# Moore "Affine" Notions

Thomas Cohn

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The extended reals are denoted as $[-\infty, +\infty] = \mathbb{R} \cup \{\pm\infty\}$.

Notation: Suppose for $\alpha \in A$ we are given $S_\alpha \subset X$, i.e., we have a function $f : A \to \mathcal{P}(X)$ where $\alpha \mapsto S_\alpha$.
$\displaystyle\bigcup_{\alpha \in A} S_\alpha = \{x \in X : x \in S_\alpha \text{ for at least one } \alpha \in A\}$
$\displaystyle\bigcap_{\alpha \in A} S_\alpha = \{x \in X : x \in S_\alpha \text{ for all } \alpha \in A\}$
If $A \neq \emptyset$, then $\displaystyle\bigcup_{\alpha \in A} S_\alpha \neq \emptyset$ and $\displaystyle\bigcap_{\alpha \in A} S_\alpha = X$. **Is that right?**

Let $V, W$ be vector spaces over $F$, a field where $1 + 1 \neq 0$. We will study functions $T : V \to W$.

Graph $T = \{(\vec{v}, \vec{w}) \in V \times W : \vec{w} = T(\vec{v})\} = \{(\vec{v}, T(\vec{v})) \in V \times W : \vec{v} \in V\}$. Note that $V \times W$ is a vector space: $(\vec{v_1}, \vec{w_1}) + (\vec{v_2}, \vec{w_2}) = (\vec{v_1} + \vec{v_2}, \vec{w_1} + \vec{w_2})$ and $t(\vec{v}, \vec{w}) = (t\vec{v}, t\vec{w})$.

The "simplest" $T$'s are those with flat graphs, i.e., a graph that is an affine subset of $V \times W$.

**Defn:** A function $T$ is affine if and only if its graph is affine.

Special Case: $T(\vec{0}) = \vec{0}$, or equivalently, $(\vec{0}, \vec{0}) \in$ Graph $T$.
Then $T$ is affine $\leftrightarrow$ Graph $T$ is a linear subspace of $V \times W$.
$\leftrightarrow$ $\vec{v_1}, \vec{v_2} \in V$, $t \in F$ implies that $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$, and $tT(\vec{v_1}) = T(t\vec{v_1})$.
$\leftrightarrow$ $T$ is linear.

General Case: $T : V \to W$ is affine $\leftrightarrow$ Graph $T$ is affine.
$\leftrightarrow$ Graph $T - (\vec{0}, T(\vec{0}))$ is a linear subspace.
Note that Graph $T - (\vec{0}, T(\vec{0})) = \left\{(\vec{v}, T(\vec{v}) - T(\vec{0})) : \vec{v} \in V\right\}$.
$\leftrightarrow$ $L$ is linear. **What was $L$?**
$\leftrightarrow$ $T$ is of the form $T(\vec{v}) = \widetilde{T}(\vec{v}) + \vec{b}$ with $\widetilde{T}$ linear.

As an exercise, prove that $\widetilde{T}$ is uniquely determined by $T$.