A Proposition on Integrals over Bounded Sets

Thomas Cohn

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Let $S^{\text{bdd}} \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$ bounded, and $f_S(\vec{x}) \stackrel{\text{def}}{=} \begin{cases} f(\vec{x}) & \vec{x} \in S \\ 0 & \vec{x} \notin S \end{cases}$

Then we define $\displaystyle\int\limits_S f \stackrel{\text{def}}{=} \int\limits_Q f_S$ for $Q^{\text{box}} \supset S$.

Prop: This integral exists, and is valid regardless of choice of Q.

Proof: To show that the choice of Q doesn't matter, choose $S \subset Q_1 \subset Q_2 \subset \operatorname{Int} Q_3 \subset Q_3$. It is enough to show $\int_{Q_1} f_S = \int_{Q_3} f_S$.

Let P partition Q_3. Refine P to P' such that Q_1 is the union of P'-boxes. Then

$$L(f_S, P) \leq L(f_S, P') = \underbrace{\sum_{P'\text{-boxes } R \subset Q_1} \left(\inf_R f_S\right) \cdot v(R)}_{\leq \int_{Q_1} f_S} + \underbrace{\sum_{P'\text{-boxes } R \subseteq Q_3 \setminus \text{rInt}\, Q_1} \left(\inf_R f_S\right) \cdot v(R)}_{\leq 0}$$

This gives us $L(f_S, P) \leq L(f_S, P') \leq \int_{Q_1} f_S$

So $\displaystyle\underline{\int}_{Q_3} f_S \leq \int\limits_{Q_1} f_S$, so $\displaystyle\int\limits_{Q_3} f_S \leq \int\limits_{Q_1} f_S$.

If we redo this all with upper sums, and combine the inequalities, we get $\int_{Q_3} f_S = \int_{Q_1} f_S$. $\square$