

CS420

Introduction to the Theory of Computation

Lecture 17: Push-down automata

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Today we will learn...

- Pushdown automata (PDA)
- Formalizing PDAs
- Union of PDAs
- Examples

■ Section 2.2

Intuition

Define an automata family \iff CFG

NFA recap

Each transition performs one input operations: read/skip an input

Examples

- **Read one input:** $q_1 \xrightarrow{a} q_2$
- **Skip one input:** $q_1 \xrightarrow{\epsilon} q_2$

Nondeterministic PushDown Automata (PDA)

- Extend NFAs with an **unbounded stack**
- Recognizes the same language as CFGs

PDA Execution

Each transition:

- input op, **pre-stack op**, **post-stack op**
- Format: $q \xrightarrow{\$INPUT, \$PRE \rightarrow \$POST} q'$

Example

$$q_a \xrightarrow{\text{READ } a, \text{SKIP} \rightarrow \text{PUSH } a} q_a$$

Possible operations

$\$INPUT$	$\$PRE$	$\$POST$
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

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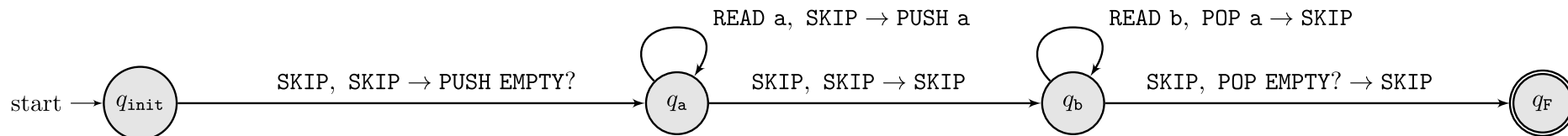
Attention!

The comma does not denote parallel edges. Instead, we stack multiple transitions **vertically**.

PDA example (intuition)

Give a PDA that recognizes $\{a^n b^n \mid n \geq 0\}$

1. $q_{\text{init}} \xrightarrow{\text{SKIP, SKIP} \rightarrow \text{PUSH EMPTY?}} q_a$
2. $q_a \xrightarrow{\text{READ a, SKIP} \rightarrow \text{PUSH a}} q_a$
3. $q_a \xrightarrow{\text{SKIP, SKIP} \rightarrow \text{SKIP}} q_b$
4. $q_b \xrightarrow{\text{READ b, POP a} \rightarrow \text{SKIP}} q_b$
5. $q_b \xrightarrow{\text{SKIP, EMPTY?} \rightarrow \text{SKIP}} q_F$



Exercising transitions

Writing transitions

Possible operations

<i>\$INPUT</i>	<i>\$PRE</i>	<i>\$POST</i>
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

Exercises

1. Test if read 0 and stack is empty (assuming we initialize the stack with a sentinel EMPTY?):

Writing transitions

Possible operations

<i>\$INPUT</i>	<i>\$PRE</i>	<i>\$POST</i>
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

Exercises

1. Test if read 0 and stack is empty (assuming we initialize the stack with a sentinel EMPTY?):
READ 0, EMPTY? \rightarrow SKIP
2. Test if stack is empty:

Writing transitions

Possible operations

<i>\$INPUT</i>	<i>\$PRE</i>	<i>\$POST</i>
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

Exercises

1. Test if read 0 and stack is empty (assuming we initialize the stack with a sentinel EMPTY?):
READ 0, EMPTY? \rightarrow SKIP
2. Test if stack is empty:
SKIP, EMPTY? \rightarrow SKIP
3. Test if a is on top and leave stack untouched:

Writing transitions

Possible operations

<i>\$INPUT</i>	<i>\$PRE</i>	<i>\$POST</i>
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

Exercises

1. Test if read 0 and stack is empty (assuming we initialize the stack with a sentinel EMPTY?):
READ 0, EMPTY? \rightarrow SKIP
2. Test if stack is empty:
SKIP, EMPTY? \rightarrow SKIP
3. Test if a is on top and leave stack untouched:
SKIP, POP a \rightarrow PUSH a
4. Read b and leave stack untouched:

Writing transitions

Possible operations

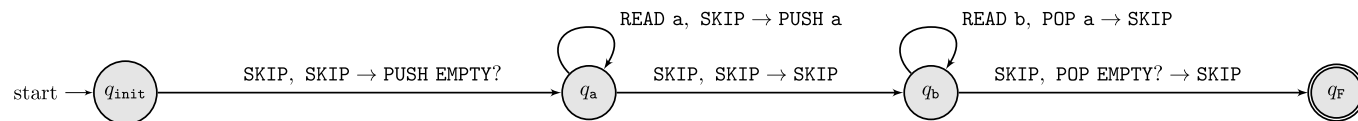
<i>\$INPUT</i>	<i>\$PRE</i>	<i>\$POST</i>
READ n	POP n	PUSH n
SKIP (ϵ)	SKIP	SKIP

Exercises

1. Test if read 0 and stack is empty (assuming we initialize the stack with a sentinel EMPTY?):
READ 0, EMPTY? \rightarrow SKIP
2. Test if stack is empty:
SKIP, EMPTY? \rightarrow SKIP
3. Test if a is on top and leave stack untouched:
SKIP, POP a \rightarrow PUSH a
4. Read b and leave stack untouched:
READ b, SKIP \rightarrow SKIP

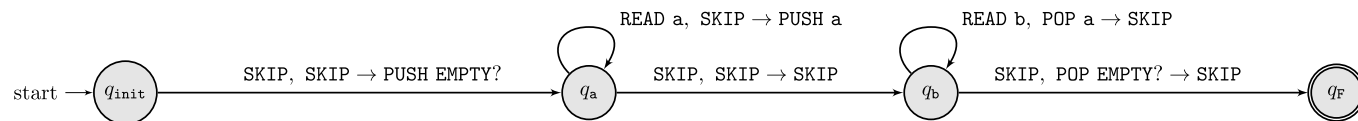
Simplifying the notation

Simplifying the notation

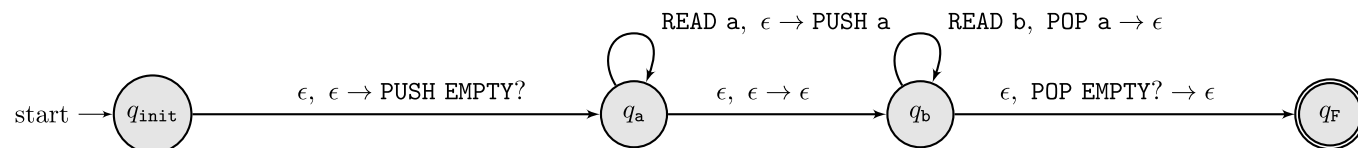


We can replace SKIP by ϵ

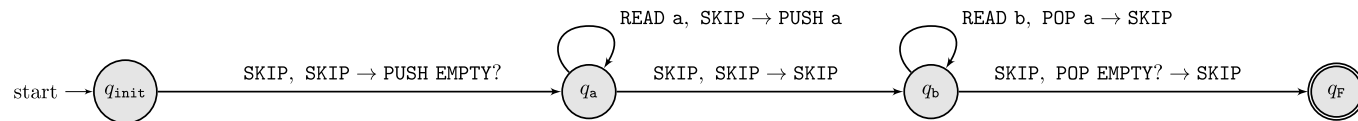
Simplifying the notation



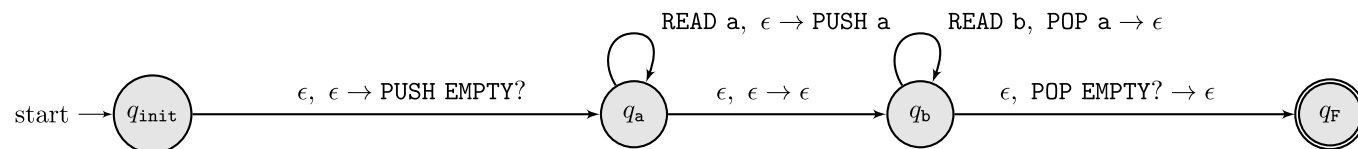
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Simplifying the notation

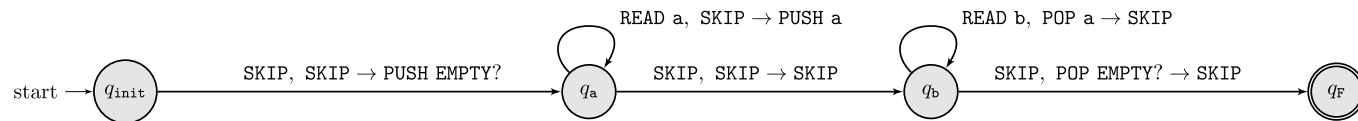


We can replace SKIP by ϵ

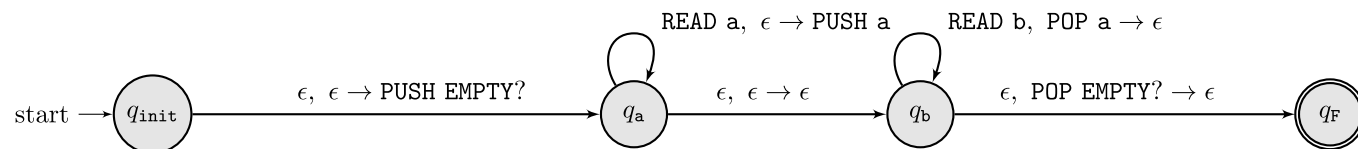


We can omit READ

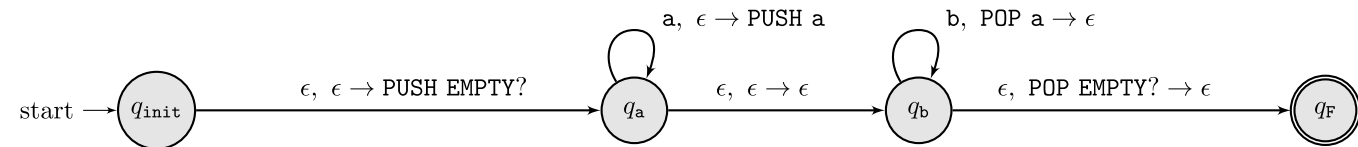
Simplifying the notation



We can replace SKIP by ϵ

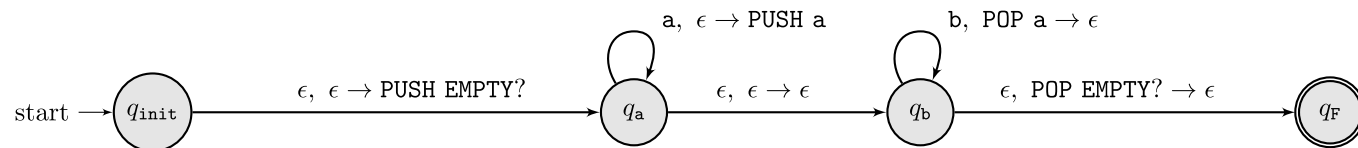


We can omit READ



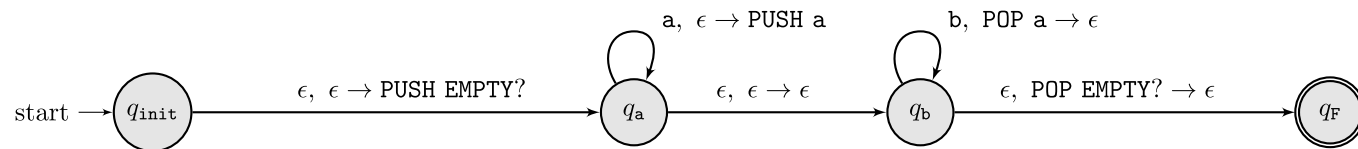
Since read always appears in the same position, we can omit it, as we do in regular DFAs/NFAs.

Simplifying the notation

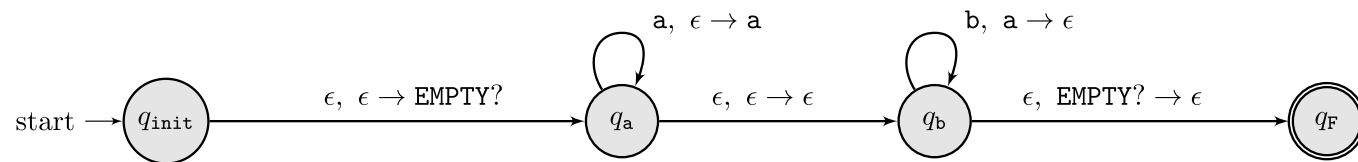


We can omit PUSH/POP

Simplifying the notation



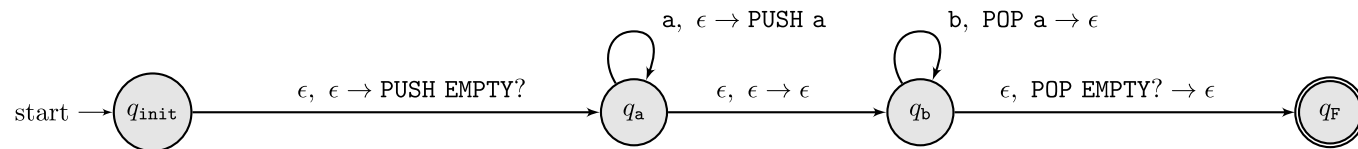
We can omit PUSH/POP



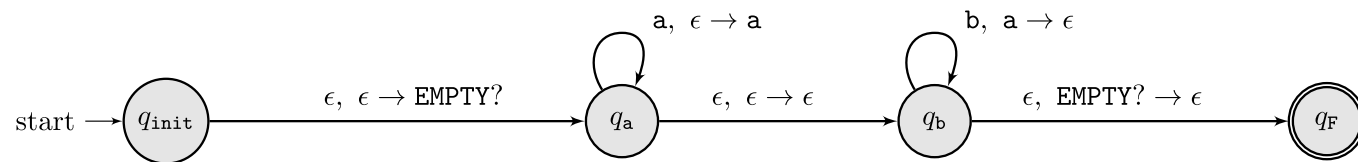
Since push/pop always appear in the same position, we can omit them.

We can replace sentinel EMPTY? by a character $\notin \Gamma$

Simplifying the notation

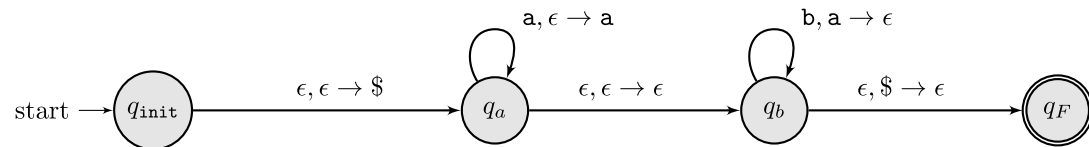


We can omit PUSH/POP



Since push/pop always appear in the same position, we can omit them.

We can replace sentinel EMPTY? by a character $\notin \Gamma$



Since empty? always appear in the same position.

Exercising transitions

(abbreviated notation)

Writing transitions

Possible operations

$\$INPUT$	$\$PRE$	$\$POST$
READ (n)	POP (n)	PUSH (n)
SKIP (ϵ)	SKIP (ϵ)	SKIP (ϵ)

Exercises

1. Test if read 0 and stack is empty, leaving stack unchanged (assume a sentinel \$)

Writing transitions

Possible operations

$\$INPUT$	$\$PRE$	$\$POST$
READ (n)	POP (n)	PUSH (n)
SKIP (ϵ)	SKIP (ϵ)	SKIP (ϵ)

Exercises

1. Test if read 0 and stack is empty, leaving stack unchanged (assume a sentinel \$)
 $0, \$ \rightarrow \$$
2. Test if stack is empty while leaving the stack unchanged (assume sentinel \$)

Writing transitions

Possible operations

$\$INPUT$	$\$PRE$	$\$POST$
READ (n)	POP (n)	PUSH (n)
SKIP (ϵ)	SKIP (ϵ)	SKIP (ϵ)

Exercises

1. Test if read 0 and stack is empty, leaving stack unchanged (assume a sentinel \$)
 $0, \$ \rightarrow \$$
2. Test if stack is empty while leaving the stack unchanged (assume sentinel \$)
 $\epsilon, \$ \rightarrow \$$
3. Test if 0 is on top of the stack and replace it by 1:

Writing transitions

Possible operations

$\$INPUT$	$\$PRE$	$\$POST$
READ (n)	POP (n)	PUSH (n)
SKIP (ϵ)	SKIP (ϵ)	SKIP (ϵ)

Exercises

1. Test if read 0 and stack is empty, leaving stack unchanged (assume a sentinel \$)
 $0, \$ \rightarrow \$$
2. Test if stack is empty while leaving the stack unchanged (assume sentinel \$)
 $\epsilon, \$ \rightarrow \$$
3. Test if 0 is on top of the stack and replace it by 1:
 $\epsilon, 0 \rightarrow 1$
4. Read 2, leave stack untouched

Writing transitions

Possible operations

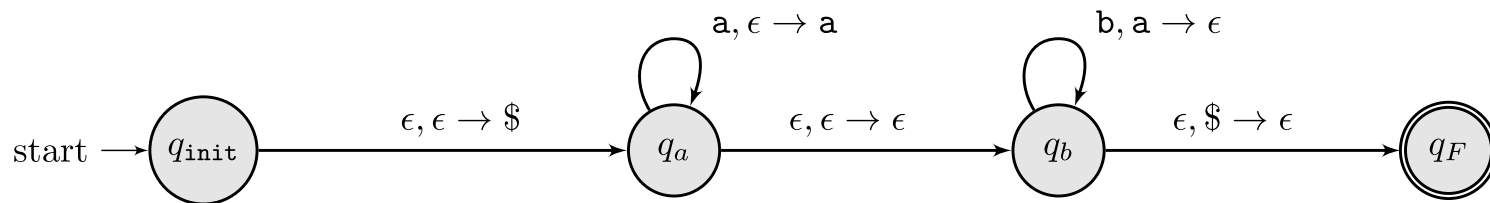
$\$INPUT$	$\$PRE$	$\$POST$
READ (n)	POP (n)	PUSH (n)
SKIP (ϵ)	SKIP (ϵ)	SKIP (ϵ)

Exercises

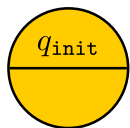
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2. Test if stack is empty while leaving the stack unchanged (assume sentinel \$)
 $\epsilon, \$ \rightarrow \$$
3. Test if 0 is on top of the stack and replace it by 1:
 $\epsilon, 0 \rightarrow 1$
4. Read 2, leave stack untouched
 $2, \epsilon \rightarrow \epsilon$

Acceptance example

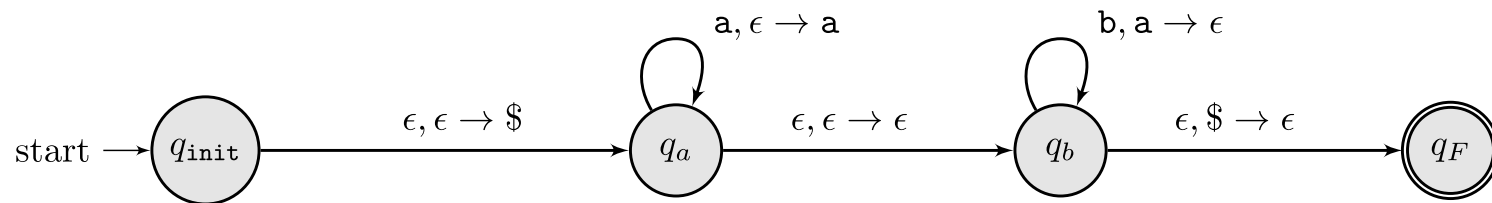
Acceptance example



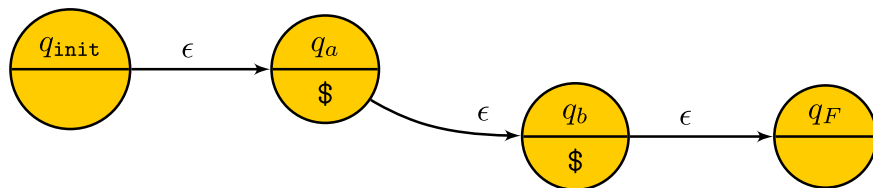
Accepting [€aabb]



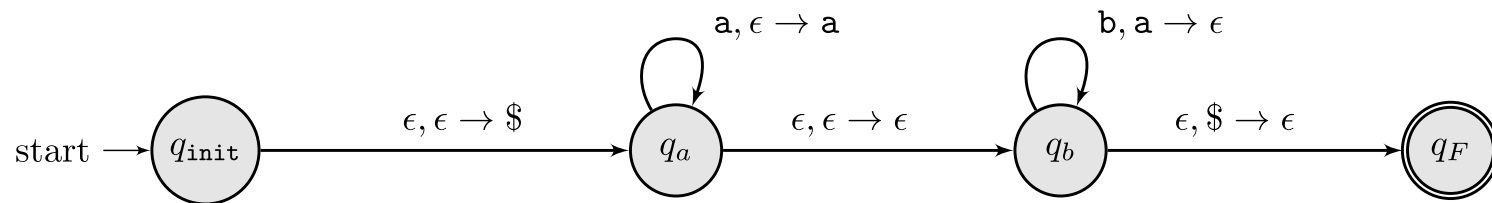
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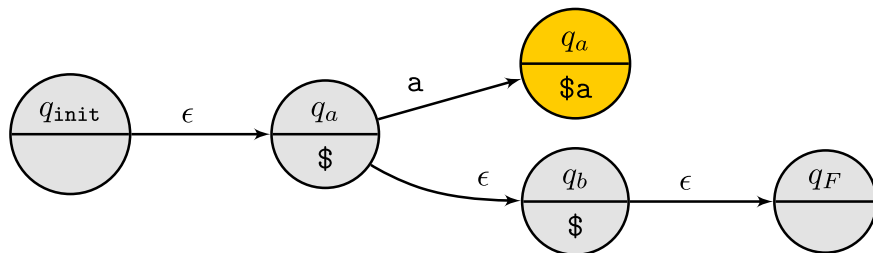
Accepting [**a**abb]



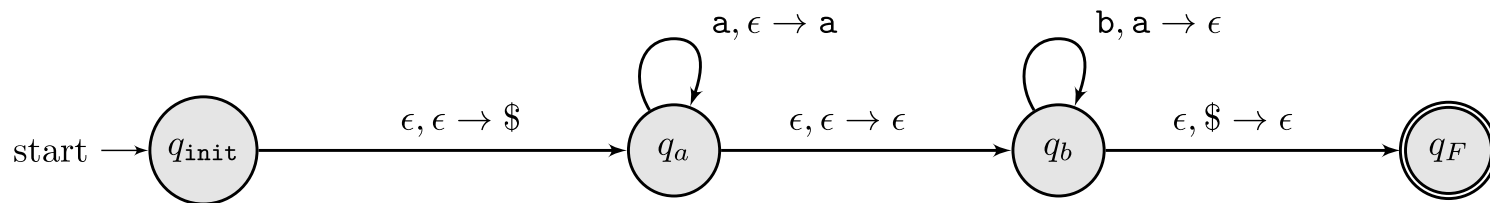
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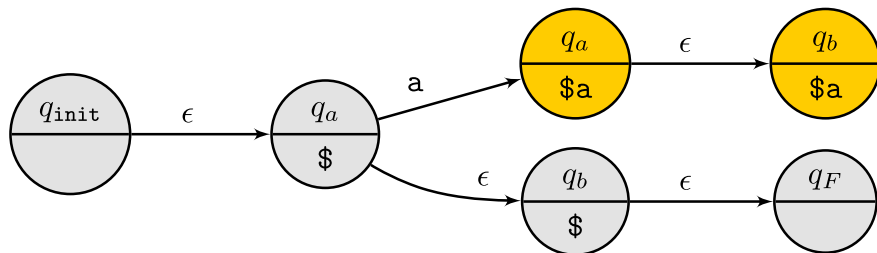
Accepting $[a\epsilon abb]$



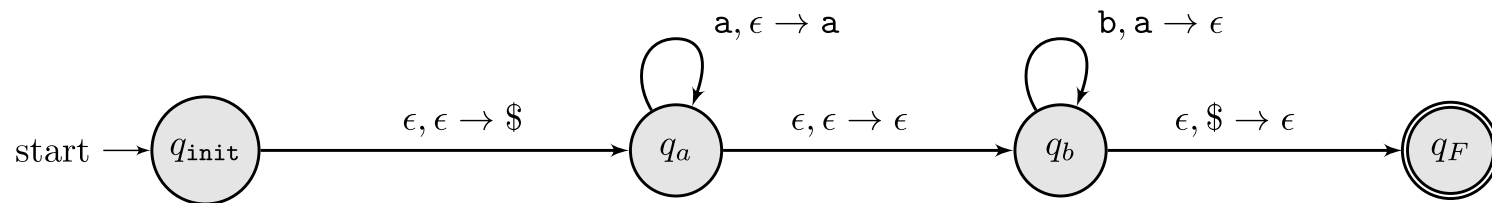
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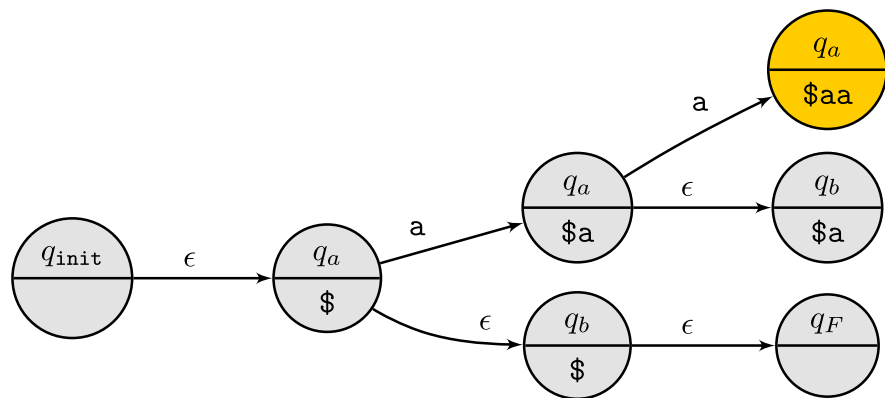
Accepting [a**a**bb]



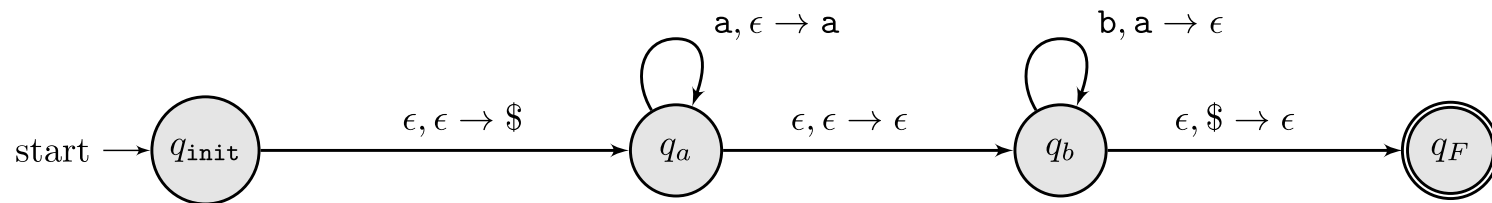
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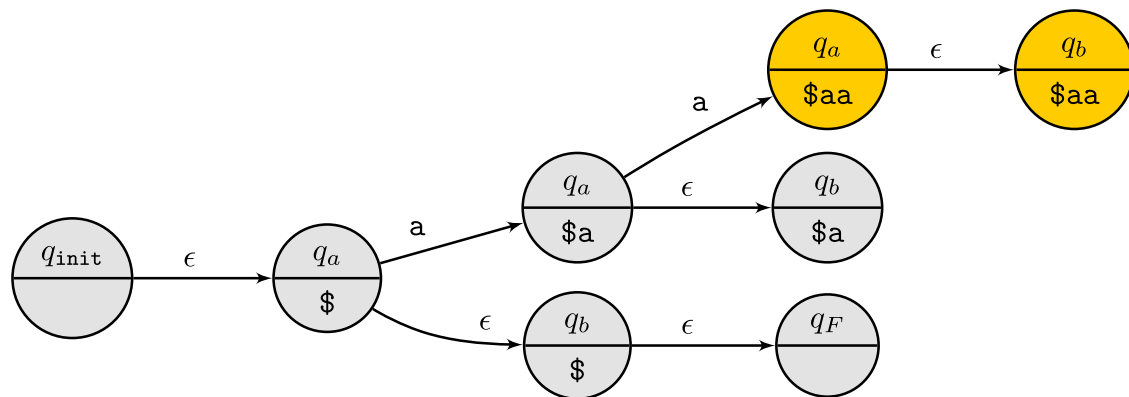
Accepting $[aa\epsilon bb]$



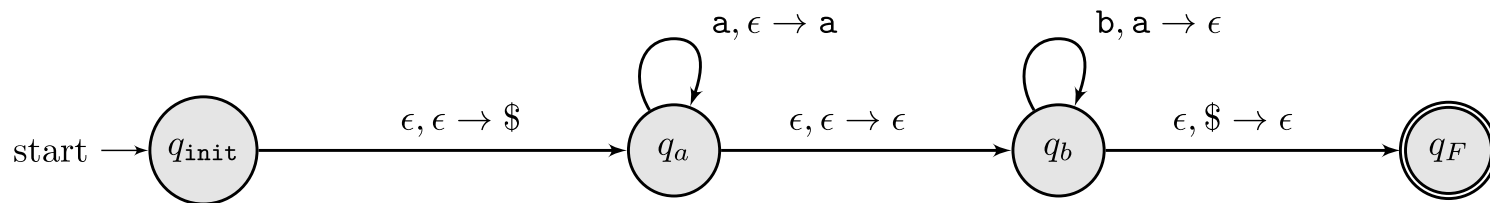
Acceptance example



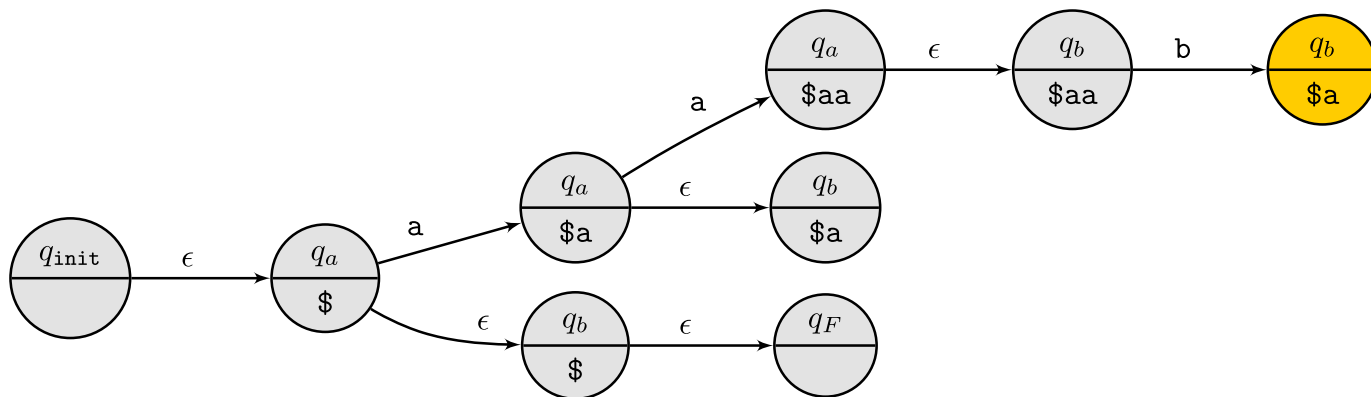
Accepting [aa**b**b]



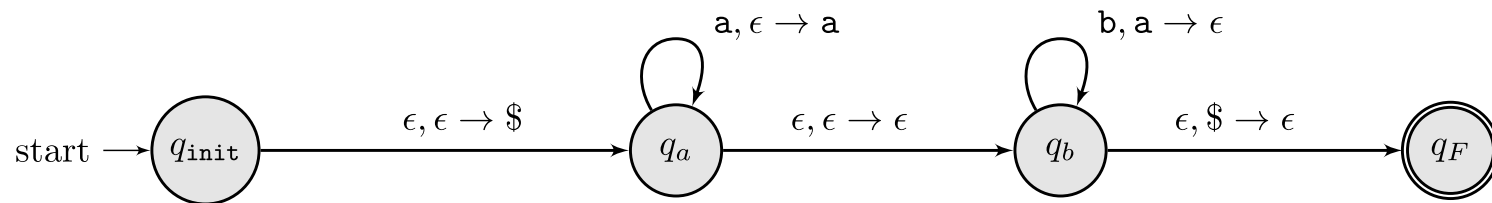
Acceptance example



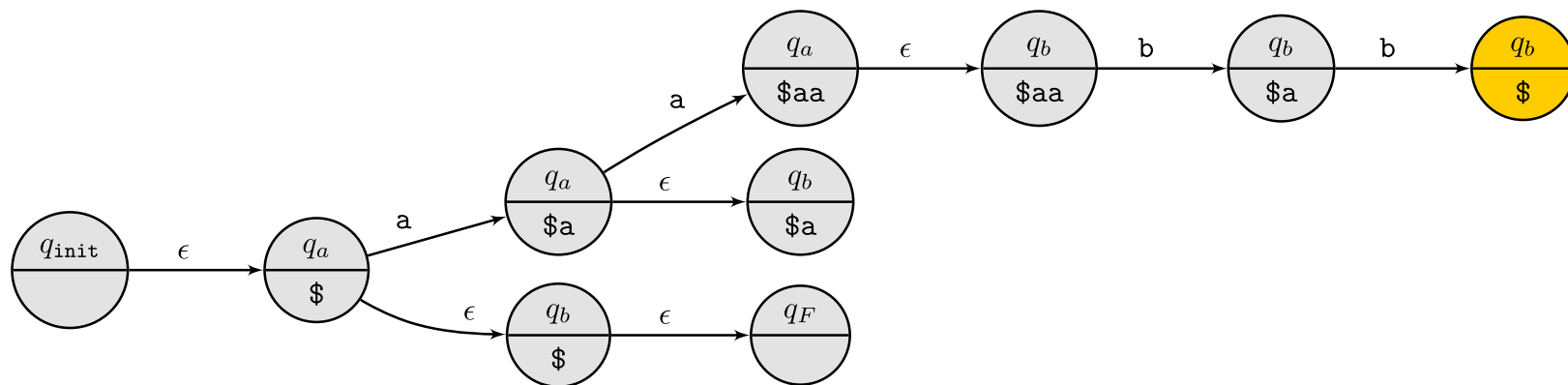
Accepting $[aab\epsilon b]$



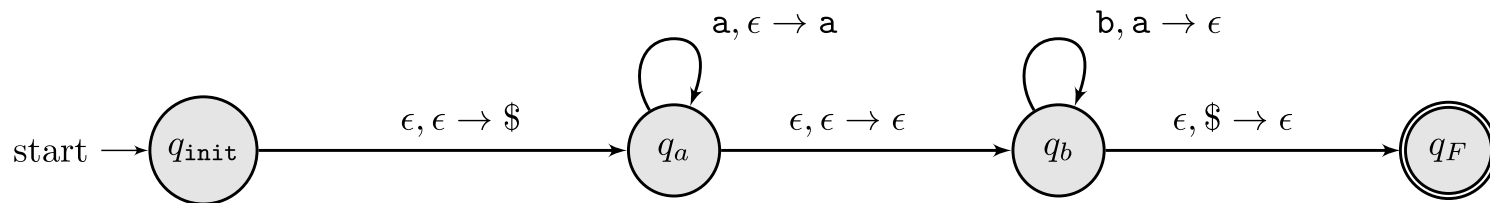
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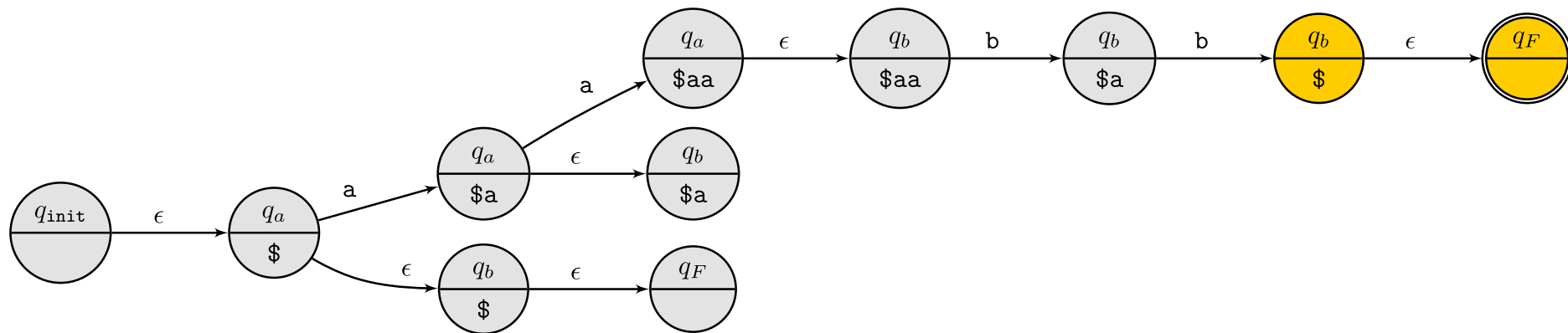
Accepting [aab**b**]



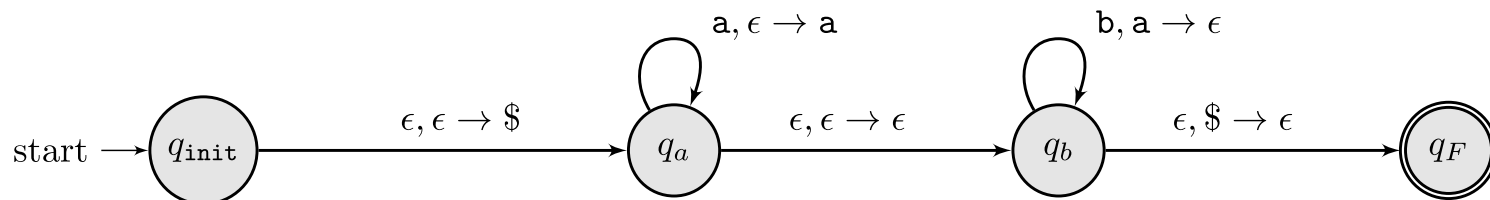
Acceptance example



Accepting [aabb€]

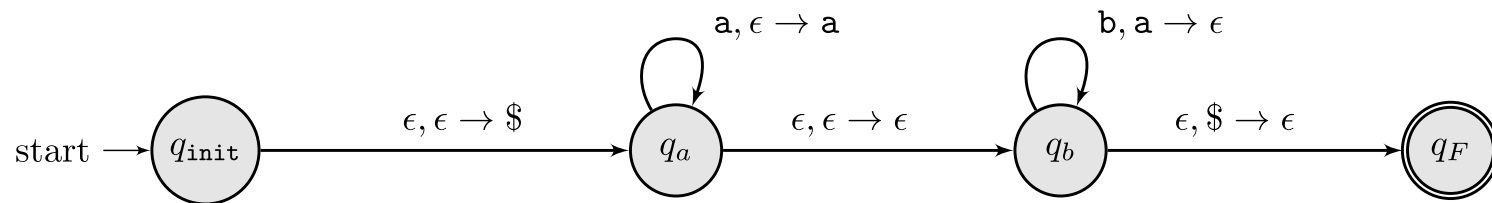


Acceptance example

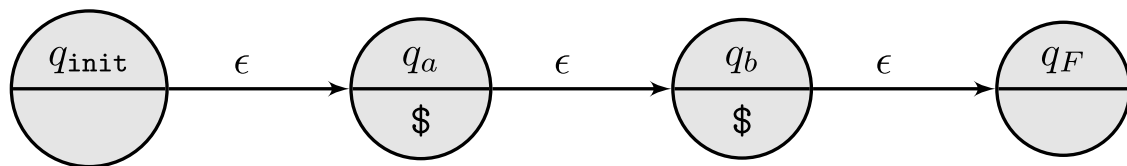


Accepting: bb

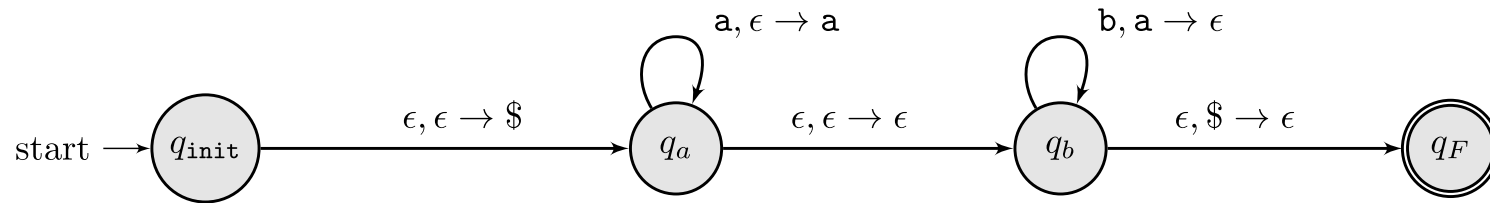
Acceptance example



Accepting: bb

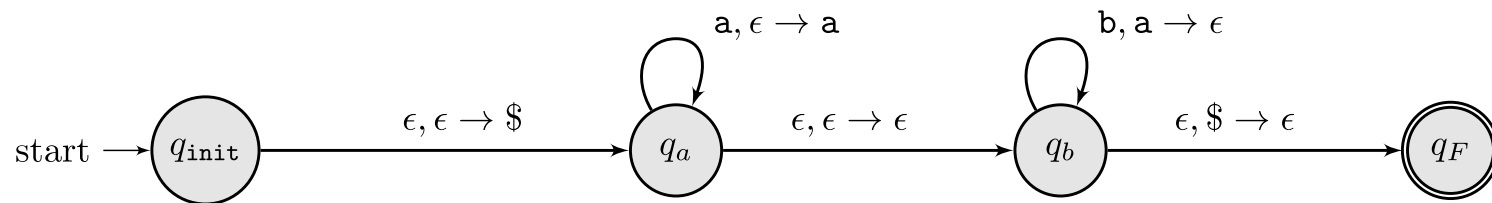


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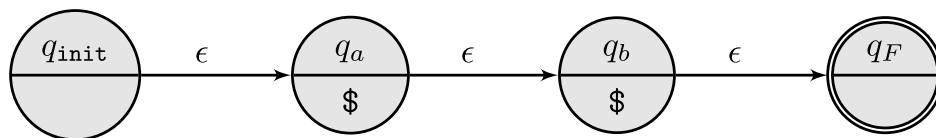


Accepting: ϵ

Acceptance example



Accepting: ϵ



Formalizing a PDA

Formalizing a PDA

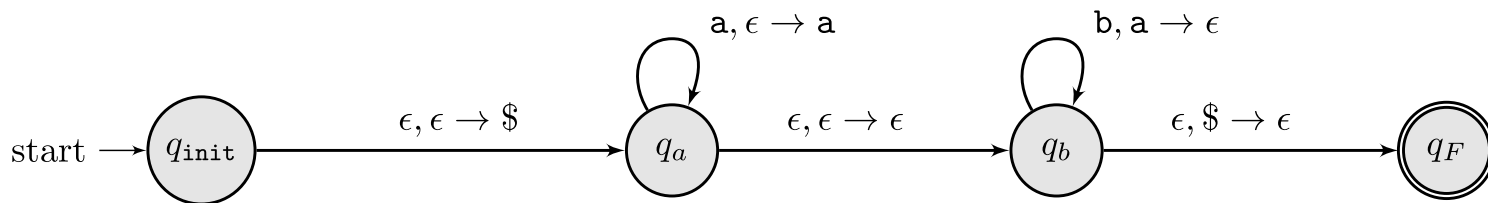
Definition 2.13

A pushdown automaton is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where

1. Q is a finite set called **states**
2. Σ is a finite set called **input alphabet**
3. Γ is a finite set called **stack alphabet**

4. $\delta: Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow \mathcal{P}(Q \times \Gamma_\epsilon)$
is the **transition function**
5. $q_0 \in Q$ is the **start state**
6. $F \subseteq Q$ is the set of **accepted states**

Example



Let $(Q, \Sigma, \Gamma, \delta, q_{init}, \{q_F\})$ be defined as: where δ is defined by branches

1. $Q = \{q_{init}, q_a, q_b, q_F\}$

2. $\Sigma = \{a, b\}$

3. $\Gamma = \{a, \$\}$

$$\delta(q_{init}, \epsilon, \epsilon) = \{(q_a, \$)\}$$

$$\delta(q_a, a, \epsilon) = \{(q_a, a)\}$$

$$\delta(q_a, \epsilon, \epsilon) = \{(q_b, \epsilon)\}$$

$$\delta(q_b, b, a) = \{(q_b, \epsilon)\}$$

$$\delta(q_b, \epsilon, \$) = \{(q_F, \epsilon)\}$$

$$\delta(q, c, s) = \{\} \quad \text{otherwise}$$

Exercise

Give a PDA for the following grammar

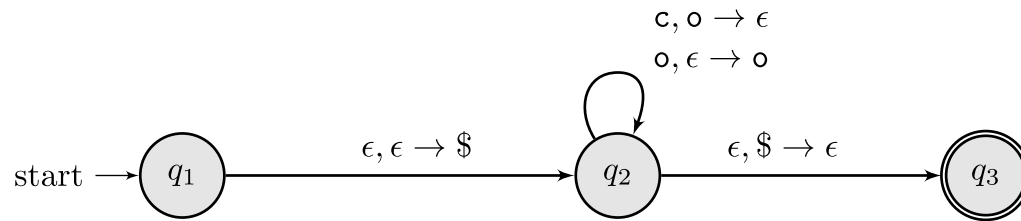
Balanced parenthesis

$$C \rightarrow \underline{o} C \underline{c} \mid CC \mid \epsilon$$

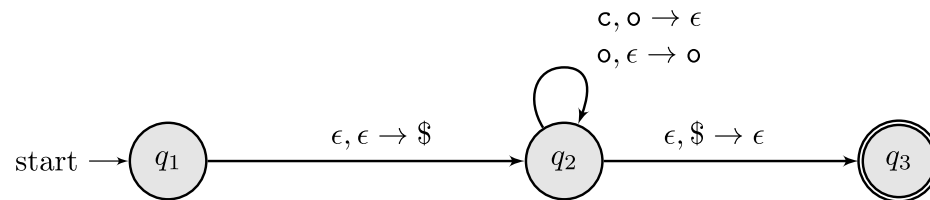
Give a PDA for the following grammar

Balanced parenthesis

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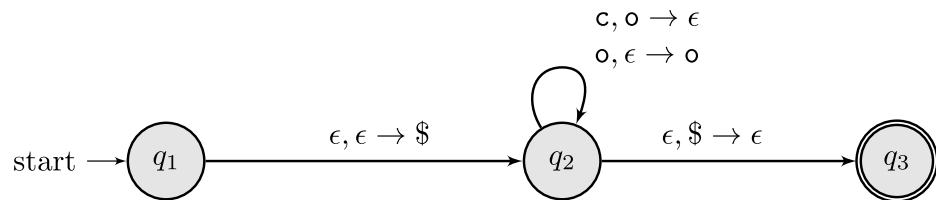


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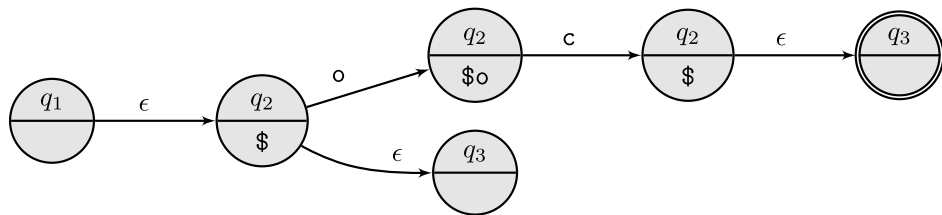


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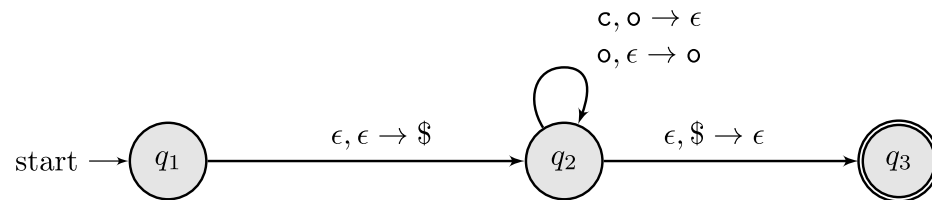
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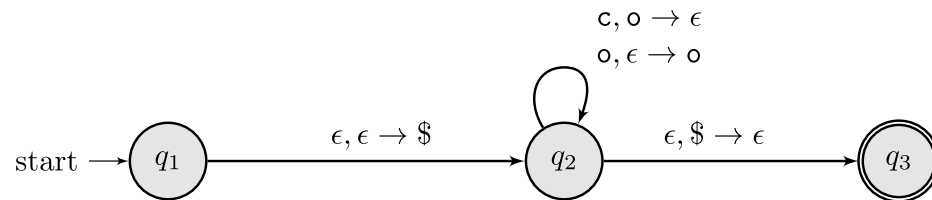


Acceptance

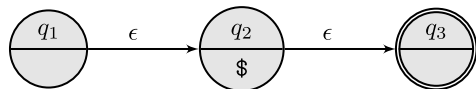


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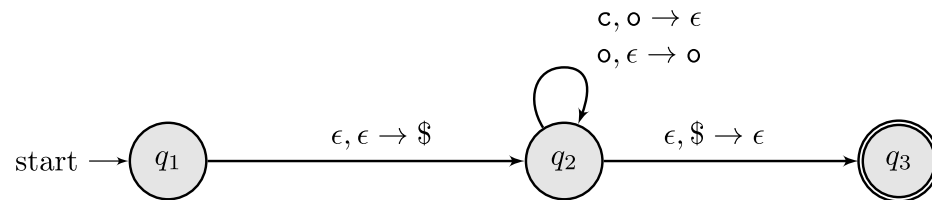
Acceptance



Acceptance: ϵ

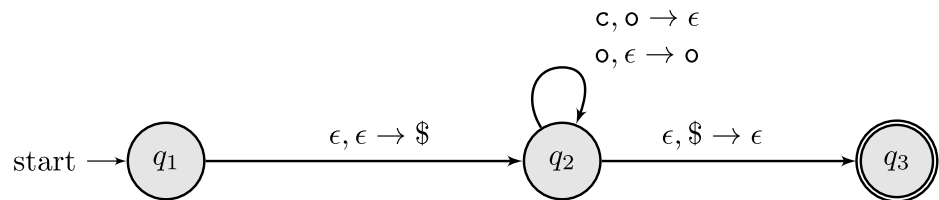


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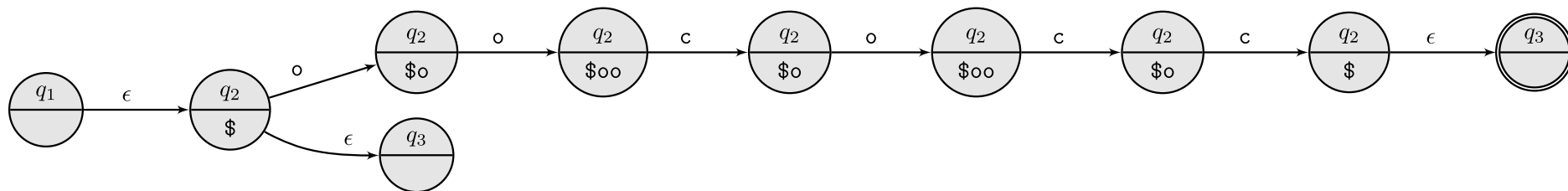


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Acceptance



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Formalization

Formalizing stack operation

Let $S(o_1, o_2, s)$ be defined as follows, where $S : \Gamma_\epsilon \times \Gamma_\epsilon \times \text{Stack}(\Gamma) \rightarrow \text{Stack}(\Gamma)$ and $\text{Stack}(\Gamma) = \text{List}(\Gamma)$:

Pop operation

$$\begin{aligned} s \triangleright \epsilon &= s \\ n :: s \triangleright n &= s \end{aligned}$$

Examples

$$\begin{aligned} [0, 1] \triangleright \epsilon &= [0, 1] \\ [0, 1] \triangleright \$ &\text{ is undefined!} \\ [0, 1] \triangleright 0 &= [1] \\ [0, 1] \triangleright 1 &\text{ is undefined!} \end{aligned}$$

Push operation

$$\begin{aligned} s \triangleleft \epsilon &= s \\ s \triangleleft n &= n :: s \end{aligned}$$

Examples

$$\begin{aligned} [0, 1] \triangleleft \epsilon &= [0, 1] \\ [0, 1] \triangleleft \$ &= [0, 1, \$] \\ [] \triangleleft \$ &= [\$] \\ [0, 1] \triangleleft 0 &= [0, 0, 1] \\ [0, 1] \triangleleft 1 &= [1, 0, 1] \end{aligned}$$

Stack operation exercises

Examples

<i>Expression</i>	<i>Result</i>
$ab \triangleright c =$	
$ab \triangleleft c =$	
$ab \triangleright a =$	
$ab \triangleleft a =$	
$ab \triangleright \$ =$	
$ab \triangleleft \$ =$	
$\epsilon \triangleright \$ =$	
$\epsilon \triangleleft \$ =$	
$\epsilon \triangleright a =$	
$\epsilon \triangleleft a =$	

Stack operation exercises

Examples

<i>Expression</i>	<i>Result</i>
$ab \triangleright c =$	undef
$ab \triangleleft c =$	cab
$ab \triangleright a =$	b
$ab \triangleleft a =$	aab
$ab \triangleright \$ =$	undef
$ab \triangleleft \$ =$	$ab\$$
$\epsilon \triangleright \$ =$	undef
$\epsilon \triangleleft \$ =$	$\$$
$\epsilon \triangleright a =$	undef
$\epsilon \triangleleft a =$	a

Formalizing acceptance

$$\frac{(q', o') \in \delta(q, y, o)}{(q, s) \xrightarrow{y, o} (q', s \triangleright o \triangleleft o')}$$

Rule 0. We can go from state q and stack s into state q' and stack s' with input $y \in \Sigma_\epsilon$ if we can construct s' from a push o and a pop o' on stack s .

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$, let the **steps through** relation, notation $q \curvearrowright_M w$, be defined as:

$$\frac{q \in F}{(q, s) \curvearrowright_M []}$$

Rule 1. State q steps through $[]$ if q is a final state.

$$\frac{(q, s) \xrightarrow{y, o} (q', s') \quad (q', s') \curvearrowright_M w}{(q, s) \curvearrowright_M y \cdot w}$$

Rule 2. If we can go from q to q' with y and q' steps through w , then q steps through $y \cdot w$.

Acceptance. We say that M accepts w if, and only if, $q_0, [] \curvearrowright_M w$.

Example of acceptance

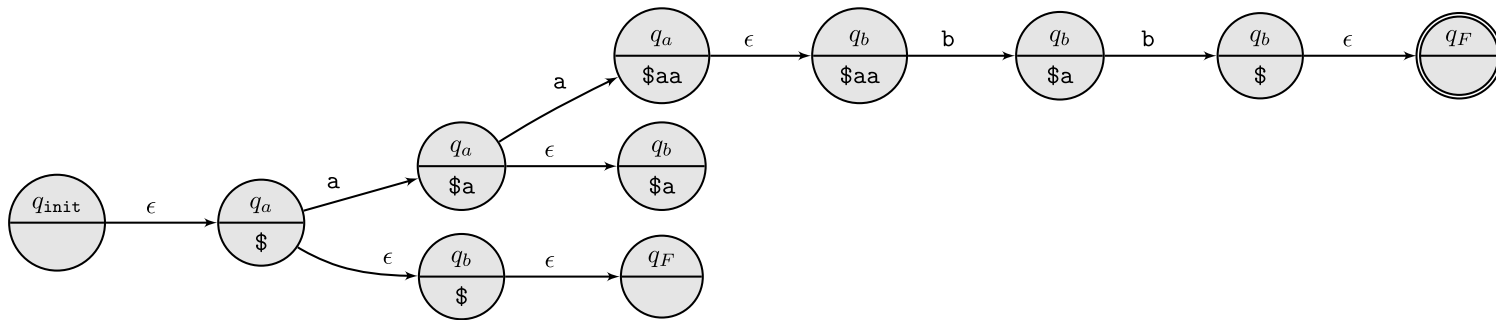
We can build a chain of states as follows

$$(q_{init}, []) \xrightarrow{\epsilon, \epsilon} (q_a, [\$]) \xrightarrow{a, \epsilon} (q_a, [a, \$]) \xrightarrow{a, \epsilon} (q_b, [a, a, \$]) \xrightarrow{\epsilon, \epsilon} (q_b, [a, a, \$]) \xrightarrow{b, a} (q_b, [a, \$]) \xrightarrow{b, a} (q_b, [\$]) \xrightarrow{\epsilon, \$} (q_F, [])$$

Since q_F is a final state, we have that

$$(q_{init}, []) \curvearrowright [a, a, b, b]$$

Recall



Example 2.16

Example 2.16

A sequence of a-s then b-s and finally c-s with as many a-s as there are b-s or as there are c-s.

$$\{a^i b^j c^k \mid i = j \vee i = k\}$$

Example 2.16

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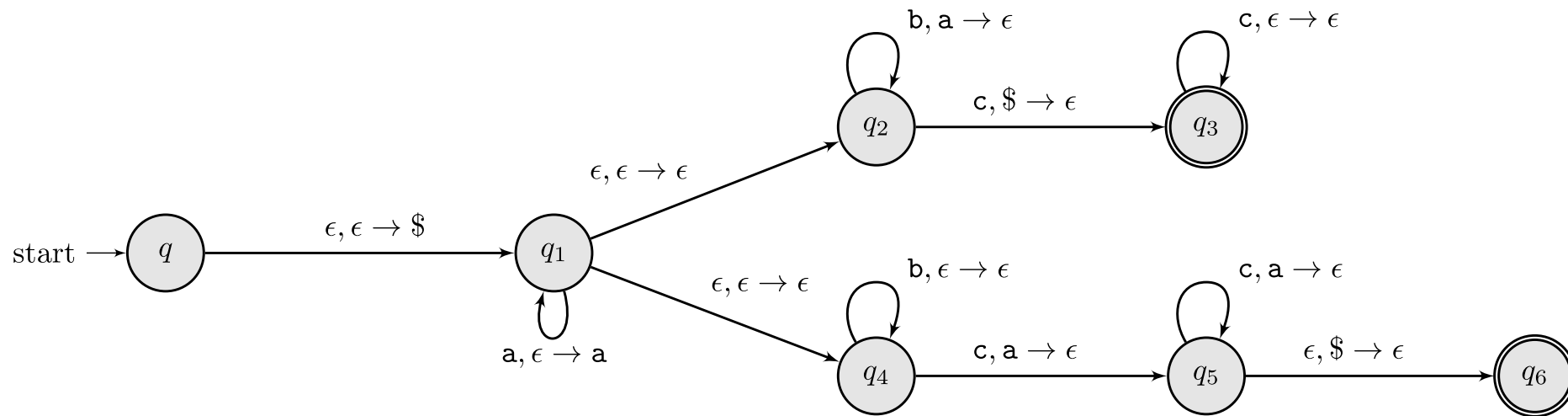
A solution

Step 1. read and push a total of N a's.

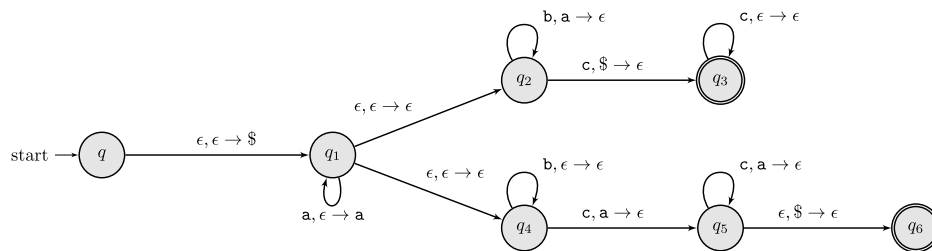
Step 2. Either:

- $(i = j)$ read N b's and pop a's; followed by reading an arbitrary number of c's
- $(i = k)$ read an arbitrary number of b's followed by read N c's and pop a's

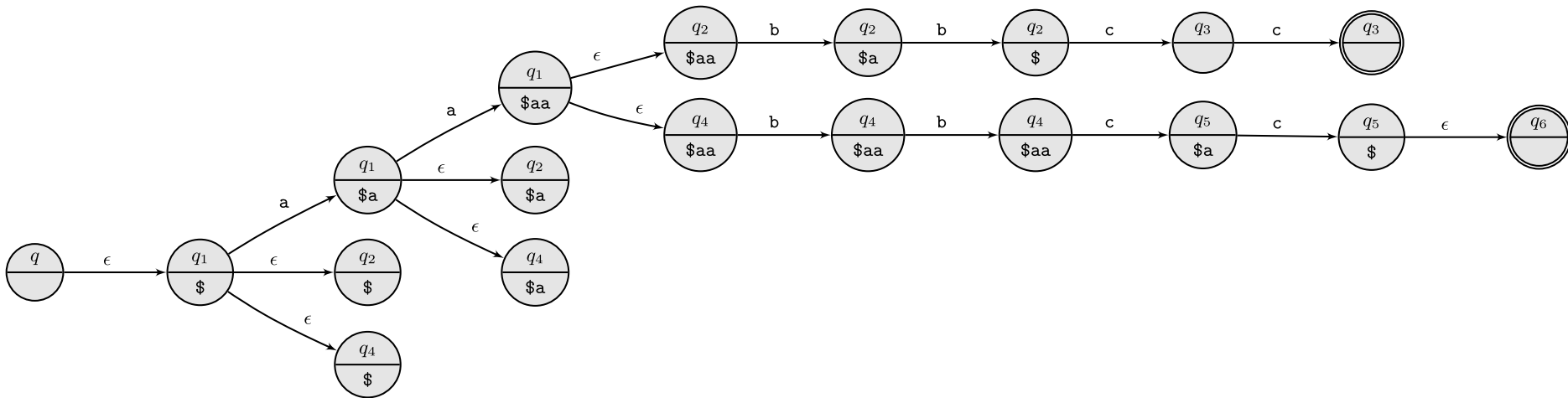
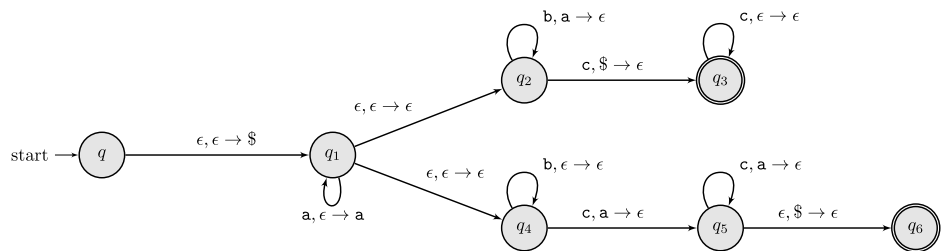
State diagram of Example 2.16



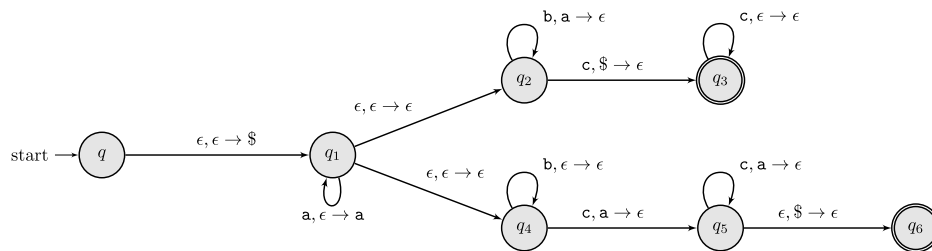
Example 2.16 accept $[a, a, b, b, c, c]$?



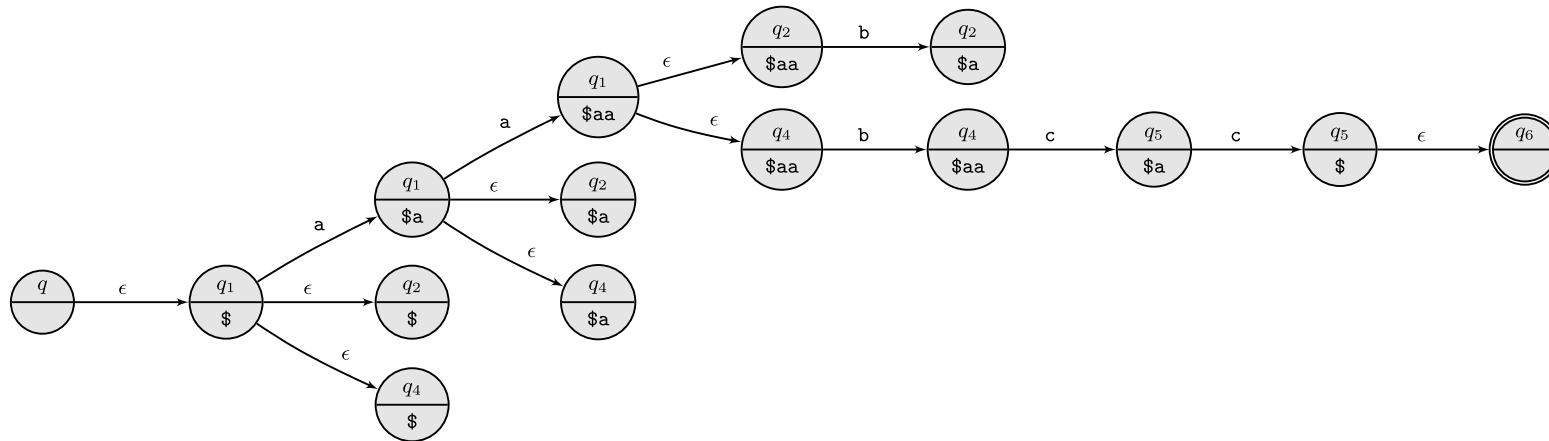
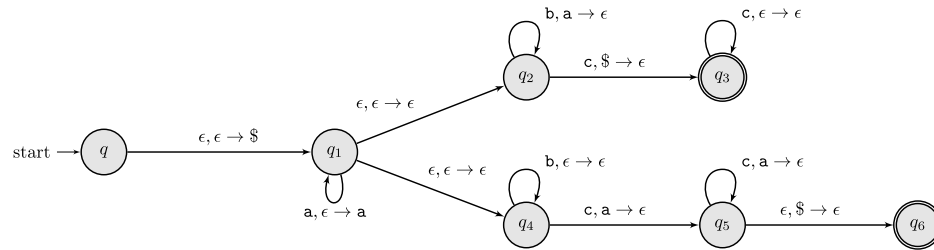
Example 2.16 accept $[a, a, b, b, c, c]$?



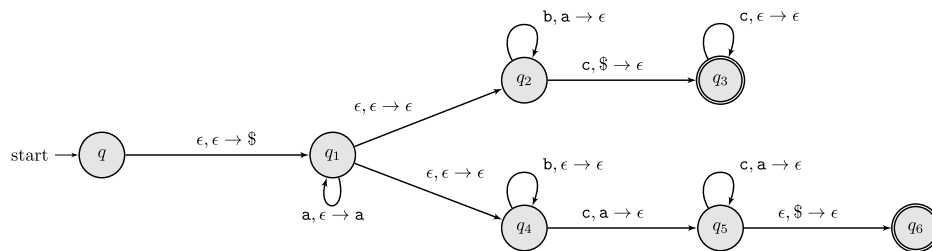
Example 2.16 accept $[a, a, b, c, c]$?



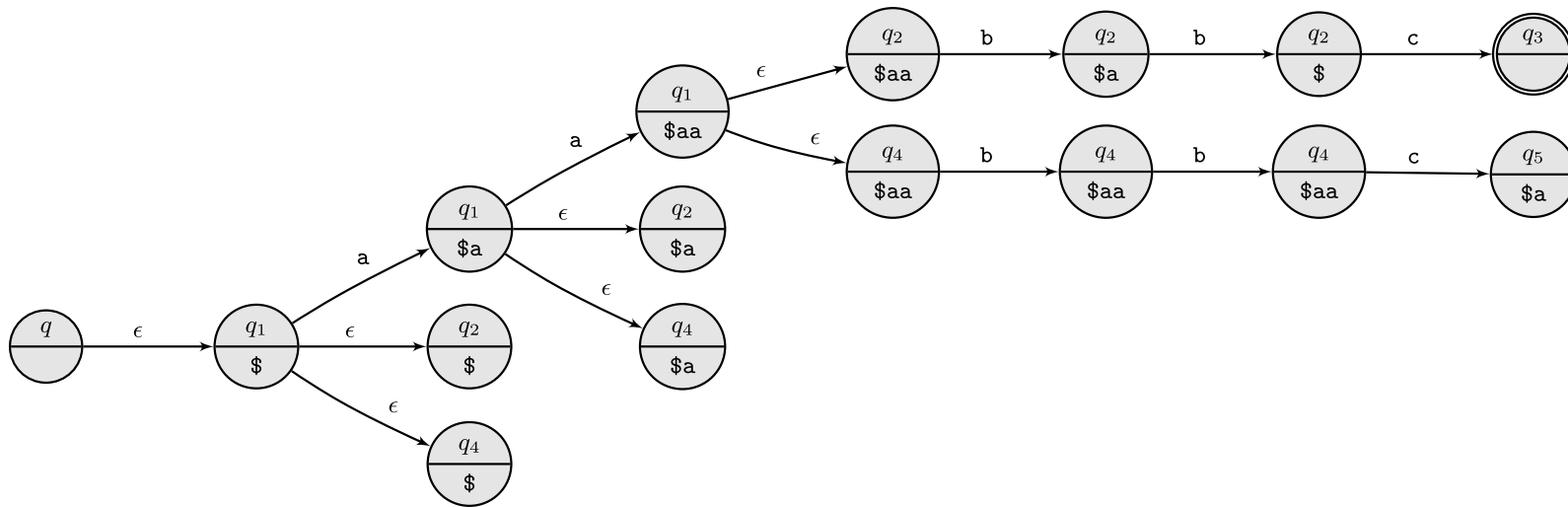
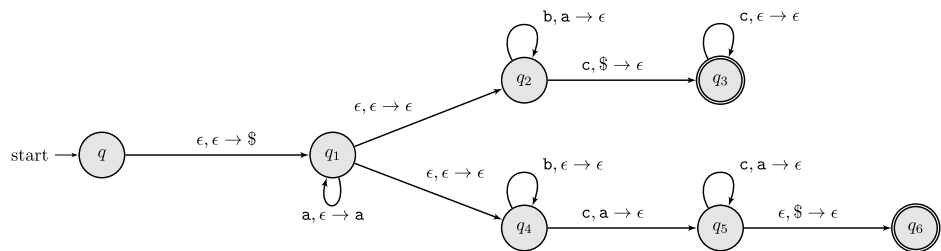
Example 2.16 accept $[a, a, b, c, c]$?



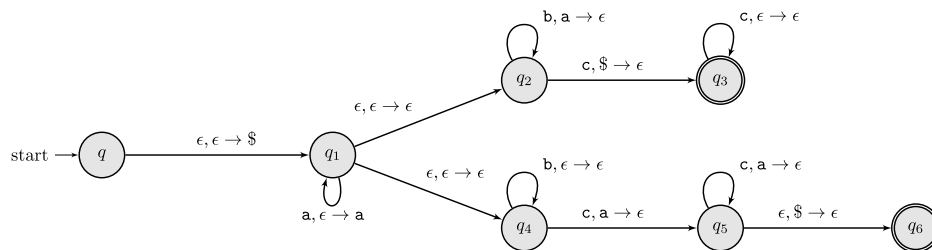
Example 2.16 accept $[a, a, b, b, c]$?



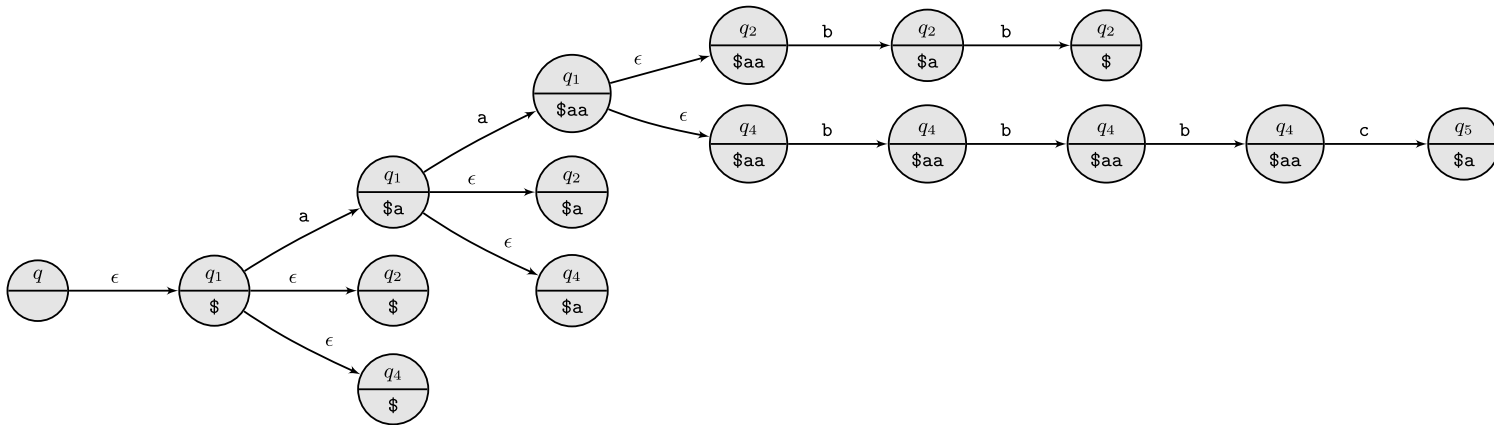
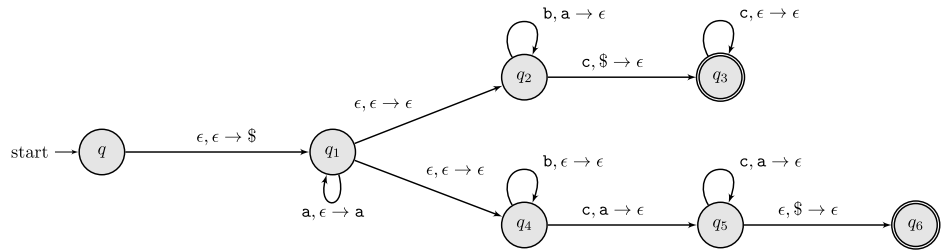
Example 2.16 accept $[a, a, b, b, c]$?



Example 2.16 rejects $[a, a, b, b, b, c]$?



Example 2.16 rejects $[a, a, b, b, b, c]$?



Union for PDAs?

Example 2.16

$$\{a^i b^j c^k \mid i = j \vee i = k\} = \{a^i b^j c^k \mid i = j\} \cup \{a^i b^j c^k \mid i = k\}$$

Example 2.16

$$\{a^i b^j c^k \mid i = j \vee i = k\} = \{a^i b^j c^k \mid i = j\} \cup \{a^i b^j c^k \mid i = k\}$$

