Binary Symmetric channel (BSC): In BSC, the probability of bit-flipping is ϵ, which is independent of all other bits and $\epsilon < \frac{1}{2}$. That is, each bit is corrupted independently and with equal probability (i.i.d).

Given a packet of size s bits, the probability of the entire packet being received correctly under BSC with bit-flipping probability ϵ is

P (packet received correctly) $= (1-\epsilon)^s$

The packet error probability $= 1 - (1-\epsilon)^s$

$= 1 - (1 - s\epsilon) = s\epsilon$ $\left[(1 \pm \epsilon)^s \simeq 1 \pm \epsilon s, \text{ when } |\epsilon| \ll 1 \text{ first order approximation}\right]$

The Simplest code: Repetition: In repetition code, each bit b is encoded as n copies of b and the result is delivered. If b is the message word, then b^n $(\underbrace{bb\ldots\ldots b}_{n \text{ times}})$ is the corresponding codeword. Here only 2 possible message word and 2 corresponding codewords

$n = 3$	$b=0$	000
	$b=1$	111

Maximum likelihood decoding: Given a received codeword r, count the number of 1's in r. Here r is a n-bit combination of 0's and 1's. If there are more than $\frac{n}{2}$

ones then decode as 1. If there are more than $\frac{n}{2}$ zeros then decode as 0. When n is odd, each codeword will be decoded unambiguously. When n is even, and has equal no. of 0's and 1's, the decoder makes arbitrary choice among 0 and 1.

$$P(\text{decoding error}) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \binom{n}{i} \epsilon^i (1-\epsilon)^{n-i} \qquad n \text{ is odd}$$

$$= \sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} \epsilon^i (1-\epsilon)^{n-i} + \underbrace{\frac{1}{2} \binom{n}{\frac{n}{2}} \epsilon^{\frac{n}{2}} (1-\epsilon)^{\frac{n}{2}}}_{(*)}$$

*: denotes that the decoder has a fifty-fifty chance of guessing correctly when it receives a codeword with an equal no. of 0's and 1's.

For a BSC with bit-flipping probability $< \frac{1}{2}$, the maximum likelihood decoding strategy maps any received ~~co~~ word to the valid codeword with smallest Hamming distance from the received word (ties may be broken arbitrarily).

3. **Binary Symmetric Channel (BSC):**
Answer the following questions about the BSC given in Figure 1; assume that $\rho = .2$ and $p(x = 0) = p(x = 1) = 0.5$.

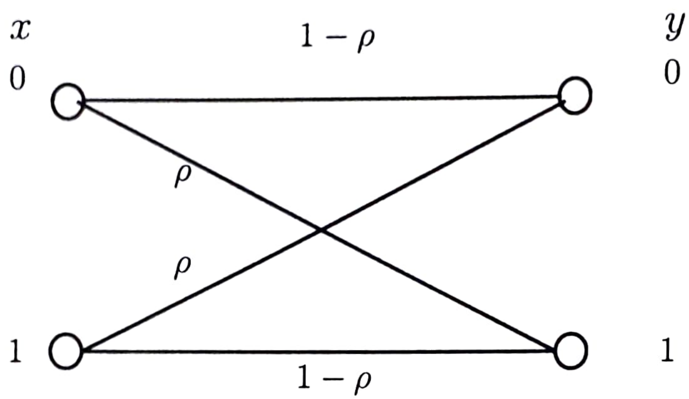

Figure 1: Binary Symmetric Channel (BSC) with crossover probability ρ, input x and output y.

(a) What is the probability that $y = 0$ if $x = 0$ (that is, $p(y = 0|x = 0)$)?

Solution:

If $x = 0$ and $y = 0$, no error occurred. The probability of no error or bit flip per bit is $1 - \rho$. Thus $p(y = 0|x = 0) = 1 - \rho = 0.8$.

(b) Send 6 bits x_1 through x_6 across the BSC. What is the probability that only 1 of the 6 received bits y_1 through y_6 is in error? What is the probability that 2 of the 6 received bits are in error? What is the probability none of the received bits are in error?

Solution: The BSC has a binomial distribution, and thus the probability of having exactly k errors in n bits is given by $P(k) = \binom{n}{k}\rho^k(1-\rho)^{n-k}$. For this question, $n = 6$.

i. What is the probability that only 1 of the 6 received bits y_1 through y_6 is in error?

Solution:
The probability $P(k=1) = \binom{6}{1}(0.2)^1(0.8)^5 = 6 * 0.2 * (0.8)^5 = 0.3932.$

ii. What is the probability that 2 of the 6 received bits are in error?

Solution:
The probability $P(k=2) = \binom{6}{2}(0.2)^2(0.8)^4 = \frac{30}{2} * (0.2)^2 * (0.8)^4 = 0.2458.$

iii. What is the probability none of the received bits are in error?

Solution:
The probability $P(k=0) = \binom{6}{0}(0.8)^6 = (0.8)^6 = 0.2621.$

(c) Now send 6000 bits x_1 through x_{6000} across the BSC. What is the probability that all 6000 bits are received correctly, that is, that none of the bits are received in error?

Solution:

Now $n = 6000$. The probability $P(k=0) = \binom{6000}{0}(0.8)^{6000} = (0.8)^{6000} \approx 0.0$. The probability is so small as to be insignificant. We can estimate it by noting that $0.8 = 8/10 = 2^3/10$ and that $10^3 = 1000 \approx 1024 = 2^{10}$. Thus $(8/10)^3 \approx 2^9/2^{10} = 2^{-1}$. Therefore, $(8/10)^{6000} = (8/10)^{3*2000} \approx 2^{-2000} = 2^{-20*100}$, which is so small as to be considered 0.

(d) Suppose $p(x=0) = 0.4$. What is $p(x=1)$? What is the probability that $y=0$ if $x=0$? What is the probability that $y=0$, regardless of what x is (that is, p(y=0))?

Solution:

i. What is $p(x=1)$?

Solution:
X is a binary variable, such that $x \in 0, 1$. Therefore $p(x=0) + p(x=1) = 1$. Thus $p(x=1) = 1 - p(x=0) = 1 - 0.4 = 0.6$.

ii. What is the probability that $y=0$ if $x=0$?

Solution:
The conditional probability $p(y=0|x=0)$ is the probability that no errors occur, given that $x=0$. This is shown on the BSC graph as $1-\rho$. Thus

$$p(y=0|x=0) = 1-\rho = 1-0.2 = 0.8$$

iii. What is the probability that $y=0$, regardless of what x is (that is, p(y=0))?

Solution:
Using marginalization, we can find $p(y=0)$.

$$\begin{aligned}
p(y=0) &= \sum_{x\in\{0,1\}} p(y=0|x)p(x) \\
&= p(y=0|x=0)p(x=0) + p(y=0|x=1)p(x=1) \\
&= (1-\rho) * 0.4 + \rho * 0.6 = 0.8 * 0.4 + 0.2 * 0.6 \\
p(y=0) &= 0.44
\end{aligned}$$