Lecture 9

Monday, 8 November 2021 2:36 PM

Is $\mathbb{R}$ countable? No.

Let $S \subseteq \mathbb{N}$

	1	2	3	4	5	
S:	Y	Y	N	N	Y	...
S:	1	1	0	0	1	...

$S_1 \neq S_2 \Rightarrow$
$\exists i \in \mathbb{N}$ s.t.
$(i \in S_1 \ \& \ i \notin S_2)$ or
$(i \notin S_1 \ \& \ i \in S_2)$

$f(S) = 0.11001\ldots$

$S_1 \neq S_2 \Rightarrow f(S_1) \neq f(S_2)$

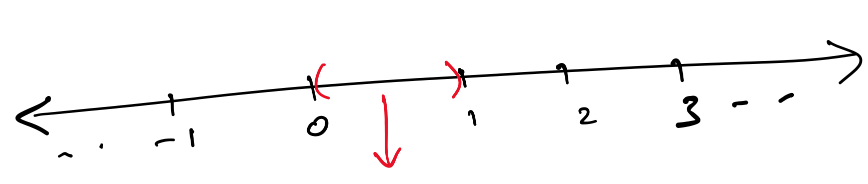
$f : 2^{\mathbb{N}} \to \mathbb{R}$

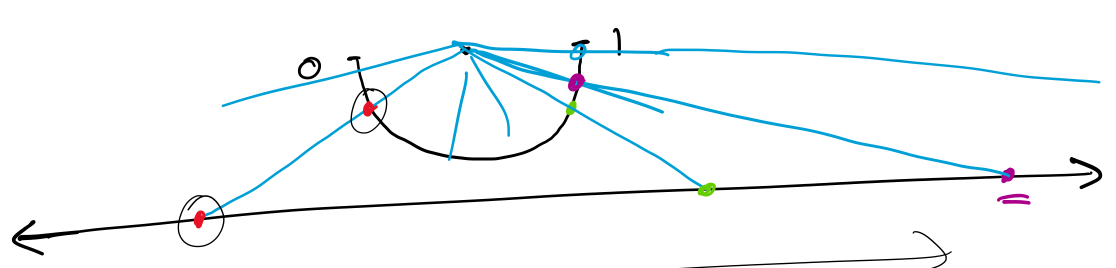
f is injective

$|2^{\mathbb{N}}| \leq |\mathbb{R}|$

$|\mathbb{R}| \leq |2^{\mathbb{N}}|$

$(0,1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \}$





This mapping is a bijection between $(0,1)$ and $\mathbb{R}$.

$|(0,1)| = |\mathbb{R}|$

Find an injective function from $(0,1)$ to $2^{\mathbb{N}}$.

$f : (0,1) \to 2^{\mathbb{N}}$

For any $x \in (0,1)$ let σ_x be the sequence of 1s and 0s obtained from one binary representation of x.

$x : \boxed{0.}\ \underbrace{11011000 1\ldots}$

$1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1 \ldots$
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$f(x) = \{1, 2, 4, 5, 9, \ldots\}$

$f(x) \in 2^{\mathbb{N}}$

f is injective : $x \neq x'$
$\Downarrow$
$f(x) \neq f(x')$

$x = 0.01111\ldots$
$2x = 0.11111\ldots$
$2x - x = 0.1000\ldots$
$x = 0.100\ldots$

$|\mathbb{R}| = |(0,1)| \leq |2^{\mathbb{N}}|$

$\Rightarrow \quad |\mathbb{R}| = |2^{\mathbb{N}}| = 2^{\aleph_0}$

$\mathbb{N} \times \mathbb{N}$ is countable
$(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ is countable
$\vdots$
$\mathbb{N}^k$ is countable for any $k \in \mathbb{N}$

Binomial Theorem

$n \in \mathbb{N}$

$(x+y)^n$

$= \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ times}}$

$= \underbrace{x \cdot x \cdot \ldots x}_{n \text{ times}} + x \cdot y \cdot \underbrace{x \cdot x \ldots x}_{(n-2) \text{ times}} + \cdots$

What is the coefficient of x^n?
" " " $x^{n-1}y$?

$\overset{y}{(x+y)} \cdot \overset{x}{(x+y)} \ldots (x+y)$

$x^{n-1}y$ can be $\underbrace{x \cdot x \ldots x}_{n-1 \text{ times}}\, y$ or

$x \cdot y \cdot \underbrace{x \cdot x \ldots x}_{n-2 \text{ times}}$ - - -

No. of terms in the summation that evaluate to $x^{n-1}y$ $= \binom{n}{1} = n$

$\binom{n}{1} x^{n-1} y$

What is the coefficient of $x^i y^{n-i}$?

$\binom{n}{i} = \binom{n}{n-i}$

$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n$

$x \to 1, \ y \to -1$

$= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = (1+1)^n = 2^n$

$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots \pm \binom{n}{n} = (1-1)^n = 0$

Multinomial Theorem

$(x+y+z)^n = \underbrace{(x+y+z)(x+y+z) \ldots (x+y+z)}_{n \text{ times}}$

$= x \cdot x \ldots x + x \cdot y \cdot z \cdot z \ldots$

Coefficient of $x^i y^j z^k$ where $i+j+k = n$ in this summation.

$\binom{n}{i}\binom{n-i}{j} =$

$\frac{n \times (n-1) \times \ldots \times (n-i+1)}{i!} \times \frac{(n-i) \times (n-i-1) \times \ldots \times (n-i-j+1)}{j!}$

$= \frac{n \times (n-1) \times \ldots \times (n-(i+j)+1)}{i!\, j!} = \frac{n \times (n-1) \times \ldots \times (k+1)}{i!\, j!}$

($n-(i+j) = n-k$)

$= \frac{n!}{i!\, j!\, k!}$

$\overbrace{\underbrace{x x x \ldots x}_{i}\ \underbrace{y y y \ldots y}_{j}\ \underbrace{z z \ldots z}_{k}}^{n}$

$(x_1 + x_2 + \ldots + x_t)^n = (x_1 + x_2 + \ldots + x_t)(x_1 + x_2 + \ldots + x_t) \ldots (x_1 + x_2 + \ldots + x_t)$

Coefficient of $x_1^{i_1} x_2^{i_2} x_3^{i_3} \ldots x_t^{i_t}$

$= \frac{n!}{i_1!\, i_2! \ldots i_t!} = \binom{n}{i_1, i_2, \ldots, i_t}$ – multinomial coefficient

$(x_1 + x_2 + \ldots + x_t)^n$

$= \sum_{\substack{i_1, i_2, \ldots, i_t \\ \text{s.t. } i_1 + i_2 + \ldots + i_t = n}} \binom{n}{i_1, i_2 \ldots i_t} x_1^{i_1} x_2^{i_2} \ldots x_t^{i_t}$

No. of terms in the summation is $\binom{n+t-1}{t-1}$

$* * | * * | * \ldots *$
n stars and $t-1$ bars

$\sum_{\substack{i_1, i_2, \ldots, i_t \\ \text{s.t. } i_1 + i_2 + \ldots + i_t = n}} \binom{n}{i_1, i_2, \ldots, i_t} = t^n$