A microfounded New Keynesian model with housing

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1 Households

The utility function is separable in consumption C_t (excl. housing), housing H_t and labor N_t.

$$U(C_t, H_t, L_t) = z_t \Gamma_c U(C_t - \zeta C_{t-1}) + j_t U(H_t) - U(N_t),$$

where Γ_c is a scalar (to insure that in SS the marginal utility from consumption is independent on habit). ζ is a degree of habit in consumption, z_t,and j_t are shocks to consumption and housing preferences. The housing bundle contains both ownership ($h_{o,t}$) and rent ($h_{r,t}$), which are imperfect substitutes.

$$H(h_{o,t}, h_{r,t}) = \left[\gamma h_{o,t}^{(\varepsilon-1)/\varepsilon} + (1-\gamma) h_{r,t}^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)},$$

where $\varepsilon > 1$ is a intratemporal elasticity of substitution between rent and ownership. $0 < \gamma < 1$ is a relative weight of ownership (determines preference-based demand for ownership relative to rent in SS for unconstrained households, $\frac{h_o}{h_r} = \left[\frac{\eta_r}{\eta_r - 1}\frac{\gamma}{1-\gamma}\right]^{\frac{1}{\vartheta}}$,while for constrained households, $\frac{h'_o}{h'_r} = \left[\frac{\frac{\eta_r}{\eta_r-1}(1+k+\beta k-\beta)}{\frac{1-\gamma'}{\gamma'}\left(1+k+\beta'(k-1)-(1-\beta'(\frac{1}{\beta}+\Psi)m^{LTV}\right)}\right]^{\frac{1}{\vartheta}}$, where $\vartheta = \frac{1}{\varepsilon}$.

1.1 Unconstrained households

The unconstrained household j ,$j \in [0, 1]$ acts under two hats: as household and also as investor. Under a hat of household, he has a demand for consumption, rent and ownership, he works and also has a demand for leisure. Under a hat of investor, he purchases houses for investment and rent them to retailers who are the final suppliers of rent services in the economy. Note that investor invests in houses to gain capital benefits to increase his consumption and does not have a direct utility from houses purchased for investment. Since all households are identical we omit index j:

The utility function is

$$E_0 \sum \beta^i \left(\Gamma_c \log(C_t - \zeta^c C_{t-1}) + j_t \log\left(\left[\gamma h_{o,t}^{(\varepsilon-1)/\varepsilon} + (1-\gamma) h_{r,t}^{(\varepsilon-1)/\varepsilon}\right]^{\varepsilon/(\varepsilon-1)}\right) - \frac{1}{1+\varrho} N_t^{1+\varrho}\right)$$

Define $\vartheta = \frac{1}{\varepsilon}$ and $\Gamma_c = \frac{1-\zeta^c}{1-\beta\zeta^c}$. Therefore:

$$E_0 \sum \beta^i \left(z_t \Gamma_c \log(C_t - \zeta^c C_{t-1}) + \frac{j_t}{1-\vartheta} \log\left[\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}\right] - \frac{1}{1+\varrho} N_t^{1+\varrho}\right)$$

Their budget constraint:

$$
\begin{aligned}
& C_t + q_t(h_{o,t} + h_{inv,t}) + kq_t(h_{o,t} + h_{inv,t}) + r_t h_{r,t} + R_{t-1}B_{t-1} = \\
= \; & B_t + q_t(h_{o,t-1} + h_{inv,t-1}) - kq_t(h_{o,t-1} + h_{inv,t-1}) + p_t^m h_{inv,t} + w_t N_t + Div_t - T_t
\end{aligned}
$$

where $h_{inv,t}(j) = h_{inv,t}$ amount of houses purchased for investment (all investors are equal). q_t is a real price of houses (in terms of price P_t of consumption bundle C_t, which excludes consumption of rent services).

The investors rent the purchased houses for investment $h_{inv,t}(j) = h_{inv,t}$ to retailers at (nominal) price P_t^m at the beginning of period t as intermediate good and receive houses back at the end of period t, therefore the capital gains or losses related to reselling $h_{inv,t}$ in period $t+1$ lay only on the investors. $p_t^m(j) = p_t^m$ is a (real) price of rent obtained from the retailers (in terms of P_t, where $p_t^m = \frac{P_t^m}{P_t}$). Notice that r_t is a (real) price of rent in the economy and $p_t^m \neq r_t$. The nominal price of rent is $P_t^{rent} = r_t P_t$.

$h_{o,t}-$amount of purchased homes (ownership), $h_{r,t}$- amount of houses for rent, B_t amount of one-period non-indexed debt, R_t is a (gross) real interest rate paid on one-period debt, $R_t = r_t^{CB} - \pi_{t+1}$ (r_t^{CB} is a nominal interest rate of central bank and π_{t+1} is an expected inflation), w_t is a (real) wage, N_t-hours worked, k is a transaction costs per 1 (real) dollar of home sold/purchased house (say, $k = 0.01$ (1%)). Div and T are dividends and taxes. We assume that there is no government, therefore $T_t = 0$.We assume that transaction costs for all households and firms are refunded (relevant for deriving resource constraint of the economy).

The unconstrained households need to decide about 6 variables: $C_t, h_{o,t}, h_{r,t}, B_t, N_t, h_{inv,t}$.

The Lagrangian:

$$
\begin{aligned}
L_t \;=\; & E_0 \sum \beta^i \left(z_{t+i}\Gamma_c \log(C_{t+i} - \zeta^c C_{t+i-1}) + \frac{j_{t+i}}{1-\vartheta} \log\left[\gamma h_{o,t+i}^{1-\vartheta} + (1-\gamma) h_{r,t+i}^{1-\vartheta}\right] - \frac{1}{1+\eta} N_{t+i}^{1+\eta} \right) - \\
& -E_0 \sum \beta^i \lambda_{t+i} \left(\begin{array}{c} C_{t+i} + q_{t+i}(h_{o,t+i} + h_{inv,t+i}) + kq_{t+i}(h_{o,t+i} + h_{inv,t+i})) + r_{t+i}h_{r,t+i} + R_{t+i-1}B_{t-} \\ -q_{t+i}(h_{o,t+i-1} + h_{inv,t+i-1}) + kq_{t+i}(h_{o,t+i-1} + h_{inv,t+i-1}) - p_{t+i}^m h_{inv,t+i} - w_{t+i}N_{t-} \end{array} \right.
\end{aligned}
$$

F.O.C of unconstrained households:

(1) $C_t : \Gamma_c(\frac{z_t}{C_t - \zeta^c C_{t-1}} - \beta\zeta^c \frac{z_{t+1}}{C_{t+1} - \zeta^c C_t}) - \lambda_t = 0 =>$

$$
\lambda_t = \Gamma_c\left(\frac{z_t}{C_t - \zeta^c C_{t-1}} - \beta\zeta^c \frac{z_{t+1}}{C_{t+1} - \zeta^c C_t}\right) \tag{1}
$$

(2) $h_{o,t} : j_t \frac{\gamma h_{o,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} - \lambda_t(1+k)q_t - \beta\lambda_{t+1}(-q_{t+1} + kq_{t+1}) = 0 =>$

$$
j_t \frac{\gamma h_{o,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} = \lambda_t(1+k)q_t + \beta\lambda_{t+1}(kq_{t+1} - q_{t+1}) \tag{2}
$$

(3): $h_{r,t} : j_t \frac{(1-\gamma) h_{r,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} - \lambda_t r_t = 0 >$

$$
j_t \frac{(1-\gamma) h_{r,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} = \lambda_t r_t \tag{3}
$$

if we divide (2) by (3) (only under $k=0$),

$$\frac{h_{o,t}}{h_{r,t}}=\left[\frac{1-\gamma}{\gamma}\underbrace{\left(\frac{q_t}{r_t}-\left(\frac{\beta\lambda_{t+1}}{\lambda_t}\right)\frac{q_{t+1}}{r_t}\right)}_{\neq 1\text{ because now }p_t^m\neq r_t}\right]^{-\frac{1}{\vartheta}}$$

*F*rom FOC of investors (see (6) below, under $k=0$): $q_t=\left(\beta\frac{\lambda_{t+1}}{\lambda_t}\right)q_{t+1}+p_t^m \Rightarrow \frac{q_t}{p_t^m}-\left(\beta\frac{\lambda_{t+1}}{\lambda_t}\right)\frac{q_{t+1}}{p_t^m}=1$ but it is now not equal to 1 when this ratio is in terms of $\frac{q_t}{r_t}$, becuase $p_t^m\neq r_t$.

Now, this ratio in SS is determined (see proof below):

$$\frac{h_o}{h_r}=\left[\frac{\eta_r}{\eta_r-1}\frac{\gamma}{1-\gamma}\right]^{\frac{1}{\vartheta}}$$

Intuitively, the more the markup of retailes is, the higher is the price of rent, therefore the higher the relative demand for ownership relative to rent (rent is very expensive).

(4) $B_t:\lambda_t-\beta\lambda_{t+1}R_t=0>$

$$\lambda_t=\beta\lambda_{t+1}R_t \tag{4}$$

note that

$$\begin{aligned}\frac{h_{o,t}}{h_{r,t}}&=\left[\frac{1-\gamma}{\gamma}\left(\frac{q_t}{r_t}-\frac{1}{R_t}\frac{q_{t+1}}{r_t}\right)\right]^{-\frac{1}{\vartheta}}\\&=\left[\frac{1-\gamma}{\gamma}\frac{q_t}{r_t}\left(1-\frac{1}{R_t}\frac{q_{t+1}}{q_t}\right)\right]^{-\frac{1}{\vartheta}}\\&=\left[\frac{1-\gamma}{\gamma}\frac{q_t}{r_t}\left(1-\frac{R_{t+1}^{ownership}}{R_t}\right)\right]^{-\frac{1}{\vartheta}}\end{aligned}$$

(5) $N_t:-\frac{1+\varrho}{1+\varrho}N_t^{\varrho}-\lambda_t(-w_t)=0 \Rightarrow$

$$N_t^{\varrho}=\lambda_t w_t \tag{5}$$

(6) $h_{inv,t}:-\lambda_t(q_t+kq_t-p_t^m)-\beta\lambda_{t+1}(-q_{t+1}+kq_{t+1})=0 \Rightarrow$

$$\lambda_t(p_t^m-q_t-kq_t)=\beta\lambda_{t+1}q_{t+1}(k-1) \tag{6}$$

for $k=0: q_t=\left(\beta\frac{\lambda_{t+1}}{\lambda_t}\right)q_{t+1}+p_t^m$

There are two possible representations of the previous equation (under $k=0$):

1.

$$p_t^m = q_t - E_t(sdf_{t,t+1}q_{t+1})$$

2.

$$q_t = E_t \sum_{i=0}^{\infty} sdf_{t,t+i} p_{t+1}^m \tag{7}$$

which is a discounted value of future payoffs from rent, where $sdf_{t,t} \equiv 1$, $sdf_{t,t+i} \equiv \prod_{l=1}^{i} sdf_{t+l}$ for $i > 0$, $sdf_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$ is a stochastic discount factor of unconstrained households.

We want to see what determines rent to price ratio **in the eyes of the investors**. Notice that the rent to price ratio below $\frac{p_t^m}{q_t}$ is not equal to the **actual** rent-to-price ratio in the economy, $\frac{r_t}{q_t}$, because $p_t^m \neq r_t$. (*Yakov: in the dynare code you have to create rent-price ratio as* $rq\_ratio = \frac{r}{q}$ *and not rely on the equations below which were in the previous version).*

Dividing previous equation by λ_t and then by q_t :

$$\frac{p_t^m}{q_t} - 1 - k = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{q_{t+1}}{q_t}(k-1)$$

The rent to price ratio is given by (using $sdf_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$):

$$\frac{p_t^m}{q_t} = sdf_{t,t+1} \frac{q_{t+1}}{q_t}(k-1) + 1 + k \tag{8}$$

For $k = 0$ and representing formally (with operator E_t), $\frac{p_t^m}{q_t} = 1 - E_t\left(sdf_{t,t+1}\frac{q_{t+1}}{q_t}\right) = 1 - E_t\left(sdf_{t,t+1}\right) E_t\left(\frac{q_{t+1}}{q_t}\right) - cov_t\left(sdf_{t,t+1}\frac{q_{t+1}}{q_t}\right)$

$$\frac{p_t^m}{q_t} = 1 - \frac{1}{R_t} E_t\left(\Delta q_{t+1}\right) - cov_t\left(sdf_{t,t+1}, \Delta q_{t+1}\right) \tag{9}$$

where $E_t \Delta q_{t+1} = \frac{q_{t+1}}{q_t}$ is an expected growth of home prices, and $E_t\left(sdf_{t,t+1}\right) = \frac{1}{R_t}$.

Ignoring the covariance term (which would be eliminated under 1'st order Taylor linearization), we can see that for given $E_t \Delta q_{t+1}$,the decline (increase) in the interest rate (R_t) results in decline (increase) in rent/price ratio, so the relationship is positive. But for given R_t,the increase in the expected price growth of homes results in decline in rent-price ratio, so so the relationship is negative. As evident in Israel, the rent-price ratio has declined in the last decade, which can be attributed both to decline in the interest rate and expectations for increase in home prices.***Note 1: the expectations here and overall all the model do not assume rationality and may take any form. Rational expectations are need only to solve the model in Dynare. ***Note 2: the equation for $\frac{p_t^m}{q_t}$ would change in we assumed that the investors are subject to LTV constraint or the interest rate is not riskless interest rate.

1.2 Retailers

The retailers buy from investors houses purchased by them for investment at price P_t^m and these houses are used as intermediate goods. Final product of houses which is used for rent in the economy Y_t^{rent} is a composite of a continuum of mass unity of differentiated retail houses that use intermediate houses from investors as the sole input (that is, $h_{inv,t}(j) = Y_{f,t}$, where $Y_{f,t}$ is a amount of houses of retailer f and $h_{inv,t}(j)$ is an intermediate product of investor j). The final product of houses Y_t^{rent} used for rent in the economy is:

$$Y_t^{rent} = \left[\int_0^1 (Y_{f,t})^{\frac{\eta_r - 1}{\eta_r}} \, dj\right]^{\frac{\eta_r}{\eta_r - 1}} \tag{10}$$

where η_r is the elasticity of substitution between differentiated houses. The demand for differentiated retail house f is given by:

$$Y_{f,t} = \left(\frac{P_{f,t}^{rent}}{P_t^{rent}}\right)^{-\eta_r} Y_t^{rent} \tag{11}$$

where the (nominal) price of Y_t^{rent} is given by:

$$P_t^{rent} = \left[\int_0^1 \left(P_{f,t}^{rent}\right)^{1-\eta_r} \, dj\right]^{\frac{1}{1-\eta_r}} \tag{12}$$

The differentiated retail firms face a price rigidity of Rotemberg (1982). The expected profits of retailer is:

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} X_{f,t+i}(j), \tag{13}$$

where $X_{f,t}(j)$ is a nominal profit of retailer j, and $sdf_{t,t} \equiv 1$, $sdf_{t,t+i} \equiv \prod_{l=1}^{i} sdf_{t+l}$ for $i > 0$, $sdf_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$, a stochastic discount factors of unconstrained households (owners of the retailers firm).

$$X_{f,t}(j) = P_{f,t}^{rent}(j) Y_{f,t}(j) - P_t^m Y_{f,t}(j) - \frac{\Theta}{2}\left(\frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} - (\pi)^{1-\phi}\left(\frac{P_{t-1}^{rent}}{P_{t-2}^{rent}}\right)^{\phi}\right)^2 P_t^{rent} Y_t^{rent}$$

where π is a steady-state inflation rate, ϕ is a degree of indexation to past inflation and Θ defines the cost associated with changing the price. We impose that $\Theta > \chi$ (*Yaakov: please pay attention in the calibration!*) meaning that the rent price stickiness is bigger that consumption price (see Section of firms below).

The expected profit is

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} \left[P_{f,t+i}^{rent}(j) Y_{f,t+i}(j) - P_{t+i}^m Y_{f,t+i}(j) - \frac{\Theta}{2}\left(\frac{P_{f,t+i}^{rent}(j)}{P_{f,t+i-1}^{rent}(j)} - (\pi)^{1-\phi}\left(\frac{P_{t+i-1}^{rent}}{P_{t+i-2}^{rent}}\right)^{\phi}\right)^2 P_{t+i}^{rent} Y_{t+i}^{rent}\right]$$

Inserting demand for $Y_{f,t}$ we get. The retailer must choose optimal price $P_{f,t}^{rent}(j)$):

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} \left[P_{f,t+i}^{rent}(j) \left(\frac{P_{f,t+i}^{rent}(j)}{P_{t+i}^{rent}} \right)^{-\eta_r} Y_{t+i}^{rent} - P_{t+i}^{m} \left(\frac{P_{f,t+i}^{rent}(j)}{P_{t+i}^{rent}} \right)^{-\eta_r} Y_{t+i}^{rent} - \frac{\Theta}{2} \left(\frac{P_{f,t+i}^{rent}(j)}{P_{f,t+i-1}^{rent}(j)} - (\pi)^{1-\phi} \right. \right.$$

After dividing by P_{t+i}^{rent} and by Y_{t+i}^{rent}

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} \left[\left(\frac{P_{f,t+i}^{rent}(j)}{P_{t+i}^{rent}} \right)^{1-\eta_r} - \frac{P_{t+i}^{m}}{P_{t+i}^{rent}} \left(\frac{P_{f,t+i}^{rent}(j)}{P_{t+i}^{rent}} \right)^{-\eta_r} - \frac{\Theta}{2} \left(\frac{P_{f,t+i}^{rent}(j)}{P_{f,t+i-1}^{rent}(j)} - (\pi)^{1-\phi} \left(\frac{P_{t+i-1}^{rent}}{P_{t+i-2}^{rent}} \right)^{\phi} \right)^2 \right]$$

First-order conditions w.r.t. price $P_{f,t}^{rent}(j)$:

$$\begin{aligned} 0 &= (1-\eta_r) \left(P_{f,t}^{rent}(j)^{-\eta_r} \right) \left(\frac{1}{P_t^{rent}} \right)^{1-\eta_r} + \eta_r \frac{P_t^m}{P_t^{rent}} \left(\frac{1}{P_t^{rent}} \right)^{-\eta_r} \left(P_{f,t}^{rent}(j) \right)^{-\eta_r - 1} - \Theta \frac{1}{P_{f,t-1}^{rent}(j)} \left(\frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} \right. \\ &\quad + sdf_{t,t+1} \Theta \frac{P_{f,t+1}^{rent}(j)}{\left(P_{f,t}^{rent}(j) \right)^2} \left(\frac{P_{f,t+1}^{rent}(j)}{P_{f,t}^{rent}(j)} - (\pi)^{1-\phi} \left(\frac{P_t^{rent}}{P_{t-1}^{rent}} \right)^{\phi} \right) \end{aligned}$$

Lets now multiply by $P_{f,t}^{rent}(j)$

$$\begin{aligned} 0 &= (1-\eta_r) \left(P_{f,t}^{rent}(j)^{1-\eta_r} \right) \left(\frac{1}{P_t^{rent}} \right)^{1-\eta_r} + \eta_r \frac{P_t^m}{P_t^{rent}} \left(\frac{1}{P_t^{rent}} \right)^{-\eta_r} \left(P_{f,t}^{rent}(j) \right)^{-\eta_r} - \Theta \frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} \left(\frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} \right. \\ &\quad + sdf_{t,t+1} \Theta \frac{P_{f,t+1}^{rent}(j)}{P_{f,t}^{rent}(j)} \left(\frac{P_{f,t+1}^{rent}(j)}{P_{f,t}^{rent}(j)} - (\pi)^{1-\phi} \left(\frac{P_t^{rent}}{P_{t-1}^{rent}} \right)^{\phi} \right) \end{aligned}$$

After rewriting

$$\begin{aligned} 0 &= (1-\eta_r) \left(\frac{P_{f,t}^{rent}(j)}{P_t^{rent}} \right)^{1-\eta_r} + \eta_r \frac{P_t^m}{P_t^{rent}} \left(\frac{P_{f,t}^{rent}(j)}{P_t^{rent}} \right)^{-\eta_r} - \Theta \frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} \left(\frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)} - (\pi)^{1-\phi} \left(\frac{P_{t-1}^{rent}}{P_{t-2}^{rent}} \right)^{\phi} \right) + \\ &\quad + sdf_{t,t+1} \Theta \frac{P_{f,t+1}^{rent}(j)}{P_{f,t}^{rent}(j)} \left(\frac{P_{f,t+1}^{rent}(j)}{P_{f,t}^{rent}(j)} - (\pi)^{1-\phi} \left(\frac{P_t^{rent}}{P_{t-1}^{rent}} \right)^{\phi} \right) \end{aligned}$$

Now lets define $\pi_t^{rent}(j) = \frac{P_{f,t}^{rent}(j)}{P_{f,t-1}^{rent}(j)}, \pi_t^{rent} = \frac{P_t^{rent}}{P_{t-1}^{rent}}$.

$$\begin{aligned} 0 &= (1-\eta_r) \left(\frac{P_{f,t}^{rent}(j)}{P_t^{rent}} \right)^{1-\eta_r} + \eta_r \frac{P_t^m}{P_t^{rent}} \left(\frac{P_{f,t}^{rent}(j)}{P_t^{rent}} \right)^{-\eta_r} - \Theta \pi_t^{rent}(j) \left(\pi_t^{rent}(j) - (\pi)^{1-\phi} \left(\pi_{t-1}^{rent} \right)^{\phi} \right) + \\ &\quad + sdf_{t,t+1} \Theta \pi_{t+1}^{rent}(j) \left(\pi_{t+1}^{rent}(j) - (\pi)^{1-\phi} \left(\pi_t^{rent} \right)^{\phi} \right) \end{aligned}$$

Since $P_{f,t}^{rent}(j) = P_t^{rent}$, and $\pi_t^{rent}(j) = \pi_t^{rent}$ (in contract to Calvo setting, here all firms are identical w.r.t. price setting)

$$0=(1-\eta_r)+\eta_r\frac{P^m_t}{P^{rent}_t}-\Theta\pi^{rent}_t\left(\pi^{rent}_t-(\pi)^{1-\phi}\left(\pi^{rent}_{t-1}\right)^{\phi}\right)+sdf_{t,t+1}\Theta\pi^{rent}_{t+1}\left(\pi^{rent}_{t+1}-(\pi)^{1-\phi}\left(\pi^{rent}_{t}\right)^{\phi}\right)$$

After simplifications

$$\frac{P^m_t}{P^{rent}_t}=\frac{\eta_r-1}{\eta_r}+\frac{\Theta}{\eta_r}\pi^{rent}_t\left(\pi^{rent}_t-(\pi)^{1-\phi}\left(\pi^{rent}_{t-1}\right)^{\phi}\right)-sdf_{t,t+1}\frac{\Theta}{\eta_r}\pi^{rent}_{t+1}\left(\pi^{rent}_{t+1}-(\pi)^{1-\phi}\left(\pi^{rent}_{t}\right)^{\phi}\right)$$

In terms of already existing variables in the model we divide numerator and denominator of the LHS by P_t

$$\frac{P^m_t/P_t}{P^{rent}_t/P_t}=\frac{\eta_r-1}{\eta_r}+\frac{\Theta}{\eta_r}\pi^{rent}_t\left(\pi^{rent}_t-(\pi)^{1-\phi}\left(\pi^{rent}_{t-1}\right)^{\phi}\right)-sdf_{t,t+1}\frac{\Theta}{\eta_r}\pi^{rent}_{t+1}\left(\pi^{rent}_{t+1}-(\pi)^{1-\phi}\left(\pi^{rent}_{t}\right)^{\phi}\right)$$

After simplification

$$\frac{p^m_t}{r_t}=\frac{\eta_r-1}{\eta_r}+\frac{\Theta}{\eta_r}\pi^{rent}_t\left(\pi^{rent}_t-(\pi)^{1-\phi}\left(\pi^{rent}_{t-1}\right)^{\phi}\right)-sdf_{t,t+1}\frac{\Theta}{\eta_r}\pi^{rent}_{t+1}\left(\pi^{rent}_{t+1}-(\pi)^{1-\phi}\left(\pi^{rent}_{t}\right)^{\phi}\right)$$

where $\pi^{rent}_t=\frac{P^{rent}_t}{P^{rent}_{t-1}}=\frac{r_tP_t}{r_{t-1}P_{t-1}}=\frac{r_t}{r_{t-1}}\pi_t$.

Finally we get the Phillips curve for rent prices:

$$\frac{p^m_t}{r_t}=\frac{\eta_r-1}{\eta_r}+\frac{\Theta}{\eta_r}\frac{r_t}{r_{t-1}}\pi_t\left(\frac{r_t}{r_{t-1}}\pi_t-(\pi)^{1-\phi}\left(\frac{r_{t-1}}{r_{t-2}}\pi_{t-1}\right)^{\phi}\right)-sdf_{t,t+1}\frac{\Theta}{\eta_r}\frac{r_{t+1}}{r_t}\pi_{t+1}\left(\frac{r_{t+1}}{r_t}\pi_{t+1}-(\pi)^{1-\phi}\left(\frac{r_t}{r_{t-1}}\pi_t\right.\right.$$

Notice that in SS:

$$\frac{p^m}{r}=\frac{\eta_r-1}{\eta_r}<1$$

Which means that in SS, $r=\frac{\eta_r}{\eta_r-1}p^m$, that is, the final price of rent on the economy is larger than the price of intermediate rent by markup.

Recall that the profit of retailers is

$$X_{f,t}(j)=P^{rent}_{f,t}(j)Y_{f,t}(j)-P^m_tY_{f,t}(j)-\frac{\Theta}{2}\left(\frac{P^{rent}_{f,t}(j)}{P^{rent}_{f,t-1}(j)}-(\pi)^{1-\phi}\left(\frac{P^{rent}_{t-1}}{P^{rent}_{t-2}}\right)^{\phi}\right)^2P^{rent}_tY^{rent}_t$$

Dividends of the retailers (in real terms of P_t) are transferred to unconstrained households (investors)

$$Div^{retail}_t=r_tY^{rent}_t-p^m_tY^{rent}_t-\frac{\Theta}{2}\left(\pi^{rent}_t-(\pi)^{1-\phi}\left(\pi^{rent}_{t-1}\right)^{\phi}\right)^2r_tY^{rent}_t$$

where $Y^{rent}_t=h_{inv,t}$ (see proof below) and $\pi^{rent}_t=\frac{r_t}{r_{t-1}}\pi_t$.

In SS

$$Div^{retail}=(r-p^m)Y^{rent}=(r-p^m)h_{inv}$$

What is relationship between $Y_t^{rent}, h_{inv,t}$ and $Y_{f,t}$?

In eq.(10) we defined the composite good of houses Y_t^{rent} which is an aggregate of intermediate retailed houses where $h_{inv,t}(j) = Y_{f,t}$. Since $h_{inv,t}(j)$ is the same for all investors j we have $\int_0^1 h_{inv,t}(j)dj = h_{inv,t}$. The main interest is to determine Y_t^{rent}. The supply of all intermediate goods of houses should be equal to total demand for retailed houses, that is:

$$\int_0^1 h_{inv,t}(j)dj = h_{inv,t} = \int_0^1 Y_{f,t}df$$

Using the definition for $Y_{f,t}$ from eq.(11) we have:

$$h_{inv,t} = \int_0^1 Y_{f,t}df = \int_0^1 \left(\frac{P_{f,t}^{rent}}{P_t^{rent}}\right)^{-\eta_r} Y_t^{rent}df = \int_0^1 \left(\frac{P_{f,t}^{rent}}{P_t^{rent}}\right)^{-\eta_r} dfY_t^{rent} = Y_t^{rent} \tag{14}$$

In simple words, since there is no price dispersion in setting of Rotemberg, $\frac{P_{f,t}^{rent}}{P_t^{rent}} = 1$, there is no loss of houses to final output of houses, therefore $Y_t^{rent} = h_{inv,t}$.

1.3 Constrained households

The utility function is (denote with symbol '). We assume for simplicity that some parameters are equal between constrained and unconstrained households: $\zeta^c = \zeta^{'c}, \vartheta = \vartheta', \varrho = \varrho'$.

$$E_0 \sum \beta'^i \left(z_t'\Gamma_c' \log(C_t' - \zeta^c C_{t-1}') + \frac{j_t'}{1-\vartheta} \log \left[\gamma' h_{o,t}'^{1-\vartheta} + (1-\gamma')h_{r,t}'^{1-\vartheta}\right] - \frac{1}{1+\varrho} N_t'^{1+\varrho} \right)$$

where $\Gamma_c' = \frac{1-\zeta^c}{1-\beta'\zeta^c}$.

The budget constraint:

$$C_t' + q_t h_{o,t}' + kq_t h_{o,t}' + r_t h_{r,t}' + R_{t-1}^L B_{t-1}' = B_t' + q_t h_{o,t-1}' - kq_t h_{o,t-1}' + w_t' N_t'$$

We assume that constrained households do not receive any dividends because they do not own any firm, $Div_t' = 0$. We assume also that there is no taxes because there is no government, $T_t' = 0$.

Constrained households are subject to (maximum) LTV restriction ($0 < m < 1$). Parameter m_t could be time-variant depending on requirement of commercial banks or macroprudential policy, and it is exogenous to the households:

$$B_t^{'} \leq m_t^{LTV} q_t h_{o,t}' \tag{15}$$

where m_t^{LTV} follows exogenous process (ϵ_t^m is a policy/commercial banks choice).

$$m_t^{LTV} = (1-\rho^m)m^{LTV} + \rho^m m_{t-1}^{LTV} + \epsilon_t^{m\_LTV}$$

In addition they are subject to PTI constraint: $m_t^{PTI} \geq \frac{r_t^L B_t^{'}}{w_t^{'} N_t^{'}}$,where r_t^L is a net interest rate on new loan, $r_t^L = R_t^L - 1$.

$$B_t^{'} \leq m_t^{PTI} \frac{w_t' N_t'}{r_t^L} \tag{16}$$

$$m_t^{PTI} = (1-\rho^m) m^{PTI} + \rho^m m_{t-1}^{PTI} + \epsilon_t^{m\_PTI}$$

where m_t^{LTV} follows exogenous process (ϵ_t^m is a policy/commercial banks choice).

We see that two constrains could be binding simultaneously.

Constrained households need to decide about 5 variables: $C_t', h_{o,t}', h_{r,t}', B_t', N_t'$.

Note: In this setting (in constrast to Jacavielo and Tsang and Tang) the interest rate on mortgages is R_t^L, where $R_t^L = R_t + \Psi_t$, which is higher by Ψ_t over the riskless real interest rate R_t. Ψ_t is a spread determined by leverage and by spread shocks

$$\Psi_t - \Psi = \alpha(m_t - m) + u_t^{\Psi}$$

where $\Psi, \alpha > 0$. u_t^{Ψ} is a shock to the spread from supply side which stems from risk considerations of banks (Beningo et al. (2020), Ilek and Cohen (2023)). Higher perceived risk aversion of banks will lead to higher spread, given leverage fixed. But in contrast to Beningo et al. (2020), Ilek and Cohen (2023) here the leverage is not a choice of the households, but decision of commercial banks or macroprudential policy. In fact, here m_t is a maximum leverage allowed, which is always binding in our model by assumption. Since m_t and u_t^{Ψ} are exogenous to the households, inclusion of leverage-spread relationship, $\Psi_t - \Psi = \alpha(m_t - m) + u_t^{\Psi}$, will not change significantly the FOC of the households, but mainly the specification of interest rate on credit R_t^L. Ilek and Cohen (2023) found elasticity of the mortgage interest rate with respect to LTV in Israel of about 0.02, namely $\alpha = 0.02$. Also note that Luo and Ma (2023) (I attached the paper in Teams) found a significant comovement in the risk premia in the housing market across many countries. We need to think how to reflect these comovement in common-shocks in the housing market in Israel. Maybe shocks to u_t^{Ψ} can reflect this (please read this paper).

The Lagrangian:

$$\begin{aligned} L_t' &= E_0 \sum \beta'^i \left(z_{t+i}' \Gamma_c' \log(C_{t+i}' - \zeta^c C_{t+i-1}') + \frac{j_{t+i}'}{1-\vartheta} \log \left[\gamma' h_{o,t+i}'^{1-\vartheta} + (1-\gamma') h_{r,t+i}'^{1-\vartheta} \right] - \frac{1}{1+\varrho} N_{t+i}'^{1+\varrho} \right) - \\ & -E_0 \sum \beta'^i \lambda_{t+i}^{'} \left(\begin{array}{c} C_{t+i}' + q_{t+i} h_{o,t+i}' + k q_{t+i} h_{o,t+i}' + r_{t+i} h_{r,t+i}' + R_{t+i-1}^L B_{t+i-1}' \\ -B_{t+i}' - q_{t+i} h_{o,t+i-1}' + k q_{t+i} h_{o,t+i-1}' - w_{t+i}' N_{t+i}' \end{array} \right) - \\ & -E_0 \sum \beta'^i \mu_{t+i}' \left(B_{t+i}^{'} - m_{t+i}^{LTV} q_{t+i} h_{o,t+i}' \right) - E_0 \sum \beta'^i \Omega_{t+i}' \left(B_{t+i}^{'} - m_{t+i}^{PTI} \frac{w_{t+i}' N_{t+i}'}{r_{t+i}^L} \right) \end{aligned}$$

F.O.C of constrained households

(1) $C_t': \Gamma_c'(\frac{z_t'}{C_t'-\zeta^c C_{t-1}'} - \beta'\zeta^c \frac{z_{t+1}'}{C_{t+1}'-\zeta^c C_t'}) - \lambda_t' = 0 =>$

$$\lambda_t' = \Gamma_c'(\frac{z_t'}{C_t' - \zeta^c C_{t-1}'} - \beta'\zeta^c \frac{z_{t+1}'}{C_{t+1}' - \zeta^c C_t'}) \tag{17}$$

(2) $h_{o,t}'$: $j_t' \frac{\gamma' h_{o,t}'^{-\vartheta}}{\gamma' h_{o,t}'^{1-\vartheta} + (1-\gamma') h_{r,t}'^{1-\vartheta}} - \lambda_t'(1+k)q_t - \beta'\lambda_{t+1}'(-q_{t+1} + kq_{t+1}) + \mu_t' m_t^{LTV} q_t = 0 =>$

$$j_t' \frac{\gamma' h_{o,t}'^{-\vartheta}}{\gamma' h_{o,t}'^{1-\vartheta} + (1-\gamma') h_{r,t}'^{1-\vartheta}} = \lambda_t'(1+k)q_t + \beta'\lambda_{t+1}' q_{t+1}(k-1) - \mu_t' m_t^{LTV} q_t \tag{18}$$

(3) $h_{r,t} : j_t' \frac{(1-\gamma') h_{r,t}'^{-\vartheta}}{\gamma' h_{o,t}'^{1-\vartheta} + (1-\gamma') h_{r,t}'^{1-\vartheta}} - \lambda_t' r_t = 0 >$

$$j_t' \frac{(1-\gamma') h_{r,t}'^{-\vartheta}}{\gamma' h_{o,t}'^{1-\vartheta} + (1-\gamma') h_{r,t}'^{1-\vartheta}} = \lambda_t' r_t \tag{19}$$

Dividing (2) by (3) leads to (under $k = 0$)

$\frac{\gamma' h_{o,t}'^{-\vartheta}}{(1-\gamma') h_{r,t}'^{-\vartheta}} = \frac{\lambda_t' q_t - \beta'\lambda_{t+1}' q_{t+1} - \mu_t' m_t^{LTV} q_t}{\lambda_t' r_t} = \frac{q_t}{r_t} - \frac{\beta'\lambda_{t+1}'}{\lambda_t'} \frac{q_{t+1}}{r_t} - \frac{\mu_t' m_t^{LTV}}{\lambda_t'} \frac{q_t}{r_t} =>$

$$\frac{h_{o,t}'}{h_{r,t}'} = [\frac{1-\gamma'}{\gamma'}(\underbrace{\frac{q_t}{r_t} - \left(\frac{\beta'\lambda_{t+1}'}{\lambda_t'}\right)\frac{q_{t+1}}{r_t}}_{\text{Similar to unc. households}} - \underbrace{\frac{\mu_t' m_t^{LTV}}{\lambda_t'}\frac{q_t}{r_t}}_{\text{Due to LTV constraint}})]^{-\frac{1}{\vartheta}} \tag{20}$$

From here it is clear that if the borrowing constraint is not binding ($\mu_t' = 0$) the specification of this ratio is the same for unconstrained households. If m_t^{LTV} decreases (the credit restriction is more tight), the demand for ownership relative to rent decreases. The same is true when the interest rate R_t^L increases. To find SS of this ratio:

$$\frac{h_o'}{h_r'} = [\frac{1-\gamma'}{\gamma'}(\frac{q}{r}(1 - \beta' - \frac{\mu' m^{LTV}}{\lambda'})]^{-\frac{1}{\vartheta}}$$

After plugging SS value of μ' (see (4) below) we get:

$$\frac{h_o'}{h_r'} = [\frac{\gamma'}{1-\gamma'} \frac{1-\beta}{(1 - \beta' - \left(1 - \frac{\beta'}{\beta} - \beta'\Psi\right) m^{LTV})}]^{\frac{1}{\vartheta}} \tag{21}$$

(4) $B_t' : \lambda_t' - \beta'\lambda_{t+1}' R_t^L - \mu_t' - \Omega_t' = 0 >$

$$\mu_t' = \lambda_t' - \beta'\lambda_{t+1}' R_t^L - \Omega_t' \tag{22}$$

(5) $N_t' : -\frac{1+\varrho}{1+\varrho} N_t'^{\varrho} - \lambda_t'(-w_t') - \Omega_t' \left(-m_t^{PTI} \frac{w_t'}{r_t^L}\right) = 0 =>$

$$N_t'^{\varrho} = \lambda_t' w_t' + \Omega_t' m_t^{PTI} \frac{w_t'}{r_t^L} \tag{23}$$

2 Firms

There is a continuum of monopolistically competitive firms indexed by $z \in [0,1]$. Each firm produces a differentiated intermediate good using labor $\bar{N}_t(z)$ with a Cobb–Douglas production function (which is linear in labor). We assume that the share of each type of labor in the production function is identical to the share in the population, τ, as in Beningo (2020)). There is no capital in production.

$$Y_t(z) = A_t \bar{N}_t(z) \tag{24}$$

where $\bar{N}_t(z) = (N_t(z))^{\tau}(N_t'(z))^{1-\tau}$ is a Cobb-Guglass composite of both types of workers. The specification of the production function is the same as in Beningo et al. (2020). The specification of labor input $\bar{N}_t(z)$ is also the same as in Sun and Tsang (2021), but the production function there also includes capital. The technology shock $\log(A_t)$ follows an $AR(1)$ process.

The differentiated-good firms sell their products in a monopolistic competition to a composite firm, producing the aggregate domestic good:

$$Y_t = \left(\int_0^1 Y_t(z)^{\frac{\eta_t - 1}{\eta_t}} dz \right)^{\frac{\eta_t}{\eta_t - 1}} \tag{25}$$

where η_t is the (time-varying) elasticity of substitution between the domestically produced differentiated goods, thus serving as a "mark-up shock" to domestic inflation π_t. The shock $\log(\eta_t)$ follows an $AR(1)$ process.

The aggregate good Y_t is then sold in perfect competition at the price $P_t = \left(\int_0^1 P_t(z)^{1-\eta_t} dz \right)^{\frac{1}{1-\eta_t}}$ and is used for private consumption (government consumption and exports in the richer model). Minimizing production costs by the composite firm implies the following demand functions for the differentiated goods:

$$Y_t(z) = [\frac{P_t(z)}{P_t}]^{-\eta_t} Y_t \tag{26}$$

We assume nominal price rigidities à la Rotemberg (1982). Each domestic firm z seeks to maximize its expected profits. The discounting factor of profits depends on firms' ownership. The simplest assumption is that firms are under the control of the unconstrained households. This assumption is in line with Gurrieri and Jacaviello (2017). The expected profits:

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} X_{t+i}(z), \tag{27}$$

where $X_t(z)$ is a nominal profit of firm z, and $sdf_{t,t} \equiv 1$, $sdf_{t,t+i} \equiv \prod_{l=1}^{i} sdf_{t+l}$ for $i > 0$, $sdf_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$, a stochastic discount factors of unconstrained households (owners of the firm).

$$X_t(z) = P_t(z)Y_t(z) - W_tN_t(z) - W_t'N_t'(z) - \frac{\chi}{2}\left(\frac{P_t(z)}{P_{t-1}(z)} - (\pi)^{1-\varpi}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\varpi}\right)^2 P_tY_t$$

where π is a steady-state inflation rate, ϖ is a degree of indexation to past inflation[1] and χ defines the cost associated with changing the price. Converting Eq. (27) in real terms the optimization problem is given by:

$$\max E_t \sum_{i=0}^{\infty} sdf_{t,t+i} \left\{ \begin{array}{c} [\frac{P_{t+i}(z)}{P_{t+i}}]^{1-\eta}Y_{t+i} - w_{t+i}N_{t+i}(z) - w_{t+i}'N_{t+i}'(z) - \frac{\chi}{2}\left(\frac{P_{t+i}(z)}{P_{t+i-1}(z)} - (\pi)^{1-\varpi}\left(\frac{P_{t+i-1}}{P_{t+i-2}}\right)^{\varpi}\right)^2 Y_{t+i} \\ -\Gamma_{t+i}(z)[(\frac{P_{t+i}(z)}{P_{t+i}})^{-\eta}Y_{t+i} - A_{t+i}(N_{t+i}(z))^{\tau}(N_{t+i}'(z))^{1-\tau}] \end{array} \right.$$

where $\Gamma_{t+i}(z)$ is a Lagrangian multiplier of the production function constraint. Real wages: $w_t = \frac{W_t}{P_t}, w_t' = \frac{W_t'}{P_t}$.

(1) **First-order conditions w.r.t. price** $P_t(z)$:

$$(1-\eta)[\frac{P_t(z)}{P_t}]^{-\eta}\frac{Y_t}{P_t} - \chi\left(\frac{P_t(z)}{P_{t-1}(z)} - (\pi)^{1-\varpi}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\varpi}\right)\frac{Y_t}{P_{t-1}(z)} + sdf_{t,t+1}\chi\left(\frac{P_{t+1}(z)}{P_t(z)} - (\pi)^{1-\varpi}\left(\frac{P_t}{P_{t-1}}\right)^{\varpi}\right)Y_{t+1}$$

Lets divide by Y_t

$$(1-\eta)[\frac{P_t(z)}{P_t}]^{-\eta}\frac{1}{P_t} - \chi\left(\frac{P_t(z)}{P_{t-1}(z)} - (\pi)^{1-\varpi}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\varpi}\right)\frac{1}{P_{t-1}(z)} + sdf_{t,t+1}\chi\left(\frac{P_{t+1}(z)}{P_t(z)} - (\pi)^{1-\varpi}\left(\frac{P_t}{P_{t-1}}\right)^{\varpi}\right)\frac{Y_{t+1}}{Y_t}$$

Now we multiply by $P_t(z)$

$$(1-\eta)[\frac{P_t(z)}{P_t}]^{-\eta}\frac{P_t(z)}{P_t} - \chi\left(\frac{P_t(z)}{P_{t-1}(z)} - (\pi)^{1-\varpi}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\varpi}\right)\frac{P_t(z)}{P_{t-1}(z)} + sdf_{t,t+1}\chi\left(\frac{P_{t+1}(z)}{P_t(z)} - (\pi)^{1-\varpi}\left(\frac{P_t}{P_{t-1}}\right)^{\varpi}\right)$$

Now lets define $\pi_t(z) = \frac{P_t(z)}{P_{t-1}(z)}, \pi_t = \frac{P_t}{P_{t-1}}$.

$$(1-\eta)[\frac{P_t(z)}{P_t}]^{1-\eta} - \chi\left(\pi_t(z) - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)\pi_t(z) + sdf_{t,t+1}\chi\left(\pi_{t+1}(z) - (\pi)^{1-\varpi}(\pi_t)^{\varpi}\right)\frac{Y_{t+1}}{Y_t}\pi_{t+1}(z) + \eta\Gamma_t(z)($$

Since $P_t(z) = P_t$, and $\pi_t(z) = \pi_t$ (in contract to Calvo setting, here all firms are identical w.r.t. price setting)

$$(1-\eta) - \chi\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)\pi_t + sdf_{t,t+1}\chi\left(\pi_{t+1} - (\pi)^{1-\varpi}(\pi_t)^{\varpi}\right)\frac{Y_{t+1}}{Y_t}\pi_{t+1} + \eta\Gamma_t(z) = 0$$

In fact, this is a Phillips curve (now we put back η_t with index t to empathize that this is time-variant and it is a source of markup shock).

$$\Gamma_t(z) = \frac{\eta_t - 1}{\eta_t} + \frac{\chi}{\eta_t}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)\pi_t - \frac{sdf_{t,t+1}\chi}{\eta_t}\left(\pi_{t+1} - (\pi)^{1-\varpi}(\pi_t)^{\varpi}\right)\frac{Y_{t+1}}{Y_t}\pi_{t+1}$$

[1] Similar to the assumption of indexation to past inflation in the Calvo price rigidity setting.

where $sdf_{t,t+1} = \beta\lambda_{t+1}/\lambda_t$ is a stochastic discount factor of unconstrained households.

Notice that in SS this is a marginal cost (inverse of markup, which is $\frac{\eta}{\eta-1}$):

$$\Gamma = \frac{\eta - 1}{\eta}$$

(2) First-order conditions w.r.t. demand for labor $N_t(z)$ and $N_t'(z)$:

$N_t(z) : -w_t + \tau\Gamma_t(z)A_t(N_t(z))^{\tau-1}(N_t'(z))^{1-\tau} = 0 => w_t = \tau\Gamma_t(z)A_t(N_t(z))^{\tau-1}(N_t'(z))^{1-\tau} =>$

$$\Gamma_t(z) = \frac{w_t}{\tau\frac{Y_t(z)}{N_t(z)}}$$

This equation is a marginal costs (real wage of unconstrained households divided by the marginal product of labor (of unconstrained households))

$N_t'(z) : -w_t' + (1-\tau)\Gamma_{t+i}(z)A_{t+i}(N_{t+i}(z))^{\tau}(N_{t+i}'(z))^{-\tau} = 0 => w_t' = (1-\tau)\Gamma_t(z)A_t(N_t(z))^{\tau}(N_t'(z))^{-\tau} =>$

$$\Gamma_t(z) = \frac{w_t'}{(1-\tau)\frac{Y_t(z)}{N_t'(z)}}$$

This equation is a marginal costs (real wage of constrained households divided by the marginal product of labor (of constrained households)).

After dividing two last equations by each other we have:

$$\frac{1-\tau}{\tau}\frac{w_t}{w_t'} = \frac{N_t'(z)}{N_t(z)}$$

We rewrite the last equation as:

$$(1-\tau)w_tN_t(z) = \tau w_t'N_t'(z)$$

Now we aggregate over all firms (over measure 1):

$$(1-\tau)w_t\int_0^1 N_t(z)dz = \tau w_t'\int_0^1 N_t'(z)dz$$

Note that $\int_0^1 N_t(z)dz = \int_0^1 N_t(j)dj = N_t$,where index j denotes the index of unconstrained households (in Beningo (2020) $\int_0^1 N_t(z)dz = \int_0^{\tau} N_t(j)dj = \tau N_t$, where index j denotes the index of unconstrained households who are of measure τ.The same is true for constrained households $\int_0^1 N_t'(z)dz = \int_{\tau}^1 N_t'(j)dj = (1-\tau)N_t')$).

Thus:

$$(1-\tau)w_tN_t = \tau w_t'N_t'$$

Now we return to the equation above which is the marginal cost of the firm ($\Gamma_t(z)$):

$$\Gamma_t(z) = \frac{w_t}{\tau \frac{Y_t(z)}{N_t(z)}}$$

or after rewriting

$$\Gamma_t(z) = \frac{w_t}{\tau} \frac{N_t(z)}{Y_t(z)}$$

Since each firm z produces the same amount of output, $Y_t(z) = Y_t$, we get

$$\Gamma_t(z) = \frac{w_t}{Y_t \tau} N_t(z)$$

Now we aggregate over all firms z of measure 1:

$$\Gamma_t = \int_0^1 \Gamma_t(z) dz = \frac{w_t}{Y_t \tau} \int_0^1 N_t(z) dz = \frac{w_t}{Y_t \tau} \int_0^1 N_t(j) dj = \frac{w_t}{Y_t \tau} N_t$$

or

$$\Gamma_t = \frac{w_t}{\tau Y_t} N_t$$

Finally

$$w_t = \Gamma_t \frac{\tau Y_t}{N_t}$$

Using previous identify we have found $(1-\tau) w_t N_t = \tau w_t' N_t'$ we get:

$$w_t' = \frac{1-\tau}{\tau} \frac{w_t N_t}{N_t'}$$

The wage income of two types of households:
$w_t N_t + w_t' N_t' = w_t N_t + \frac{1-\tau}{\tau} w_t N_t = \frac{1}{\tau} w_t N_t = \frac{1}{\tau} \Gamma_t \frac{\tau Y_t}{N_t} N_t = \Gamma_t Y_t$.
Real dividends (since $P_t^D(z) = P_t^D(z), Y_t(z) = Y_t$) :

$$Div_t = Y_t - w_t N_t - w_t' N_t' - \frac{\chi}{2} \left(\pi_t - (\pi)^{1-\varpi} (\pi_{t-1})^{\varpi} \right)^2 Y_t$$

In SS, $Div = (1 - \Gamma)Y = (1 - \frac{\eta - 1}{\eta})Y = \frac{1}{\eta} Y$.

3 Commercial Banks (intermediates)

Commercial banks act as intermediates: they take deposits from the unconstrained households and lend resources to the constrained households. We assume that α in the spread-leverage relationship is chosen optimally by commercial banks to maximize their profits subject to possible default risk of the borrowers. Moreover, α could be affected by macropridential policy, as in Ilek and Cohen (2023).

$$\Psi_t - \Psi = \alpha(m_t - m) + u_t^{\Psi}$$

where the maximum allowed leverage m_t could be also banks choice or macro-prudential policy choice.

$$m_t = (1-\rho^m)m + \rho^m m_{t-1} + \epsilon_t^m$$

The profits of the banks in period t are transferred to the unconstrained households in period $t+1$[2] (we assume that the unconstrained households are only owners of the banks).

$$Div_t^b = X_{t-1}^b = B'_{t-1}(R_{t-1}^L - R_{t-1})$$

4 Central bank

The central bank sets the one-period nominal interest rate according to the following Taylor rule:

$$r_t^{CB} = \rho r_{t-1}^{CB} + (1-\rho)\left(r^{CB} + \theta_1(\pi_t^{CPI} - \pi) + \theta_2 \Delta y_t\right) + \epsilon_t^{CB} \tag{28}$$

where variables without time subscript denote values at the non-stochastic steady state, and ϵ_t^{CB} is a monetary policy shock. Define CPI inflation as $\pi_t^{CPI} = (1-0.25)\pi_t + 0.25\pi_t^{rent}$, where π_t is a rate of inflation excluding rent and π_t^{rent} is a inflation of rent services (*Yakov: please change in the code the definition of inflation in the Taylor rule).*

***Note: we also can express the policy rule that includes the gap $\Gamma_t - \Gamma$, instead of Δy_t. The marginal cost gap is proportional to output gap (actual output minus natural output (under flexible prices)).

4.0.1 Aggregate resource constraint

Combining the household and the clearing conditions in all markets we can arrive at the economy's aggregate resource constraint:

The budget of unconstrained:

$$\begin{aligned} & C_t + q_t(h_{o,t} + h_{inv,t}) + kq_t(h_{o,t} + h_{inv,t}) + r_t h_{r,t} + R_{t-1}B_{t-1} = \\ = \quad & B_t + q_t(h_{o,t-1} + h_{inv,t-1}) - kq_t(h_{o,t-1} + h_{inv,t-1}) + p_t^m h_{inv,t} + w_t N_t + Div_t \end{aligned}$$

The budget of constrained households

$$C'_t + q_t h'_{o,t} + kq_t h'_{o,t} + r_t h'_{r,t} + R_{t-1}^L B'_{t-1} = B'_t + q_t h'_{o,t-1} - kq_t h'_{o,t-1} + w'_t N'_t$$

Dividends of retailes, consumption production firms and intermidiates:

$$Div_t^{retail} = r_t h_{inv,t} - p_t^m h_{inv,t} - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}$$

$$Div_t = Y_t - w_t N_t - w'_t N'_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}\left(\pi_{t-1}\right)^{\varpi}\right)^2 Y_t$$

[2] This assumption regarding to timing is needed for the simplification of derivation of economy 's resource constraint.

$$Div_t^b = B'_{t-1}(R^L_{t-1} - R_{t-1})$$

Summing up two first equations (and taking into account that k costs are refunded)

$$\begin{aligned}
& C_t + q_t(h_{o,t} + h_{inv,t}) + r_t h_{r,t} + R_{t-1}B_{t-1} + C'_t + q_t h'_{o,t} + r_t h'_{r,t} + R^L_{t-1}B'_{t-1} \\
= \quad & B_t + q_t(h_{o,t-1} + h_{inv,t-1}) + p_t^m h_{inv,t} + w_t N_t + Div_t + B'_t + q_t h'_{o,t-1} + w'_t N'_t
\end{aligned}$$

Plug in the dividends:

$$\begin{aligned}
& C_t + q_t(h_{o,t} + h_{inv,t}) + r_t h_{r,t} + R_{t-1}B_{t-1} + C'_t + q_t h'_{o,t} + r_t h'_{r,t} + R^L_{t-1}B'_{t-1} \\
= \quad & B_t + q_t(h_{o,t-1} + h_{inv,t-1}) + p_t^m h_{inv,t} + w_t N_t + Y_t - w_t N_t - w'_t N'_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t + B'_t + \\
& + B'_{t-1}(R^L_{t-1} - R_{t-1}) + r_t h_{inv,t} - p_t^m h_{inv,t} - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}
\end{aligned}$$

After simplification (taking account that $B_t + B_t^{'} = 0, B_{t-1} + B_{t-1}^{'} = 0$)

$$\begin{aligned}
& C_t + q_t h_{o,t} + q_t h_{inv,t} + r_t h_{r,t} + C'_t + q_t h'_{o,t} + r_t h'_{r,t} \\
= \quad & q_t h_{o,t-1} + q_t h_{inv,t-1} + Y_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t + q_t h'_{o,t-1} \\
& + r_t h_{inv,t} - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}
\end{aligned}$$

Now take into account that $h_{inv,t} = h_{r,t} + h'_{r,t}$,and $h_{inv,t-1} = h_{r,t-1} + h'_{r,t-1}$

$$\begin{aligned}
& C_t + C'_t + q_t h_{o,t} + q_t h_{r,t} + q_t h'_{r,t} + r_t h_{r,t} + q_t h'_{o,t} + r_t h'_{r,t} \\
= \quad & q_t h_{o,t-1} + q_t h_{inv,t-1} + Y_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t + q_t h'_{o,t-1} \\
& + r_t h_{r,t} + r_t h'_{r,t} - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}
\end{aligned}$$

After simplifications

$$\begin{aligned}
& C_t + C'_t + q_t(h_{o,t} + h_{r,t} + h'_{r,t} + h'_{o,t}) \\
= \quad & q_t(h_{o,t-1} + h_{r,t-1} + h'_{r,t-1} + h'_{o,t-1}) + Y_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t \\
& - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}
\end{aligned}$$

Using $h_{r,t} + h'_{r,t} + h_{o,t} + h'_{o,t} = h$ and $h_{r,t-1} + h'_{r,t-1} + h_{o,t-1} + h'_{o,t-1} = h$)

$$C_t + C'_t = Y_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t}$$

That means that the total resources are equal to total product net of adjustment costs

5 Exogenous processes

The two exogenous processes are modeled as:

$$\log z'_t = \rho_z \log z'_t + u_{z,t} + u'_{z,t}$$

$$\log z_t = \rho_z \log z_t + u_{z,t}$$

There is a common innovation $u_{z,t}$ as well as possible idiosyncractic for constrained group $u'_{z,t}$.

$$\log j'_t = \rho_j \log j'_t + u_{j,t} + u'_{j,t}$$

$$\log j_t = \rho_j \log j_t + u_{j,t}$$

There is a common innovation $u_{j,t}$ as well as possible idiosyncractic innovation for constrained group $u'_{j,t}$.

The $\log(A_t)$ is a (stationary) technological shock given by:

$$\log(A_t) = (1 - \rho^A) \log(A) + \rho^A \log(A_{t-1}) + \epsilon_t^a \tag{29}$$

There is also markup shock

$$\log(\mu_t) = (1 - \rho^\mu) \log(\mu) + \rho^\mu \log(\mu_{t-1}) + \epsilon_t^\mu \tag{30}$$

The monetary policy shock

$$u_t^i = \rho^i u_{t-1}^i + \epsilon_t^{u_i}$$

The leverage LTV

$$m_t^{LTV} = (1 - \rho^m) m^{LTV} + \rho^m m_{t-1}^{LTV} + \epsilon_t^{LTV}$$

The PTI

$$m_t^{PTI} = (1 - \rho^m) m^{PTI} + \rho^m m_{t-1}^{PTI} + \epsilon_t^{PTI}$$

The spread shock

$$u_t^\Psi = \rho^\Psi u_{t-1}^\Psi + e_t$$

6 Summary of FOC and equilibrium:

6.1 Unconstrained households

$$\lambda_t = \Gamma_c(\frac{z_t}{C_t - \zeta^c C_{t-1}} - \beta\zeta^c \frac{z_{t+1}}{C_{t+1} - \zeta^c C_t}) \tag{31}$$

$$j_t \frac{\gamma h_{o,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} = \lambda_t(1+k)q_t + \beta\lambda_{t+1}(kq_{t+1} - q_{t+1}) \tag{32}$$

$$j_t \frac{(1-\gamma) h_{r,t}^{-\vartheta}}{\gamma h_{o,t}^{1-\vartheta} + (1-\gamma) h_{r,t}^{1-\vartheta}} = \lambda_t r_t \tag{33}$$

$$\lambda_t = \beta\lambda_{t+1} R_t \tag{34}$$

$$R_t = r_t^{CB} - \pi_{t+1} \tag{35}$$

$$N_t^{\varrho} = \lambda_t w_t \tag{36}$$

$$\frac{p_t^m}{q_t} = \frac{1}{R_t}\frac{q_{t+1}}{q_t}(k-1) + 1 + k \tag{37}$$

$$C_t + q_t(h_{o,t} + h_{inv,t}) + r_t h_{r,t} + R_{t-1}B_{t-1} = B_t + q_t(h_{o,t-1} + h_{inv,t-1}) + p_t^m h_{inv,t} + w_t N_t + Div_t \tag{38}$$

6.2 Constrained households

$$\lambda'_t = \Gamma'_c(\frac{z'_t}{C'_t - \zeta^c C'_{t-1}} - \beta'\zeta^c \frac{z'_{t+1}}{C'_{t+1} - \zeta^c C'_t}) \tag{39}$$

$$j'_t \frac{\gamma' h'^{-\vartheta}_{o,t}}{\gamma' h'^{1-\vartheta}_{o,t} + (1-\gamma') h'^{1-\vartheta}_{r,t}} = \lambda'_t(1+k)q_t + \beta'\lambda'_{t+1}q_{t+1}(k-1) - \mu'_t m_t^{LTV} q_t \tag{40}$$

$$j'_t \frac{(1-\gamma') h'^{-\vartheta}_{r,t}}{\gamma' h'^{1-\vartheta}_{o,t} + (1-\gamma') h'^{1-\vartheta}_{r,t}} = \lambda'_t r_t \tag{41}$$

$$\mu'_t = \lambda'_t - \beta'\lambda'_{t+1} R_t^L - \Omega'_t \tag{42}$$

$$N'^{\varrho}_t = \lambda'_t w'_t + m_t^{PTI}\Omega'_t \frac{w'_t}{r_t^L} \tag{43}$$

$$B'_t = m_t^{LTV} q_t h'_{o,t} \tag{44}$$

$$B_t^{'} = m_t^{PTI} \frac{w_t' N_t'}{r_t^L} \tag{45}$$

$$C_t' + q_t h_{o,t}' + r_t h_{r,t}' + R_{t-1}^L B_{t-1}' = B_t' + q_t h_{o,t-1}' + w_t' N_t' \tag{46}$$

6.3 Retailers

$$\frac{p_t^m}{r_t} = \frac{\eta_r - 1}{\eta_r} + \frac{\Theta}{\eta_r} \pi_t^{rent} \left(\pi_t^{rent} - (\pi)^{1-\phi} \left(\pi_{t-1}^{rent} \right)^{\phi} \right) - sdf_{t,t+1} \frac{\Theta}{\eta_r} \pi_{t+1}^{rent} \left(\pi_{t+1}^{rent} - (\pi)^{1-\phi} \left(\pi_t^{rent} \right)^{\phi} \right) \tag{47}$$

$$\pi_t^{rent} = \frac{r_t}{r_{t-1}} \pi_t \tag{48}$$

$$Div_t^{retail} = r_t h_{inv,t} - p_t^m h_{inv,t} - \frac{\Theta}{2} \left(\pi_t^{rent} - (\pi)^{1-\phi} \left(\pi_{t-1}^{rent} \right)^{\phi} \right)^2 r_t h_{inv,t} \tag{49}$$

6.4 Firms

$$\Gamma_t = \frac{\eta_t - 1}{\eta_t} + \frac{\chi}{\eta_t} \left(\pi_t - (\pi)^{1-\varpi} (\pi_{t-1})^{\varpi} \right) \pi_t - \frac{m_{t,t+1} \chi}{\eta_t} \left(\pi_{t+1} - (\pi)^{1-\varpi} (\pi_t)^{\varpi} \right) \frac{Y_{t+1}}{Y_t} \pi_{t+1} \tag{50}$$

$$Y_t = A_t (N_t)^{\tau} (N_t')^{1-\tau} \tag{51}$$

$$Div_t = Y_t - w_t N_t - w_t' N_t' - \frac{\chi}{2} \left(\pi_t - (\pi)^{1-\varpi} (\pi_{t-1})^{\varpi} \right)^2 Y_t \tag{52}$$

$$w_t = \Gamma_t \frac{\tau Y_t}{N_t} \tag{53}$$

$$w_t' = \frac{1-\tau}{\tau} \frac{w_t N_t}{N_t'} \tag{54}$$

6.5 Commercial banks

$$R_t^L = R_t + \Psi_t \tag{55}$$

$$\Psi_t - \Psi = \alpha (m_t - m) + u_t^{\Psi} \tag{56}$$

$$m_t = (1 - \rho^m) m + \rho^m m_{t-1} + \epsilon_t^m \tag{57}$$

$$Div_t^b = B_{t-1}' (R_{t-1}^L - R_{t-1}) \tag{58}$$

6.6 Central bank

$$r_t^{CB} = \rho r_{t-1}^{CB} + (1-\rho)\left(r^{CB} + \theta_1(\pi_t^{CPI} - \pi) + \theta_2 \Delta y_t\right) + \epsilon_t^{CB} \tag{59}$$

$$\pi_t^{CPI} = (1-0.25)\pi_t + 0.25\pi_t^{rent} \tag{60}$$

6.7 Market clearing

$$C_t + C_t' = Y_t - \frac{\chi}{2}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)^2 Y_t - \frac{\Theta}{2}\left(\pi_t^{rent} - (\pi)^{1-\phi}\left(\pi_{t-1}^{rent}\right)^{\phi}\right)^2 r_t h_{inv,t} \tag{61}$$

$$B_t + B_t^{'} = 0 \tag{62}$$

$$h_{inv,t} = h_{r,t} + h_{r,t}' \tag{63}$$

$$h_{r,t} + h_{r,t}' + h_{o,t} + h_{o,t}' = h \tag{64}$$

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7 Deriving SS

From unconstrained households: $\lambda_t = \frac{1-\zeta^c}{1-\beta\zeta^c}(\frac{z_t}{C_t-\zeta^c C_{t-1}} - \beta\zeta^c \frac{z_{t+1}}{C_{t+1}-\zeta^c C_t}) =>$

$$\lambda = \frac{1}{C}$$

The same is from constrained households

$$\lambda' = \frac{1}{C'}$$

From $\lambda_t = \beta\lambda_{t+1}R =>$

$$R = \frac{1}{\beta}$$

From firms $\Gamma_t = \frac{\eta_t-1}{\eta_t} + \frac{\chi}{\eta_t}\left(\pi_t - (\pi)^{1-\varpi}(\pi_{t-1})^{\varpi}\right)\pi_t - \frac{\beta\chi}{\eta_t}\left(\pi_{t+1} - (\pi)^{1-\varpi}(\pi_t)^{\varpi}\right)\frac{Y_{t+1}}{Y_t}\pi_{t+1}$

$$\Gamma = \frac{\eta - 1}{\eta}$$

From demand for labor

$$\Gamma = \frac{w'}{(1-\tau)\frac{Y}{N'}}$$

$$(1-\tau)\Gamma Y = w'N'$$

From supply of labor

$$N'^{\varrho} = \frac{1}{C'}w'$$

From budget constraint of constrained households

$$C' + rh'_r + (R^L - 1)B' = w'N'$$

where $R^L = R + \Psi$.
Using $B' = mqh'_o$ and $(1-\tau)\Gamma Y = w'N'$

$$C' + rh'_r + (R^L - 1)mqh'_o = (1-\tau)\Gamma Y$$

But $qh'_o = \varkappa C'$ (see proof below)

$$C' + rh'_r + (R^L - 1)m\varkappa C' = (1-\tau)\Gamma Y$$

$$C'\left(1 + (R^L - 1)m\varkappa\right) + rh'_r = (1-\tau)\Gamma Y$$

Using $rh'_r = \psi_1\psi_2 C'$ (see proof below)

$$\begin{aligned} C'\left(1+(R^L-1)m\varkappa\right)+\psi_1\psi_2 C' &= (1-\tau)\Gamma Y => \\ C'\left(1+(R^L-1)m\varkappa+\psi_1\psi_2\right) &= (1-\tau)\Gamma Y => \\ \frac{C'}{Y} &= \digamma = \frac{(1-\tau)\Gamma}{1+(R^L-1)m\varkappa+\psi_1\psi_2} \end{aligned}$$

From labor supply: $N'^{1+\varrho} = \frac{1}{C'}w'N' = \frac{(1-\tau)\Gamma Y}{C'}$

$$C' = \frac{(1-\tau)\Gamma Y}{N'^{1+\varrho}}$$

At the same time

$$\begin{aligned} \digamma &= \frac{C'}{Y} = \frac{(1-\tau)\Gamma}{N'^{1+\varrho}} => \\ N'^{1+\varrho} &= \frac{(1-\tau)\Gamma}{\digamma} => \\ N' &= \left[\frac{(1-\tau)\Gamma}{\digamma}\right]^{\frac{1}{1+\varrho}} \end{aligned}$$

$$\begin{aligned} C'+C &= Y => \\ \text{Using } N^{1+\varrho} &= \lambda w N => \frac{1}{C}\tau\Gamma Y => C = \frac{\tau\Gamma Y}{N^{1+\varrho}}, C' = \frac{(1-\tau)\Gamma Y}{N'^{1+\varrho}} \\ \frac{\tau\Gamma Y}{N^{1+\varrho}} + \frac{(1-\tau)\Gamma Y}{N'^{1+\varrho}} &= Y => \frac{\tau}{N^{1+\varrho}} + \frac{1-\tau}{N'^{1+\varrho}} = \frac{1}{\Gamma} => \\ N &= \left[\frac{\frac{1}{\Gamma} - \frac{1-\tau}{N'^{1+\varrho}}}{\tau}\right]^{-\frac{1}{1+\varrho}} \end{aligned}$$

where N' was found before.
From here we find the output

$$Y = (N)^\tau (N')^{1-\tau}$$

Then $\frac{C'}{Y} = \frac{(1-\tau)\Gamma}{1+(R-1)m\varkappa+\psi_1\psi_2} = \digamma =>$

$$C' = \digamma Y$$

$$B' = mqh'_o C'$$

$$C = Y - C'$$

Now we find wages: $\tau\Gamma Y = wN =>$

$$w = \frac{\tau\Gamma Y}{N}$$

$$w' = \frac{(1-\tau)\Gamma Y}{N'}$$

From demand for house 'ownership'

$$\begin{aligned}\frac{\gamma h_o^{-\vartheta}}{\gamma h_o^{1-\vartheta} + (1-\gamma) h_r^{1-\vartheta}} &= \frac{1}{C}(1+k)q + \beta \frac{1}{C}(kq - q) => \\ \frac{\gamma h_o^{-\vartheta}}{\gamma h_o^{1-\vartheta} + (1-\gamma) h_r^{1-\vartheta}} &= \frac{q}{C}\left(1 + k + \beta(k-1)\right)\end{aligned}$$

From above

$$\frac{q}{C} = \frac{\frac{\gamma h_o^{-\vartheta}}{\gamma h_o^{1-\vartheta} + (1-\gamma) h_r^{1-\vartheta}}}{(1 + k + \beta(k-1))} \tag{65}$$

Let us multiply by h_0 and then divide the LHS by $\gamma h_o^{1-\vartheta}$

$$\frac{q h_0}{C} = \frac{1}{1 + k + \beta(k-1)} \frac{1}{1 + \frac{1-\gamma}{\gamma}\left(\frac{h_r}{h_o}\right)^{1-\vartheta}} \tag{66}$$

From demand for rent:

$$\frac{r}{C} = \frac{(1-\gamma) h_r^{-\vartheta}}{\gamma h_o^{1-\vartheta} + (1-\gamma) h_r^{1-\vartheta}} \tag{67}$$

Combining previous equation for $\frac{r}{C}$ with $\frac{q}{C}$ above we derive $\frac{r}{q}$ for unconstrained households:

$$\frac{r}{q} = \frac{1-\gamma}{\gamma}\left(1 + k + \beta(k-1)\right)\left(\frac{h_o}{h_r}\right)^{\vartheta} \tag{68}$$

Constrained households

From demand for "o"

$$\frac{\gamma' h_o'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma') h_r'^{1-\vartheta}} = \frac{1}{C'}(1+k)q + \beta' \frac{1}{C'}(kq - q) - \mu' m q$$

if $k = 0$, $\frac{\gamma' h_o'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma') h_r'^{1-\vartheta}} = \frac{1}{C'} q - \beta' \frac{1}{C'} q - \mu' m q$,where

$$\lambda' = \frac{1}{C'} \tag{69}$$

From $\lambda_t^{'} = \beta' \lambda_{t+1}^{'} R_t^L + \mu_t' =>$The Lagrangian multiplier (in calibration must hold that $\frac{\beta'}{\beta} + \beta' \Psi < 1$, (critical that $\beta' < \beta$) (the same assumption is in Guerreri and Iacavielo).

$$\lambda'(1 - \frac{\beta'}{\beta} - \beta' \Psi) = \mu'$$

Combining with previous equation for demand for "o":

$$\frac{\gamma' h_o'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma')h_r'^{1-\vartheta}} = \frac{1}{C'}(1+k)q + \beta'\frac{1}{C'}q(k-1) - \frac{1}{C'}q(1-\beta'(\frac{1}{\beta}+\Psi)m$$

After simplifications:

$$\frac{\gamma' h_o'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma')h_r'^{1-\vartheta}} = \left(1+k+\beta'(k-1) - (1-\frac{\beta'}{\beta} - \beta'\Psi)m\right)\frac{q}{C'}$$

We get expression for $\frac{q}{C'}$

$$\frac{q}{C'} = \frac{\frac{\gamma' h_o'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma')h_r'^{1-\vartheta}}}{\left(1+k+\beta'(k-1) - (1-\beta'(\frac{1}{\beta}+\Psi)m\right)} \tag{70}$$

For further purposes we multiply $\frac{q}{C'}$ by h_0'

$$\frac{\gamma' h_o'^{1-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma')h_r'^{1-\vartheta}} = \left(1+k+\beta'(k-1) - (1-\beta'(\frac{1}{\beta}+\Psi)m\right)\frac{qh_o'}{C'}$$

Dividing the LHS by $\gamma' h_o'^{1-\vartheta}$

$$\frac{1}{1+\frac{1-\gamma'}{\gamma'}\left(\frac{h_r'}{h_o'}\right)^{1-\vartheta}} = \left(1+k+\beta'(k-1) - (1-\beta'(\frac{1}{\beta}+\Psi)m\right)\frac{qh_o'}{C'}$$

Finally

$$\frac{qh_o'}{C'} = \frac{1}{\left(1+k+\beta'(k-1) - (1-\beta'(\frac{1}{\beta}+\Psi)m\right)}\frac{1}{1+\frac{1-\gamma'}{\gamma'}\left(\frac{h_r'}{h_o'}\right)^{1-\vartheta}} \tag{71}$$

From demand for "rent"

$$\frac{r}{C'} = \frac{(1-\gamma')h_r'^{-\vartheta}}{\gamma' h_o'^{1-\vartheta} + (1-\gamma')h_r'^{1-\vartheta}} \tag{72}$$

We combine expression for $\frac{q}{C'}$ shown above and expression for $\frac{r}{C'}$ to derive $\frac{r}{q}$ of constrained households:

$$\frac{r}{q} = \frac{1-\gamma'}{\gamma'}\left(1+k+\beta'(k-1) - (1-\beta'(\frac{1}{\beta}+\Psi)m\right)\left(\frac{h_o'}{h_r'}\right)^{\vartheta} \tag{73}$$

From investor FOC we have $\lambda_t(p_t^m - q_t - kq_t) = \beta\lambda_{t+1}(kq_{t+1} - q_{t+1})$, thus in SS:

$$\frac{p^m}{q} = 1 + k + \beta k - \beta \tag{74}$$

where

$$p^m = \left(\frac{\eta_r - 1}{\eta_r}\right) r$$

Combining two previous equations we get:

$$\psi_2 = \frac{r}{q} = \frac{1 + k + \beta k - \beta}{\frac{\eta_r - 1}{\eta_r}} \tag{75}$$

Now we return to the ratios $\frac{r}{q}$ from two types of households ($\frac{r}{q} = \frac{1-\gamma}{\gamma}\left(1 + k + \beta(k-1)\right)\left(\frac{h_o}{h_r}\right)^{\vartheta}$ and $\frac{r}{q} = \frac{1-\gamma'}{\gamma'}\left(1 + k + \beta'(k-1) - (1 - \beta'(\frac{1}{\beta} + \Psi)m\right)\left(\frac{h'_o}{h'_r}\right)^{\vartheta}$).

First we derive for unconstrained households:

$$\frac{\eta_r}{\eta_r - 1} = \frac{1-\gamma}{\gamma}\left(\frac{h_o}{h_r}\right)^{\vartheta}$$

Or

$$c_1 = \frac{h_o}{h_r} = \left[\frac{\eta_r}{\eta_r - 1}\frac{\gamma}{1-\gamma}\right]^{\frac{1}{\vartheta}} \tag{76}$$

Now we derive for constrained households:

$$c'_1 = \frac{h'_o}{h'_r} = \left[\frac{\frac{1+k+\beta k-\beta}{\frac{\eta_r - 1}{\eta_r}}}{\frac{1-\gamma'}{\gamma'}\left(1 + k + \beta'(k-1) - (1 - \beta'(\frac{1}{\beta} + \Psi)m\right)}\right]^{\frac{1}{\vartheta}} \tag{77}$$

For unconstrained, given $\frac{h_o}{h_r}$ above and therefore given ratio above $\frac{qh_0}{C}$, we have qh_0 :

$$c_0 = \frac{qh_0}{C} = \frac{1}{1 + k + \beta(k-1)}\frac{1}{1 + \frac{1-\gamma}{\gamma}\left(\frac{h_r}{h_o}\right)^{1-\vartheta}} \tag{78}$$

For constrained, given $\frac{qh'_o}{C'}$ and $\frac{h'_o}{h'_r}$,we can have qh'_o :

$$c'_0 = \frac{qh'_o}{C'} = \frac{1}{\left(1 + k + \beta'(k-1) - (1 - \beta'(\frac{1}{\beta} + \Psi)m\right)}\frac{1}{1 + \frac{1-\gamma'}{\gamma'}\left(\frac{h'_r}{h'_o}\right)^{1-\vartheta}} \tag{79}$$

Now we are going to find $\frac{qh_r}{C}$ and $\frac{qh'_r}{C'}$.

Starting with $\frac{qh_r}{C}$, we use equation for $\frac{qh_0}{C}$, ($\frac{qh_0}{C} = c_0$) and $\frac{h_o}{h_r}$, ($\frac{h_o}{h_r} = c_1$) above, noting that $\frac{qh_r}{C} = \frac{c_0}{c_1}$).

$$\frac{qh_r}{C} = \frac{c_0}{c_1} \tag{80}$$

Regarding$\frac{qh'_r}{C'}$ we use equation for $\frac{qh'_0}{C'}$, $(\frac{qh'_0}{C'} = c'_0)$ and $\frac{h'_o}{h'_r}$, $(\frac{h'_o}{h'_r} = c'_{1)}$ above, noting that $\frac{qh'_r}{C'} = \frac{c'_0}{c'_1}$).

$$\psi_1 = \frac{qh'_r}{C'} = \frac{c'_0}{c'_1} \tag{81}$$

Now we want to derive rh'_r. Note that $\frac{qh'_r}{C'} = \psi_1, \frac{r}{q} = \psi_2$:

$$rh'_r = \psi_1 \psi_2 C'$$

From $B' = mqh'_o =>$

$$B' = mqh'_o = mc'_0 C'$$

c'_0 is a parameter defined above.

Now we find amount of houses for ownership and rent for all

$$h_r + h'_r + h_o + h'_o = h \tag{82}$$

What is left to find is $r, q, h^o, h^r, h'^o, h'^r$.

We have derive above:$\frac{h_o}{h_r} = c_1$, $\frac{h'_o}{h'_r} = c'_1$, where c_1, c'_1 are defined above.

We also derived above $\frac{qh_r}{C} = \varrho_1, \varrho_1 = \frac{c_0}{c_1}$,and $\frac{qh'_r}{C'} = \varrho_2, \varrho_2 = \frac{c'_0}{c'_1}$.

If we derive $\frac{\frac{qh_r}{C}}{\frac{qh'_r}{C'}} = \frac{\varrho_1}{\varrho_2} => \frac{h_r}{h'_r}\frac{C'}{C} = \frac{\varrho_1}{\varrho_2} =>$

$$\frac{h_r}{h'_r} = \frac{\varrho_1}{\varrho_2}\frac{C}{C'}$$

where $\frac{C}{C'}$ we found before. Define $\frac{h_r}{h'_r} = \chi$,where $\chi = \frac{\varrho_1}{\varrho_2}\frac{C}{C'}$.

$$h_r = \chi h'_r$$

Now we use $h_r + h'_r + h_o + h'_o = h$

$$\chi h'_r + h'_r + c_1 \chi h'_r + c'_1 h'_r = h$$

Finally

$$h'_r = \frac{h}{1 + \chi + c_1\chi + c'_1}$$

From here

$$h_r = \chi \frac{h}{1 + \chi + c_1\chi + c'_1}$$

$$h'_o = c'_1 \frac{h}{1 + \chi + c_1\chi + c'_1}$$

$$h_o = c_1 \chi \frac{h}{1 + \chi + c_1\chi + c'_1}$$

From $qh_0 = c_0C$ =>

$$q = c_0\frac{C}{h_0}$$

and the price of rent

$$r = \frac{1 + k + \beta k - \beta}{\frac{\eta_r - 1}{\eta_r}}q$$