Discrete Mathematics

Problem sheet 1 - Solutions

1. Induction on n. Base case $n = 0$ is true since $r^0 + 1/r^0 = 2$ is an integer and case $n = 1$ is true since $r + \frac{1}{r}$ is given to be an integer. Let $n > 1$. Inductive hypothesis: $r^t + \frac{1}{r^t}$ is an integer for all $t < n$. Note that $r^n + \frac{1}{r^n} = (r^{n-1} + \frac{1}{r^{n-1}})(r + \frac{1}{r}) - (r^{n-2} + \frac{1}{r^{n-2}})$. Since we have by the inductive hypothesis that $r^{n-1} + \frac{1}{r^{n-1}}$, $r + \frac{1}{r}$, and $r^{n-2} + \frac{1}{r^{n-2}}$ are all integers, it follows that $r^n + \frac{1}{r^n}$ is also an integer.

3. Consider the set of numbers $S = \{1, 11, 111, \ldots, \sum_{i=0}^{n} 10^i\}$. Since any number when divided by n leaves a remainder from the set $\{0, 1, \ldots, n-1\}$ and $|S| = n+1$, by the pigeonhole principle, there exist two distinct numbers $i, j \in S$ such that $i \equiv j \mod n$ (i.e. i and j leave the same remainder when divided by n). Then assuming that $i > j$, the number $i - j$ is divisible by n and its decimal representation consists only of 1s and 0s.

4. For any $i \in \{1, 2, \ldots, m\}$, let $\overline{S}_i$ denote the set $\{1, 2, \ldots, n\} - S_i$. As for any $i, j \in \{1, 2, \ldots, m\}$, $S_i = S_j$ if and only if $\overline{S}_i = \overline{S}_j$, we have that $|\{\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_m\}| = m$. Also, since for any $i \in \{1, 2, \ldots, m\}$, we have $S_i \cap \overline{S}_i = \emptyset$, we know that none of the sets $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_m$ belong to the collection $\{S_1, S_2, \ldots, S_m\}$. Thus $|\{S_1, S_2, \ldots, S_m\} \cup \{\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_m\}| = 2m$, implying that $2m \leq 2^n$ (the size of the power set of $\{1, 2, \ldots, n\}$). We then get $m \leq 2^{n-1}$. The collection of sets $\{\{1\} \cup X \colon X \subseteq \{2, 3, \ldots, n\}\}$ contains 2^{n-1} sets and satisfies the property that no two sets in the collection are disjoint.

5. For any $i \in \{1, 2, \ldots, 2n\}$, let $f(i)$ denote the highest number in $\{1, 2, \ldots, 2n\}$ of the form $2^t i$ for some integer t. Note that for any i, $f(i) \in \{n+1, n+2, \ldots, 2n\}$. Thus by the pigeonhole principle, in any set $S \subseteq \{1, 2, \ldots, 2n\}$ containing at least $n+1$ elements, there exist two distinct numbers i and j such that $f(i) = f(j)$. Then there exist s, t such that $f(i) = 2^s i$ and $f(j) = 2^t j$. Assuming that $i < j$, we get $s > t$, and further that $j = 2^{s-t} i$. Then i divides j.

7. Let S be a set of 52 integers. If any two integers in S leave the same remainder when divided by 100, then their difference is divisible by 100. So let us assume that no two integers in S leave the same remainder when divided by 100. For each $i \in \{0, 1, \ldots, 49\}$, define $T_i = \{i, 100 - i\}$ and $T_{50} = \{50\}$. Then by pigeonhole principle, there exist two integers in S such that their remainders when divided by 100 belong to T_i, for some $i \in \{0, 1, \ldots, 50\}$. The sum of these two integers is divisible by 100.

8. Divide the square into four squares, each of side length 1/2. Out of the five points, by pigeonhole principle, at least two must lie inside the same square. Any two points inside a square of side length 1/2 are at a distance of at most $1/\sqrt{2}$.

9. Let $R_0 = \{1, 2, \ldots, (s-1)n^{(t-1)n+1} + 1\}$. Let v_1 denote a value that occurs the most number of times in column 1 of the matrix. Clearly, there exist at least $|R_0|/n$ rows in column 1 of the matrix that contain the value v_1. We shall denote the set of rows in R_0 that contain v_1 by R_1. For each $i \in \{2, 3, \ldots, (t-1)n+1\}$, we inductively define v_i and R_i in a similar way. That is, once v_{i-1} and R_{i-1} have been defined, we let v_i denote a value that occurs the most number of times in column i in the submatrix formed by the rows in R_{i-1} and we denote by R_i the set of at least $|R_{i-1}|/n$ rows in R_{i-1} that contain the value v_i in column i. Clearly, $|R_{(t-1)n+1}| \geq |R_0|/n^{(t-1)n+1} > s-1$, which implies that $|R_{(t-1)n+1}| \geq s$. Let M be the submatrix formed by the rows in $R_{(t-1)n+1}$ and all the columns of the original matrix. Notice that for every $i \in \{1, 2, \ldots, (t-1)n+1\}$, column i of M contains only the value v_i. Let x denote a value that occurs the most number of times in $v_1, v_2, \ldots, v_{(t-1)n+1}$ and let $C = \{j \colon v_j = x\}$. It follows from the pigeonhole principle that $|C| \geq t$. Now the submatrix of M formed by the columns in C is a submatrix of the original matrix that contains only the value x and has at least s rows and at least t columns.

10. For a fixed cell at location (i, j), we denote by $Q(i, j)$ the "positive quadrant" with origin at (i, j)—i.e., all the cells (x, y) such that $x \geq i$ and $y \geq j$ (the cells that lie to the top-right of (i, j)). We use induction on n. Inductive hypothesis: "Let (i, j) be any cell of the infinite grid. For any $t < n$, if at some point of time $Q(i, j)$ contains t black cells, then $Q(i, j)$ contains only white cells after t rounds." The base case when $n = 1$, that is when $Q(i, j)$ contains just one black cell, is clearly true (note that the presence of black cells outside $Q(i, j)$ does not affect the fact that $Q(i, j)$ becomes completely white after just one round). Suppose that $n > 1$. Let i' denote the lowest row that contains a black cell in $Q(i, j)$ and let j' denote the leftmost column that contains a black cell in $Q(i, j)$. Since both the quadrants $Q(i'+1, j)$ and $Q(i, j'+1)$ contain less than n black cells, it follows from the inductive hypothesis that both these quadrants are fully white after at most $n-1$ rounds. Thus, at the beginning of the n-th round, the only location in $Q(i, j)$ where there can be a black square is the position (i', j'). It is clear that even this square will become white after the n-th round.