More Notes on Derivatives

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Refresher from Monday: Given $A^{\text{open}} \subset V$ normed vector space, with W normed vector space, and $f : A \to W$.
f is $C^1 \leftrightarrow f$ is continuously differentiable
$\leftrightarrow f$ is differentiable at each $\vec{a} \in A$ and $Df : A \to B(V, W)$, $\vec{a} \mapsto Df(\vec{a})$.

$C^1(A, W)$ is the set of all C^1 $f : A \to W$.

Special Case: $V = \mathbb{R}^m$, $W = \mathbb{R}^n$, so we have $B(V, W) \leftrightarrow \text{Mat}\,(n, m)$.
Then the (j, k) entry of $Df(\vec{a})$ is $D_k f_j(\vec{a}) = f_j'(\vec{a}; \vec{e_k})$
f is C^1 iff it is differentiable at each $\vec{a} \in A$ and each $D_k f_j : A \to \mathbb{R}$ is continuous.

Thm: Given $f : A^{\text{osso}\mathbb{R}^n} \to \mathbb{R}^n$, and all $D_k f_j$ exist and are continuous on A, then $f \in C^1(A, \mathbb{R}^n)$.
Proof: It is enough to show that f is differentiable at each $\vec{a} \in A$. From last wednesday, it is enough to show that each component is differentiable.

Some board work for $m = 2$:
Fix $\vec{a} = (a_1, a_2) \in A$, and consider small $\vec{h} = (h_1, h_2)$.

$$\begin{array}{ccccl} & & & \cdot & (a_1 + h_1, a_2 + h_2) \\ & & & \uparrow & \vec{q} \\ \cdot & & \to & \cdot & \\ (a_1, a_2) & & \vec{p} & (a_1 + h_1, a_2) & \end{array}$$

$f(a_1 + h_1, a_2) - f(a_1, a_2) = D_1 f(\vec{p}) h_1$ for some $\vec{p}$ by MVT.
$f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2) = D_2 f(\vec{q}) h_2$ for some $\vec{q}$ by MVT.

If $D_f(\vec{a})$ exists, it must be $\begin{pmatrix} D_1 f(\vec{a}) & D_2 f(\vec{a}) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} f(\vec{a} + \vec{h}) - f(\vec{a}) = D_1 f(\vec{p}) h_1 + D_2 f(\vec{q}) h_2$.

Goal: $(\star)$ $\dfrac{f(\vec{a} + \vec{h}) - f(\vec{a}) - (D_1 f(\vec{a}) h_1 + D_2 f(\vec{a}) h_2)}{\left|\left|\vec{h}\right|\right|} \to 0$ as $\vec{h} \to \vec{0}$.

Well, this is $= (D_1 f(\vec{p}) - D_1 f(\vec{a})) \frac{h_1}{||\vec{h}||} + (D_2 f(\vec{q}) - D_2 f(\vec{a})) \frac{h_2}{||\vec{h}||}$.

As $\vec{h} \to \vec{0}$, $\frac{h_1}{||\vec{h}||}$ is bounded, since $\left| \frac{h_1}{||\vec{h}||} \right| \leq 1$. The same applies for $\frac{h_2}{||\vec{h}||}$.

As $\vec{h} \to \vec{0}$, $\vec{p}, \vec{q} \to \vec{a}$. So $D_1 f(\vec{p}) \to D_1 f(\vec{a})$ and $D_2 f(\vec{q}) \to D_2 f(\vec{a})$. So $D_1 f(\vec{p}) - D_1 f(\vec{a}) \to 0$ and $D_2 f(\vec{q}) - D_2 f(\vec{a}) \to 0$.

Therefore, $(D_1 f(\vec{p}) - D_1 f(\vec{a})) \frac{h_1}{||\vec{h}||} + (D_2 f(\vec{q}) - D_2 f(\vec{a})) \frac{h_2}{||\vec{h}||} \to 0$. $\square$

Ex: Generalize the above proof (Munkres 6.2).

Thm: If $A \subset V$ is open (V is a normed vector space), $f : A \to W$ (another normed vector space), and $D_f(\vec{a}; \vec{u})$ exist and are continuous for $(\vec{a}, \vec{u}) \in A \times V$, then f is differentiable, and $f \in C^1(A, W)$.

$f \in C^1(A^{\text{osso}V}, W)$ leads to $D_f : A \to B(V, W)$ continuous. It might be differentiable.
If so, have $D^2 f = D(Df) : A \to B(V, B(V, W))$.

$f \in C^2(A, W) \leftrightarrow Df \in C^1(A, B(V, W)) \leftrightarrow Df$ is differentiable at each $\vec{a}$ and $D^2 f$ is continuous.
$\leftarrow V = \mathbb{R}^m, W = \mathbb{R}^m$, and f' is C^1 and each $D_\ell D_k f_j$ exists and is continuous on A.

$C^r(A, W)$ follows similarly. $C^r(A) = C^r(A, W)$.

Thm: If $f \in C^2(A^{\text{osso}\mathbb{R}^2}, \mathbb{R})$, then $D_2 D_1 f(a, b) = \displaystyle\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}$

Proof: Let $\varphi(s) = f(s, b+k) - f(s, b)$. It's differentiable by the chain rule.
$\varphi'(s) = D_1 f(s, b+k) - D_1 f(s, b)$.

So the numerator is $\varphi(a+h) - \varphi(a) = \varphi'(s_0)h$ for some s_0 by the MVT.
So this is equal to $(D_1 f(s_0, b+k) - D_1 f(s_0, b))h$.
Applying the MVT again gives us $(D_2 D_1 f(s_0, t_0)kh$ for some s_0, t_0.

So $\displaystyle\lim_{(h,k)\to(0,0)} \frac{D_2 D_1 f(s_0, t_0)hk}{kh} = D_2 D_1 f(a, b)$. $\square$

Cor: f as above $\to D_2 D_1 f = D_1 D_2 f$ (Clairaut's Theorem). Note that the existence of $D^2 f$ is not enough for this result. See Munkres §6 #10.

Notation: For differentiable $f : A^{\text{osso}V} \to W$, $\vec{u} \in V$, set $\begin{array}{l} D_{\vec{u}} f : A \to W \\ \vec{a} \mapsto f'(\vec{a}; \vec{u}) = Df(\vec{a})(\vec{u}) \end{array}$.

If $f \in C^2(A, \mathbb{R})$, then $D_{\vec{u_1}} D_{\vec{u_2}} f = D_{\vec{u_2}} D_{\vec{u_1}} f$.

Proof 1: Apply chain rule twice to $(x_1, x_2) \mapsto f(\vec{a} + x_1 \vec{u_1} + x_2 \vec{u_2})$.

Proof 2: Study $\displaystyle\lim_{(h,k)\to(0,0} \frac{f(\vec{a} + h\vec{u_1} + k\vec{u_2}) - f(\vec{a} + h\vec{u_1}) - f(\vec{a} + k\vec{u_2}) + f(\vec{a})}{hk}$.

Spoiler! It equals both $D_{\vec{u_1}} D_{\vec{u_2}} f$ and $D_{\vec{u_2}} D_{\vec{u_1}} f$.

Cor: This also works for $f \in C^2(A, \mathbb{R}^m)$.