Inverse Functions

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Let $A^{\text{open}} \subset V$, $B^{\text{open}} \subset W$. Suppose the following:
$f : A \to B$ is differentiable at $\vec{a}$
$g : B \to A$ is differentiable at $\vec{b} = f(\vec{a})$
$g \circ f = \text{Id}_A$ (i.e. $g(f(x)) = x$)

Then $Dg(\vec{b}) \circ Df(\vec{a}) = \text{Id}_A$
$Dg(\vec{b})$ is a left inverse of $Df(\vec{a})$
$\dim V \leq \dim W$.

1. If also $f \circ g = \text{Id}_B$, then
   - $Df(\vec{a}) \circ Dg(\vec{b}) = \text{Id}_B$
   - $Dg(\vec{b})$ is a 2-sided inverse of $Df(\vec{a})$
   - $\dim V \geq \dim W$, so $\dim V = \dim W$
2. If instead we have $\dim V = \dim W < +\infty$, then
   - $Dg(\vec{b})$ is $\not{a}$ the 2-sided inverse of $Df(\vec{a})$
3. If A, B, f, g as above, $\dim V, \dim W < +\infty$, and f, g are continuous, then
   - $g \circ f = \text{Id}_A$ and $f \circ g = \text{Id}_B \to \dim V = \dim W$. Proof of this is very hard. It requires new tools, so we'll return to it another time.
4. $\exists f : \mathbb{R} \to \mathbb{R}^2$ continuous and surjective.

Defn: A homeomorphism is a continuous bijection $f : A \to B$ (A, B topological spaces) such that f^{-1} is continuous.

Ex: $f : [0, 2\pi) \to S' = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$
$t \mapsto (\cos t, \sin t)$
f is a continuous bijection, but not a homeomorphism.
$f^{-1}\left(\cos \frac{1}{n}, \sin \frac{1}{n}\right) = 2\pi - \frac{1}{n}$, so as $n \to +\infty$, $f^{-1} \to 2\pi$. But $f^{-1}(1, 0) = 0$, so f^{-1} is not continuous.

Defn: A C^r-diffeomorphism is a C^r bijection $f : A^{\text{osso}V} \to B^{\text{osso}W}$ (V, W normed vector spaces) such that f^{-1} is also C^r.

Ex: $f : \mathbb{R} \to \mathbb{R}$
$t \mapsto t^3$ is a homeomorphism, but not a C^1-diffeomorphism.

Defn: A complete, normed vector space is called a Banach space.

Thm: (Inverse Function) Given $\vec{a} \in A^{\text{open}} \subset \mathbb{R}^n$, $f \in C^r(A, \mathbb{R}^n)$ for $r \in \mathbb{N}$, $Df(\vec{a})$ is invertible, then there is a $\mathcal{U}^{\text{open}}$ with $\vec{a} \in \mathcal{U}$ such that $f|_{\mathcal{U}}$ is a C^r-diffeomorphism, i.e., f maps $\mathcal{U}$ injectively to an open set, f^{-1} is C^r.

It turns out this is ok if the dimension is infinite, so long as V and W are Banach spaces.

Ex: $A = \left\{ \begin{pmatrix} r \\ \theta \end{pmatrix} \in \mathbb{R}^2 : r > 0 \right\}$

$f \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$

$Df \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ $\quad$ $\det\left(Df \begin{pmatrix} r \\ \theta \end{pmatrix} \right) = r\cos^2\theta - (-r\sin^2\theta) = r > 0$

So $Df \begin{pmatrix} r \\ \theta \end{pmatrix}$ is invertible for all $\begin{pmatrix} r \\ \theta \end{pmatrix} \in A$, <u>but</u> f is not injective on A.

$f \begin{pmatrix} r \\ \theta \end{pmatrix} = f \begin{pmatrix} r \\ \theta + 2\pi \end{pmatrix}$. So we can get local C^∞ inverses, but no global inverse.

$f[A] = \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Some notes: $E = \{\vec{x} \in A : Df(\vec{x}) \text{ invertible}\} = \{\vec{x} \in A : \deg()\, Df(\vec{x}) \neq 0\}$.
E is an open set containing $\vec{a}$. The inverse function theorem doesn't assume $E = A$, but it could.

Proof of the Inverse Function Theorem

Preliminaries: Let $T_{\vec{a}} : \vec{x} \mapsto \vec{x} + \vec{a}$.
$DT_{\vec{a}} = \mathrm{Id}$
$g = Df(\vec{a})^{-1} \circ T_{-f(\vec{a})} \circ f \circ T_{\vec{a}}$

Check: $\left. \begin{array}{l} g(\vec{0}) = \vec{0} \\ Dg(\vec{0}) = \mathrm{Id} \\ f = T_{f(\vec{a})} \circ Df(\vec{a}) \circ g \circ T_{-\vec{a}} \end{array} \right\}$ E.T.S. g is a C^r-diffeomorphism on some open set containing $\vec{0}$.

In proving the inverse function theorem, we may assume $\vec{a} = \vec{0}$, and $Dg(\vec{0}) = \mathrm{Id}$.

Let $h = g - \mathrm{Id}$, $Dh = Dg - \mathrm{Id}$, $Dh(\vec{0}) = 0$, $Dh : A \to \mathrm{Mat}\,(n, m)$ is continuous.
Fix $0 < \varepsilon < 1$. Then $\exists \delta > 0$ s.t. $||Dh|| < \varepsilon$ on $\mathcal{U}(\vec{0}, \delta)$. ($||Dh||$ is defined in the HW3 handout.)

Lemma: Given $A^{\text{convex open}} \subset V$, $\varphi : V \to W$ differentiable, $||D\varphi(\vec{p})|| \leq M\ \forall \vec{p} \in A$.
Then $||\varphi(\vec{y}) - \varphi(\vec{x})|| \leq M\, ||\vec{y} - \vec{x}||\ \forall \vec{x}, \vec{y} \in A$.
Proof: HW 5. $\square$

So, for $\vec{x} \in \mathcal{U}(\vec{0}, \delta)$, we have $||h(\vec{x})|| = \left|\left| h(\vec{x}) - h(\vec{0}) \right|\right| \leq \varepsilon\, ||\vec{x}||$ $(\star)$.

Also, for $\vec{x}, \vec{y} \in \mathcal{U}(\vec{0}, \delta)$, we have
$(1 - \varepsilon)\, ||\vec{y} - \vec{x}|| \leq ||\vec{y} - \vec{x}|| - ||h(\vec{y}) - h(\vec{x})|| \leq ||(\vec{y} - \vec{x}) + (h(\vec{y}) - h(\vec{x}))|| = ||g(\vec{y}) - g(\vec{x})||$
$||(\vec{y} - \vec{x}) + (h(\vec{y}) - h(\vec{x}))|| \leq ||\vec{y} - \vec{x}|| + ||h(\vec{y}) - h(\vec{x})|| \leq (1 + \varepsilon)\, ||\vec{y} - \vec{x}||$.

So $(1 - \varepsilon)\, ||\vec{y} - \vec{x}|| \leq ||g(\vec{y}) - g(\vec{x})|| \leq (1 + \varepsilon)\, ||\vec{y} - \vec{x}||$. Thus g is Bi-Lipschitz on $\mathcal{U}(\vec{0}, \delta)$.
And $g(\mathcal{U}(\vec{0}, \delta)) \subset \mathcal{U}(\vec{0}, (1 + \varepsilon)\delta)$.
And g is injective on $\mathcal{U}(\vec{0}, \delta)$.

TO BE CONTINUED...