# Discrete Mathematics

## Problem sheet 2 - Solutions

1. We know that the set $\mathbb{N} \times \mathbb{N}$ is countable (here $\mathbb{N}$ is the set of natural numbers including zero). Let $\mathbb{Q}$ denote the set of rational numbers. Define $f\colon \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$ in the following way: for $q \in Q$, define $f(q) = (a, b)$, where $a, b \in \mathbb{N}$ and $q = \frac{a}{b}$. Clearly, $f$ is injective. Thus $|\mathbb{Q}| \leq |\mathbb{N} \times \mathbb{N}|$ and therefore, $\mathbb{Q}$ is countable.

2. Let $r$ and $s$ be any two real numbers. Without loss of generality, assume $r < s$. Let $\mathbf{r}$ and $\mathbf{s}$ be decimal expansions of $r$ and $s$ that do not contain an infinite sequence of 9s. Let $i$ be the first (most significant) position where $\mathbf{r}$ and $\mathbf{s}$ differ. Clearly, the digit in the $i$-th position of $\mathbf{r}$ is lesser than the digit in the $i$-th position of $\mathbf{s}$. Let $j$ be the first position after $i$ in $\mathbf{r}$ that contains a digit other than 9. Increment the digit in the $j$-th position of $\mathbf{r}$ by one and replace every digit after that position by 0 to obtain the decimal expansion of another number, say $p$. Notice that the decimal expansion of $p$ that we constructed contains only a finite number of digits (there are only $j$ digits in it). Thus $p$ is rational, and clearly, we have $r < p < s$.

3. If we think of any subset $S$ of $\{1, 2, \dots, n\}$ as being represented by an $n$-bit binary string (where the $i$-th position of the string contains a 1 if and only if $i \in S$), then the number of subsets of $\{1, 2, \dots, n\}$ that do not contain two consecutive numbers is exactly the same as the number of $n$-bit binary strings that do not contain two consecutive 1s. This can be easily seen to be the same as the number of $n$-bit strings with no consecutive 0s. As seen in class, this is the $n$-th Fibonacci number.

4. By the previous argument, this is the same as the number of $n$-bit strings containing exactly $k$ 1s in which there are no consecutive 1s. To find this number we can arrange the $n-k$ 0s in a sequence, leaving gaps between two consecutive 0s and also before the first 0 and the last 0. Thus, we have a total of $n-k+1$ gaps. The number of ways in which we can choose $k$ of these gaps to place 1s is exactly the number of $n$-bit strings containing exactly $n-k$ 0s (and hence exactly $k$ 1s) and containing no two consecutive 1s. This number is equal to $\binom{n-k+1}{k}$. Notice that if $k > n-k+1$, then the answer is zero, since in that case, two consecutive 1s cannot be avoided.

5. Consider any monotone function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. For $i \in \{1, 2, \dots, n\}$, let $f^{-1}(i) = \{j \in \{1, 2, \dots, n\} \colon f(j) = i\}$. Since $f$ is monotone, we know that if $j \in f^{-1}(i)$ and $j' \in f^{-1}(i')$, where $i < i'$, then $j < j'$. This means that $f^{-1}(i) = \{x+1, x+2, \dots, x+|f^{-1}(i)|\}$, where $x = \sum_{t=1}^{i-1} |f^{-1}(t)|$. Thus, the values $|f^{-1}(1)|, |f^{-1}(2)|, \dots, |f^{-1}(n)|$ uniquely define the function $f$. This implies the number of different monotone functions is the same as the number of ways to write the integer $n$ as a sum of $n$ integers, where the order of summation matters and some of the integers in the summation can be 0. Or in other words,

it is the number of ways to put $n$ indistinguishable balls into $n$ distinguishable bags (where the bags can be empty). (This is also the same as the number of functions from $A = \{1, 2, \ldots, n\}$ to $B = \{1, 2, \ldots, n\}$ up to permutations of the elements in $A$.) So using the stars and bars argument, we have the answer as $\binom{2n-1}{n}$.

6. By the multinomial theorem,

$$(3x^2 + y + 2z)^5 = \sum_{\substack{i,j,k \\ i+j+k=5}} \frac{5!}{i!j!k!}(3x^2)^i y^j (2z)^k$$

The term corresponding to $x^2yz^3$ in this expression is the one corresponding to $i = 1$, $j = 1$ and $k = 3$. This term is $\frac{5!}{1!1!3!}(3x^2)^1 y^1 (2z)^3 = 480x^2yz^3$. Thus the coefficient of $x^2yz^3$ in the summation is 480.

7. One way to think about this is to think of all permutations of $\{1, 2, \ldots, kn\}$, i.e. all possible ways to write the elements of $\{1, 2, \ldots, kn\}$ in a sequence and then consider the first $k$ elements to form the first set, the next $k$ elements to form the next set and so on. Since by permuting the elements inside each set, and also interchanging the subsequences corresponding to two different sets as a whole, one gets permutations that give rise to the same partition of $\{1, 2, \ldots, kn\}$ into sets of size $k$, the number of permutations that give rise to distinct partitions of $\{1, 2, \ldots, kn\}$ into sets of size $k$ is equal to $\frac{(kn)!}{(k!)^n n!}$.

   Another way to solve this problem is by writing a recurrence relation. Let $f(n)$ denote the number of ways to partition the set $\{1, 2, \ldots, kn\}$ into sets of size $k$. Then consider the element 1. We can choose the $k-1$ other elements in the set containing 1 in $\binom{nk-1}{k-1}$ ways and the $nk - k$ elements in the other sets can be chosen in $f(n-1)$ ways. So we have $f(n) = \binom{nk-1}{k-1} f(n-1)$. Solving, we get

$$\begin{aligned}
f(n) &= \binom{nk-1}{k-1}\binom{nk-k-1}{k-1}\binom{nk-2k-1}{k-1}\cdots\binom{k-1}{k-1} \\
&= \frac{(nk-1)!}{(k-1)!(nk-k)!}\frac{(nk-k-1)!}{(k-1)!(nk-2k)!}\frac{(nk-2k-1)!}{(k-1)!(nk-3k)!}\cdots\frac{(nk-(n-1)k-1)!}{(k-1)!} \\
&= \frac{(nk-1)!}{((k-1)!)^n(nk-k)(nk-2k)\cdots(nk-(n-1)k)} \\
&= \frac{(nk)!}{nk((k-1)!)^n k^{n-1}(n-1)!} = \frac{(nk)!}{(k!)^n n!}
\end{aligned}$$

8.

$$\begin{aligned}
\binom{n}{r}\binom{r}{k} &= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\cdot\frac{r(r-1)(r-2)\cdots(r-k+1)}{k!} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\cdot\frac{(n-k)(n-k-1)\cdots(n-r+1)r!}{r!(r-k)!} \\
&= \binom{n}{k}\binom{n-k}{r-k}
\end{aligned}$$

9. A combinatorial proof can be given as follows. Consider a set of $2n$ people consisting of $n$ women and $n$ men. The total number of ways to choose $n$ people from this set is clearly $\binom{2n}{n}$. Another way to calculate this number is to sum up the ways in which we can choose $i$ women and $n-i$ men from this set, over all $i \in \{0, 1, \ldots, n\}$. This number is clearly equal to $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i}$. Thus $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.

10. This is the same as the number of $n$-bit strings with an even number of 1s. Let $f(n)$ denote the number of $n$-bit strings with an even number of 1s. We shall show that $f(n) = 2^{n-1}$. We assume that $n > 1$ since it is easy to see that $f(1) = 1$. The number of such strings which have 0 as the first bit is $f(n-1)$ and the number of such strings which have 1 as the first bit is equal to the number of $(n-1)$-bit strings which contain an odd number of 1s, which is equal to $2^{n-1} - f(n-1)$. Then $f(n) = f(n-1) + 2^{n-1} - f(n-1) = 2^{n-1}$.

11. Notice that $\binom{n+m}{k}$ is the coefficient of $x^k$ in $(1+x)^{n+m}$. We rewrite this coefficient in terms of the coefficients from the expansion of $(1+x)^n$ and the coefficients from the expansion of $(1+x)^m$. That is,

$$\begin{aligned}\binom{n+m}{k} &= [x^k](1+x)^{n+m} = [x^k](1+x)^n(1+x)^m \\ &= \sum_{i=0}^{k} [x^i](1+x)^n \cdot [x^{k-i}](1+x)^m = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}\end{aligned}$$

12. Since $n$ objects are indistinguishable, only the number of such objects chosen matters. If we decide to choose $i$ of the indistinguishable objects, then we can choose the remaining $n-i$ objects in $\binom{2n+1}{n-i}$ ways. Thus the total number of ways to choose $n$ objects is $\sum_{i=0}^{n} \binom{2n+1}{n-i} = \binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n}$. Since this summation is equal to $\binom{2n+1}{2n+1} + \binom{2n+1}{2n} + \cdots + \binom{2n+1}{n+1}$, we have that $\sum_{i=0}^{n} \binom{2n+1}{n-i} = \frac{1}{2}\left(\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{2n+1}\right) = \frac{1}{2} \cdot 2^{2n+1} = 2^{2n} = 4^n$.

13. For any natural number $i$, we let $M_i$ denote the multiples of $i$ in $\{1, 2, \ldots, 100\}$. We want to find $|\{1, 2, \ldots, 100\} - (M_2 \cup M_3 \cup M_5 \cup M_7)| = 100 - |M_2 \cup M_3 \cup M_5 \cup M_7|$. By the inclusion-exclusion principle, $|M_2 \cup M_3 \cup M_5 \cup M_7| = |M_2| + |M_3| + |M_5| + |M_7| - |M_2 \cap M_3| - |M_2 \cap M_5| - |M_2 \cap M_7| - |M_3 \cap M_5| - |M_3 \cap M_7| - |M_5 \cap M_7| + |M_2 \cap M_3 \cap M_5| + |M_2 \cap M_3 \cap M_7| + |M_2 \cap M_5 \cap M_7| + |M_3 \cap M_5 \cap M_7| - |M_2 \cap M_3 \cap M_5 \cap M_7| = |M_2| + |M_3| + |M_5| + |M_7| - |M_6| - |M_{10}| - |M_{14}| - |M_{15}| - |M_{21}| - |M_{35}| + |M_{30}| + |M_{42}| + |M_{70}| + |M_{105}| - |M_{210}| = 50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 + 0 - 0 = 82$. Thus there are 18 numbers in $\{1, 2, \ldots, 100\}$ that are not divisible by any of 2, 3, 5 or 7.

14. Let $M_i$ denote the multiples of $i$ in $\{1, 2, \ldots, n\}$. An integer $x \in \{1, 2, \ldots, n\}$ is relatively prime to $n$ if and only if none of $p_1, p_2, \ldots, p_t$ are factors of $x$. By the inclusion-exclusion principle, the number of integers in $\{1, 2, \ldots, n\}$ that are

relatively prime to $n$ is equal to

$$\begin{aligned}
& n - |M_{p_1} \cup M_{p_2} \cup \cdots \cup M_{p_t}| \\
= \; & n - \left( |M_{p_1}| + |M_{p_2}| + \cdots + |M_{p_t}| - \sum_{\substack{p_i,p_j \\ i \neq j}} |M_{p_i} \cap M_{p_j}| + \sum_{\substack{p_i,p_j,p_k \\ \text{distinct } i,j,k}} |M_{p_i} \cap M_{p_j} \cap M_{p_k}| - \cdots \right) \\
= \; & n - \left( \frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_t} - \sum_{\substack{p_i,p_j \\ i \neq j}} \frac{n}{p_i p_j} + \sum_{\substack{p_i,p_j,p_k \\ \text{distinct } i,j,k}} \frac{n}{p_i p_j p_k} - \cdots \right) \\
= \; & n \left( 1 - \frac{1}{p_1} - \frac{1}{p_2} - \cdots - \frac{1}{p_t} + \sum_{\substack{p_i,p_j \\ i \neq j}} \frac{1}{p_i p_j} - \sum_{\substack{p_i,p_j,p_k \\ \text{distinct } i,j,k}} \frac{1}{p_i p_j p_k} + \cdots \right) \\
= \; & n \left( 1 - \frac{1}{p_1} \right) \left( 1 - \frac{1}{p_2} \right) \cdots \left( 1 - \frac{1}{p_t} \right)
\end{aligned}$$

15. Let $A_{LEFT}, A_{TURN}, A_{SIGN}, A_{CAR}$ denote respectively the arrangements of the 26 letters of the alphabet in which the words LEFT, TURN, SIGN, and CAR appear. If $A$ denote all the arrangements of the 26 letters of the alphabet, then we have to compute $|A - (A_{LEFT} \cup A_{TURN} \cup A_{SIGN} \cup A_{CAR})| = 26! - |A_{LEFT} \cup A_{TURN} \cup A_{SIGN} \cup A_{CAR}|$. We use the inclusion-exclusion principle to compute $|A_{LEFT} \cup A_{TURN} \cup A_{SIGN} \cup A_{CAR}|$. Note that $|A_{LEFT}| = 23!$ (considering "LEFT" to be a single token that can be permuted along with the other letters), $|A_{TURN} = 23!$, $|A_{SIGN}| = 23!$, $|A_{CAR}| = 24!$, $|A_{LEFT} \cap A_{TURN}| = 20!$ (considering "LEFTURN" to be a single token), $|A_{LEFT} \cap A_{SIGN}| = 20!$ (considering both "LEFT" and "SIGN" to be single tokens), $|A_{LEFT} \cap A_{CAR}| = 21!$, $|A_{TURN} \cap A_{SIGN}| = 0$ (no ordering can contain both "TURN" and "SIGN"), $|A_{TURN} \cap A_{CAR}| = 0$, $|A_{SIGN} \cap A_{CAR}| = 21!$, $|A_{LEFT} \cap A_{TURN} \cap A_{SIGN}| = 0$, $|A_{LEFT} \cap A_{TURN} \cap A_{CAR}| = 0$, $|A_{LEFT} \cap A_{SIGN} \cap A_{CAR}| = 18!$, $|A_{TURN} \cap A_{SIGN} \cap A_{CAR}| = 0$, and $|A_{LEFT} \cap A_{TURN} \cap A_{SIGN} \cap A_{CAR}| = 0$. By the inclusion-exclusion principle, $|A_{LEFT} \cup A_{TURN} \cup A_{SIGN} \cup A_{CAR}| = 23! + 23! + 23! + 24! - 20! - 20! - 21! - 21! + 18!$. Thus the answer we need is $26! - 3 \cdot 23! - 24! + 2 \cdot 20! + 2 \cdot 21! - 18!$.

16. Let $A_{TA}$ denote the number of arrangements with both Ts before both As, $A_{AM}$ the number of arrangements with both As before both Ms, and $A_{ME}$ the number of arrangements with both Ms before the E. To compute $|A_{TA}|$, we can just compute the number of permutations of the letters in MATHEMATICS, where we replace the two Ts and two As with four tokens which are indistinguishable from each other. In every such permutation, we can simply replace the tokens with T,T,A,A in that order to obtain a permutation in which both Ts occur before both As, and every permutation in which both the Ts come before both the As can be obtained in this way. Thus $|A_{TA}| = \frac{11!}{4!2!}$ (note that the two Ms are also indistinguishable). Similarly, $|A_{AM}| = \frac{11!}{4!2!}$, $|A_{ME}| = \frac{11!}{3!2!2!}$, $|A_{TA} \cap A_{AM}| = \frac{11!}{6!}$, $|A_{TA} \cap A_{ME}| = \frac{11!}{4!3!}$ (here we use four indistinguishable red tokens for the As and Ts, and three indistinguishable blue tokens for the Ms and the E), $|A_{AM} \cap$

$A_{ME}| = \frac{11!}{5!2!}$, and $|A_{TA} \cap A_{AM} \cap A_{ME}| = \frac{11!}{7!}$. Then by the inclusion-exclusion principle, $|A_{TA} \cup A_{AM} \cup A_{ME}| = 11!\left(\frac{1}{4!2!} + \frac{1}{4!2!} + \frac{1}{3!2!2!} - \frac{1}{6!} - \frac{1}{4!3!} - \frac{1}{5!2!} + \frac{1}{7!}\right) = \frac{11!}{4!}\left(2 - \frac{1}{30} - \frac{1}{6} - \frac{1}{10} + \frac{1}{210}\right)$.

17. Let $g(n)$ denote the number of $n$-bit binary strings that contain two consecutive 0s. The number of such strings in which the first bit is 1 is exactly $g(n-1)$, the first two bits being 0 are exactly $2^{n-2}$ and the first two bits being 01 is exactly $g(n-2)$. Thus we have the recurrence relation $g(n) = g(n-1) + g(n-2) + 2^{n-2}$. As boundary conditions, we have $g(1) = 0$, $g(2) = 1$. We get the same sequence $g(0), g(1), g(2), \ldots$ if we let the boundary conditions be $g(0) = 0$ and $g(1) = 0$. Let $F_n$ denote the $n$-th Fibonacci number. We know that since $F_{n+2}$ is the number of $n$-bit strings containing no two consecutive zeroes, we must have $g(n) = 2^n - F_{n+2}$. Let us anyway solve the recurrence using generating functions. Let $G(x)$ denote the generating function of the sequence $g(0), g(1), g(2), \ldots$.

$$\begin{aligned}
g(n) &= g(n-1) + g(n-2) + 2^{n-2} \\
\sum_{n=2}^{\infty} g(n)x^n &= \sum_{n=2}^{\infty} g(n-1)x^n + \sum_{n=2}^{\infty} g(n-2)x^n + \sum_{n=2}^{\infty} 2^{n-2}x^n \\
G(x) - g(0) - g(1)x &= x(G(x) - g(0)) + x^2 G(x) + \frac{x^2}{1-2x} \\
G(x)(1 - x - x^2) &= \frac{x^2}{1-2x} \\
G(x) &= \frac{x^2}{(1-2x)(1-x-x^2)} = \frac{-x^2}{(1-2x)(\frac{-1-\sqrt{5}}{2} - x)(\frac{-1+\sqrt{5}}{2} - x)}
\end{aligned}$$

Using partial fractions, this becomes

$$\begin{aligned}
G(x) &= \frac{1}{1-2x} + \frac{1-\sqrt{5}}{2\sqrt{5}(\frac{-1-\sqrt{5}}{2} - x)} - \frac{\sqrt{5}+1}{2\sqrt{5}(\frac{-1+\sqrt{5}}{2} - x)} \\
&= \frac{1}{1-2x} + \frac{\sqrt{5}-1}{\sqrt{5}(1+\sqrt{5}+2x)} - \frac{\sqrt{5}+1}{\sqrt{5}(-1+\sqrt{5}-2x)} \\
&= \frac{1}{1-2x} + \frac{\sqrt{5}-1}{\sqrt{5}(1+\sqrt{5})(1+\frac{2x}{1+\sqrt{5}})} - \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)(1-\frac{2x}{\sqrt{5}-1})} \\
&= \frac{1}{1-2x} + \frac{\sqrt{5}-1}{\sqrt{5}(1+\sqrt{5})(1-\frac{(1-\sqrt{5})x}{2})} - \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)(1-\frac{(1+\sqrt{5})x}{2})}
\end{aligned}$$

$$\begin{aligned}
[x^n]G(x) &= 2^n + \frac{\sqrt{5}-1}{\sqrt{5}(1+\sqrt{5})}\left(\frac{1-\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)}\left(\frac{1+\sqrt{5}}{2}\right)^n \\
&= 2^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} \\
&= 2^n - F_{n+2}
\end{aligned}$$

18. Let $A(x)$ denote the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Then

$$\begin{aligned} A(x) &= (1+x+x^2+\cdots)(x^3+x^6+x^9+\cdots)(x^3+x^6+x^9+\cdots) \\ &= \frac{1}{1-x}\left(\frac{x^3}{1-x^3}\right)^2 = \frac{x^6}{(1-x)(1-x^3)^2} \end{aligned}$$

We can apply the method of partial fractions to the above expression, but it turns out to be too cumbersome. Instead, we use the fact that $(1-x)^3 = (1-x)(1+x+x^2)$ and rewrite $A(x)$ as follows.

$$A(x) = \frac{x^6}{(1-x)(1-x^3)^2} = \frac{x^6(1+x+x^2)}{(1-x^3)^3} = \frac{x^6}{(1-x^3)^3} + \frac{x^7}{(1-x^3)^3} + \frac{x^8}{(1-x^3)^3}$$

Recall that $[x^n]\frac{1}{(1-x)^3} = \binom{n+2}{2}$. Thus $[x^n]\frac{1}{(1-x^3)^3} = \begin{cases} 0 & \text{if } n \not\equiv 0 \mod 3 \\ \binom{\frac{n}{3}+2}{2} & \text{if } n \equiv 0 \mod 3 \end{cases}$.

Then

$$a_n = [x^n]A(x) = \begin{cases} \binom{\frac{n}{3}}{2} & \text{if } n \equiv 0 \mod 3 \\ \binom{\frac{n-1}{3}}{2} & \text{if } n \equiv 1 \mod 3 \\ \binom{\frac{n-2}{3}}{2} & \text{if } n \equiv 2 \mod 3 \end{cases}$$

19. Let $A(x)$ denote the generating function for $(a_0, a_1, a_2, \ldots)$.

$$\begin{aligned} A(x) &= (1+x+x^2+\cdots+x^5)(1+x^5+x^{10}+x^{15}+x^{20})(1+x^{10}+x^{20}+x^{30}) \\ &= \left(\frac{1}{1-x} - \frac{x^6}{1-x}\right)\left(\frac{1}{1-x^5} - \frac{x^{25}}{1-x^5}\right)\left(\frac{1}{1-x^{10}} - \frac{x^{40}}{1-x^{10}}\right) \\ &= \frac{(1-x^6)(1-x^{25})(1-x^{40})}{(1-x)(1-x^5)(1-x^{10})} = \frac{1-x^{40}-x^{25}+x^{65}-x^6+x^{46}+x^{31}-x^{71}}{(1-x)(1-x^5)(1-x^{10})} \end{aligned}$$

Again, using partial fractions seems to be too cumbersome here. Also, we have not been asked to find a general formula for $a_n$. So let us first observe that $\frac{1}{(1-x^5)(1-x^{10})} = \frac{1}{(1-x^5)(1-(x^5)^2)}$. Since $\frac{1}{1-(x^5)^2} = 1 + (x^5)^2 + (x^5)^4 + \cdots$, we can conclude that $\frac{1}{1-x^5} \cdot \frac{1}{1-(x^5)^2} = 1 + x^5 + 2(x^5)^2 + 2(x^5)^3 + 3(x^5)^4 + 3(x^5)^5 + \cdots$. Thus $\frac{1}{(1-x^5)(1-x^{10})}$ is the generating function of the sequence

$$(1,0,0,0,0,1,0,0,0,0,2,0,0,0,0,2,0,0,0,0,3,\ldots)$$

Then $G(x) = \frac{1}{(1-x)(1-x^5)(1-x^{10})}$ is the generating function of the sequence

$$(1,1,1,1,1,2,2,2,2,2,4,4,4,4,4,6,6,6,6,6,9,\ldots)$$

Thus $a_{20} = [x^{20}]\frac{1-x^{40}-x^{25}+x^{65}-x^6+x^{46}+x^{31}-x^{71}}{(1-x)(1-x^5)(1-x^{10})} = [x^{20}]G(x) - [x^{14}]G(x) = 9 - 4 = 5.$

20. We shall use the method of generating functions. Let $(a_0, a_1, a_2, \ldots)$ be the

sequence $(f(0), f(1), f(2), \ldots)$. Then

$$\begin{aligned}
a_n &= 5a_{n-1} - 6a_{n-2} + 4n - 4 \text{ (where } a_0 = 7, \text{ and } a_1 = 16) \\
\sum_{n=2}^{\infty} a_n x^n &= \sum_{n=2}^{\infty} 5a_{n-1}x^n - \sum_{n=2}^{\infty} 6a_{n-2}x^n + \sum_{n=2}^{\infty} 4(n-1)x^n \\
A(x) - a_1x - a_0 &= 5x(A(x) - a_0) - 6x^2A(x) + 4x^2(1 + 2x + 3x^2 + \cdots) \\
A(x)(1 - 5x + 6x^2) &= 7 - 19x + \frac{4x^2}{(1-x)^2} \\
A(x)(1-2x)(1-3x) &= \frac{7 - 33x + 49x^2 - 19x^3}{(1-x)^2} \\
A(x) &= \frac{7 - 33x + 49x^2 - 19x^3}{(1-2x)(1-3x)(1-x)^2}
\end{aligned}$$

Let $A(x) = \frac{B}{1-2x} + \frac{C}{1-3x} + \frac{D}{1-x} + \frac{E}{(1-x)^2}$. Then we get the system of equations

$$\begin{aligned}
B + C + D + E &= 7 \\
5B + 4C + 6D + 5E &= 33 \\
7B + 5C + 11D + 6E &= 49 \\
3B + 2C + 6D &= 19
\end{aligned}$$

Solving this, we get $B = -3$, $C = 5$, $D = 3$ and $E = 2$. Thus $A(x) = \frac{-3}{1-2x} + \frac{5}{1-3x} + \frac{3}{1-x} + \frac{2}{(1-x)^2}$. Then

$$\begin{aligned}
a_n = [x^n]A(x) &= -3[x^n]\frac{1}{1-2x} + 5[x^n]\frac{1}{1-3x} + 3[x^n]\frac{1}{1-x} + 2[x^n]\frac{1}{(1-x)^2} \\
a_n &= 5 \cdot 3^n - 3 \cdot 2^n + 2n + 5
\end{aligned}$$

21. Let $f(n)$ denote the minimum number of moves required to move a stack of $n$ discs from peg 1 to peg 3 such that no disc is ever moved directly from peg 1 to peg 3 or from peg 3 to peg 1.

    (a) Note that we can always follow these steps to accomplish our goal:

    i. Move the smallest $n-1$ discs from peg 1 to peg 3 using $f(n-1)$ moves.
    ii. Move the largest disc from peg 1 to peg 2.
    iii. Move the smallest $n-1$ discs from peg 3 to peg 1 using $f(n-1)$ moves.
    iv. Move the largest disc from peg 2 to peg 3.
    v. Move the smallest $n-1$ discs from peg 1 to peg 3 using $f(n-1)$ moves.

    Thus we have $f(n) \leq 3f(n-1) + 2$. Now consider any strategy that takes $f(n)$ moves to accomplish the goal. Let $i$ denote the step at which the largest disc is first moved. Clearly, this move must be from peg 1 to peg 2, which means that the smallest $n-1$ discs must be on peg 3 at this point. For this to happen, clearly $f(n-1)$ moves must have been made before step $i$. Now let $j > i$ be the last step at which the largest disc was moved. This move must be from peg 2 to peg 3. When this happens, clearly the stack of $n-1$ smallest discs must be on peg 1. So between steps $i$ and $j$, the stack of smallest $n-1$ discs, which were on peg 3 at step $i$, must have moved to peg 1. This means that at least $f(n-1)$ moves must have been made between steps $i$ and $j$. Now after step $j$, it will take at least another $f(n-1)$ moves to bring the stack of smallest $n-1$ discs from peg 1 to peg 3, so that the goal is accomplished. Thus we can conclude that $f(n) \geq 3f(n-1)+2$. We thus have the following recurrence relation for $f(n)$.

$$f(n) = 3f(n-1) + 2$$

It is easy to see that the boundary condition for this recurrence is $f(1) = 2$.

(b)

$$\begin{aligned}
f(n) &= 3f(n-1) + 2 = 3(3f(n-2) + 2) + 2 \\
&= 3^2 f(n-2) + 3 \cdot 2 + 2 = 3^2(3(f(n-3) + 2)) + 3 \cdot 2 + 2 \\
&= 3^3 f(n-3) + 3^2 \cdot 2 + 3 \cdot 2 + 2 \\
&\vdots \\
&= 3^{n-1} f(1) + 2(1 + 3 + 3^2 + \cdots + 3^{n-2}) \\
&= 2(1 + 3 + 3^2 + \cdots + 3^{n-1}) \\
&= 3^n - 1
\end{aligned}$$

(c) If a strategy takes $i$ steps to move the stack of $n$ discs from peg 1 to peg 3, then clearly, the number of configurations seen by it is $i+1$. If the $i+1$ configurations seen by it are not distinct, i.e. if some configuration repeats, then the strategy can be shortened to one with fewer than $i$ steps. Since any strategy takes at least $f(n) = 3^n - 1$ steps, it is clear that any strategy sees at least $3^n$ different configurations. The number of allowable arrangements of $n$ discs on 3 pegs is equal to the number of ways to partition a set of $n$ (distinguishable) elements into three subsets, where the subsets are also allowed to be empty (once you choose the subset of discs to place on a peg, there is only one way of arranging them on the peg). This is simply the number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, 3\}$, which is equal to $3^n$. Thus every possible configuration of discs on the three pegs must be encountered during any strategy that moves the stack of $n$ discs from peg 1 to peg 3.