CS420

Introduction to the Theory of Computation

Lecture 22: Mapping reducibility

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Mini Test 3 overview

- 50 points for Sections 4.1 and 4.2 (HW7 + Exercises in Lesson 20)
- around 10 points for Section 5.1
- around 40 points for Section 5.3

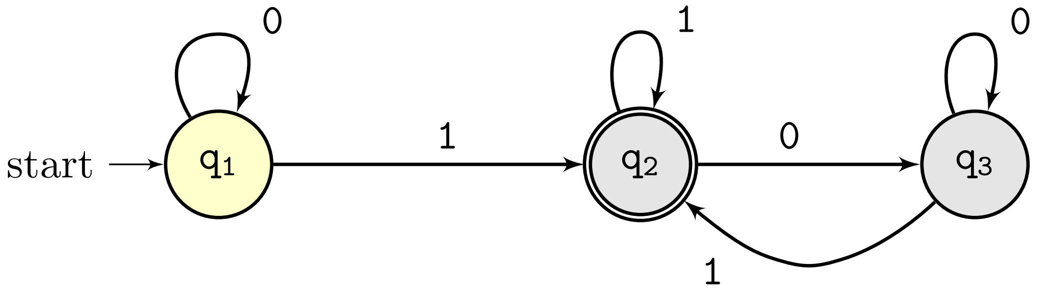
- Level 1: 60 points
- Level 2: 25 points
- Level 3: 15 points

> Today we will be doing exercises of Level 2 and Level 3.

Exercise 1 (Level 1)

Know why membership tests fail and succeed; explain **why** certain membership fails.

Let D be the DFA below




```
def A_DFA(D, w): return D accept w
def E_DFA(D): return L(D) == {}
def EQ_DFA(D1, D2): return L(D1) == L(D2)
```

- Exercise 2.1: Is $\langle D, 0100\rangle \in A_{DFA}$?
- Exercise 2.2: Is $\langle D, 101\rangle \in A_{DFA}$?
- Exercise 2.3: Is $\langle D\rangle \in A_{DFA}$?
- Exercise 2.4: Is $\langle D, 101\rangle \in A_{REX}$?
- Exercise 2.5: Is $\langle D\rangle \in E_{DFA}$?
- Exercise 2.6: Is $\langle D, D\rangle \in EQ_{DFA}$?
- Exercise 2.7: Is $101 \in A_{REX}$?

Exercise 2 (Level 1)

> Know how to compose decidable algorithms as new decidable algorithms.

Give an algorithm that decides EQ_{REX} is undecidable.

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> Know how to compose decidable algorithms as new decidable algorithms.

Give an algorithm that decides EQ_{REX} is undecidable.

```
def EQ_REX(R1, R2):
  return EQ_DFA(REX_TO_DFA(R1), REX_TO_DFA(R2))
```

Similar examples: give a decider for

- A_{NFA}, A_{REX}, A_{PDA} (Lesson 17)
- EQ_{DFA} (Lesson 18)
- EQ_{DFAREX} (Exercise 4.2) (or any combination therein)
- ALL_{DFA} (Exercise 4.3)
- $A\epsilon_{\text{CFG}}$ (Exercise 4.4)
- $\{\langle R, S\rangle \mid R, S \text{ are regex} \wedge L(R) \subseteq L(S)\}$ is decidable (Problem 4.13)
- $\{\langle R\rangle \mid R \text{ is regex over } \{0,1\} \wedge w \text{ contains } 111 \wedge w \in L(G)\}$ (Exercise 4.16)

Exercise 3 (Level 1)

> Know examples of recognizable, decidable, unrecognizable, undecidable languages.

Give an example of a recognizable and undecidable language.

Exercise 3 (Level 1)

Know examples of recognizable, decidable, unrecognizable, undecidable languages.

Give an example of a recognizable and undecidable language.

Solution: A_{TM} is recognizable (in proof of Theorem 4.11, page 202) and undecidable (Theorem 4.11).

Tip: build a table of (co-)recognizable, decidable, undecidable, and (co-)unrecognizable languages

- Think of A, E, EQ for DFA, CFG, and TM

Exercise 4 (Level 2)

> Map-reducible: Use decidability (Theorem 5.22 and Corollary 5.23) and recognizability (Theorem 5.28 and Corollary 5.29) to derive conclusions about the languages we studied ($A, E, EQ + DFA, CFG, TM$).

Given that $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$, show that $HALT_{\mathsf{TM}}$ is undecidable.

Exercise 4 (Level 2)

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Given that $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$, show that $HALT_{\mathsf{TM}}$ is undecidable.

Proof. Apply Corollary 5.23 since A_{TM} is undecidable (Theorem 4.11) and $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$ (hypothesis).

More examples

- Show that $\overline{HALT}_{\mathsf{TM}}$ is unrecognizable.
- Show that $HALT_{\mathsf{TM}}$ is undecidable. (Exercise 5.24/Lesson 22)
- Show that A_{TM} is recognizable via mapping reducibility. (Lesson 22)

Exercise 5 (level 2)

Relate facts on map-reducible.

- Exercise 5.6: $\leq_{\mathrm{m}}$ is a transitive relation.
- Exercise 5.22: A is recognizable iff $A \leq_{\mathrm{m}} A_{\mathsf{TM}}$.

Let (H1) $A_{CFG} \leq_{\mathrm{m}} A_{TM}$, (H2) $A_{DFA} \leq_{\mathrm{m}} A_{CFG}$, and (H3) A_{TM} is recognizable.
Prove that we can conclude that A_{DFA} is recognizable using map-reducibility.

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Let (H1) $A_{CFG} \leq_{\mathrm{m}} A_{TM}$, (H2) $A_{DFA} \leq_{\mathrm{m}} A_{CFG}$, and (H3) A_{TM} is recognizable. Prove that we can conclude that A_{DFA} is recognizable using map-reducibility.

1. $A_{DFA} \leq_{\mathrm{m}} A_{TM}$ by Exercise 5.6, (H1) $A_{CFG} \leq_{\mathrm{m}} A_{TM}$, (H2) $A_{DFA} \leq_{\mathrm{m}} A_{CFG}$.
2. A_{DFA} is recognizable, by Exercise 5.22, (1) $A_{DFA} \leq_{\mathrm{m}} A_{TM}$, and (H3).

Exercise 6 (Level 2)

Relate facts on map-reducible.

- Lemma R.1: If $A \leq_{\mathrm{m}} B$, then $\overline{A} \leq_{\mathrm{m}} \overline{B}$.
- Lemma R.2: If $A \leq_{\mathrm{m}} \overline{B}$ and B recognizable, then $\overline{A} \leq_{\mathrm{m}} B$.
- Lemma R.3: If A recognizable and $\overline{A} \leq_{\mathrm{m}} B$, then $A \leq_{\mathrm{m}} \overline{B}$.

Let (H1) $B \leq \overline{A}_{TM}$. Show that $\overline{B}$ is recognizable.

Exercise 6 (Level 2)

Relate facts on map-reducible.

- Lemma R.1: If $A \leq_{\rm m} B$, then $\overline{A} \leq_{\rm m} \overline{B}$.
- Lemma R.2: If $A \leq_{\rm m} \overline{B}$ and B recognizable, then $\overline{A} \leq_{\rm m} B$.
- Lemma R.3: If A recognizable and $\overline{A} \leq_{\rm m} B$, then $A \leq_{\rm m} \overline{B}$.

Let (H1) $B \leq \overline{A}_{TM}$. Show that $\overline{B}$ is recognizable.

Proof.

1. $\overline{B} \leq A_{TM}$, by Lemma R.2, (H1) $\overline{A}_{TM} \leq B$, and A_{TM} recognizable (pp 202).
2. $\overline{B}$ is recognizable, by Exercise 5.22 and (1) $\overline{B} \leq A_{TM}$.

Exercise 7 (Level 2)

> Relate facts on map-reducible.

Show that $\overline{HALT}_{\mathsf{TM}}$ is unrecognizable.

Exercise 7 (Level 2)

Relate facts on map-reducible.

Show that $\overline{HALT}_{\text{TM}}$ is unrecognizable.

Proof.

1. $\overline{A}_{\text{TM}} \leq_{\text{m}} \overline{HALT}_{\text{TM}}$, by Theorem R.1 and $A_{\text{TM}} \leq_{\text{m}} HALT_{\text{TM}}$ (exercise 5.24)
2. $\overline{HALT}_{\text{TM}}$ is unrecognizable, by Corollary 5.29, $\overline{A}_{\text{TM}} \leq_{\text{m}} \overline{HALT}_{\text{TM}}$ (1), and $\overline{A}_{\text{TM}}$ is unrecognizable (Corollary 4.23)

Exercise 8 (Level 3)

(Exercise 4.2 in the book.)

$$EQ_{DFAREX}\{\langle D, R\rangle \mid D \text{ is a DFA} \land R \text{ is a regex} \land L(D) = L(R)\}$$

Exercise 8 (Level 3)

(Exercise 4.2 in the book.)

$$EQ_{DFAREX}\{\langle D, R\rangle \mid D \text{ is a DFA} \land R \text{ is a regex} \land L(D) = L(R)\}$$

Let $r2n$ be the function that converts a regular expression into an NFA and $n2d$ be the function that converts an NFA into a DFA.

1. $EQ_{DFAREX} \leq_m EQ_{DFA}$ with $F(\langle D, R\rangle) = \langle D, n2d(r2n(R))\rangle$.
   - Unfold $\leq_m$. Goal: $\langle D, R\rangle \in EQ_{DFAREX} \iff F(\langle D, R\rangle) \in EQ_{DFA}$
   - Unfold EQ_{DFAREX}, EQ_{DFA}, and F. Goal: $L(D) = L(R) \iff L(D) = n2d(r2n(R))$
   - Rewrite goal with $\forall N, L(n2d(N)) = L(N)$ and $\forall R, r2n(R) = L(R)$. Goal: $L(D) = L(R) \iff L(D) = L(R)$. Proof: trivial, since $\forall P, P \iff P$.
2. EQ_{DFAREX} is decidable, by Theorem 5.22, (1) $EQ_{DFAREX} \leq_m EQ_{DFA}$, and EQ_{DFA} decidable (Theorem 4.5).

> The proof has two main parts: 1) showing that the given language map-reduces to a decidable language and 2) use Theorem 5.22 to conclude.

Exercise 8 (Level 3)

Continuation...

- The proof has two main parts: 1) showing that the given language map-reduces to a decidable language and 2) use Theorem 5.22 to conclude.
- Whenever you say that $A \leq_m B$ be clear about which **function** reduces A to B.

More examples

- See HW7

Exercise 9 (Level 3)

Hint...

> Combine Lemma R.1, R.2, R.3, Exercise 5.6, Exercise 5.22, and decidability, recognizability to relate the recognizability/decidability between mapping-reducible languages.