Extended Riemann Integrals

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Recall: $f \in C(A^{\text{osso}\mathbb{R}^n}, \mathbb{R})$, $f \geq 0$
$\text{ext} \int_A f \stackrel{\text{def}}{=} \sup \left\{ \int_E f : E^{\text{cpt,rect}} \subset A \right\}$
$\text{ext} \int_A f =$ "ordinary" $\int_A f$ if $\int_A f$ exists
$\text{ext} \int_A f = \lim_{j\to\infty} \int_{E_j} f$ if $E_j^{\text{cpt,rect}} \subset A$, $E_1 \subset E_2 \subset \cdots$, and $\bigcup_{j=1}^\infty \operatorname{Int} E_j = A$.
$\text{ext} \int_A f = \lim_{j\to\infty} \text{ext} \int_{U_j} f$ if $U_j^{\text{open}} \subset A$, $U_1 \subset U_2 \subset \cdots$, and $\bigcup_{j=1}^\infty U_j = A$.

Proof of the last one: $\text{ext} \int_{U_j} f \leq \text{ext} \int_A f$, so $\lim_{j\to\infty} \text{ext} \int_{U_j} f = \sup \left\{ \text{ext} \int_{U_j} f \right\} \leq \text{ext} \int_A f$.
Each compact rectifiable $E \subset A$ lies in some U_j. So $\int_E f \leq \text{ext} \int_{U_j} f \leq \lim_{j\to\infty} \text{ext} \int_{U_j} f$.
Then, take the supremum over the E_j. So $\text{ext} \int_A f \leq \lim_{j\to\infty} \text{ext} \int_{U_j} f$.

Defn: For $x \in [-\infty, +\infty]$, $x_+ \stackrel{\text{def}}{=} \max\{x, 0\} = \frac{|x|+x}{2}$ and $x_- \stackrel{\text{def}}{=} \max\{-x, 0\} = \frac{|x|-x}{2}$.

Then $x_+, x_- \geq 0$, $x_+ \cdot x_- = 0$, $x = x_+ - x_-$, and $|x| = x_+ + x-$.

Defn: For $f : X \to [-\infty, \infty]$, $f_+(x) \stackrel{\text{def}}{=} (f(x))_+$ is the <u>positive part of f</u>, and $f_-(x) \stackrel{\text{def}}{=} (f(x))_-$ is the <u>negative part of f</u>.

$f_+, f_- \geq 0$, $f_+ \cdot f_- = 0$, $f = f_+ - f_-$, and $|f| = f_+ + f_-$.

Consider $f \in C(A^{\text{osso}\mathbb{R}^n}, \mathbb{R})$ (with f not necessarily non-negative). Then we say f is "extended integrable on A" or "integrable in the extended sense" if $\text{ext} \int_A f_+, \text{ext} \int_A f_- < +\infty$.

$\text{ext} \int_A f$ exists if at least one of $\text{ext} \int_A f_+$ and $\text{ext} \int_A f_-$ is finite. Set $\text{ext} \int_A f = \text{ext} \int_A f_+ - \text{ext} \int_A f_-$.

$\text{ext} \int_A af + bg = a\text{ext} \int_A f + b\text{ext} \int_A g$
$f \geq g$ on $A \Rightarrow \text{ext} \int_A f \leq \text{ext} \int_A g$ if they exist.
For compact, rectifiable $E_1 \subset E_2 \subset \cdots \subset A$ with $\bigcup_{j=1}^\infty \operatorname{Int} E_j = A$, $\text{ext} \int_A f = \lim_{j\to\infty} \int_{E_j} f$.
For open $U_1 \subset U_2 \subset \cdots \subset A$, with $\bigcup_{j=1}^\infty U_j = A$, $\text{ext} \int_A f = \lim_{j\to\infty} \text{ext} \int_{U_j} f$

Consider Q box $\stackrel{\vec{x} \mapsto M\vec{x}+\vec{b}}{\to}$ P parallelopiped,
$A^{\text{open}} \subset \mathbb{R}^n \stackrel{g \text{ diffeo}}{\to} B^{\text{open}} \subset \mathbb{R}^n \stackrel{f \text{ cts}}{\to} \mathbb{R}$.
Then we want to prove P is rectifiable, $v(P) = |\det M| \cdot v(Q)$, and $\text{ext} \int_B f = \text{ext} \int_A f$.

Thm: (Change of Variable Thm) Given f, g as above, then either $\text{ext} \displaystyle\int_B f = \text{ext} \int_{A=g^{-1}[B]} f \circ g \, |\det Dg|$, or the integral on neither side exists.

Special case: $n = 1$, A connected (i.e. an interval), $A = (\alpha, \beta)$ for $\alpha < \beta \in [-\infty, \infty]$. Then g monotonic.
Case 1: $B = (g(\alpha), g(\beta))$. Then $\text{ext} \int_B f = \text{ext} \int_A (f \circ g) g'$

Case 2: $B = (g(\beta), g(\alpha))$. Then $\text{ext} \int_B f = -\text{ext} \int_A (f \circ g) g' \stackrel{\text{calc } 1/2}{=} -\text{ext} \int_{g(\beta)}^{g(\alpha)} f$