Derivatives

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Defn: $\lim_{t\to 0} \frac{f(\vec{a}+t\vec{u})-f(\vec{a})}{t} = T(\vec{u})$
We call this $f'(\vec{a};\vec{u})$ a directional derivative.

Defn: $T \in \text{Hom}(V,W)$ is said to be bounded if $\exists M \in \mathbb{R}_{\geq 0}$ s.t. $||T(\vec{v})|| \leq M\vec{v}$, $\forall \vec{v} \in V$.

Defn: $B(V,W)$ is the set of bounded linear maps $V \to W$.

With normed vector space V, for $\vec{a}, \vec{u} \in V$, define $g_{\vec{a},\vec{u}} : \mathbb{R} \to V$, $t \mapsto \vec{a} + t\vec{u}$. Note that $g'_{\vec{a},\vec{u}}(t) = \vec{u}$.

Return to $f : V \to W$. We have $(f \circ g_{\vec{a},\vec{u}})'(0) = \frac{(f\circ g)(t)-f(g(0))}{t} = \frac{f(\vec{a}+\vec{u}t)-f(\vec{a})}{t}$, the directional derivative. This requires $\vec{a}$ be an interior point of the domain of f.

If we λ-dilate Graph f centered at $(\vec{a}, f(\vec{a}))$, and let $\lambda \to \infty$, then the graph converges pointwise to some affine graph (if the limit exists).

To get theory on this, we need
(1) $f'(\vec{a};\vec{u}) = T(\vec{u})$ linear in $\vec{u}$.
(2) $f(\vec{a}) + T(\vec{y} - \vec{a}) \approx f(\vec{y})$.

Formally, for bounded T, $Df(\vec{a}) = T \to \lim_{h\to 0} \frac{||f(a+h)-f(h)-T(h)||}{||h||} = 0$. **I need to check with Nikhil to make sure I have this written down right.**

Prop: If $Df(veca)$ exists and $\vec{u} \in V$, then $Df(\vec{a})(\vec{u})$ is $f'(\vec{a};\vec{u})$.

Cor: $Df(\vec{a})$ is unique.
Proof: Directional derivatives are unique because they are limits. □

Prop: $Df(\vec{a}))$ exists implies that f is continuous at $\vec{a}$.
Proof: It is enough to show that $\vec{x} \to \vec{a}$ implies that $f(\vec{x}) \to f(\vec{a})$. So does $f(\vec{x}) - f(\vec{a}) \to 0$? **I need to check with Nikhil on this part too.**

Special Cases:

(A) $W = \mathbb{R}^n$, $f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$, $A \subset \mathbb{R}^n$.

Prop: f is differentiable at $\vec{a} \leftrightarrow$ each f_j is differentiable at $\vec{a}$.

(B) $V = \mathbb{R}^m$. $Df(\vec{a}) = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = \sum_{j=1}^{m} u_j T(e_j) = \sum_{j=1}^{m} u_j f'(\vec{a_j};\vec{e_j})$ for $T = Df(\vec{a})$, and f' directional derivative. This is the partial derivative.