Fubini's Theorem

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Recall that $\underline{\int_Q} f \leq \overline{\int_Q} f$. We will refer to this result as $(*)$

According to §10#1, $f \leq g$ on Q^{box} means $\underline{\int_Q} f \overset{(a)}{\leq} \underline{\int_Q} g$ and $\overline{\int_Q} f \overset{(b)}{\leq} \overline{\int_Q} g$.

Proof (a): $f \leq g$ on Q. Then $L(f,P) \leq L(g,P) \leq \underline{\int_Q} g$. Now take the supremum over P. $\square$

Thm: (Fubini's Theorem) Given $A^{\text{box}} \subseteq \mathbb{R}^k$, $B^{\text{box}} \subseteq \mathbb{R}^n$, $Q = A \times B$, $f : Q \to \mathbb{R}$ bounded, then

$$\underline{\int_Q} f \overset{(1)}{\leq} \underline{\int}_{\vec{x}\in A} \underline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y}) \underset{(2\beta)}{\overset{(2\alpha)}{\leq}} \left\{ \begin{array}{c} \underline{\int}_{\vec{x}\in A} \overline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y}) \\ \overline{\int}_{\vec{x}\in A} \underline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y}) \end{array} \right\} \underset{(3\beta)}{\overset{(3\alpha)}{\leq}} \overline{\int}_{\vec{x}\in A} \overline{\int}_{\vec{y}\in B} f(\vec{x},\vec{y}) \overset{(4)}{\leq} \overline{\int_Q} f$$

Cor: f integrable on Q implies that all of these are equal and $\int_Q f$ is defined to be equal to all of these terms. Note that $\int_{\vec{y}\in B} f(\vec{x},\vec{y})$ may not exist.

Cor: f integrable on $Q = I_1 \times \cdots \times I_n$ for intervals I_j implies that $\int_Q f = \underline{\int}_{x_1\in I_1} \cdots \overline{\int}_{x_n\in I_n} f(x_1,\ldots,x_n)$. These can be upper or lower integrals, but the first must exist.

Ex: $Q = [-1,1] \times [-1,1]$
$f = \mathbb{1}_{\{0\}\times\mathbb{Q}}$
$\mathcal{D}_f$ is the set of points where f is discontinuous (so $\mathcal{D}_f = \{0\} \times [-1,1]$).

Then we proved on Friday that f is integrable on Q.

$$\overline{\int}_{y\in[-1,1]} f(x,y) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$$\underline{\int}_{y\in[-1,1]} f(x,y) = \begin{cases} 0 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\int_Q f = 0.$$

Proof of Fubini's Theorem:
$(*)(a) \Rightarrow (2\alpha)$
$(*) \Rightarrow (2\beta), (3\alpha)$
$(*)(b) \Rightarrow (3\beta)$

(4) follows from (1) using $\overline{\int} f = -\underline{\int}(-f)$

We still need to prove (1).

Proof (1): Partitions of Q correspond with partitions of A, B.

$$\underline{\int}_{\vec{x} \in B} f(\vec{x_0}, \vec{y}) \geq \sum_{R_B} \inf_{\vec{y} \in R_B} (f(\vec{x_0}, \vec{y}) \cdot V(R_B)) \geq \sum_{R_B} \inf_{R_A \times R_B} (f) \cdot V(R_B)$$

$$\inf_{x_\phi \vec{\in} R_A} \left(\underline{\int}_{\vec{y} \in B} f(\vec{x_\phi}, \vec{y}) \right) \geq \sum_{R_B} \inf_{R_A \times R_B} (f) \cdot V(R_B)$$

$$\sum_{R_A} \left(\inf_{\vec{x} \in R_A} \left(\underline{\int}_{\vec{y} \in B} f(\vec{x}, \vec{y}) \right) \right) V(R_A) \geq \sum_{R_A} \sum_{R_B} \inf_{R_A \times R_B} f \cdot V(R_A \times R_B) = L(f, P)$$

$$L(f, P) \leq \underline{\int}_{\vec{x} \in A} \underline{\int}_{\vec{y} \in B} f(\vec{x}, \vec{y})$$

Take the supremum over the possible choices of P, and then (1) follows.

Defn: For $I \subseteq \mathbb{R}^n$, set $m^*(E) \stackrel{\text{def}}{=} \inf \left\{ \sum_{j=1}^\infty V(Q_j) : E \subset \bigcup_{j=1}^\infty Q_j^{\text{box}} \right\}$.
$m^*(E)$ is called the outer Lebesgue measure of E.

For E bounded, we also set $m^{*,J}(E) \stackrel{\text{def}}{=} \inf \left\{ \sum_{j=1}^k V(Q_j) : E \subset \bigcup_{j=1}^k Q_j^{\text{box}} \right\}$.
$m^{*,J}(E)$ is called the outer Jordan measure of E.

Note that $m^*(E) \leq m^{*,J}(E)$.

Prop: $m^*(E_j) = 0$ for $j = 1, 2, \ldots \Rightarrow m^*(\bigcup_{j=1}^\infty E_j) = 0$.
Proof: (3) $\to$ (4). Last Wednesday/Thm 11.1 (b)

Similar for $m^{*,J}$:
Prop: $m^{*,J}(E_j) = 0$ for $j = 1, 2, \ldots, n \Rightarrow m^{*,J}(\bigcup_{j=1}^n E_j) = 0$.

Lemma: $m^*(E) = \inf \left\{ \sum_{j=1}^\infty V(Q_j) : E \subset \bigcup_{j=1}^\infty \text{rInt}\, Q_j^{\text{box}} \right\}$
If E bounded, then $m^{*,J}(E) = \inf \left\{ \sum_{j=1}^k V(Q_j) : E \subset \bigcup_{j=1}^k \text{rInt}\, Q_j^{\text{box}} \right\}$
Proof: Suppose we have Q_js covering E, $\varepsilon > 0$.
Pick $\widetilde{Q_j} \supset \text{rInt}\, \widetilde{Q_j} \supset Q_j$, $V(\widetilde{Q_j}) < v(Q_j + \frac{\varepsilon}{2j}$.
Get rInt $(\widetilde{Q_j})$ covering E with $\sum V_{\widetilde{Q_j}} < \sum(V(Q_j)) + \varepsilon$. $\square$