Bolzano-Weierstrass

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Thm: (Bolzano-Weierstrass) (X, d) compact $\leftrightarrow$ X sequentially compact – i.e. every sequence admits a convergent subsequence.

Proof: $\Rightarrow$ Suppose to the contrary, we have (v_j) in X with no convergent subsequence. Then for every $y \in X$, $\exists U_y$ open with $y \in U_y$ such that U_y only contains finitely many v_j.
So $X = \bigcup_{y \in X} U_y = \bigcup_{i<k} U_{y_i}$. But that only contains finitely many v_j. Oops!

$\Leftarrow$ Lemma: If X is a sequentially compact metric space, then $\forall \epsilon > 0$ we can cover X by finitely many ϵ-balls. We then call X "totally bounded".
Proof: suppose not. Pick $v_1 \in X$

$$\begin{aligned} &v_2 \notin B_\epsilon(v_1) \\ &\quad\vdots \\ &v_n \notin B_\epsilon(v_{n-1}) \cup \cdots \cup B_\epsilon(v_1). \text{ Oops. } \square \end{aligned}$$

Suppose $X = \bigcup_{\alpha \in A} X_\alpha$ with each X_α open. Call $S \subset X$ "good" if S contained in a finite union of X_α. Otherwise, call S "bad". Note that the finite union of "good" sets is "good". Our goal is to show that X is good. Suppose, to the contrary, that X is bad. Lemma + note $\rightarrow$ $\forall m \in \mathbb{N}$, $\exists v_m \in X$ s.t. $B_{1/m}(v_m)$ is bdd.
Choose $v_{m_k} \to v \in X$ by hypothesis. Choose α s.t. $v \in X_\alpha$. Then $\exists \epsilon > 0$ s.t. $B_\epsilon(v) \subset X_\alpha$. Pick $k \in \mathbb{N}$ s.t. $m_k > 2/\epsilon$ and $d(v, v_{m_k}) < \epsilon/2$. But then $B_{1/m_k}(v_{m_k})$ is good. Oops! $\square$

This proves the Bolzano-Weierstrass Theorem. $\square$

It is also true that (X, d) is compact if and only if (X, d) is totally bounded and complete. We will receive a handout on this.

Let $f : X \to Y$ with X, Y metric spaces with metrics d_X and d_Y, respectively.

Defn: We say f is <u>Lipschitz</u> if $\exists C \in \mathbb{R}_{>0}$ s.t. $d_Y(f(x_1), f(x_2)) \le C d_X(x_1, x_2)$, implying continuity.

Ex: The inf of Lipschitz constants of f is a Lipschitz constant of f.

f is uniformly continuous means we can choose δ independently of x_0.

Prop: f is Lipschitz $\to$ f is uniformly continuous.

Defn: f is a <u>contraction</u> if f is a Lipschitz mapping with $C < 1$.

Defn: A <u>bi-Lipschitz</u> map $f : X \to Y$ is a bijection s.t. f, f^{-1} are Lipschitz.

Ex: $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x + \frac{\sin x}{37}$ is bi-Lipschitz.

Defn: f is said to be an isometry iff $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$.

Defn: Suppose X is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and is also a metric space with metric d. d is said to be translation invariant $\leftrightarrow d(x_1, x_2) = d(x_1 + y, x_2 + y)$.

Ex: Show that d is translation-invariant iff $d(x_1, x_2) = d(0, x_2 - x_1)$.

Defn: d is homogenous $\leftrightarrow d(tx_1, tx_2) = |t|\, d(x_1, x_2)$.

Defn: If d is both translation-invariant and homogenous, then d defines a norm.