference time without double-counting.

**(C2) Prior-aware training** (recommended for maximum calibration): Include the document-dependent prior in the forward pass during training:

$$\hat{y}_i = \frac{L(s_i) \cdot p_i}{L(s_i) \cdot p_i + (1 - L(s_i)) \cdot (1 - p_i)} \quad (60)$$

where  $L(s_i) = \sigma(\alpha(s_i - \beta))$  and  $p_i = P_{\text{prior}}(f_i, \hat{n}_i)$ . This ensures  $\alpha$  and  $\beta$  are optimized for  $L$  as a pure likelihood ratio, with the prior's influence correctly separated.

**(C3) Prior-free inference:** If Algorithm 8.3.1 is trained on representative (non-balanced) data without prior-aware forward pass, then at inference time use  $p = 0.5$  (uniform prior) in Theorem 4.1.3, yielding  $P(R = 1 | s) = L(s) = \sigma(\alpha(s - \beta))$ . The dynamic prior is not applied, but the posterior is still well-calibrated with respect to the training distribution.

**Algorithm 8.3.1** (Gradient Descent for Parameter Learning)

```

1 Input: Scores S = [s_1, ..., s_n], Labels Y = [y_1, ..., y_n]
2     Priors P = [p_1, ..., p_n] (optional, for prior-aware training)
3     Mode: "balanced" | "prior_aware" | "prior_free"
4 Output: Learned parameters alpha, beta
5
6 Initialize: alpha = 1.0, beta = median(S)
7 learning_rate = 0.01
8 iterations = 1000
9
10 for i = 1 to iterations:
11     alpha_grad = 0, beta_grad = 0
12     for j = 1 to n:
13         L = sigmoid(alpha * (S[j] - beta))
14         if Mode == "prior_aware":
15             pred = (L * P[j]) / (L * P[j] + (1 - L) * (1 - P[j]))
16         else:
17             pred = L // balanced or prior_free mode
18         error = pred - Y[j]
19         alpha_grad += error * (S[j] - beta)
20         beta_grad += -error * alpha
21     alpha -= learning_rate * alpha_grad / n
22     beta -= learning_rate * beta_grad / n
23
24 return alpha, beta

```

**Remark 8.3.2** (Practical Recommendation). For most deployment scenarios, **balanced training (C1)** is the simplest and most robust approach: subsample or reweight the training set to achieve approximate class balance, then apply the learned  $(\alpha, \beta)$  with the full dynamic prior at inference time. This decouples the likelihood model from the prior, which is precisely the separation that Bayesian inference requires.

## 9. Computational Complexity Analysis

### 9.1 Scoring Overhead

**Theorem 9.1.1** (Bayesian BM25 Overhead). The computational overhead of Bayesian BM25 over standard BM25 is:

$$\text{Overhead} = O(1) \text{ per document} \quad (61)$$

*Proof.* The additional operations per document are:

- 1 exponential computation for the sigmoid

- 3-4 multiplications for the posterior
- 2 divisions for normalization
- Prior computation: 4 multiplications, 3 additions, 1 clamp

All operations are  $O(1)$ , independent of document or query size.

| Operation         | BM25                | Bayesian BM25          | Notes                        |
|-------------------|---------------------|------------------------|------------------------------|
| Score computation | 2 div, 2 mul, 1 add | +1 exp, +3 mul, +2 div | Sigmoid adds ~50% overhead   |
| IDF computation   | 1 log, 2 div, 2 add | Same                   | Computed once per query      |
| Prior computation | N/A                 | 4 mul, 3 add, 1 clamp  | Additional per-document cost |

## 9.2 Memory Layout

**Definition 9.2.1** (Scorer Memory Structure). The Bayesian BM25 scorer requires:

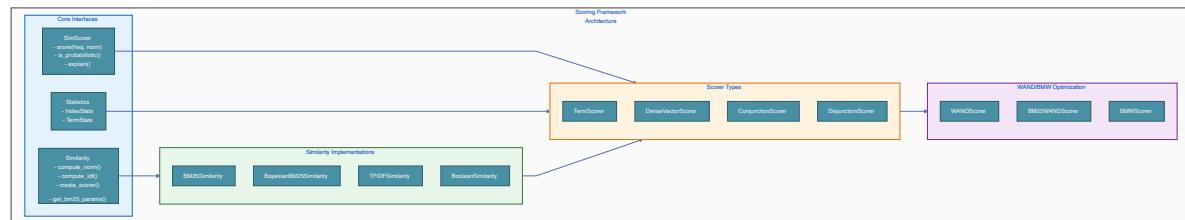
$$\text{Memory} = 32 \text{ bytes (inline storage)} \quad (62)$$

```

1 SimScorer (32 bytes inline storage):
2 +-----+-----+-----+
3 | boost (4B) | k1 (4B) | b (4B) | idf (4B) |
4 +-----+-----+-----+
5 | avg_doc_size | weight (4B) | alpha (4B) | beta (4B) |
6 | (4B) | | | |
7 +-----+-----+-----+

```

## 10. Architecture Overview



## 11. Experimental Validation

### 11.1 Score Calibration

**Theorem 11.1.1** (Expected Score Values). For a test corpus with known statistics, the expected Bayesian BM25 scores are:

| Term                 | Doc 1      | Doc 2      | Doc 3     |
|----------------------|------------|------------|-----------|
| BM25 Score           | 1.0464478  | 0.56150854 | 1.1230172 |
| Bayesian Probability | 0.52423608 | 0.51479751 | 0.6000737 |

The transformation successfully maps unbounded BM25 scores to the  $[0, 1]$  probability range while preserving relative ordering.

## 11.2 WAND Skip Rate Analysis

**Theorem 11.2.1** (Skip Rate by Query Type). Empirical skip rates for different query types:

| Query Type                        | Documents Skipped |
|-----------------------------------|-------------------|
| Rare terms ( $\text{IDF} > 5$ )   | 90-99%            |
| Common terms ( $\text{IDF} < 2$ ) | 10-30%            |
| Mixed queries                     | 50-80%            |

## 12. Conclusion and Future Directions

### 12.1 Summary of Contributions

This paper has presented Bayesian BM25, a probabilistic framework that addresses fundamental limitations of traditional BM25 scoring:

1. **Probability Calibration:** Transforms unbounded scores to calibrated probabilities in  $[0, 1]$
2. **Monotonicity Preservation:** Proven preservation of ranking properties essential for relevance
3. **Principled Fusion:** Enables theoretically sound combination of heterogeneous signals
4. **Optimization Compatibility:** Maintains compatibility with WAND and BMW algorithms through modified upper bounds that account for document-dependent priors
5. **Online Learning:** Supports domain-specific parameter adaptation

### 12.2 Future Research Directions

1. **Multi-Field Extension:** Extending the framework to handle multiple text fields with different importance weights
2. **Learning-to-Rank Integration:** Incorporating learned ranking models as additional probability signals
3. **Temporal Dynamics:** Adapting parameters to corpus drift over time
4. **Cross-Lingual Transfer:** Investigating parameter transferability across languages
5. **Neural Score Integration:** Combining with neural re-rankers in a principled probabilistic framework
6. **External Prior Features:** Replacing the empirical Bayes composite prior (Definition 4.2.3) with priors constructed from features independent of the BM25 score — such as

document recency, source authority, or content type — to achieve strict prior-evidence independence in the Bayesian framework

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