Discrete Mathematics

Problem sheet 1

1. Let r be a real number such that $r + \frac{1}{r}$ is an integer. Then prove that $r^n + \frac{1}{r^n}$ is also an integer for any natural number $n \geq 1$.
2. Prove that the following is true for every natural number $n \geq 1$ using mathematical induction:
   (a) $\sum_{i=0}^{n-1} 2^i = 2^n - 1$.
   (b) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.
   (c) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$.
   (d) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.
3. Prove that every natural number $n \geq 1$ has a multiple whose decimal representation contains only 1s and 0s.
4. Let $S_1, S_2, \ldots, S_m$ be a collection of subsets of $\{1, 2, \ldots, n\}$ such that for all $i, j \in \{1, 2, \ldots, m\}$, $S_i \cap S_j \neq \emptyset$. Show that $m \leq 2^{n-1}$ and that this bound is tight.
5. Show if S is any subset of $\{1, 2, \ldots, 2n\}$ such that $|S| > n$, then S contains two numbers such that one of them divides the other.
6. Prove DeMorgan's laws using truth tables: Show that if p and q are two propositions, then $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$, and $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$.
7. Prove that in any set of 52 integers, there are two numbers such that either their sum or difference is divisible by 100.
8. Show that if one selects five points that lie inside a square of side length 1, then some two points among them will be at a distance of at most $1/\sqrt{2}$.
9. Show that every $((s-1)n^{(t-1)n+1} + 1) \times ((t-1)n + 1)$ matrix whose elements are from the set $\{1, 2, \ldots, n\}$ contains an $s \times t$ submatrix whose elements are all the same.
10. Consider the following game played on an infinite grid with square cells (like an infinite chessboard in which all squares are white). First, some n cells that are chosen arbitrarily are coloured black. The game now begins. In each round, some cells coloured black change to white, some cells coloured white change to black and other cells remain the same according to the following rule: the colour of a cell is toggled if and only if its current colour is different from the colour of the cell to its right and the colour of the cell above it. Prove that at the end of the n-th round, all cells of the grid are white.