

The Extended Reimann Integral

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Consider $f \in C(A^{\text{osso}\mathbb{R}^n}, \mathbb{R})$ (with f and/or A possibly unbounded). For now, assume $f \geq 0$.

Defn: We define the Extended Reimann Integral $\text{ext} \int_A f \stackrel{\text{def}}{=} \sup \left\{ \int_E f : E^{\text{cpt, rect}} \subset A \right\}$.

Lemma: $B^{\text{open}} \subset A^{\text{open}} \Rightarrow \text{ext} \int_B f \leq \text{ext} \int_A f$

What if A and f are bounded?

(i) The old $\int_A f$ may not exist. (It exists if A is rectifiable.)

(ii) $\int_E f = \int_{\overline{E}} f \leq \int_A f \stackrel{\text{def}}{=} \int_{\overline{Q}} f_A$. So $\text{ext} \int_A f \leq \int_A f$

(iii) Let P be a partition of $Q^{\text{box}} \supset A$. Then $L(f_A, P) \leq \int_{\text{union of } P\text{-boxes}} f \leq \text{ext} \int_A f$

(iv) Thus, $\text{ext} \int_A f = \int_{\overline{A}} f$. So $\int_{\overline{A}} f \leq \text{ext} \int_A f$. So $\text{ext} \int_A f = \text{old} \int_A f$ if old $\int_A f$ exists.

How do we compute?

Suppose we have an infinite sequence of compact rectifiable sets $E_1 \subset E_2 \subset E_3 \subset \dots \subset A$, and

$\bigcup_{j=1}^{\infty} \text{Int } E_j = A$. Then we claim $\text{ext} \int_A f = \lim_{j \rightarrow \infty} \int_{E_j} f$

Ex: $E_j = [-j, 0] \cup [\frac{1}{j}, j]$. Then $\bigcup_{j=1}^{\infty} E_j = \mathbb{R}$, and $\bigcup_{j=1}^{\infty} \text{Int } E_j = \mathbb{R} \setminus \{0\}$.

Proof of claim: $\int_{E_j} f \leq \text{ext} \int_A f$, so $\lim_{j \rightarrow \infty} \int_{E_j} f \leq \text{ext} \int_A f$.

If $E \subset A$ is compact and rectifiable, then $E \subset E_j$ for some j . Thus, $\int_E f \leq \int_{E_j} f \leq \lim_{j \rightarrow \infty} \int_{E_j} f$.

Therefore, $\text{ext} \int_E f \leq \lim_{j \rightarrow \infty} \int_{E_j} f$. \square

Alternate proof (outline): Let E_j be the union of all closed (hyper) cubes with side length $\frac{1}{2^j}$, subsets of A , with each vertex having coords in $\mathbb{Z}/2^j \cap [-j, j]$.

Ex: $\int_{\mathbb{R}} \frac{1}{1+x^2}$

Let $E_j = [-j, j]$. Then $\int_{\mathbb{R}} \frac{1}{1+x^2} = \lim_{j \rightarrow \infty} \int_{-j}^j \frac{1}{1+x^2} = \lim_{j \rightarrow \infty} [\arctan x]_{x=-j}^{x=j} = \lim_{j \rightarrow \infty} 2 \arctan j = \pi$

Ex: $\int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} = \lim_{j \rightarrow \infty} \int_{-j}^j \int_{-j}^j \frac{1}{1+x^2+y^2} dx dy = \dots = \lim_{j \rightarrow \infty} \int_{-j}^j \frac{2 \arctan \frac{j}{\sqrt{1+y^2}}}{\sqrt{1+y^2}} dy$. Ew.

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} &= \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{\substack{-k \leq x \leq k \\ -j \leq y \leq j}} \frac{1}{1+x^2+y^2} = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{-j}^j \int_{-k}^k \frac{1}{1+x^2+y^2} dx dy = \\ &= \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{-j}^j \frac{2 \arctan \frac{k}{\sqrt{1+y^2}}}{\sqrt{1+y^2}} = \lim_{j \rightarrow \infty} \int_{-j}^j \frac{\pi}{\sqrt{1+y^2}} dy = \pi \lim_{j \rightarrow \infty} \int_{-j}^j \frac{1}{\sqrt{1+y^2}} dy \stackrel{\text{use } y=\sinh u}{=} \dots = \\ &= \pi \lim_{j \rightarrow \infty} [\operatorname{arcsinh} y]_{y=-j}^{y=j} = +\infty - (-\infty) = +\infty \end{aligned}$$

We “computed” this extended integral.