Math 221 Homework 7

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Demmel 4.5

Let $UTU^* = A$ be the Schur form. By part 2 of Question 4.4, $f(T)$ and $f(A)$ have the same eigenvalues. Since

$$(f(T))_{ii} = f(T_{ii}) = f(\lambda_i),$$

the eigenvalues of $f(T)$ are $f(\lambda_i)$. Therefore, the eigenvalues of $f(A)$ are $f(\lambda_i)$ with the same multiplicity as λ_i.

Demmel 4.6

1. *Proof.* Let $A = U_1 T_1 U_1^*$ and $B = U_2 T_2 U_2^*$ be the Schur decompositions of A and B, so T_1 and T_2 are upper triangular and U_1, U_2 are unitary. Substituting these into

$$AX - XB = C$$

gives

$$U_1 T_1 U_1^* X - X U_2 T_2 U_2^* = C.$$

Multiply on the left by U_1^* and on the right by U_2:

$$T_1(U_1^* X U_2) - (U_1^* X U_2) T_2 = U_1^* C U_2.$$

Define

$$Y = U_1^* X U_2 \quad \text{and} \quad C' = U_1^* C U_2.$$

Then

$$T_1 Y - Y T_2 = C'.$$

If we set $A' = T_1$ and $B' = T_2$, this has the form

$$A'Y - YB' = C',$$

with A' and B' upper triangular, as required. $\square$

2. *Proof.* We have the system $A'Y - YB' = C'$, where $A' = (a_{ij}) \in \mathbb{C}^{m \times m}$ and $B' = (b_{ij}) \in \mathbb{C}^{n \times n}$ are upper triangular, $Y = (y_{ij}) \in \mathbb{C}^{m \times n}$ is unknown, and $C' = (c'_{ij})$. Looking at the (k, j) entry of $A'Y - YB'$ gives

$$\sum_{s=1}^{m} a_{ks} y_{sj} \; - \; \sum_{t=1}^{n} y_{kt} b_{tj} \; = \; c'_{kj}.$$

Since A' is upper triangular, $a_{ks} = 0$ for $s < k$, so the first sum is $\sum_{s=k}^{m} a_{ks} y_{sj}$. Since B' is upper triangular, $b_{tj} = 0$ for $t > j$, so the second sum is $\sum_{t=1}^{j} y_{kt} b_{tj}$. Separating out the terms $s = k$ and $t = j$ gives

$$a_{kk} y_{kj} + \sum_{s=k+1}^{m} a_{ks} y_{sj} \; - \; \left(\sum_{t=1}^{j-1} y_{kt} b_{tj} + y_{kj} b_{jj} \right) = c'_{kj}.$$

Rearranging, we obtain the linear equation

$$(a_{kk} - b_{jj}) y_{kj} = c'_{kj} - \sum_{s=k+1}^{m} a_{ks} y_{sj} + \sum_{t=1}^{j-1} y_{kt} b_{tj}. \qquad (*)$$

This gives a back-substitution procedure. First take $k = m$, the last row. Then the sum over $s = k+1, \ldots, m$ is empty, so for $j = 1$,

$$(a_{mm} - b_{11}) y_{m1} = c'_{m1},$$

which determines y_{m1} if $a_{mm} \neq b_{11}$. For $j = 2$,

$$(a_{mm} - b_{22})y_{m2} = c'_{m2} + y_{m1}b_{12},$$

which determines y_{m2}, and so on across the row. Thus the last row $y_{m1}, \ldots, y_{mn}$ is determined from left to right. Now take $k = m - 1$. In (**??**), the terms y_{sj} with $s > k$ are already known from the previous step, so we can again solve for $y_{k1}, y_{k2}, \ldots, y_{kn}$ in order. Continuing upward in k and within each row increasing in j, we solve all of Y. At each step we divide by $(a_{kk} - b_{jj})$, so the system is nonsingular exactly when $a_{kk} \neq b_{jj}$ for all k, j. Since the a_{kk} are the eigenvalues of A' (hence of A) and the b_{jj} are the eigenvalues of B' (hence of B), this condition is that no eigenvalue of A equals an eigenvalue of B. □

3. *Proof.* From part 1 we introduced $Y = U_1^* X U_2$, where U_1 and U_2 are the unitary matrices from the Schur decompositions $A = U_1 T_1 U_1^*$ and $B = U_2 T_2 U_2^*$. Solving the triangular system gave us Y. To recover X from Y, multiply $Y = U_1^* X U_2$ on the left by U_1 and on the right by U_2^*, which gives

$$U_1 Y U_2^* = U_1(U_1^* X U_2)U_2^* = (U_1 U_1^*)X(U_2 U_2^*) = X.$$

Thus the solution of the original Sylvester equation is $X = U_1 Y U_2^*$. □

Demmel 4.7

Since $S^{-1} = \begin{bmatrix} I & -R \\ 0 & I \end{bmatrix}$, we have

$$S^{-1}TS = \begin{bmatrix} I & -R \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & R \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AR + C - RB \\ 0 & B \end{bmatrix}$$

Thus, the question is equivalent to solving

$$AR - RB = -C$$

which is the type of equation discussed in Question 4.6.

Demmel 4.8

Consider the nonsingular matrix $S = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$. We have

$$S \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} S^{-1} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

Thus, $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ are similar, which means that they have the same eigenvalues. Then, by Question 4.1, we have

$$\lambda^n \det(\lambda I - AB) = \lambda^m \det(\lambda I - BA)$$

Thus, AB and BA have the same nonzero eigenvalues.

Additional question (corrected Demmel 4.9)

The statement of this question is still false with "min" and is correct only with "inf". Take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Delta = ad - bc \neq 0.$$

A direct calculation gives

$$S^{-1}AS = \frac{1}{\Delta} \begin{bmatrix} ad - ac - bc & a^2 \\ -c^2 & ac + ad - bc \end{bmatrix},$$

hence

$$\left\| S^{-1}AS \right\|_F^2 = \frac{a^4 + c^4}{\Delta^2} + 2.$$

The eigenvalues of A are $1, 1$, so $\sum_{i=1}^{2} |\lambda_i|^2 = 2$. Since S is invertible, so $a = c = 0$ cannot be true, hence the extra term $(a^4 + c^4)/\Delta^2$ is strictly positive and therefore we have that $\|S^{-1}AS\|_F^2 > 2 = \sum_i |\lambda_i|^2$ for every invertible S. Consequently the minimum is not attained. However, the infimum is 2: for example, fixing $\Delta = 1$ and taking $a = \varepsilon,\ c = 0,\ d = 1/\varepsilon$ gives $\|S^{-1}AS\|_F^2 = 2 + \varepsilon^4 \to 2$. Thus the correct statement is

$$\sum_{i=1}^{n} |\lambda_i|^2 = \inf_{\det(S) \neq 0} \left\| S^{-1}AS \right\|_F^2.$$

Proof. Let $A \in \mathbb{C}^{n \times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_n$. By the Schur decomposition there exists a unitary matrix U such that

$$U^*AU = T,$$

where T is upper triangular with diagonal entries $T_{ii} = \lambda_i$. For any invertible S there is a one-to-one correspondence between S and US, so the set of invertible matrices does not change if we replace S by US. Using the unitary invariance of the Frobenius norm, we obtain

$$\|S^{-1}AS\|_F^2 = \|S^{-1}(UU^*)A(UU^*)S\|_F^2 = \|(U^*S)^{-1}(U^*AU)(U^*S)\|_F^2 = \|(U^*S)^{-1}T(U^*S)\|_F^2.$$

Thus

$$\inf_{\det(S) \neq 0} \|S^{-1}AS\|_F^2 = \inf_{\det(S) \neq 0} \|S^{-1}TS\|_F^2.$$

Now let S be any invertible matrix and write the QR decomposition of S^{-1} as $S^{-1} = QR$, where Q is unitary and R is invertible upper triangular. Then

$$S^{-1}TS = QRTR^{-1}Q^{-1}.$$

Since Q is unitary and the Frobenius norm is unitarily invariant on the left and right, we have

$$\|S^{-1}TS\|_F = \|RTR^{-1}\|_F.$$

Therefore
$$\inf_{\det(S)\neq 0} \|S^{-1}TS\|_F^2 = \inf_{\det(R)\neq 0} \|RTR^{-1}\|_F^2,$$
where the infimum is now taken over all invertible upper triangular R.

Because R is upper triangular and invertible, its inverse R^{-1} is also upper triangular, so RTR^{-1} is upper triangular. Moreover, conjugation by R preserves the diagonal of T, so
$$(RTR^{-1})_{ii} = T_{ii} = \lambda_i, \quad i = 1, \dots, n.$$
Hence
$$\|RTR^{-1}\|_F^2 = \sum_{i=1}^n |\lambda_i|^2 + \sum_{i<j} |(RTR^{-1})_{ij}|^2 \geq \sum_{i=1}^n |\lambda_i|^2.$$
This shows
$$\inf_{\det(R)\neq 0} \|RTR^{-1}\|_F^2 \geq \sum_{i=1}^n |\lambda_i|^2,$$
and therefore
$$\inf_{\det(S)\neq 0} \|S^{-1}AS\|_F^2 \geq \sum_{i=1}^n |\lambda_i|^2.$$
To complete the proof we show that this lower bound can be approached arbitrarily closely. Let
$$c = \max_{i<j} |T_{ij}|.$$
Fix $\varepsilon \in (0,1)$ and set
$$a = \frac{\varepsilon}{\max(c,1)}, \qquad R = \operatorname{diag}(a^{n-1}, a^{n-2}, \dots, a, 1).$$
Then R is invertible and upper triangular. For $i < j$ we have
$$(RTR^{-1})_{ij} = R_{ii}\, T_{ij}\, R_{jj}^{-1} = a^{n-i}\, T_{ij}\, a^{-(n-j)} = a^{j-i} T_{ij}.$$
Since $0 < a < 1$, we have $|a^{j-i}T_{ij}| \leq a|T_{ij}| \leq \varepsilon$. Thus every strict upper triangular entry of RTR^{-1} has magnitude at most ε. There are at most $n(n-1)/2$ such entries, so the sum of their squared magnitudes is at most $\frac{n(n-1)}{2}\varepsilon^2 \leq \frac{n(n-1)}{2}\varepsilon$. It follows that
$$\|RTR^{-1}\|_F^2 \leq \sum_{i=1}^n |\lambda_i|^2 + \frac{n(n-1)}{2}\,\varepsilon.$$
Since $\varepsilon > 0$ was arbitrary, this shows
$$\inf_{\det(R)\neq 0} \|RTR^{-1}\|_F^2 \leq \sum_{i=1}^n |\lambda_i|^2.$$
Combining both inequalities gives
$$\inf_{\det(S)\neq 0} \|S^{-1}AS\|_F^2 = \sum_{i=1}^n |\lambda_i|^2,$$
and the infimum is achieved in the limit by a sequence of diagonal scalings as above. This proves the corrected statement. $\square$

Demmel 4.10

1. Suppose $A = H + S$, then we have $A^* = H - S$. Thus, we can solve H and S as
$$H = \frac{1}{2}(A + A^*),$$
$$S = \frac{1}{2}(A - A^*),$$
where $A = H + S, H$ is Hermitian and S is skew-Hermitian.

2. Let $U^*AU = T$ be the Schur form of A. Then, for H, we have
$$\|H\|_F^2 = \|U^*HU\|_F^2 = \left\|\tfrac{1}{2}(T + T^*)\right\|_F^2 = \left\|\begin{bmatrix} \Re\lambda_1 & * & \cdots & * \\ * & \Re\lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Re\lambda_n \end{bmatrix}\right\|_F^2 \geq \sum_{i=1}^n |\Re\lambda_i|^2.$$

3. Again, let $U^*AU = T$ be the Schur form of A. Then, for S, we have
$$\|S\|_F^2 = \|U^*SU\|_F^2 = \left\|\tfrac{1}{2}(T - T^*)\right\|_F^2 = \left\|\begin{bmatrix} \Im\lambda_1 & * & \cdots & * \\ * & \Im\lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Im\lambda_n \end{bmatrix}\right\|_F^2 \geq \sum_{i=1}^n |\Im\lambda_i|^2.$$

4. If A is normal, according to Question 4.2, there exists unitary matrix U such that $UAU^* = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Thus, we have
$$\|A\|_F^2 = \|UAU^*\|_F^2 = \sum_{i=1}^n |\lambda_i|^2$$
On the other hand, if A is not normal, denote its Schur form as $UAU^* = T$. We can claim that T is not diagonal, otherwise T will be normal and consequently A is normal, which contradicts our assumption. So, we get
$$\|A\|_F^2 = \|UAU^*\|_F^2 = \|T\|_F^2 > \sum_{i=1}^n \lambda_i$$
Thus, if $\sum_{i=1}^n |\lambda_i|^2 = \|A\|_F^2, A$ must be normal.