Manifolds-Without-Boundary

Thomas Cohn

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Defn: $M \subseteq \mathbb{R}^n$ is a $\underline{C^r\ k\text{-manifold-without-boundary}} \overset{\text{def}}{\Leftrightarrow} U^{\text{open}} \subseteq \mathbb{R}^k$, $p \in V^{\text{rel. open}} \subseteq M$, and $\alpha \in C^r(U,V)$ homeomorphic with rank $D\alpha(\vec{x}) = k$ for all $\vec{x} \in U$ (α is a coordinate patch).

Ex: $M = \{(x,y) : x^2 = y^3\}$. Let $\begin{array}{rcl} \alpha : \mathbb{R} & \to & M \\ t & \mapsto & (t^3, t^2) \end{array}$ homeomorphic, but $\alpha'(0) = (0,0)$. So this fails! We will show M is not a manifold-without-boundary.

Ex: $M = \{(x,y) : y = |x|\}$. Let $\begin{array}{rcl} \alpha : \mathbb{R} & \to & M \\ t & \mapsto & (t^3, t^6) \end{array}$ homeomorphic, but $\alpha'(0) = (0,0)$. So this fails! We will show this M is also not a manifold-without-boundary.

Ex: $M = \{(x,y) : y = x^2\}$. Let $\begin{array}{rcl} \alpha : \mathbb{R} & \to & M \\ t & \mapsto & (t^3, t^6) \end{array}$ homeomorphic, but $\alpha'(0) = (0,0)$. So this fails.

But let $\begin{array}{rcl} \beta : \mathbb{R} & \to & M \\ t & \mapsto & (t, t^2) \end{array}$ homeomorphic, and $\alpha'(0) = (1,0)$. So it is a manifold-without-boundary.

Ex: Interval to a figure 8 symbol (really only makes sense with the picture).

Ex: $M = \{(x,y) : x^2 + y^2 = 1\}$. We can't handle this with one coordinate patch, or else $\exists \alpha^{-1} : M \to U^{\text{osso}\mathbb{R}}$ continuous. M is compact, so α^{-1} can't exist. But we can use 2 trig parameters.

Thm: For $M \subseteq \mathbb{R}^n$, the following are equivalent:

(1) M is a k-mfd-wob

(2) $\forall p \in M$, $\exists V^{\text{rel. open}} \subset M$ with $V \ni p$ and $\exists g \in C^r(\text{osso}\mathbb{R}^k, \mathbb{R}^{n-k})$, $\exists \text{Perm} : \mathbb{R}^n \to \mathbb{R}^n$ coord permutation, such that $v = \text{Perm}(\text{Graph}(g))$.

Proof (2)$\Rightarrow$(1): Take $\alpha : \text{dom}(g) \to V$, $\vec{x} \mapsto \text{Perm}(\vec{x}, g(\vec{x}))$. Then $\alpha^{-1} = (\text{proj onto } \mathbb{R}^k) \circ \text{Perm}^{-1}$.
Proof (1)$\Rightarrow$(2): For $1 < s_1 < \cdots < s_k \leq n$, let $S = \{s_1, \ldots, s_k\}$. Define $P_S : \mathbb{R}^n \to \mathbb{R}^k$ by $\vec{x} \mapsto (x_{s_1}, \ldots, x_{s_k})$. Then $D(P_S \circ \alpha)(q) = DP_S(p) \cdot D\alpha(a) = P_S \cdot D\alpha(p)$. $D\alpha(p)$ has n rows, k columns, and is of rank k, so by the rank condition, choose S such that $D(P_S \circ \alpha)(g)$ invertible. By the inverse function theorem, we can convert U, V, Ω to $\tilde{U}, \tilde{V}, \tilde{\Omega}$ s.t. $P_S \circ \alpha$ is differentiable from $\tilde{U} \to \tilde{\Omega}$.

Let $h = \alpha \circ (P_S \circ \alpha)^{-1}$. Then $P_S \circ h = P_S \circ \alpha \circ (P_S \circ \alpha)^{-1} = \text{Id}$. Choose a permutation such that $\text{Perm}(\{1, \ldots, k\}) = S$. Then $\tilde{V} = \text{Perm}(\text{Graph}(P_S \circ h))$.

Fact 1: $\alpha^{-1} : \tilde{V} \to \tilde{U}$ extends $(P_S \circ \alpha)^{-1} \circ P_S \in C^r(\text{Perm}(\tilde{\Omega} \times \mathbb{R}^{n-k}), \tilde{U})$.
Fact 2: Because of fact 1, $\alpha_2^{-1} \circ \alpha_1$ (where α_2 maps from U_2 to M) is C^r where defined. So $\alpha_2^{-1} \circ \alpha_1$ is a C^r diffeomorphism. Hence, $D(\alpha_2^{-1} \circ \alpha_1)$ is invertible.
Thus, $p \in M$, and M is a k-mfd-wob. $\square$

After “B IG ZOOM”, M looks more like an affine set $p + \tau_p(M)$ (specifically, the tangent space to M at p).

Defn: $\tau_p(M) \stackrel{\text{def}}{=} D\alpha_1(q_1)[\mathbb{R}^k]$ is the tangent space to M at p.

$D\alpha_2(q_2[\mathbb{R}^k]) \stackrel{(2)}{=} D\alpha_2(q_2) \cdot D(\alpha_2^{-1} \circ \alpha_1)(q_1)[\mathbb{R}^k] = D\alpha_1(q_1)[\mathbb{R}^k]$