Discrete Mathematics Midsem Model Solution

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Question 1

Let L be a set of lines in the plane.

1. For lines $l_1, l_2 \in L$, we say that $l_1 \sim l_2$ if the lines l_1 and l_2 intersect. Discuss whether the relation $\sim$ is reflexive, symmetric, anti-symmetric, or transitive.

2. For $l_1, l_2 \in L$, we say that $l_1 \nsim l_2$ if the lines l_1 and l_2 do not intersect. Discuss whether the relation $\nsim$ is reflexive, symmetric, anti-symmetric, or transitive.

Solution.

1. For lines $l_1, l_2 \in L$, we say that $l_1 \sim l_2$ if the lines l_1 and l_2 intersect.
   - *Reflexivity:* Every line coincides with itself, intersecting itself at every point. Hence, for any line l, $l \sim l$ holds true.
   - *Symmetric:* If for lines $l_1, l_2 \in L$, we have $l_1 \sim l_2$, then the lines l_1 and l_2 intersect, which means that $l_2 \sim l_1$ is also true. Hence $\sim$ is symmetric.
   - *Anti-symmetric:* Consider two lines l and l' such that they intersect but $l \neq l'$. Hence $l \sim l'$ and $l' \sim l$, but they are distinct. Therefore $\sim$ is not anti-symmetric.
   - *Transitive:* Consider three lines l_1, l_2, l_3, such that l_2 and l_3 are parallel and l_1 intersects both l_2 and l_3. Hence, $l_2 \sim l_1$ and $l_1 \sim l_3$, but $l_2 \sim l_3$ is not valid. Hence $\sim$ is not transitive.

2. For $l_1, l_2 \in L$, we say that $l_1 \nsim l_2$ if the lines l_1 and l_2 do not intersect.
   - *Reflexivity:* Any line l coincides with itself on all points. Hence $l \nsim l$ is not valid. Hence $\nsim$ is not reflexive.
   - *Symmetric:* For any two lines l and l' such that $l \nsim l'$, the lines l and l' do not intersect, implying that $l' \nsim l$ is valid. Therefore $\nsim$ is symmetric.
   - *Anti-symmetric:* Consider two distinct parallel lines l and l'. Then $l \nsim l'$ and $l' \nsim l$ but $l \neq l'$. Hence $\nsim$ is not anti-symmetric.
   - *Transitive:* Consider two distinct parallel lines l and l'. By definition and symmetry, $l \nsim l'$ and $l' \nsim l$. But $\nsim$ is not reflexive, that is, $l \nsim l$ is not valid. Hence $\nsim$ is not transitive.

Question 2

Recall that $\left\{ {n \atop m} \right\}$ denotes the number of different ways to partition a set of n elements into m non-empty subsets. Show that $\left\{ {n \atop m} \right\} \leq m^n/m!$.

Solution.

Consider a surjective function $f : X \to Y$, where $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$. Note that f defines a labelled partition of X into m sets. (Formally, for $y \in Y$, let $f^{-1}(y) = \{x \in X \colon f(x) = y\}$. Then

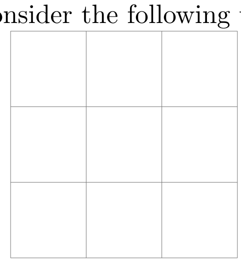
$\mathcal{P}_f = \{f^{-1}(y) \colon y \in Y\}$ is a partition of X into m sets. For two surjective functions f, f' from X to Y, we have $\mathcal{P}_f = \mathcal{P}_{f'}$ if and only if there exists a bijection $g : Y \to Y$ (a permutation of Y) such that $f' = g \circ f$. Also note that for a surjective function $f : X \to Y$ and two different permutations g, g' of Y, the functions $g \circ f$ and $g' \circ f$ are different. Thus, for a partition $\mathcal{P}$ of X into m sets, there are exactly $m!$ surjective functions f from X to Y such that $\mathcal{P}_f = \mathcal{P}$, since this is the number of different possible permutations of Y.) Let p^n_m = number of possible surjective functions $f : X \to Y$. Then the number of partitions of X into m sets is $\left\{ {n \atop m} \right\} = p^n_m / m!$.
Number of possible surjective functions from X to Y is at most the number of possible functions from X to Y. Therefore, $p^n_m \leq m^n$. Hence, $\left\{ {n \atop m} \right\} \leq m^n / m!$.

Question 3

Suppose that you have a 3×3 chessboard in which all nine cells are initially white. In how many different ways can you color some of the cells black so that no 2×2 square of cells is completely black?

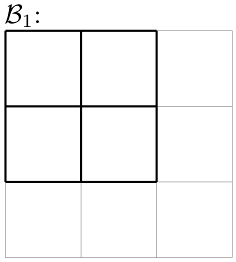
Solution.

Consider the following white 3×3 board:

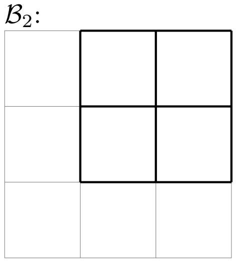


Let us label all possible 2×2 squares in the 3×3 chessboard. There are total 4 possible such 2×2 squares. They are

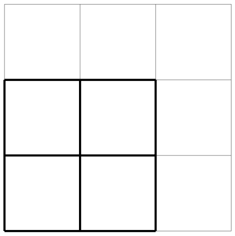
- $\mathcal{B}_1$:



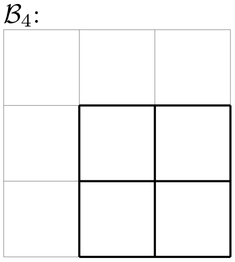
- $\mathcal{B}_2$:



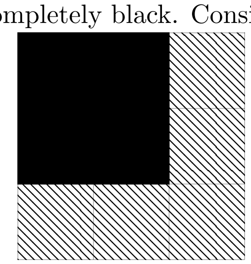
- $\mathcal{B}_3$:



- $\mathcal{B}_4$:



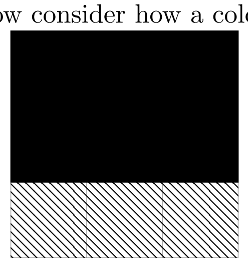
For $i \in \{1,2,3,4\}$, let A_i denote the set of colorings of the 3×3 chessboard in which the square $\mathcal{B}_i$ is completely black. Consider the figure that shows a coloring in A_1 (the shaded cells can be black or white):



$|A_1|$ is the number of ways the 5 shaded cells can be colored, which is 2^5. Hence $|A_1| = 32$. Similarly, each of $|A_2|$, $|A_3|$ and $|A_4|$ is 32.
By the principle of inclusion-exclusion,

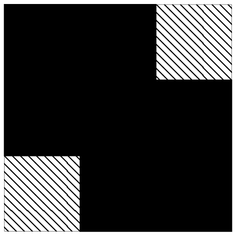
$$\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\
& - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_4| \\
& - |A_1 \cap A_4| - |A_2 \cap A_4| - |A_1 \cap A_3| \\
& + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\
& + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}$$

Now consider how a coloring in $A_1 \cap A_2$ looks:



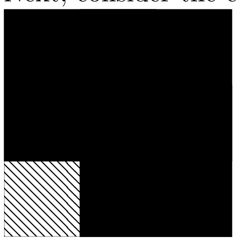
Thus the number of ways to color the grid such that A_1 and A_2 are both black is the total number of ways the 3 shaded cells can be colored, so $|A_1 \cap A_2| = 2^3 = 8$. Symmetrically, we have $|A_1 \cap A_3| = |A_3 \cap A_4| = |A_2 \cap A_4| = 8$.

Consider the case of a coloring in $A_1 \cap A_4$:



Hence, similarly, the total number of colorings of the 3×3 grid such that A_1 and A_4 are both colored black is the number of ways the two shaded cells can be colored, that is, $|A_1 \cap A_4| = 2^2 = 4$ ways. Symmetrically, we have $|A_2 \cap A_3| = 4$.

Next, consider the case of a coloring in $A_1 \cap A_2 \cap A_4$:



Using similar arguments as above, number of ways the 3×3 grid cells can be colored black such that A_1, A_2, and A_4 are all colored black is $|A_1 \cap A_2 \cap A_4| = 2^1 = 2$. Symmetrically, $|A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3| = 2$.

Finally, the number of ways coloring such that all the 2×2 squares are black is 1.

Therefore,

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = \\
& (4 \times 32) \\
& - (4 \times 8) - (2 \times 4) \\
& + (4 \times 2) \\
& - 1 \\
& = 95
\end{aligned}
$$

Now, the total number of different ways in which the 3×3 grid can be colored is $2^9 = 512$. Therefore, the required number of ways of coloring such that no 2×2 squares are completely colored black is $512 - 95 = 417$.

Question 4

Show that in a group of 37 people, at least 4 people have the same month in their date of birth.

Solution.

There are 12 months in a year. Using pigeonhole principle, consider the months as pigeonhole and people's birthdates as pigeons. Each birthdate is put in the pigeonhole corresponding to the month in the date. There are 12 holes and 37 pigeons, and by pigeonhole principle (generalized), there should exist a hole, or month, where there are at least $\lceil \frac{37}{12} \rceil = 4$ objects or people's birthdates.

Question 5

Let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is a bijection, then which of the functions f and g are necessarily required to be injective? Which of the functions f and g are necessarily required to be surjective?

Solution.

Let us prove the following claims:

- **f is necessarily injective.**

  Proof. Consider $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

  $$\begin{aligned} & f(x_1) = f(x_2) \\ \implies & g(f(x_1)) = g(f(x_2)), \text{ since } g \text{ is a function} \\ \implies & g \circ f(x_1) = g \circ f(x_2) \\ \implies & x_1 = x_2, \text{ since } g \circ f \text{ is a bijection} \end{aligned}$$

  Therefore f is necessarily injective. □

- **g is not necessarily injective.**

  Proof. Consider the following example:
  $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2\}$. Consider the following mapping:
  $f(x) = x$, and $g(y) = \begin{cases} y, & \text{if } y \in \{1, 2\} \\ 2, & \text{otherwise} \end{cases}$.

  Therefore, $g \circ f$ is the identity function which is bijective, but g is not injective. □

- **g is necessarily surjective.**

  Proof. Consider any $z \in C$. Since $g \circ f$ is a bijection, there exists an $x \in A$ such that $g \circ f(x) = z$. This implies there exists a $y \in B$, such that $f(x) = y$ and $g(y) = z$. Hence g is necessarily surjective. □

- **f is not necessarily surjective.**

  Proof. Consider the following example:
  $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and $C = \{1, 2\}$. Consider the following mapping:
  $f(x) = x$, and $g(y) = \begin{cases} y, & \text{if } y \in \{1, 2\} \\ 2, & \text{otherwise} \end{cases}$.

  Therefore, $g \circ f$ is the identity function which is bijective, but f is not surjective. □

Question 6

Consider the $n \times n$ grid of points in the plane formed by the set of points $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. Show that we cannot choose $2n + 1$ points from this set of points such that no three of our chosen points are collinear.

Solution.

This can be shown using Pigeonhole Principle. Consider n columns formed by the grid of points $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ as n pigeonholes. Also, consider the $2n+1$ points to be chosen as pigeons. Each pigeon (point) is put in the hole corresponding to the column the point belongs to. Since there are n columns, by generalized pigeonhole principle, there has to be a column with at least $\lceil (2n + 1)/n \rceil = 3$ points in it.

Question 7

How many twelve bit binary strings consisting of seven 1s and five 0s are there that contain no two consecutive 0s?

Solution.

This can be seen as a stars and bars problem with the 1s being considered as the bars. Since there can be no consecutive 0s, hence there has to be at most one 0 in between and at the two terminal positions. Since there are seven 1s, there are 8 slots for 0s (6 in between, and 2 at the two ends). And among those 8 slots, 5 has to be chosen for 0s. Hence number of such required binary strings is $\begin{pmatrix} 8 \\ 5 \end{pmatrix} = \frac{8!}{5!3!} = \frac{8\times7\times6}{6} = 56$.

Question 8

In a collection of $2n$ objects, n are indistinguishable. In how many ways can n objects be chosen from this collection?

Solution.

Let us consider r objects are chosen from the n distinguishable items, and $n-r$ items from the remaining n indistinguishable items.
Hence, the total number of ways this can be chosen = Number of ways r items can be chosen$\times$ Number of ways $n-r$ items can be chosen.
Now, number of ways $n-r$ indistinguishable items can be chosen from n indistinguishable items is 1, and number of ways r distinguishable items that can be chosen from n distinguishable items is $\binom{n}{r}$. Hence, number of ways n items can be chosen from n distinguishable and n indistinguishable items $= \sum_{r=0}^{n} \binom{n}{r} \times 1 = \sum_{r=0}^{n} \binom{n}{r} = 2^n$.

Question 9

If $G(x)$ is the generating function of a sequence $(g_0, g_1, g_2, \ldots)$, then what is the generating function of the sequence $(g_0^2, 2g_0g_1, g_1^2 + 2g_0g_2, 2(g_0g_3 + g_1g_2), g_2^2 + 2(g_0g_4 + g_1g_3), 2(g_0g_5 + g_1g_4 + g_2g_3), \ldots)$?

Solution.

Consider two generating functions $A(x)$, with sequence $(a_0, a_1, \ldots)$, and $B(x)$ with the sequence $(b_0, b_1, \ldots)$.
Then the coefficient of x^n in $A(x)B(x)$ is $\sum_{i=0}^{n} a_i b_{n-i}$.
Thus the coefficient of x^n in $G(x)G(x)$ is $\sum_{i=0}^{n} g_i g_{n-i}$, which is our required coefficient.
Therefore our required generating function is $G(x)G(x) = G^2(x)$.

Question 10

Solve the recurrence relation $f(n) = 4f(n-1) - 2^n$ with boundary condition $f(0) = 2$.

Solution.

This will be proved by the the following claim:

Claim. *Given $f(n) = 4f(n-1) - 2^n$ with the boundary condition $f(0) = 2$, $f(n) = 2^{2n} + 2^n$*

Proof. This can be proved by induction on n.
Base Case.
$f(0) = 2^{2\times0} + 2^0 = 2$
Induction Hypothesis.
$f(m) = 2^{2m} + 2^m$ for all $m < n$.

Inductive Step.

$$\begin{aligned}
f(n) &= 4f(n-1) - 2^n \\
&= 4(2^{2(n-1)} + 2^{(n-1)}) - 2^n \text{(by the induction hypothesis, since } n-1 < n) \\
&= 2^{2(n-1)+2} + 2^{n-1+2} - 2^n \\
&= 2^{2n} + 2^{n+1} - 2^n \\
&= 2^{2n} + 2.2^n - 2^n \\
&= 2^{2n} + 2^n
\end{aligned}$$

□

Question 11

What is the coefficient of xy^2z^4 in the expansion of $(2x + y + 3z^2)^5$?

Solution.

By multinomial theorem,

$$(2x + y + 3z^2)^5 = \sum_{i+j+k=5; i,j,k \geq 0} \frac{5!}{i!j!k!}(2x)^i.y^j.(3z^2)^k$$

Since, it required to find coefficient of xy^2z^4, we substitute $i = 1, j = 2$ and $k = 2$ in the above equation. Hence the coefficient is:

$$\frac{5!}{1!2!2!}.2^1.3^2 = \frac{5 \times 4 \times 3}{2} \times 2 \times 9 = 30 \times 18 = 540$$

Question 12

Solve the recurrence relation $f(n) = 3f(n-1) - 3f(n-2) + f(n-3)$ with boundary conditions $f(0) = 1, f(1) = 3$, and $f(2) = 6$.

Solution.

Consider $f(n) = x^n$ to be the solution of the linear recurrence relation. Hence,

$$\begin{aligned}
& x^n = 3x^{n-1} - 3x^{n-2} + x^{n-3} \\
\implies & x^n - 3x^{n-1} + 3x^{n-2} - x^{n-3} = 0 \\
\implies & x^{n-3}(x^3 - 3x^2 + 3x - 1) = 0 \\
\implies & x^{n-3}(x-1)^3 = 0
\end{aligned}$$

Since $f(n)$ is not equal to zero for all n, it is clear that $x \neq 0$. So we divide by x^{n-3} on both sides to get the characteristic equation of the recurrence, which is $(x-1)^3 = 0$. The characteristic polynomial (the LHS of the characteristic equation) has the single root 1 with multiplicity 3. Therefore $f(n)$ can be written as:

$$\begin{aligned}
f(n) &= A.1^n + B.n.1^n + C.n^2.1^n \\
&= A + Bn + Cn^2
\end{aligned}$$

where A, B, C are constants. We shall determine them using boundary condition. Since $f(0) = 1$,

$$f(0) = A = 1$$

Now for $n = 2$,

$$f(2) = 1 + 2B + 4C = 6 \implies 2B + 4C = 5$$

For $n = 1$,

$$f(1) = 1 + B + C = 3 \implies B + C = 2 \implies 2B + 2C = 4$$

Substracting the equation for $n = 1$ from $n = 2$,

$$4C - 2C = 5 - 4 \implies C = \frac{1}{2}$$

Hence using $C = \frac{1}{2}$ in the equation for $n = 1$,

$$B + \frac{1}{2} = 2 \implies B = \frac{3}{2}$$

Therefore, $f(n) = 1 + \frac{3}{2}n + \frac{1}{2}n^2$.