Partitions of Unity and Proving the Change of Variables Theorem

Thomas Cohn

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Recall
Type (1) diffeomorphisms: coordinate transposition

Type (2) diffeomorphisms: $\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ \alpha(\vec{x}) \end{pmatrix}$

We can combine these to obtain type (3) diffeomorphisms: "generalized shears"

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{j-1} \\ x_j \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{j-1} \\ \eta(\vec{x}) \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix}$$

Prop: Any invertible affine map may be factored into a composition of affine maps of type (1) or (2) (equivalently, type (1) or (3)).
Proof: For linear maps, use "elementary matrix factorization" (Thm 2.4)
For translations, move coord at a time.
$\square$

Fun fact: type (1) is just the composition of three type (3) maps.

Ex: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Thm: In some neighborhood of $\vec{p}$, g can be factored into a composition of diffeomorphisms of type (1) or (2) (equivalently (1) or (3)).

Step 1: Pick $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ invertible, affine such that $g = T_1 \circ \tilde{g} \circ T_2$ with $T_2(\vec{p}) = \vec{0}$ and $T_1(\vec{0}) = \vec{q}$, $D\tilde{g}(\vec{0}) = \text{Id}$. So $\tilde{g}(\vec{0}) = \vec{0}$, and we can take $DT_2 = Dg(\vec{p})$ and $DT_1 = \text{Id}$.

$$\begin{array}{lcccl}
 & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} & \mapsto & \begin{pmatrix} \tilde{g}_1(\vec{x}) \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} & \text{So } D\tilde{g}_k(\vec{0}) = \vec{e_k}\text{, so the derivative at } \vec{0} \text{ is Id, so it's locally diffeomorphic.} \\
 & & & \downarrow & \text{local diffeomorphism*} \\
\text{Step 2:} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} & \mapsto & \begin{pmatrix} \tilde{g}_1(\vec{x}) \\ \tilde{g}_2(\vec{x}) \\ x_3 \\ \vdots \\ x_n \end{pmatrix} & \text{Also has derivative at } \vec{0} \text{ is Id, so locally diffeomorphic.} \\
 & & & \downarrow & \text{local diffeomorphism*} \\
 & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} & \mapsto & \begin{pmatrix} \tilde{g}_1(\vec{x}) \\ \tilde{g}_2(\vec{x}) \\ \tilde{g}_3(\vec{x}) \\ \vdots \\ x_n \end{pmatrix} & \text{''} \\
 & & & \downarrow & \text{local diffeomorphism*} \\
 & & & \downarrow & \text{local diffeomorphism*}
\end{array}$$

So $\vec{x} \mapsto \tilde{g}(\vec{x})$ has derivative Id at $\vec{0}$, so it's locally diffeomorphic.

* These diffeomorphisms preserve $n-1$ coordinates, so they're type (3).

And T_1, T_2 are type (1) diffeomorphisms.
$\square$

Defn: Consider $f : X^{\text{metric space}} \to V^{\text{vector space}}$. $\operatorname{supp} f \overset{\text{def}}{=} \overline{\left\{\vec{x} : f(\vec{x}) \neq \vec{0}\right\}}$.
So $\vec{x} \notin \operatorname{supp} f \Leftrightarrow \exists \varepsilon > 0$ s.t. $f \equiv \vec{0}$ on $U(\vec{x}, \varepsilon)$.

Cor: (of factorization and results from Monday):
There exists a neighborhood U of $\vec{q}$ such that the COVT holds when $\operatorname{supp} f \subset U$.
Proof: Picture. $\square$

We now have a local version of the COVT!

Thm: (Partition of Unity) Given $\Omega^{\text{open}} \subset \mathbb{R}^n$ with $\Omega = \bigcup_{\alpha \in \Gamma} U_\alpha^{\text{open}}$, then
$\exists \varphi_1, \varphi_2, \ldots \in C^\infty(\Omega, [0, +\infty))$ s.t.

(i) each $\operatorname{supp} \varphi_j \subset$ some U_{α_j}

(ii) each $\operatorname{supp} \varphi_j$ compact

(iii) each $\vec{x} \in \Omega$ has a nbd meeting (i.e., non-empty intersection) only finitely many $\operatorname{supp} \varphi_j$

(iv) $\sum_{j=1}^\infty \varphi_j(\vec{x}) = 1$ for al $\vec{x} \in \Omega$ (locally finite sum)

Then $\{\varphi_j\}$ is a "partition of unity dominated by $\{U_\alpha\}$".
Proof: Nov 21, or read §16.

Lemma: Given $f \in C(B^{\text{osso}\mathbb{R}^n}, \mathbb{R})$, ext $\int_B f$ exists, $\{\varphi_j\}$ satisfies (ii), (iii), (iv).

$$\text{Then ext} \int\limits_B f = \sum_{j=1}^\infty \int\limits_B \varphi_j \cdot f.$$

Proof: strategy is tackle $f \geq 0$, then apply previous result to f_+, f_-, and combine.

Assume $f \geq 0$. Then $E^{\text{cpt,rect}} \subset B \Rightarrow \exists M$ s.t. $U_j \equiv 0$ on E for $j \geq M$. This is compact using (iii).

Thus, $\int\limits_E f = \int\limits_E \sum\limits_{i=1}^M \varphi_j \cdot f = \sum\limits_{i=1}^M \int\limits_E \varphi_j \cdot f \leq \sum\limits_{j=1}^M \int\limits_B \varphi_j \cdot f \leq \sum\limits_{j=1}^\infty \int\limits_B \varphi_j \cdot f.$

Take the supremum over E, obtain ext $\int\limits_E f \leq \sum\limits_{j=1}^\infty \int\limits_B \varphi_j \cdot f$

Also, $\sum\limits_{j=1}^\infty \int\limits_B \varphi_j \cdot f = \lim\limits_{M\to\infty} \sum\limits_{j=1}^M \int\limits_B \varphi_j \cdot f = \lim\limits_{M\to\infty} \int\limits_B \sum\limits_{j=1}^M \varphi_j \cdot f \leq \text{ext} \int\limits_B f.$

Apply to f_+, f_-.

Combine. □

Proof of the Change of Variables Theorem: For $\vec{y} \in B$, choose $U_{\vec{y}}$ s.t. g factors on $U_{\vec{y}}$. Choose partition of unity dominated by $\{U_{\vec{y}} : \vec{y} \in B\}$. Then

$$\text{ext} \int\limits_B f_+ \overset{\text{lemma}}{=} \sum \int\limits_B \varphi_j f_+ = \sum \int\limits_A (\varphi_1 \circ g)(f_+ \circ g)\,|\det Dg| \overset{\text{lemma}}{=} \text{ext} \int\limits_A (f_+ \circ g) \cdot |\det Dg|$$

Similarly, ext $\int\limits_A f_- = \text{ext} \int\limits_A (f_- \circ g) \cdot |\det Dg|$.

Now combine (unles both terms are infinite). □