Discrete Mathematics

Assignment 1

Give justifications for all your answers. The set of all real numbers is denoted by $\mathbb{R}$. The letter n always denotes a positive integer. Each question carries 5 marks.

1. Prove by induction on n:
   (a) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$
   (b) $1^2 - 2^2 + 3^2 - 4^2 + \cdots (-1)^{n-1} n^2 = \frac{(-1)^{n-1} n(n+1)}{2}$
2. Show using truth tables that if p and q are two propositions, the proposition $p \Rightarrow q$ is equivalent to the proposition $\neg p \vee q$.
3. For any two real numbers $i, j \in \mathbb{R}$, where $i \leq j$, the "interval" $[i, j]$ is defined to be the set $\{x \in \mathbb{R} \colon i \leq x \leq j\}$. Let $\mathcal{F}$ be a finite family of intervals.
   (a) Define a relation $\sim$ on $\mathcal{F}$ as follows: for two intervals $A, B \in \mathcal{F}$, we define $A \sim B$ if and only if $A \cap B \neq \emptyset$. Discuss whether the relation $\sim$ reflexive, symmetric, anti-symmetric or transitive.
   (b) For an interval $A \in \mathcal{F}$, define $N(A) = \{B \in \mathcal{F} \colon A \sim B\}$. Show that there is an interval $A \in \mathcal{F}$ with the property that if $B, C \in N(A)$, then $B \sim C$.
   (c) Let $\nsim$ denote the relation defined as follows: for two intervals $A, B \in \mathcal{F}$, $A \nsim B$ if and only if $A \cap B = \emptyset$. Discuss what kind of a relation $\nsim$ is.
4. Show that given $n^2 + 1$ data points in the XY-plane, one can always draw a monotonically increasing or decreasing curve (one whose slope is always non-negative or is always non-positive) that passes through at least n of them.
5. Let p be some prime number other than 2 and 5. Show that there is some power of p whose decimal representation has 01 as its last two digits.
6. Let $f : A \to B$ and $g : B \to C$ be two functions and let $h = g \circ f$.
   (a) Is h injective if both f and g are injective? What about the converse ("Are both f and g injective if h is injective?") ?
   (b) Is h surjective if both f and g are surjective? What about the converse?
7. Let $S \subseteq 2^{\mathbb{N}}$ be a family of subsets of natural numbers such that any two sets in S are disjoint. Is S necessarily a countable set?
8. In how many ways can the aadhaar cards of n people be distributed among them in such a way that nobody receives his/her own card? (The answer can be in the form of a formula involving summation.)
9. How many integral solutions are there for $x_1 + x_2 + x_3 + x_4 = 20$ which satisfy the constraints $2 \leq x_1 \leq 6$, $3 \leq x_2 \leq 7$, $5 \leq x_3 \leq 8$, and $2 \leq x_4 \leq 9$?
10. What is the least number of colours required to color the integers $1, 2, \ldots, 2^n - 1$ such that in any set of consecutive integers, there is a colour that occurs exactly once? That is, the colouring has to satisfy the condition that for any integers i, j such that $1 \leq i \leq j \leq 2^n - 1$, there is a colour that is given to exactly one integer in the set $\{i, i+1, \ldots, j-1, j\}$.