Change of Variable Theorem

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11/12/18

Thm: (Change of Variables Theorem) Given $A^{\text{osso}\mathbb{R}^n} \xrightarrow[C^1\text{-diffeo}]{g} B^{\text{osso}\mathbb{R}^n} \xrightarrow[\text{cts}]{f} \mathbb{R}$, then (ext) $\int_B f \overset{\text{def}}{=}$ (ext) $\int_A (f \circ g)\,|\det Dg|$.

Special Case: $g : \vec{x} \mapsto r\vec{x}$ for $r > 0$. Then (ext) $\int_{\vec{x}\in B} f(\vec{x}) =$ (ext) $\int_{\vec{x}\in A} f(r\vec{x})r^n$.

Ex: Let $B^n(r) = \{\vec{x} \in \mathbb{R}^n : ||\vec{x}|| < r\} = U(\vec{0}, r)$.
Then $v(B^n(r)) = r^n v(B^n(1)) = r^n \lambda_n$.

Goal: compute λ_n. Guess the behavior of $\frac{v(B^n(1)}{v([-1,1]^n)}$ for large n.

$$\lambda_n = \int\limits_{B^n(1)} 1 = \int\limits_{\vec{x}\in B^k(1)} \left(\int\limits_{\vec{y}\in B^{n-k}(\sqrt{1-||\vec{x}||^2})} 1 \right) = \int\limits_{\vec{x}\in B^k(1)} \left(1 - ||\vec{x}||^2\right)^{\frac{n-k}{2}} \lambda_{n-k}$$

This formula becomes the nicest for $k = 2$. So

$$\lambda_n = \lambda_{n-2} \int\limits_{B^2(1)} \left(1 - ||\vec{x}||^2\right)^{\frac{n}{2}-1} = \lambda_{n-2} \int\limits_{B^2(1)\setminus((-1,0]\times\{0\})} \left(1 - ||\vec{x}||^2\right)^{\frac{n}{2}-1} + \underbrace{\lambda_{n-2} \int\limits_{(-1,0]\times\{0\}} \left(1 - ||\vec{x}||^2\right)^{\frac{n}{2}-1}}_{=0,\text{ because its over a set of measure 0}}$$

Thus,

$$\lambda_n = \lambda_{n-2} \int\limits_{B^2(1)\setminus((-1,0]\times\{0\})} \left(1 - ||\vec{x}||^2\right)^{\frac{n}{2}-1} = \lambda_{n-2} \int\limits_{\substack{0<r<1 \\ -\pi<\theta<\pi}} \left(1 - r^2\right)^{\frac{n}{2}-1} r$$

$$= \lambda_{n-2} \int\limits_0^1 \int\limits_{-\pi}^{\pi} \left(1 - r^2\right)^{\frac{n}{2}-1} r d\theta dr$$

$$= \lambda_{n-2} \int\limits_0^1 2\pi r \left(1 - r^2\right)^{\frac{n}{2}-1} dr \qquad \begin{matrix} u = 1 - r^2 \\ du = -2rdr \end{matrix}$$

$$= \lambda_{n-2} \int\limits_1^0 \pi u^{\frac{n}{2}-1} du$$

$$= \pi\lambda n - 2 \int\limits_0^1 u^{\frac{n}{2}-1} du$$

$$= \pi\lambda_{n-2} \left[\frac{u^{\frac{n}{2}}}{\frac{n}{2}}\right]_{u=0}^{u=1}$$

$$= \pi\lambda_{n-2} \cdot \frac{1}{\frac{n}{2}}$$

$$= \frac{2\pi\lambda_{n-2}}{n}$$

$\lambda_2 \stackrel{\text{def}}{=} \pi$: the area of the unit circle. So $\lambda_4 = \frac{\pi^2}{2}$, $\lambda_6 = \frac{\pi^3}{6}$, $\lambda_{2n} = \frac{\pi^n}{n!}$.
$\lambda_1 \stackrel{\text{def}}{=} 2$: the length of the unit interval $(-1, 1)$. So $\lambda_3 = \frac{4\pi}{3}$, $\lambda_5 = \frac{8\pi^2}{15}$, $\lambda_{2n+1} = \frac{2^{n+1}\pi^n}{(2n+1)(2n-1)\cdots 3\cdot 1}$

So $\lambda_n = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & 2 \mid n \\ \frac{2^{\frac{n+1}{2}}\pi^{\frac{n-1}{2}}}{n\cdot(n-2)\cdot\cdots\cdot 3\cdot 1} & 2 \nmid n \end{cases}$

$\lambda_n = v(B^n(1))$
$\mu_n = v([-1, 1]^n) = v\left(\left\{\vec{x} \in \mathbb{R}^n : ||\vec{x}||_{\sup} < 1\right\}\right) = 2^n$
$K_n = v\left(\{\vec{x} \in \mathbb{R}^n : |x_1| + |x_2| + \cdots + |x_n| < 1\}\right) = \frac{2^n}{n!}$. Note that $|x_1| + \cdots + |x_n| \stackrel{\text{def}}{=} ||\vec{x}||_1$.

$K_n \leq \lambda_n \leq \mu_n$. Check that $\frac{\lambda_n}{\mu_n} \to 0$, and that strict inequalities for $n > 1$.

Proof the Change of Variables Theorem:
(With the temporarily added assumptions A, B bounded and rectifiable, g a diffeomorphism from a neighborhood of $\overline{A}$ to a neighborhood of $\overline{B}$)

Special case 1: $g : \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}$ coordinate transposition. $\deg Dg = -1$, so $|\det Dg| = 1$.

Thus, $\int\limits_B f = \int\limits_A f \circ g$. This works because reorienting boxes doesn't change their volume.

Special case 2: $E^{\text{bdd,open,rect}} \subset \mathbb{R}^{n+1}$, $\varphi, \psi \in C(\overline{E}, \mathbb{R})$, $\varphi < \psi$ on E,

$$B = \left\{ \vec{x} \in \mathbb{R}^n : \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \varphi\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} < x_n < \psi\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\}, \text{ and } g^{\text{diffeo}} : \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \alpha(\vec{x}) \end{pmatrix}.$$

Then

(a) A and B are rectifiable (by study ex 5, HW9, or lemma 14.3)

(b) For fixed $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, we have $\vec{x} \in A \Leftrightarrow x_n \in I$ where I is an interval determined by $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$.

(c) $\det Dg = D_n\alpha$, so $|\deg Dg| = |\det D_n\alpha|$.

Thus

$$\int\limits_B f = \int\limits_{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in E} \left(\int\limits_{x_n \in \left(\varphi\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \psi\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}\right)} f \right) = \int\limits_{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in E} \left(\int\limits_{\alpha(\vec{x}) \in (\cdots, \cdots)} (f \circ g) \cdot |D_n\alpha| \right) = \int\limits_A (f \circ g) \det Dg$$

Prop: Given $A \xrightarrow[\text{diffeo}]{g} B \xrightarrow[\text{diffeo}]{h} C \xrightarrow[\text{cts}]{f} \mathbb{R}$, Then the COVT holds for g and h.

Proof:

ext $\int_C f =$ ext $\int_B f \circ h\, |\det Dh| =$ ext $\int_A ((f \circ h) \circ g)\, |\det Dh| \cdot g\, |\det Dg| =$ ext $\int_A f \circ h \circ g\, |\det D(h \circ g)|$

$\square$

Strategy: Factor general diffeomorphic maps into composition of maps of types (1) and (2).
Good news: we can do this!
Bad news: good news is only local.