Some Directional Derivatives

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Recall that X is path-connected if $\forall \alpha, \beta \in X$, $\exists \varphi : [0,1] \to X$ continuous with $\varphi(0) = \alpha$ and $\varphi(1) = \beta$.

Some More Special Cases

(4) Is $\mathrm{GL}\,(n, \mathbb{R})$, the set of invertible $n \times n$ real matrices, connected?

No! Consider $f : \mathrm{GL}\,(n\mathbb{R}) \to \{0, 1\}$, with $M \mapsto \dfrac{\frac{\det M}{|\det M|} + 1}{2}$. f is both continuous and surjective.

(5) Exercise: $\mathrm{GL}_+(n, \mathbb{R}) = \{M \in \mathrm{GL}\,(n, \mathbb{R}) : \det M > 0\}$. Show that $\mathrm{GL}_+(n, \mathbb{R})$ is path-connected.

(6) $X \subseteq \mathbb{R} \leftrightarrow X$ is an interval or $X = \emptyset$ or X is a singleton. (Note that some people consider $\emptyset$ and singletons to be an interval.)

Prop: For X a topological space, we have $f = (f_1, \dots, f_n) : X \to \mathbb{R}^n$ continuous $\leftrightarrow$ each f_i is continuous.
Proof: $\Rightarrow$ Let $f_j = p_j \circ f$ where $p_j : \mathbb{R}^n \to \mathbb{R}$ $(y_1, \dots, y_n) \mapsto y_j$. The composition of two continuous functions is continuous, and p_j is continuous, so f_j is continuous.
$\Leftarrow$ Assume X is a metric space. Fix $x_0 \in X$, $\varepsilon > 0$. There is $\delta_j > 0$ s.t. $|f_j(x) - f_j(x_0)| \leq \frac{\varepsilon}{\sqrt{n}}$ for $1 \leq j \leq n$, when $d(x, x_0) < \delta_j$. Then let $\delta = \min\{\delta_1, \dots, \delta_n\}$.
$d(f(x), f(x_0)) = ||f(x_0) - f(x)|| \leq \sqrt{n}\,|f(x) - f(x_0)|_{\text{sup norm}} \leq \sqrt{n}\frac{\varepsilon}{\sqrt{n}} = \varepsilon$ when $d(x, x_0) < \delta$.
$\square$

This is not true the other way!

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$

$$(x_1, x_2) \mapsto \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

This is a continuous function of x_1 if x_2 is fixed.
This is a continuous function of x_2 if x_1 is fixed.
But f is not continuous on $\mathbb{R}^2$ since $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \to \frac{1}{2} \neq f(0,0)$. So f is not sequentially continuous, and we're in a metric space, so f is not continuous.

Now, we present a potential paradox. Consider $f : \mathbb{R} \to \mathbb{R}$.
We say f is continuous at $a \leftrightarrow$ the graph of f is "almost horizontal" when magnified.
We say f is differentiable at $a \leftrightarrow$ the graph of f is "almost affine" (and not vertical) when magnified.
But continuous at $a \not\to$ differentiable at a.

Try this in a vector space V.
$f : V \to V$, $x \mapsto \lambda x$. This is the dilation centered at $\vec{0}$.
The dilation centered at $\vec{p}$:

$$\begin{array}{ccccccc} V & \to & V & \to & V & \to & V \\ \vec{x} & \mapsto & \vec{x} - \vec{p} & \mapsto & \lambda(\vec{x} - \vec{p}) & \mapsto & \lambda(\vec{x} - \vec{p}) + \vec{p} \end{array}.$$

Put concisely, $\vec{x} \mapsto \lambda\vec{x} + (1 - \lambda)\vec{p}$

Given $f : V \to W$ with V, W vector spaces over $\mathbb{R}$ (or perhaps $\mathbb{C}$), we define $\text{Graph} f = \{(\vec{x}, f(\vec{x}) \in V \times W : \vec{x} \in V\}$.

Dilation about $(\vec{a}, f(\vec{a}))$ is $(\vec{x}, f(\vec{x})) \mapsto (\lambda(\vec{x} - \vec{a}) + \vec{a}, \lambda(f(\vec{x}) - f(\vec{a})) + f(\vec{a}))$.

Set $t = \frac{1}{\lambda}$, $\vec{u} = \frac{\vec{x}-\vec{a}}{t}$. So $\vec{x} = \vec{a} + t\vec{u}$. So the dilated grpah now looks like
$\left\{ \left(\vec{a} + \vec{u}, f(\vec{a}) \dfrac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} \right) : \vec{u} \in V \right\}$
With $t \to 0$ (i.e. $\lambda \to \infty$), we want $f(\vec{a}) \frac{f(\vec{a}+t\vec{u})-f(\vec{u})}{t}$ to be an affine function of $\vec{a} + \vec{u}$.
i.e. a linear function of $(\vec{a} + \vec{u})$ plus a constant.
i.e. a linear function of $\vec{u}$ plus some other constant.
i.e. $T(\vec{u}) + \vec{b}$ with $\vec{b} = f(\vec{a})$.

Defn: This reduces to $\displaystyle\lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t} = f'(\vec{a}; \vec{u})$, the directional derivative of f at $\vec{a}$ in direction $\vec{u}$.

We could try to make this theorem the core definition for multivariable differential calculus, but we won't!
(1) Munkres §5 EX 2: all $f'(\vec{a}; \vec{u})$ with fixed $\vec{a}$ exist but not linear in $\vec{u}$.
(2) $f(x, y) = \begin{cases} x^3/y & y \neq 0 \\ 0 & y = 0 \end{cases}$ $f'(\vec{0}; \vec{v}) = \vec{0}\ \forall \vec{u}$. But $f(\frac{1}{n}, \frac{1}{n}u) \not\to 0$ as $n \to \infty$. So f is not continuous.
Does that mean differentiability $\not\to$ continuity? No, we just need a stronger definition of differentiable.
(3) The Chain Rule (Munkres §7) will fail without a stronger assumption.

Something easier: vector-valued functions of a scalar.
$f : I \subset \mathbb{R} \to W$ where I is an open interval and W is a vector space.
$f'(x) = \lim_{t \to \infty} \frac{f(x+t)-f(x)}{t}$, but we need a topology on W.
Choose W to be a normed vector space, thus giving us a topology.

Fact: $\dim W < \infty \to$ all norms on W induce the same topology.