

# Riemann Integrability (Continued)

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(Continued from Wednesday)

Proof 5  $\Rightarrow$  2: Given  $\varepsilon > 0$ , let  $\tilde{\varepsilon} = \frac{\varepsilon}{2(M + V(Q))}$ . Write  $\mathcal{D} \subset \bigcup_{p=1}^{\infty} \text{rInt}(R_p^{\text{box}})$  with  $\sum_{p=1}^{\infty} V(R_p) < \tilde{\varepsilon}$ . For  $\vec{a} \in Q \setminus \mathcal{D}$ ,  $\exists Q_{\vec{a}}^{\text{box}}$  with  $\vec{a} \in \text{rInt}(Q_{\vec{a}})$  and  $|f(\vec{x}) - f(\vec{a})| < \tilde{\varepsilon}$  for  $\vec{x} \in Q_{\vec{a}}$ .

$$Q = \left( \bigcup_{p=1}^{\infty} \text{rInt}(R_p) \right) \cup \left( \bigcup_{\vec{a} \in Q \setminus \mathcal{D}} \text{rInt}(Q_{\vec{a}}) \right)$$

$Q$  is compact. So pick a finite subcover:

$$Q \subset \text{rInt}(R_{p_1}) \cup \cdots \cup \text{rInt}(R_{p_s}) \cup \text{rInt}(Q_{\vec{a}_1}) \cup \cdots \cup \text{rInt}(Q_{\vec{a}_u})$$

Pick partition  $P$  such that each box  $R$  defined by  $P$  satisfying  $R \subset Q_{\vec{a}_j}$  for some  $\vec{a}_j$  “good” or  $R \subset R_{p_k}$  “bad”. If both hold, we make a choice. Then

$$U(f, P) - L(f, P) = \sum_{R \text{ det'd by } P} \left( \sup_R f - \inf_R f \right) v(R) = \sum_{R \text{ "good"}} + \sum_{R \text{ "bad"}} \leq \sum_{R \text{ "good"}} 2\tilde{\varepsilon}V(R) + \sum_{R \text{ "bad"}} 2MV(R)$$

Using a previous lemma, this is less than  $2\tilde{\varepsilon}V(Q) + 2M\tilde{\varepsilon} < \varepsilon$ .  $\square$

Given  $B_1 \cup \cdots \cup B_j \subset X_1 \cup \cdots \cup X_k$ , and the interiors of the  $B_i$ s are disjoint, we have  $V(B_1) + \cdots + V(B_j) \leq V(X_1) + \cdots + V(X_k)$ .

Proof: TODO