A Little Linear Algebra

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Let $\vec{p}, \vec{q}, \vec{x}$ be vectors in the vector space V, with $\vec{p} \neq \vec{q}$.

$\vec{p}, \vec{q}, \vec{x}$ are collinear $\leftrightarrow \vec{x} - \vec{p} = t(\vec{q} - \vec{p})$ for some scalar t.
$\leftrightarrow \vec{x} = (1-t)\vec{p} + t\vec{q}$ for some scalar t.

So we can say that the line through $\vec{p}, \vec{q}$ is the set of points $(1-t)\vec{p} + t\vec{q}$.

Defn: $S \subset V$ is <u>affine</u> if and only if S contains all lines joining any two of its points if and only if $\vec{x}, \vec{y} \in S$, scalar t implies that $(1-t)\vec{x} + t\vec{y} \in S$.

Ex: $S_1 = \{(t+1, t, 2t) : t \text{ scalar}\}$
S_1 is the line through $(1,0,0)$ and $(2,1,2)$.
S_1 is an affine set.

Ex: $S_2 = \{(x, y, 3) : x, y \text{ scalar}\}$
S_2 is the plane through $(0,0,3)$ parallel to the x-y plane.
S_2 is an affine set.

Thm: If $\vec{0} \in S$ (and $1+1 \neq 0$, that is, our vector space is not on the field of characteristic 2), then S is affine if and only if S is a vector subspace of V.
Proof: Assume S is a linear subspace, $\vec{x}, \vec{y} \in S$, t scalar. Then $(1-t)\vec{x} + t\vec{y} \in S$, so S is affine. Assume $\vec{0} \in S$, $\vec{y} \in S$, t scalar. Then $t\vec{y} + (1-t)\vec{0} = t\vec{y} \in S$, so S is closed under scalar multiplication. Let $\vec{x}, \vec{y} \in S$. Then $2\vec{x}, 2\vec{y} \in S$. So $(1-t)(2\vec{x}) + t(2\vec{y}) \in S$. Let $t = \frac{1}{2}$, so $\vec{x} + \vec{y} \in S$. So S is closed under addition, and is therefore a vector space. $\square$

Special Case: F is a field of characteristic 2. For example, $F = \{0, 1\} = {}^{\mathbb{Z}}/_{2\mathbb{Z}}$. Then the line through $\vec{p}, \vec{q}$ is just $\{\vec{p}, \vec{q}\}$. So all 2-point sets are lines. And therefore, all subsets of V are affine – the set of affine sets is just $\mathcal{P}(V)$.

Henceforth, for convenience (and our collective sanity), we will assume that $1+1 \neq 0$. We're almost always working with $F = \mathbb{R}$ in 395. In 396, we'll deal a bit with $F = \mathbb{C}$.

Defn: $S - \vec{x} = \{\vec{y} - \vec{x} | \vec{y} \in S\}$

Important Note! For $A, B \subset V$, we say $A \setminus B = \{\vec{a} \in A | \vec{a} \notin B\}$, $A - B = \left\{\vec{a} - \vec{b} \middle| \vec{a} \in A, \vec{b} \in B\right\}$

Ex: If $S \subset V$, $\vec{x} \in V$, then S is affine if and only if $S - \vec{x}$ is affine.

Hence, if $\vec{x} \in S$, then S is affine iff $S - \vec{x}$ is affine iff $S - \vec{x}$ is a linear subspace.

With $S \subset V$, let $\widetilde{S} = \left\{\vec{a} - \vec{b} \middle| \vec{a}, \vec{b} \in S\right\} = S - S$.

Thm: If S is affine, $\vec{x} \in S$, then $S - \vec{x} = \widetilde{S}$.
Proof: $\vec{y} \in S - \vec{x} \to \vec{y} = \vec{a} - \vec{x}$ for some $\vec{a} \in S$, so $\vec{y} \in S$, so $S - \vec{x} \subset \widetilde{S}$.
$\vec{y} \in \widetilde{S} \to \vec{y} = \vec{a} - \vec{b}$ for some $\vec{a}, \vec{b} \in S$, so $\vec{y} = \vec{a} - \vec{b} + \vec{x} - \vec{x} = (\vec{a} - \vec{x}) - (\vec{b} - \vec{x})$. $\vec{a} - \vec{x}, \vec{b} - \vec{x} \in S - \vec{x}$, so $\vec{y} \in S - \vec{x}$, so $\widetilde{S} \subset S - \vec{x}$. $\square$

Corrolary: If S is affine, $\vec{x_1}, \vec{x_2} \in S$, then $S - \vec{x_1} = S - \vec{x_2}$. We say that $\widetilde{S}$ is the unique linear subspace associated to S.

Ex: $S_1 = \{(t+1, t, 2t) : t \in F\}$
$\widetilde{S_1} = \{(t, t, 2t) : t \in F\}$

$S_2 = \{(x, y, 3) : x, y \in F\}$
$\widetilde{S_2} = \{(x, y, 0) : x, y \in F\}$

Defn: If S is affine, the <u>dimension</u> of S, $\dim(S) = \dim(\widetilde{S})$.

Note: If $\vec{a_1}, \ldots, \vec{a_k}$ basis for $\widetilde{S}$, $\widetilde{S} = \{t_1\vec{a_1} + \cdots + t_k\vec{a_k} | t_1, \ldots, t_k \in F\}$, and for some $\vec{x} \in S$,
$S = \{\vec{x} + t_1\vec{a_1} + \cdots + t_k\vec{a_k} | a_1, \ldots, a_k \in F\}$.

Defn: $S \subset V$ is <u>convex</u> if it contains all line segments joining any two of its points. If $\vec{x}, \vec{y} \in S$, then $(1-t)\vec{x} + t\vec{y} \in S$ for all $0 \le t \le 1$.

Ex: $S_3 = \left\{(x, y, 3) | x^2 + y^2 \le 1\right\}$ is convex.

Ex: If $S \subset V$, $\dim V = 1$, S is convex $\leftrightarrow$ S is connected.

Ex: S is convex $\leftrightarrow$ the intersection with each affine line is connected.