Discrete Mathematics
Assignment 1 - Solutions

3. (b) Let A be an interval in $\mathcal{F}$ with left-most right end-point. Or in other words, let $A = [i, j]$ be an interval in $\mathcal{F}$ such that for every interval $[i', j'] \in \mathcal{F}$, we have $j \leq j'$. We claim that if $B, C \in N(A)$, then $B \sim C$. Suppose that $B, C \in N(A)$. Let $B = [b_1, b_2]$ and $C = [c_1, c_2]$. Since $B \sim A$, we know that $b_1 \leq j$, as otherwise, the intervals A and B cannot intersect. Similarly, we have $c_1 \leq j$. Since A is an interval with left-most right end-point, we also know that $j \leq b_2$ and $j \leq c_2$. We then have $j \in [b_1, b_2]$ and $j \in [c_1, c_2]$. Thus, $j \in B \cap C$, which implies that $B \sim C$.

4. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_{n^2+1}, y_{n^2+1})$ be the given set of points. Assume without loss of generality that the x-coordinates are in a non-decreasing order, i.e. $x_1 \leq x_2 \leq \cdots \leq x_{n^2+1}$. Now we only need to show that there is a subsequence of $n+1$ points in the sequence $(x_1, y_1), (x_2, y_2), \ldots, (x_{n^2+1}, y_{n^2+1})$ whose y-coordinates are either in non-decreasing order or in non-increasing order. To show this we only need to show that the sequence $y_1, y_2, \ldots, y_{n^2+1}$ contains either a non-increasing subsequence of $n+1$ elements or a non-decreasing sequence of $n+1$ elements. This can be deduced by just applying the Erdos-Szekeres theorem to the sequence. (Note that there being multiple repeating values in this sequence does not affect our conclusion. It is not difficult to modify the proof of the Erdos-Szekeres theorem presented in class so that we can have repeating values in the sequence—only thing is that we should now only ask for non-decreasing or non-increasing subsequence, instead of strictly decreasing or strictly increasing subsequence. Another way to see it is as follows: If the set of points contains many points with the same y-coordinate, you can slightly perturb the points so that no two of them have the same y-coordinate. Now we can use the Erdos-Szekeres theorem to find a strictly increasing or strictly decreasing curve containing at least $n+1$ points. Now if we undo our perturbation and move the points back to their original position, the curve will still be non-increasing or non-decreasing, provided the initial perturbations were not too much.)

5. Let p be a prime number other than 2 or 5. Consider the numbers $p, p^2, p^3, \ldots, p^{101}$. Now consider the remainders when each of these is divided by 100. Since there can be at most 100 different remainders when dividing any number by 100, we have by the pigeonhole principle that some two numbers in $p, p^2, \ldots, p^{101}$ leave the same remainder when divided by 100. Or in other words, we have $i, j \in \{1, 2, \ldots, 101\}$ such that $p^i \equiv p^j \mod 100$. Assuming $i < j$, this means that $p^j - p^i$ is divisible by 100, which means that $p^i(p^{j-i} - 1)$ is divisible by 100. Since p^i and 100 have no common factors other than 1 (notice that the only prime factors of 100 are 2 and 5, and that neither of them are factors of p^i since $p \notin \{2, 5\}$), this means that $p^{j-i} - 1$ is divisible by 100. It follows that the last two digits in the decimal representation of p^{j-i} are 01 (note that $j - i > 0$).

7. For any set $A \in S$, define $f(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ \min A + 1 & \text{otherwise} \end{cases}$. Here $\min A$ denotes the smallest natural number that is an element of A. We claim that f is an injective function from S to $\mathbb{N}$. Suppose for the sake of contradiction that f is not injective. Then there exists $A, B \in S$ such that $A \neq B$, but $f(A) = f(B)$. Then it must be the case that A and B are both nonempty and that $\min A+1 = \min B + 1$, which implies that $\min A = \min B = t$ (say). But then the natural number t is contained in both A and B, contradicting the assumption that any two sets from S are disjoint. Thus f is injective, implying that $|S| \leq |\mathbb{N}|$. So S is countable. (The proof will also work if you define $f(A)$ for any non-empty set A to be an arbitrary element in A so that f is an injective function from $S - \{\emptyset\}$ to $\mathbb{N}$. It is not difficult to see that if $S - \{\emptyset\}$ is countable, then S is also countable.)

10. The least number of colours using which we can colour the set of integers $\{1, 2, \ldots, 2^n - 1\}$ in the required way is n. First, we shall prove that we can colour this set with just n colours. We prove this by induction on n. As the base case, notice that when $n = 1$, we can colour the set $\{1, 2, \ldots, 2^n - 1\} = \{1\}$ with just $n = 1$ colours (this is trivially a valid colouring). Now let us assume that we have proved that the set $\{1, 2, \ldots, 2^{n-1} - 1\}$ can be coloured in the required way using $n - 1$ colours. We need to show that the set $\{1, 2, \ldots, 2^n - 1\}$ can be coloured in the required way using n colours. By the inductive hypothesis, we can colour the set $\{1, 2, \ldots, 2^{n-1} - 1\}$ using $n - 1$ colours so that each consecutive set of integers in that set contains a colour that appears only once. Let us denote the $n - 1$ colours used in this colouring as $c_1, c_2, \ldots, c_{n-1}$. Again by the inductive hypothesis, we can colour the set of integers $\{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1\}$ using the same $n - 1$ colours $c_1, c_2, \ldots, c_{n-1}$ so that every consecutive set of integers in that set also has the property that it contains a colour that appears only once. Now we give a new colour c_n to the integer 2^{n-1}. We now have got a colouring of $\{1, 2, \ldots, 2^n - 1\}$ using n colours. We show that this colouring satisfies the required property. Consider any consecutive set of integers from $\{1, 2, \ldots, 2^n - 1\}$. If this set contains the integer 2^{n-1}, then the colour c_n occurs on exactly one integer in the set, and we are done. If the set does not contain the integer 2^{n-1}, then it is a set of consecutive integers from $\{1, 2, \ldots, 2^{n-1} - 1\}$ or $\{2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1\}$. In either case, we have by the inductive hypothesis that there is a colour that appears only once in the set. This proves that we can always colour the integers in $\{1, 2, \ldots, 2^n - 1\}$ using n colours in such a way that every set of consecutive integers contains an integer whose colour is not given to any other integer.

    Now we have to show that it is not possible to colour the set $\{1, 2, \ldots, 2^n - 1\}$ using less than n colours satisfying our requirements. We again prove this by induction on n. As the base case, it can be easily seen that when $n = 1$, the set $\{1, 2, \ldots, 2^n - 1\} = \{1\}$ cannot be coloured with less than 1 colour. Let us assume inductively that the set $\{1, 2, \ldots, 2^{n-1}\}$ cannot be coloured in the required way using less than $n - 1$ colours. We now show that then the set $\{1, 2, \ldots, 2^n - 1\}$ cannot be coloured in the required way using less than n colours. Suppose for the sake of contradiction that the set $\{1, 2, \ldots, 2^n - 1\}$

has been coloured using less than n colours, satisfying the requirements. Then any set of consecutive integers contains an integer whose colour has not been given to any other integer in that set. So we know that any set of consecutive integers contained in the set $\{1, 2, \ldots, 2^{n-1}-1\}$ also satisfies this property. Then by the inductive hypothesis, we know that at least $n-1$ colours must have been used on the integers in $\{1, 2, \ldots, 2^{n-1}-1\}$. Similarly, by the inductive hypothesis, we can conclude that at least $n-1$ colours must have been used on the integers in $\{2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n-1\}$. Since we have assumed that less than n colours have been used overall, this means that the $n-1$ colours used on $\{1, 2, \ldots, 2^{n-1}-1\}$ are the same as the $n-1$ colours used on $\{2^{n-1}+1, 2^{n-1}+2, \ldots, 2^n-1\}$, and also that the colour given to 2^{n-1} is also a colour from these $n-1$ colours. Now if we consider the set of consecutive integers that is the whole set, i.e. the set $\{1, 2, \ldots, 2^n-1\}$, every colour has appeared twice in this set, meaning that there is no integer whose colour has not been given to any other integer in the set. This contradicts our assumption that the colouring satisfies our requirements.