Riemann Integrability (Continued)

Thomas Cohn

10/26/18

(Continued from Wednesday)

Proof 5 $\Rightarrow$ 2: Given $\varepsilon > 0$, let $\widetilde{\varepsilon} = \dfrac{\varepsilon}{2(M+V(Q))}$. Write $\mathcal{D} \subset \bigcup_{p=1}^{\infty} \text{rInt}\,(R_p^{\text{box}})$ with $\sum_{p=1}^{\infty} V(R_p) < \widetilde{\varepsilon}$. For $\vec{a} \in Q \setminus \mathcal{D}$, $\exists Q_{\vec{a}}^{\text{box}}$ with $\vec{a} \in \text{rInt}\,(Q_{\vec{a}})$ and $|f(\vec{x}) - f(\vec{a})| < \widetilde{\varepsilon}$ for $\vec{x} \in Q_{\vec{a}}$.

$$Q = \left(\bigcup_{p=1}^{\infty} \text{rInt}\,(R_p) \right) \cup \left(\bigcup_{\vec{a} \in Q \setminus \mathcal{D}} \text{rInt}\,(Q_{\vec{a}}) \right)$$

Q is compact. So pick a finite subcover:

$$Q \subset \text{rInt}\,(R_{p_1}) \cup \cdots \cup \text{rInt}\,(R_{p_s}) \cup \text{rInt}\,(Q_{\vec{a}_1}) \cup \cdots \cup \text{rInt}\,(Q_{\vec{a}_u})$$

Pick partition P such that each box R defined by P satisfying $R \subset Q_{\vec{a}_j}$ for some $\vec{a_j}$ "good" or $R \subset R_{P_k}$ "bad". If both hold, we make a choice. Then

$$U(f,P) - L(f,P) = \sum_{R \text{ det'd by } P} \left(\sup_R f - \inf_R f \right) v(R) = \sum_{R \text{ "good"}} + \sum_{R \text{ "bad"}} \leq \sum_{R \text{ "good"}} 2\widetilde{\varepsilon} V(R) + \sum_{R \text{ "bad"}} 2MV(R)$$

Using a prevous lemma, this is less than $2\widetilde{\varepsilon}V(Q) + 2M\widetilde{\varepsilon} < \varepsilon$. $\square$

Given $B_1 \cup \cdots \cup B_j \subset X_1 \cup \cdots \cup X_k$, and the interiors of the B_is are disjoint, we have $V(B_1) + \cdots + V(B_j) \leq V(X_1) + \cdots + V(X_k)$.
Proof: TODO