Parallelopipeds and the Pythagorean Theorem

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Let $A \in \mathrm{Mat}(n,k)$. Consider $A^TA \in \mathrm{Mat}(k,k)$.

Claim: $\ker A^TA = \ker A$
Proof: $\supset$ trivial
$\subset$ $A^TA\vec{x} = \vec{0} \Rightarrow ||A\vec{x}||^2 = \langle A\vec{x}, A\vec{x}\rangle = (\vec{x}A)^T A\vec{x} = \vec{0}$.
Cor: $\operatorname{rank} A^TA = k - \dim(\ker(A^TA)) = k - \dim(\ker(A)) = \operatorname{rank}(A)$
<u>or</u> $\det(A^TA) = 0 \Leftrightarrow \operatorname{rank}(A) < k$.

Cor: $k > n \Rightarrow \det(A^TA) = 0$

Claim: All eigenvalues of A^TA are non-negative.
Proof: If $A^TA\vec{x} = \lambda\vec{x}$ (with $\vec{x} \neq \vec{0}$), then $\left\langle A^TA\vec{x}, \vec{x}\right\rangle = ||A\vec{x}||^2$, and $\left\langle A^TA\vec{x}, \vec{x}\right\rangle = \langle \lambda\vec{x}, \vec{x}\rangle = \lambda\,||\vec{x}||^2$.
So $\lambda = \frac{||A\vec{x}||^2}{||\vec{x}||^2} = \left(\frac{||A\vec{x}||}{||\vec{x}||}\right)^2$. So $\lambda \geq 0$. $\square$

Cor: $\det A^TA \geq 0$

Recall: Thm 21.2 $A \in \mathrm{Mat}(n,k) \Rightarrow \exists B \in O_n(\mathbb{R})$ (i.e. $B^TB = \mathrm{Id}$) with $BA = \begin{pmatrix} M \\ 0 \end{pmatrix} \in \mathrm{Mat}(k,n)$.

Note: $M^TM = \begin{pmatrix} M^T & 0 \end{pmatrix}\begin{pmatrix} M \\ 0 \end{pmatrix} = A^TB^TBA = A^TA$.
So $(\det M)^2 = \det A^TA$, and $|\det M| = \sqrt{\det A^TA}$.

Given $T : Q^{\text{box in } \mathbb{R}^k} \to \mathbb{R}^n$ injective, affine (i.e. $T : \vec{x} \mapsto A\vec{x} + b$), then $T[Q]$ is a "k-parallelopiped".
We want $V_k : \{k - \text{p'pipeds}\} \to (0, +\infty)$ unique s.t.

(1) $A = \begin{pmatrix} M \\ 0 \end{pmatrix} \Rightarrow v_k(T[Q]) = |\det M|\, v_k(Q)$.

(2) $h : \vec{x} \mapsto B^{\text{orthogonal}}\vec{x} + \vec{p} \Rightarrow v_k((h \circ T)[Q]) = v_k(T[Q])$.

Choose B as in Thm 21.2, suitable $\vec{p}$. Then we get $(h \circ T) : \vec{x} \mapsto \begin{pmatrix} M \\ 0 \end{pmatrix}\vec{x} = BA\vec{x}$.
Thus, $v_k(T[Q]) = \sqrt{\det(A^TA)}v(Q)$.

Defn: $V_k(T[Q]) = \sqrt{\det A^TA}v(Q)$

Check (1) holds: $A = \begin{pmatrix} M \\ 0 \end{pmatrix} \to \sqrt{\det(A^Ta)} = |\det M| \Rightarrow V_k(T[Q]) = |\det M|\, v(Q)$. ✓

Check (2) holds: $\vec{x} \overset{h}{\mapsto} B\vec{x} + \vec{p}$ ($\mathbb{R}^n \to \mathbb{R}^n$) $\Rightarrow \det((BA)^T(BA)) = \det(A^TA) \Rightarrow V_k((h \circ T)[Q]) = V_k(T[Q])$.

Useful observation: $V(A) \overset{\text{def}}{=} \sqrt{\det(A^TA)}$.

Thm: (Pythagorean Theorem) $(V(A))^2$ is the sum of the squares of all k-by-k sub-determinants of A.
Proof: Theorem 21.4. $\square$

Defn: Given $U^{\text{open}} \subset \mathbb{R}^k$, $\alpha \in C^1(U, \mathbb{R}^n)$, $Y = \alpha[U]$, then Y_α is a parameterized manifold.

Think of $V(D_\alpha) = \sqrt{\det(D_\alpha^T D_\alpha)}$ as the "volume magnification factor".

Defn: $V_k(Y_\alpha) \stackrel{\text{def}}{=} \text{ext} \int_U V(D_\alpha)$

Does this only depend on Y and not on α?
No, it depends on both.

But suppose...

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & \\ g \text{ difffeo} \downarrow & & Y \text{ Manifold}, \\ V & \xrightarrow{\beta} & \end{array} \quad \text{with } Y = \beta[V] = \alpha[U]. \text{ Then}$$

$$\begin{aligned} V(Y_\alpha) &= \int_U \sqrt{\det(D(\beta \circ g))^T D(\beta \circ g))} \\ &= \int_U \sqrt{\det(Dg^T (D\beta \circ g)^T (D\beta \circ g) Dg)} \\ &= \int_U \sqrt{\det(D\beta^T D\beta)} \circ g \, |\det Dg| \\ &= \int_V V(D\beta) \circ g \, |\det Dg| \\ &= \int_V V(D\beta) = V(Y_\beta) \end{aligned}$$