Homework 1 Solutions

1. **Estimating logarithm function.** For $x \in [0, 1)$, we shall use the identity that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

   (a) **(5 points)** Prove that $\ln(1-x) \leqslant -x - \frac{x^2}{2}$.

   Solution.

   It is clearly that $\frac{x^k}{k} \geqslant 0$ for every $x \in [0, 1)$ and for every positive integer k. Therefore, we have

$$-\frac{x^3}{3} - \frac{x^4}{4} - \ldots \leqslant 0,$$

   which implies that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \leqslant -x - \frac{x^2}{2}.$$

(b) **(10 points)** For $x \in [0, 1/2]$, prove that

$$\ln(1-x) \geqslant -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \frac{x^2}{2 \cdot 2^3} - \cdots = -x - x^2.$$

Solution.
First we shall show that $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}$ for every $x \in [0, 1/2]$ and for every positive integer $k \geqslant 2$. It suffices to show that $x^k \cdot 2^{k-1} \leqslant x^2 \cdot k$, which is equivalent to $x^{k-2} \cdot 2^{k-1} = (2x)^{k-2} \cdot 2 \leqslant k$. Since $x \in [0, 1/2]$, we have $0 \leqslant 2x \leqslant 1$, so $(2x)^{k-2} \leqslant 1$, which implies that $(2x)^{k-2} \cdot 2 \leqslant 2 \leqslant k$. Therefore $\frac{x^k}{k} \leqslant \frac{x^2}{2^{k-1}}$, in other words, $-\frac{x^k}{k} \geqslant \frac{x^2}{2^{k-1}}$. Applying this inequality for $k = 2, 3, \ldots$, we have

$$\begin{aligned}
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \\
&\geqslant -x - \frac{x^2}{2} - \frac{x^2}{2^3} - \frac{x^2}{2^4} - \cdots \\
&= -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \cdots \\
&= -x - \frac{x^2}{2}\left(\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} \cdots\right) \\
&= -x - \frac{x^2}{2} \cdot 2 && \text{By simplifying the geometric sum} \\
&= -x - x^2
\end{aligned}$$

2. **Tight Estimations.** Provide meaningful upper-bounds and lower-bounds for the following expressions.

   (a) **(10 points)** $S_n = \sum_{i=1}^{n} \ln i$,
   Solution.

$$\int_{i}^{i+1} \ln(t)\,\mathrm{d}t \geqslant \ln(i)$$

$$\int_{i-1}^{i} \ln(t)\,\mathrm{d}t \leqslant \ln(i)$$

Using the first inequality for values $i = 1, 2, \ldots, n$, we get the following inequality:

$$S_n = \sum_{i=1}^{n} \ln(i) \leqslant \sum_{i=1}^{n} \int_{i}^{i+1} \ln(t)\,\mathrm{d}t = \int_{1}^{n+1} \ln(t)\,\mathrm{d}t = [t\ln(t) - t]_{t=1}^{n+1} = (n+1)\ln(n+1) - n$$

To find a lower bound, first notice that $\ln(1) = 0$ and so $S_n = \sum_{i=2}^{n} \ln(i)$. Now, by using the second inequality for values $i = 2, 3, \ldots, n$, we can find a lower bound for S_n:

$$\begin{aligned} S_n = \sum_{i=2}^{n} \ln(i) &\geqslant \sum_{i=2}^{n} \int_{i-1}^{i} \ln(t)\,\mathrm{d}t = \sum_{i=1}^{n-1} \int_{i}^{i+1} \ln(t)\,\mathrm{d}t = \int_{1}^{n} \ln(t)\,\mathrm{d}t = [t\ln(t) - t]_{1}^{n} \\ &= n\ln(n) - n + 1 \end{aligned}$$

(b) **(10 points)** $A_n = n!$
Solution.
According to part (b), we have:

$$n\ln(n) - n + 1 \leqslant \ln(n!) = \sum_{i=1}^{n} \ln(i) \leqslant (n+1)\ln(n+1) - n$$

and so, we have:

$$\begin{aligned} & e^{n\ln(n)-n+1} \leqslant A_n = e^{\ln(n!)} \leqslant e^{(n+1)\ln(n+1)-n} \\ \iff & \frac{n^n}{e^{n-1}} \leqslant A_n \leqslant \frac{(n+1)^{n+1}}{e^n} \end{aligned}$$

If we use the more precise upper bound we found in part a, then we get the following upper bound which is more precise:

$$A_n = e^{\ln(n!)} \leqslant e^{\frac{\ln n}{2} + n\ln(n) - n + 1} = \frac{n^{n+\frac{1}{2}}}{e^{n-1}}$$

(c) **(10 points)** $B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$

Solution.

Let $A_n = n!$, then we observe that $B_n = \frac{A_{2n}}{A_n^2}$. According to part b, we have

$$\frac{(2n)^{2n}}{e^{2n-1}} \leqslant A_{2n} \leqslant \frac{(2n+1)^{2n+1}}{e^{2n}}$$

$$\frac{n^{2n}}{e^{2n-2}} \leqslant A_n^2 \leqslant \frac{(n+1)^{2n+2}}{e^{2n}}$$

Therefore, we have:

$$\frac{1}{\frac{n^{2n}}{e^{2n-2}}} \geqslant \frac{1}{A_n^2} \geqslant \frac{1}{\frac{(n+1)^{2n+2}}{e^{2n}}}$$

This implies that:

$$\frac{(2n+1)^{2n+1}}{e^2 \cdot n^{2n}} = \frac{\frac{(2n+1)^{2n+1}}{e^{2n}}}{\frac{n^{2n}}{e^{2n-2}}} \geqslant \frac{A_{2n}}{\frac{n^{2n}}{e^{2n-2}}} \geqslant B_n = \frac{A_{2n}}{A_n^2} \geqslant \frac{A_{2n}}{\frac{(n+1)^{2n+2}}{e^{2n}}} \geqslant \frac{\frac{(2n)^{2n}}{e^{2n-1}}}{\frac{(n+1)^{2n+2}}{e^{2n}}} = \frac{e \times (2n)^{2n}}{(n+1)^{2(n+1)}}$$

3. **Understanding Joint Distribution.** Recall that in the lectures we considered the joint distribution $(\mathbb{T}, \mathbb{B})$ over the sample space $\{4, 5, \ldots, 10\} \times \{\mathsf{T}, \mathsf{F}\}$, where $\mathbb{T}$ represents the time I wake up in the morning, and $\mathbb{B}$ represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.01
4	F	0.05
5	T	0
5	F	0.04
6	T	0.1
6	F	0.20
7	T	0.25
7	F	0.10
8	T	0.10
8	F	0.05
9	T	0.03
9	F	0.05
10	T	0.01
10	F	0.01

Calculate the following probabilities.

(a) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier, but do not have breakfast. That is, calculate $\mathbb{P}[\mathbb{T} \leqslant 8, \mathbb{B} = \mathsf{F}]$,
Solution.

$$\begin{aligned}\mathbb{P}[\mathbb{T} \leqslant 8, \mathbb{B} = F] &= \mathbb{P}[\mathbb{T} = 8, \mathbb{B} = F] + \mathbb{P}[\mathbb{T} = 7, \mathbb{B} = F] \\ &\quad + \mathbb{P}[\mathbb{T} = 6, \mathbb{B} = F] + \mathbb{P}[\mathbb{T} = 5, \mathbb{B} = F] + \mathbb{P}[\mathbb{T} = 4, \mathbb{B} = F] = 0.44\end{aligned}$$

(b) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier. That is, calculate $\mathbb{P}\left[\mathbb{T} \leqslant 8\right]$,
Solution.

$$\mathbb{P}[\mathbb{T} \leqslant 8] = 1 - \mathbb{P}[\mathbb{T} = 9] - \mathbb{P}[\mathbb{T} = 10] = 0.9$$

(c) **(5 points)** Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 8 a.m. or earlier. That is, compute $\mathbb{P}\left[\mathbb{B} = \mathsf{F} \mid \mathbb{T} \leqslant 8\right]$.
Solution.

$$\mathbb{P}[\mathbb{B} = F|\mathbb{T} \leqslant 8] = \frac{\mathbb{P}[\mathbb{B} = F, \mathbb{T} \leqslant 8]}{\mathbb{P}[\mathbb{T} \leqslant 8]} = \frac{0.44}{0.9} = 0.49$$

4. **Random Walk.** There is a frog sitting at the origin $(0, 0)$ in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point $(\mathbb{X}, 0)$, where $\mathbb{X} \in \{1, 2, 3, 4, 5, 6\}$. Then, it jumps uniformly at random along the Y-axis to some point $(\mathbb{X}, \mathbb{Y})$, where $\mathbb{Y} \in \{1, 2, 3, 4, 5, 6\}$. So $(\mathbb{X}, \mathbb{Y})$ represents the final position of the frog after these two jumps. Note that $\mathbb{X}$ and $\mathbb{Y}$ are two independent random variables that are uniformly distributed over their respective sample spaces.

   (a) **(5 points)** What is the probability that the frog jumps more than 3 units along the Y-axis. That is, compute $\mathbb{P}[\mathbb{Y} > 3]$.
   Solution. Note that $\mathbb{Y}$ is a uniform random variable over sample space $\{1, 2, 3, 4, 5, 6\}$. This means that $\mathbb{P}[\mathbb{Y} = i] = \frac{1}{6}$ for each $i \in \{1, 2, 3, 4, 5, 6\}$. So we have:

$$\mathbb{P}[\mathbb{Y} > 3] = \mathbb{P}[\mathbb{Y} = 4] + \mathbb{P}[\mathbb{Y} = 5] + \mathbb{P}[\mathbb{Y} = 6] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

   (b) **(10 points)** What is the probability that the final position of the frog is above the line $X + Y = 7$. That is compute $\mathbb{P}[\mathbb{X} + \mathbb{Y} > 7]$?
   Solution. Note that $\mathbb{X}$ and $\mathbb{Y}$ are uniform random variable and are independent. So, for each $(i, j) \in \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$, we have the following:

$$\mathbb{P}[\mathbb{X} = i, \mathbb{Y} = j] = \mathbb{P}[\mathbb{X} = i] \times \mathbb{P}[\mathbb{Y} = j] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

   There are 15 points (i, j) in the set $\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ such that $i + j > 7$. Thus, we have

$$\mathbb{P}[\mathbb{X} + \mathbb{Y} > 6] = \frac{15}{36} = \frac{5}{12}$$

(c) **(10 points)** What is the probability that the frog has jumped 4 units along X-axis conditioned on the fact that its final position is above the line $X + Y = 7$? That is, compute $\mathbb{P}\left[\mathbb{X} = 4 \mid \mathbb{X} + \mathbb{Y} > 7\right]$?

$$\begin{aligned}\mathbb{P}\left[\mathbb{X} = 4 \mid \mathbb{X} + \mathbb{Y} > 7\right] &= \frac{\mathbb{P}\left[\mathbb{X} = 4, \mathbb{X} + \mathbb{Y} > 7\right]}{\mathbb{P}\left[\mathbb{X} + \mathbb{Y} > 7\right]} = \frac{\mathbb{P}\left[\mathbb{X} = 4, \mathbb{Y} > 3\right]}{\mathbb{P}\left[\mathbb{X} + \mathbb{Y} > 7\right]} \\ &= \frac{\mathbb{P}\left[\mathbb{X} = 4, \mathbb{Y} = 4\right] + \mathbb{P}\left[\mathbb{X} = 4, \mathbb{Y} = 5\right] + \mathbb{P}\left[\mathbb{X} = 4, \mathbb{Y} = 6\right]}{\mathbb{P}\left[\mathbb{X} + \mathbb{Y} > 7\right]} \\ &= \frac{\frac{1}{36} + \frac{1}{36} + \frac{1}{36}}{\frac{15}{36}} \\ &= \frac{1}{5}\end{aligned}$$

5. **Coin Tossing Word Problem.** We have three (independent) coins represented by random variables $\mathbb{C}_1, \mathbb{C}_2$, and $\mathbb{C}_3$.

   (i) The first coin has $\mathbb{P}[\mathbb{C}_1 = H] = \frac{1}{4}, \mathbb{P}[\mathbb{C}_1 = T] = \frac{3}{4}$,

   (ii) The second coin has $\mathbb{P}[\mathbb{C}_2 = H] = \frac{3}{4}$ and $\mathbb{P}[\mathbb{C}_2 = T] = \frac{1}{4}$, and

   (iii) The third coin has $\mathbb{P}[\mathbb{C}_3 = H] = \frac{1}{4}$ and $\mathbb{P}[\mathbb{C}_3 = T] = \frac{3}{4}$.

   Consider the following experiment.

   (A) Toss the first coin. Let the outcome of the first coin-toss be ω_1.

   (B) If $\omega_1 = H$, then we toss the second coin twice. Otherwise, (i.e., if $\omega_1 = T$) toss the third coin twice. Let the two outcomes of this step be represented by ω_2 and ω_3.

   (C) Output $(\omega_1, \omega_2, \omega_3)$.

   Based on this experiment, compute the probabilities below.

   (a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are H (head)?
   Solution.
   The majority of the outcomes is H if and only if

   $$(\omega_1, \omega_2, \omega_3) \in \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$$

   $$\mathbb{P}[\omega_1 = H, \omega_2 = H, \omega_3 = H] = \mathbb{P}[\omega_1 = H] \times \mathbb{P}\left[\omega_2 = H, \omega_3 = H | \omega_1 = H\right] = \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}$$

   because when the first outcome is H, we toss the second coin twice. Similarly, we have:

   $$\mathbb{P}[\omega_1 = H, \omega_2 = H, \omega_3 = T] = \mathbb{P}[\omega_1 = H] \times \mathbb{P}\left[\omega_2 = H, \omega_3 = T | \omega_1 = H\right] = \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}$$

   $$\mathbb{P}[\omega_1 = H, \omega_2 = T, \omega_3 = H] = \mathbb{P}[\omega_1 = H] \times \mathbb{P}\left[\omega_2 = T, \omega_3 = H | \omega_1 = H\right] = \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4}$$

   But when the first outcome is T, we toss the second coin twice, so, we have:

   $$\mathbb{P}[\omega_1 = T, \omega_2 = H, \omega_3 = H] = \mathbb{P}[\omega_1 = T] \times \mathbb{P}\left[\omega_2 = H, \omega_3 = H | \omega_1 = T\right] = \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}$$

   adding all four probabilities together, we get $\frac{18}{64} = \frac{9}{32}$.

(b) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are H, conditioned on the fact that the first outcome was T?

Solution.

Assuming the first outcome is T, the majority of three outcomes is H if and only if the second and third outcome are both H. So, we need to find the following probability:

$$\mathbb{P}[\omega_2 = H, \omega_3 = H | \omega_1 = T] = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

because when the first outcome is T, we toss the third coin twice.

(c) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are different from the first outcome?

Solution.

$\mathbb{P}\left[\omega_1 = H, \omega_2 = T, \omega_3 = T\right] + \mathbb{P}\left[\omega_1 = T, \omega_2 = H, \omega_3 = H\right] = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} + \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$.