Contraction Mapping Theorem

Thomas Cohn

10/3/18

$f : X^{\text{complete metric space}} \to X \Rightarrow f$ has a unique fixed point.

We define $T_{\vec{y}} : \vec{x} \mapsto \vec{x} + \vec{y}$.

From Monday:
Lemma: Given $\vec{0} \in \mathcal{U}^{\text{open}} \subset \mathbb{R}^n$
$g : \mathcal{U} \to \mathbb{R}^n$ is at least C^1
$g(\vec{0}) = \vec{0}$
$Dg(\vec{0}) = \text{Id}$
$0 < \varepsilon < 1$
Then $\exists \delta > 0$ s.t. $h = g - \text{Id}$ satisfies (1) $||h(\vec{y}) - h(\vec{a})|| \leq \varepsilon\, ||\vec{y} - \vec{x}||$ for $\vec{x}, \vec{y} \in \mathcal{U}(\vec{0}, \delta)$
(2) $(1-\varepsilon)\, ||\vec{y} - \vec{x}|| \leq ||g(\vec{y}) - g(\vec{x})|| \leq (1+\varepsilon)\, ||\vec{y} - \vec{x}||$
(3) g is injective on $\mathcal{U}(\vec{0}, \delta)$.

Let $f : A^{\text{t.s.}} \to B^{\text{t.s.}}$.
Defn: f is <u>continuous</u> $\leftrightarrow f^{-1}(\mathcal{U})$ open in A when $\mathcal{U}$ open in B.
f is <u>open</u> $\leftrightarrow f(\mathcal{U})$ open in B when $\mathcal{U}$ open in A.

Suppose f is a bijection.
Then f is continuous $\leftrightarrow f^{-1}$ is open
f is open $\leftrightarrow f^{-1}$ is continuous
f is a homeomorphism $\leftrightarrow f$ is open and continuous.

Let $\psi_{\vec{y}} : \vec{x} \mapsto \vec{y} - h(\vec{x})$ $(\vec{x} \in \mathcal{U})$. Pick $0 < \widetilde{\delta} < \delta$.
Then $\vec{y} \in \mathcal{U}(\vec{0}, (1-\varepsilon)\widetilde{\delta})$, $\vec{x} \in \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})} \Rightarrow ||\psi_{\vec{y}}(\vec{x})|| \leq ||\vec{y}|| + ||h(\vec{x})|| \leq (1-\varepsilon)\widetilde{\delta} + \varepsilon\widetilde{\delta} = \widetilde{\delta}$.
So $\psi_{\vec{y}} : \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})} \to \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})}$ with $||\psi_{\vec{y}}(\vec{x_1}) - \psi_{\vec{y}}(\vec{x_2})|| = ||h(\vec{x_2}) - h(\vec{x_1})|| \leq \varepsilon\, ||\vec{x_2} - \vec{x_1}||$.

Therefore, $\psi_{\vec{y}} : \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})} \to \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})}$ is a contraction. (We use the closure because $\mathcal{U}(\vec{0}, \widetilde{\delta})$ is not a complete metric space, but $\overline{\mathcal{U}(\vec{0}, \widetilde{\delta})}$ is.)
And so, by the contraction mapping theorem, $\exists \vec{x} \in \overline{\mathcal{U}(\vec{0}, \widetilde{\delta})}$ s.t. $\psi_{\vec{y}}(\vec{x}) = \vec{x}$. But $\psi_{\vec{y}}(\vec{x}) = \vec{y} - h(\vec{x}) = \vec{y} - g(\vec{x}) + \vec{x}$. So $\vec{y} - g(\vec{x}) + \vec{x} = \vec{x}$. So $\vec{y} = g(\vec{x})$. And thus, $g(\mathcal{U}) \supset g(\overline{\mathcal{U}(\vec{0}, \widetilde{\delta})}) \supset \mathcal{U}(\vec{0}, (1-\varepsilon)\widetilde{\delta})$.

Ex: Upgrade to $g(\mathcal{U}(\vec{0}, \delta)) \supset \mathcal{U}(\vec{0}, (1-\varepsilon)\delta)$.

Conclude: Add $g(\mathcal{U})] \supset g(\vec{0}, (1-\varepsilon)\delta)$ to the lemma. In particular, $\vec{0} \in \text{Int}\, g(\vec{u})$.

Thm: (Cousin of Inverse Function Theorem) Given $E^{\text{open}} \subset \mathbb{R}^n$, $f \in C^1(E, \mathbb{R}^n)$, and $\det Df \neq 0$ on E, then the following are true:

1. $\vec{a} \in E \to f(\vec{a}) \in \text{Int } f[E]$.
2. $f[E]$ is open in $\mathbb{R}^n$.
3. $f : E \to f[E]$ is an open map.

Cor: f as above and injective $\to f : E \to f[E]$, is a homeomorphism.

Proof:
(1) Apply lemma to $g = Df(\vec{a})^{-1} \circ T_{-f(\vec{a})} \circ f \circ T_{\vec{a}}$. We get $\vec{0} \in \text{Int}\,(\text{im}(g))$. $\vec{0} = Df(\vec{a})(\vec{0}) \in \text{im}(T_{-\vec{a}} \circ f \cancel{\circ T_{\vec{a}}})$.
We ignore $T_{\vec{a}}$ because it's bijective.
Apply $T_{f(\vec{a})}$. Then $f(\vec{a}) \in \text{Int}\,(\text{im}(f))$, so $f(\vec{a}) \in \text{Int}\,(f[E])$.
(2) Since all $f(\vec{a}) \in \text{Int}\,(f[E])$, $f[E]$ is open.
(3) For $\mathcal{U}^{\text{open}} \subset E$, apply (2) to $f|_{\mathcal{U}}$.
$\square$

Prop: f as in cor and $f \in C^r \Rightarrow f^{-1} \in C^r$.

The inverse function theorem follows from the lemma, the corollary, and the proposition.

Proof: For $r = 1$, $g = f^{-1}$, $\vec{b} = f(\vec{a})$, $M = Df(\vec{a})$, we need $\dfrac{g(\vec{b}+\vec{k}) - g(\vec{b}) - M^{-1}\vec{k}}{\left|\left|\vec{k}\right|\right|} \to \vec{0}$ as $\vec{k} \to \vec{0}$.

$$\frac{g(\vec{b}+\vec{k}) - g(\vec{b}) - M^{-1}\vec{k}}{\left|\left|\vec{k}\right|\right|} - \frac{\vec{h} - M^{-1}\vec{k}}{\left|\left|\vec{k}\right|\right|} = -M^{-1}\left(\frac{\vec{k} - M\vec{h}}{\left|\left|\vec{k}\right|\right|}\right) = -M^{-1}\left(\frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - M\vec{h}}{\left|\left|\vec{h}\right|\right|}\right)\left(\frac{\left|\left|\vec{h}\right|\right|}{\left|\left|\vec{k}\right|\right|}\right).$$

$\left(\frac{||\vec{h}||}{||\vec{k}||}\right)$ is bounded for $\left|\left|\vec{k}\right|\right| < \delta$ by lemma (2). So it follows that $\vec{k} \to \vec{0}$ as $\vec{h} \to \vec{0}$. So $\frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - M\vec{h}}{||\vec{h}||} \to \vec{0}$.
So $M^{-1}\vec{0} = \vec{0}$. Thus, as $\vec{k} \to \vec{0}$, the other thing goes to $\vec{0}$.

We've now shown that g is differentiable. $Dg(\vec{b}) = Df(g(\vec{b}))^{-1}$. Still need Dg continuous.

$$f[\mathcal{U}] \underset{\text{cts}}{\overset{g}{\to}} \mathcal{U} \underset{\text{cts}}{\overset{Df}{\to}} \text{GL}\,(n, \mathbb{R}) \underset{\text{Thm 2.14}}{\overset{\text{inversion}}{\to}} \text{GL}\,(n, \mathbb{R})$$

So Dg is the composition of continuous maps, and is therefore continuous. So Dg is continuous, and we're done for $r = 1$. For $r > 1$, recall that $g \in C^r \Leftrightarrow Dg \in Cr - 1$.

Lemma: C^{r-1} mapping closed under composition.
Proof: Will be done on Friday.

Induction on r. Assume that prop holds for C^{r-1}. Then

$$f(\mathcal{U}) \underset{C^{r-1}}{\overset{g}{\to}} \mathcal{U} \underset{C^{r-1}}{\overset{Df}{\to}} \text{GL}\,(n, \mathbb{R}) \underset{C^{\infty}}{\overset{\text{inversion}}{\to}} \text{GL}\,(n, \mathbb{R})$$

So $Dg \in C^{r-1}$. $\square$