Discrete Mathematics

Problem sheet 2

1. Show that the set of rational numbers is countable.
2. Show that there is a rational number between any two real numbers.
3. How many subsets of $\{1, 2, \ldots, n\}$ exist that contain no two consecutive numbers?
4. How many k-element subsets of $\{1, 2, \ldots, n\}$ exist containing no two consecutive numbers?
5. How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are *monotone*; that is, for $i < j$ we have $f(i) \leq f(j)$?
6. What is the coefficient of x^2yz^3 in $(3x^2 + y + 2z)^5$?
7. How many different partitions of the set $\{1, 2, \ldots, kn\}$ into sets of size k are there?
8. Prove that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.
9. Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.
10. Prove that the number of subsets of $\{1, 2, \ldots, n\}$ containing an even number of elements is 2^{n-1}.
11. Prove that $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$ using the binomial theorem.
12. In a collection of $3n + 1$ objects, n are indistinguishable. Find the number of ways to choose n of these objects.
13. How many numbers in $\{1, 2, \ldots, 100\}$ are not divisible by 2, 3, 5 or 7?
14. For a positive natural number n, let $\phi(n)$ denote the number of natural numbers less than n that are relatively prime to n. Prove that $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_t})$, where $p_1, p_2, \ldots, p_t$ are the prime factors of n.
15. How many arrangements of the 26 letters of the alphabet are there that do not contain the letters of any of the words LEFT, TURN, SIGN, or CAR in the correct order?
16. How many arrangements are there of the letters of the word MATHEMATICS with both T's before both A's, or both A's before both M's, or both M's before the E?

17. Write a recurrence relation for the number of n-bit binary strings that contain two consecutive 0's. What are the boundary conditions for this recurrence relation? Find a closed form expression for the number of n-bit binary strings that contain two consecutive 0's.

18. Let a_n be the number of ordered triples (i, j, k) of integer numbers such that $i \geq 0$, $j \geq 1$, $k \geq 1$, and $i + 3j + 3k = n$. Find the generating function of the sequence $(a_0, a_1, a_2, \ldots)$ and calculate a formula for a_n.

19. You have 5 one-rupee coins, 4 five-rupee notes and 3 ten-rupee notes. Let a_n denote the number of ways in which you can give change for a sum of Rs. n. Write a generating function for the sequence $(a_0, a_1, a_2, \ldots)$. Using the generating function, calculate in how many ways you can give change for a sum of Rs. 20.

20. Suppose that f is a function such that $f(0) = 7$, $f(1) = 16$, and $f(n) = 5f(n-1) - 6f(n-2) + 4n - 4$ for every natural number $n \geq 2$. Derive a closed form expression for $f(n)$.

21. Consider the following variation of the game of Towers of Hanoi. As before, there are n discs of decreasing radii stacked on peg 1 and the aim is to transfer all the discs to peg 3, following the usual rules of the game. But in addition, we have an additional rule that no disc should be transferred between pegs 1 and 3 directly. That is, every move of a disc must be to or from peg 2.

    (a) Determine a recurrence relation for the minimum number of moves required to transfer a stack of n discs from peg 1 to peg 3 following these rules.

    (b) Solve the recurrence to find a closed form formula for the minimum number of moves required.

    (c) Show that every allowable arrangement of n discs on the three pegs occurs during a strategy that moves n discs from peg 1 to peg 3 using the minimum number of moves.