SpOoOoOky Halloween Lecture

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10/31/18

Prop: $K^{\text{cpt}} \subset \mathbb{R}^n \Rightarrow M^{*,J}(K) = m^*(K)$

Proof: We know $m^*(K) \leq m^{*,J}(K)$ is always true. So it is enough to show $m^*(K) \geq m^{*,J}(K)$.
Pick boxes Q_j $(j = 1, 2, \ldots)$ with $\bigcup_{j=1}^{\infty} \text{rInt}\, Q_j \supset K$.
Compactness implies that $\bigcup_{j=1}^{M} \text{rInt}\, Q_j \supset K$. So

$$\sum_{j=1}^{\infty} v(Q_j) \geq \sum_{j=1}^{M} v(Q_j) \geq m^{*,J}(K)$$

Now we take the inf over the choice of Q_j's.
Therefore $m^*(K) \geq M^{*,J}(K)$. $\square$

Recall the theorem from Friday:
For bounded $f : Q^{\text{box}} \to \mathbb{R}$, the following are equivalent

1. f is integrable
2.
3.
4. $\mathscr{D} \stackrel{\text{def}}{=} \{\vec{x} \in Q : f \text{ not cts at } \vec{x}\}$ has $m^*(\mathscr{D}) = 0$.
5.

Prop: $|S_1 \cup \cdots \cup S_k| \leq |S_1| + \cdots + |S_k|$. (Cardinality maps to $\mathbb{N} \cup \{0, +\infty\}$).

We call this property finite subadditivity.
Proof 1: Induction.
Proof 2: Use $S_1 \cup \cdots \cup S_k = S_1 \sqcup (S_2 \setminus S_1) \sqcup (S_3 \setminus (S_1 \cup S_2)) \sqcup \cdots \sqcup (S_k - \bigcup_{i=1}^{k-1} S_i)$
$|S_1| + \cdots + |S_k| \leq |RHS| = |S_1 \cup \cdots \cup S_k|$. $\square$

Lemma: Given $B_1 \cup \cdots \cup B_j \subset X_1 \cup \cdots \cup X_k$ (all boxes), with Int B_ℓ disjoint.

Then $v(B_1) + \cdots + v(B_j) \leq v(X_1) + \cdots + v(X_n)$.
Proof 1: Chop into smaller pieces.
Proof 2: Exercise: R box $\to v(R) = m_{\text{pixel}}(R) = m_{\text{pixel}}(\text{Int}\, R) = m_{\text{pixel}}(\text{rInt}\, R)$
Where $m_{\text{pixel}}(E) = \lim\limits_{N \to \infty} \left| E \cap \dfrac{\mathbb{Z}^n}{2^N} \right|$

$$\sum_{\ell=1} \left| \text{Int}\, B_\ell \cap \frac{\mathbb{Z}^n}{2^{nN}} \right| \leq \left| \bigcup X_p \cap \frac{\mathbb{Z}^n}{2^{nN}} \right| \leq \sum_{p=1}^{k} \left| X_p \cap \frac{\mathbb{Z}^n}{2^{nN}} \right|$$

Therefore $\sum m_{\text{pixel}}(\text{Int}\, B_\ell) = v(B_\ell)$. $\square$

Prop: Given f integrable on Q^{box}, $f \geq 0$ on Q.

Then $\int_Q f = 0$ iff $m^*(f^{-1}[(0, +\infty)]) = 0$.

Proof $\Rightarrow$: $f^{-1}[(0,+\infty)] \subset \mathscr{D} \cup \{\vec{a} \in Q : f \text{ is cts and positive at } \vec{a}\}$.
For $\vec{a}$ in the second set, $\exists B^{\text{box}} \supset \text{rInt } B \ni \vec{a}$ with $f \geq \frac{f(\vec{a})}{2}\mathbb{I}_B$.
Then $\int_Q f \geq \int_Q \frac{f(\vec{a})}{2}\mathbb{I}_B \overset{\text{exer}}{=} \frac{f(\vec{a})}{2}v(B) > 0$. Oops! Hence no such $\vec{a}$ exists.

Proof $\Leftarrow$: $f^{-1}[(0,+\infty)]$ contains no boxes of positive volume. To show this, it's enough to prove $m^*(B^{\text{positive volume box}}) > 0$. This is equal to $m^{*,J}(B)$, so this is all true by previous lemma.
So each $L(f,P) = 0$, so $\underline{\int_Q} f = 0$, so $\int_Q f = 0$.

Consider $S^{\text{bdd}} \subset \mathbb{R}^n$, with $f : S \to \mathbb{R}$ bounded, $f_S(\vec{x}) = \begin{cases} f(\vec{x}) & \vec{x} \in S \\ 0 & \vec{x} \notin S \end{cases}$, and
$\int_S f \overset{\text{def}}{=} \int_Q f_S$ for $Q^{\text{box}} \supset S$.

Prop: Existence and value of $\int_Q f_s$ do not depend on the choice of Q.
Proof: Choose Q_3 s.t. $\overline{S}, Q_1, Q_2 \supset \text{Int } Q_3$. Then the discontinuities set for Q_3 is equal to the discontinuities set for f on S.