

Change of Variable Theorem

Professor David Barrett

Transcribed by Thomas Cohn

11/12/18

Thm: (Change of Variables Theorem) Given $A \xrightarrow[C^1\text{-diffeo}]{g} B \xrightarrow[\text{cts}]{f} \mathbb{R}$, then $(\text{ext}) \int_B f \stackrel{\text{def}}{=} (\text{ext}) \int_A (f \circ g) |\det Dg|$.

Special Case: $g : \vec{x} \mapsto r\vec{x}$ for $r > 0$. Then $(\text{ext}) \int_{\vec{x} \in B} f(\vec{x}) = (\text{ext}) \int_{\vec{x} \in A} f(r\vec{x}) r^n$.

Ex: Let $B^n(r) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < r\} = U(\vec{0}, r)$.
Then $v(B^n(r)) = r^n v(B^n(1)) = r^n \lambda_n$.

Goal: compute λ_n . Guess the behavior of $\frac{v(B^n(1))}{v([-1,1]^n)}$ for large n .

$$\lambda_n = \int_{B^n(1)} 1 = \int_{\vec{x} \in B^k(1)} \left(\int_{\vec{y} \in B^{n-k}(\sqrt{1-\|\vec{x}\|^2})} 1 \right) = \int_{\vec{x} \in B^k(1)} \left(1 - \|\vec{x}\|^2\right)^{\frac{n-k}{2}} \lambda_{n-k}$$

This formula becomes the nicest for $k = 2$. So

$$\lambda_n = \lambda_{n-2} \int_{B^2(1)} \left(1 - \|\vec{x}\|^2\right)^{\frac{n}{2}-1} = \lambda_{n-2} \int_{B^2(1) \setminus ((-1,0] \times \{0\})} \left(1 - \|\vec{x}\|^2\right)^{\frac{n}{2}-1} + \underbrace{\lambda_{n-2} \int_{(-1,0] \times \{0\}} \left(1 - \|\vec{x}\|^2\right)^{\frac{n}{2}-1}}_{=0, \text{ because its over a set of measure 0}}$$

Thus,

$$\begin{aligned}
\lambda_n &= \lambda_{n-2} \int_{B^2(1) \setminus ((-1,0] \times \{0\})} \left(1 - \|\vec{x}\|^2\right)^{\frac{n}{2}-1} = \lambda_{n-2} \int_{\substack{0 < r < 1 \\ -\pi < \theta < \pi}} (1 - r^2)^{\frac{n}{2}-1} r \\
&= \lambda_{n-2} \int_0^1 \int_{-\pi}^{\pi} (1 - r^2)^{\frac{n}{2}-1} r d\theta dr \\
&= \lambda_{n-2} \int_0^1 2\pi r (1 - r^2)^{\frac{n}{2}-1} dr \quad \begin{array}{l} u = 1 - r^2 \\ du = -2r dr \end{array} \\
&= \lambda_{n-2} \int_1^0 \pi u^{\frac{n}{2}-1} du \\
&= \pi \lambda_{n-2} - 2 \int_0^1 u^{\frac{n}{2}-1} du \\
&= \pi \lambda_{n-2} \left[\frac{u^{\frac{n}{2}}}{\frac{n}{2}} \right]_{u=0}^{u=1} \\
&= \pi \lambda_{n-2} \cdot \frac{1}{\frac{n}{2}} \\
&= \frac{2\pi \lambda_{n-2}}{n}
\end{aligned}$$

$\lambda_2 \stackrel{\text{def}}{=} \pi$: the area of the unit circle. So $\lambda_4 = \frac{\pi^2}{2}$, $\lambda_6 = \frac{\pi^3}{6}$, $\lambda_{2n} = \frac{\pi^n}{n!}$.

$\lambda_1 \stackrel{\text{def}}{=} 2$: the length of the unit interval $(-1, 1)$. So $\lambda_3 = \frac{4\pi}{3}$, $\lambda_5 = \frac{8\pi^2}{15}$, $\lambda_{2n+1} = \frac{2^{n+1} \pi^n}{(2n+1)(2n-1)\cdots 3 \cdot 1}$

$$\text{So } \lambda_n = \begin{cases} \frac{\pi^{n/2}}{(n/2)!} & 2 \mid n \\ \frac{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}}}{n \cdot (n-2) \cdots 3 \cdot 1} & 2 \nmid n \end{cases}$$

$$\lambda_n = v(B^n(1))$$

$$\mu_n = v([-1, 1]^n) = v\left(\left\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\|_{\text{sup}} < 1\right\}\right) = 2^n$$

$$K_n = v(\{\vec{x} \in \mathbb{R}^n : |x_1| + |x_2| + \cdots + |x_n| < 1\}) = \frac{2^n}{n!}. \text{ Note that } |x_1| + \cdots + |x_n| \stackrel{\text{def}}{=} \|\vec{x}\|_1.$$

$K_n \leq \lambda_n \leq \mu_n$. Check that $\frac{\lambda_n}{\mu_n} \rightarrow 0$, and that strict inequalities for $n > 1$.

Proof the Change of Variables Theorem:

(With the temporarily added assumptions A, B bounded and rectifiable, g a diffeomorphism from a neighborhood of \bar{A} to a neighborhood of \bar{B})

$$\text{Special case 1: } g : \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} \text{ coordinate transposition. } \deg Dg = -1, \text{ so } |\det Dg| = 1.$$

Thus, $\int_B f = \int_A f \circ g$. This works because reorienting boxes doesn't change their volume.

Special case 2: $E^{\text{bdd,open,rect}} \subset \mathbb{R}^{n+1}$, $\varphi, \psi \in C(\overline{E}, \mathbb{R})$, $\varphi < \psi$ on E ,

$$B = \left\{ \vec{x} \in \mathbb{R}^n : \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \varphi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} < x_n < \psi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \right\}, \text{ and } g^{\text{diffeo}} : \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \alpha(\vec{x}) \end{pmatrix}.$$

Then

(a) A and B are rectifiable (by study ex 5, HW9, or lemma 14.3)

(b) For fixed $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, we have $\vec{x} \in A \Leftrightarrow x_n \in I$ where I is an interval determined by $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$.

(c) $\det Dg = D_n \alpha$, so $|\deg Dg| = |\det D_n \alpha|$.

Thus

$$\int_B f = \int_{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in E} \left(\int_{x_n \in \left(\varphi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \psi \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \right)} f \right) = \int_{\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} \in E} \left(\int_{\alpha(\vec{x}) \in (\dots, \dots)} (f \circ g) \cdot |D_n \alpha| \right) = \int_A (f \circ g) \det Dg$$

Prop: Given $A \xrightarrow[\text{diffeo}]{g} B \xrightarrow[\text{diffeo}]{h} C \xrightarrow[\text{cts}]{f} \mathbb{R}$, Then the COVT holds for g and h .

Proof:

$$\text{ext} \int_C f = \text{ext} \int_B f \circ h |\det Dh| = \text{ext} \int_A ((f \circ h) \circ g) |\det Dh| \cdot g |\det Dg| = \text{ext} \int_A f \circ h \circ g |\det D(h \circ g)|$$

□

Strategy: Factor general diffeomorphic maps into composition of maps of types (1) and (2).

Good news: we can do this!

Bad news: good news is only local.