Rectifiable Sets

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Recall from Friday that if $S^{\text{bdd}} \subset \mathbb{R}^n$ and $f : S \to \mathbb{R}$ is a bounded function, then we say f is integrable over S if and only if $\int_S f = \int_Q f_S$ is defined for some/all $Q^{\text{box}} \supset S$.

Some rules:

(a) f, g integrable over S implies that $\displaystyle\int\limits_S af + bg = a\int\limits_S f + b\int\limits_S g$

(b) f, g integrable over S, and $f \leq g$ on S implies that $\displaystyle\int\limits_S f \leq \int\limits_S g$

(b′) f integrable over S implies that $|f|$ is integrable over S.
Also, $\displaystyle\left|\int\limits_S f\right| = \max\left\{\int\limits_S f, -\int\limits_S f\right\} \leq \int\limits_S |f|$

(c) $T \subseteq S$, $f \geq 0$ integrable on T, S implies that $\displaystyle\int\limits_T f \leq \int\limits_S f$

(d) f integrable over S_1 and S_2 implies that f is integrable over $S_1 \cup S_2$ and $S_1 \cap S_2$, and
$\displaystyle\int\limits_{S_1\cup S_2} f = \int\limits_{S_1} f + \int\limits_{S_2} f - \int\limits_{S_1\cap S_2} f$

Proof (a): Let $A = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0 \vee x = y\right\}$. Define $\varphi : A \to \mathbb{R}$ by $(x, 0) \mapsto x$, $(0, y) \mapsto y$, and $(x, x) \mapsto x$.

Exercise 1: Show that φ is continuous.
Exercise 2: Show that $\varphi \circ (f_{S_1}, f_{S_2}) = f_{S_1\cup S_2}$. This tells us that $f_{S_1\cup S_2}$ is continuous at points where f_{S_1} and f_{S_2} are continuous. Hence, $f_{S_1\cup S_2}$ is cotntinuous.

Now, use $f_{S_1\cup S_2} + f_{S_1\cap S_2} = f_{S_1} + f_{S_2}$.

Ex: $S = \{(x_1, x_2, x_3) : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}$. Let $Q = [0,1] \times [0,1] \times [0,1]$.

Check that $\int_S 1$ exists (use study exercise)

$$\begin{aligned}
\int\limits_S 1 = \int\limits_Q \mathbb{1}_S &= \int\limits_{0 \leq x_1 \leq 1} \left(\int\limits_{0 \leq x_2 \leq 1} \left(\int\limits_{0 \leq x_3 \leq 1} \mathbb{1}_S \right) \right) \\
&= \int\limits_{0 \leq x_1 \leq 1} \left(\int\limits_{0 \leq x_2 \leq 1} \max\{1 - x_1 - x_2, 0\} \right) \\
&= \int\limits_{0 \leq x_1 \leq 1} \left(\int\limits_{0 \leq x_2 \leq 1 - x_1} 1 - x_1 - x_2 \right) \\
&= \int\limits_{0 \leq x_1 \leq 1} \left[(1 - x_1)x_2 - \frac{x_2^2}{2} \right]_{x_2 = 0}^{x_2 = 1 - x_1} \\
&= \int\limits_{0 \leq x_1 \leq 1} \frac{(1 - x_1)^2}{2} \\
&= \left[-\frac{(1 - x_1)^3}{6} \right]_{x_1 = 0}^{x_1 = 1} \\
&= \frac{1}{6}
\end{aligned}$$

Thm: Given $S^{\text{bdd}} \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$ bounded and continuous, $E \stackrel{\text{def}}{=} \left\{ \vec{x_0} \in \operatorname{Bd} S : \text{it is false that } \lim\limits_{\vec{x} \to \vec{x_0}\ (\vec{x} \in S)} f(\vec{x}) = 0 \right\}$, and $m^*(E) = 0$, then f is integrable on S.
Proof: $\mathscr{D} f_S \subset E$. $\square$

Cor: Given S^{bdd}, $f : S \to \mathbb{R}$ bounded and continuous, and $m^*(\operatorname{Bd} S) = 0$, then f is integrable over S.

Let's further study the condition that $m^*(\operatorname{Bd} S) = 0$:
Defn: S is <u>rectifiable</u>
$\stackrel{\text{def}}{\Leftrightarrow}$ 1 integrable over S
$\Leftrightarrow$ $\mathbb{1}_S$ integrable on $Q^{\text{box}} \supset S$
$\Leftrightarrow$ $m^*(\operatorname{Bd} S) = 0 \stackrel{\text{Cor}}{\Rightarrow}$ all bounded $f \in C(S, \mathbb{R})$ are integrable over S
$\Leftrightarrow$ $m^{*,J}(\operatorname{Bd} S) = 0$

For S rectifiable, we define $v(S) = \int_S 1$

S rectifiable, $A = \operatorname{Int} S \Rightarrow \operatorname{Bd} A \subset \operatorname{Bd} S$, $m^*(\operatorname{Bd} A) \leq m^*(\operatorname{Bd} S) = 0$. This implies that $\mathbb{1}_S, \mathbb{1}_A$ are integrable. So $\mathbb{1}_{S \setminus A} = \mathbb{1}_S - \mathbb{1}_A$ is integrable (on Q). Thus, $S \setminus A$ is rectifiable, and $\operatorname{Int}(S \setminus A) = \emptyset$.

All $L(\mathbb{1}_{S \setminus A}, P) = 0$, so $\underline{\int\limits_Q} \mathbb{1}_{S \setminus A} = 0$. Thus, $\int \mathbb{1}_A = \int \mathbb{1}_S \Rightarrow v(A) = v(S)$.

But what if S and or f are not bounded? **Improper Integrals**

Munkres starts to focus on integrals over open sets. Start with $f \geq 0$.

Defn: The <u>extended integral</u> of f over set A, $\text{ext}\displaystyle\int_A f \stackrel{\text{def}}{=} \sup\left\{\int_E f : E \text{ cpt, rect}\right\}$