The Local Second Derivative Test

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For $n = 1$: Given $f \in C^2(I^{\text{interval}\subset\mathbb{R}}, \mathbb{R})$, $f'' \geq 0$ on I, and $f'(x_0) = 0$, then $f(x) \geq f(x_0)$ for all $x \in I$.

For any n: Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$, $Hf(\vec{x}) \geq 0\ \forall \vec{x} \in \Omega$ (that is, $\vec{a}^T \cdot Hf(\vec{x}) \cdot \vec{a} \geq 0$ for all $\vec{a}$), and $Df(\vec{x_0}) = \vec{0}$, then $f(\vec{x}) \geq f(\vec{x_0})$ for all $\vec{x} \in \Omega$.

Proof: Let $\varphi(t) = (1-t)\vec{x_0} + t\vec{x}$. Then $\varphi'(t) = \vec{x} - \vec{x_0} \overset{\text{def}}{=} \vec{a}$.
So $(f \circ \varphi)'(t) = f'(\varphi(t)) \cdot \vec{a} = \sum_j D_j f(\varphi(t)) \cdot \varphi'(t) \cdot a_j$
$(f \circ \varphi)''(t) = \sum_j DD_j f(\varphi(t)) \cdot \varphi'(t) \cdot a_j = \sum_{j,k} D_k D_j f(\varphi(t)) \cdot a_k \cdot a_j = \vec{a}^T Hf(\varphi(t))\vec{a} \geq 0$

The one dimensional result implies that $f(\vec{x}) = (f \circ \varphi)(1) \geq (f \circ \varphi)(0) = f(\vec{x_0})$.

If we also have $Hf(\vec{x}) > 0$, then $\vec{a}^T Hf(\vec{x})\vec{a} > 0$ for $\vec{a} \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$.

Consider $f \in C^2(\text{osso}\mathbb{R}^n, \mathbb{R})$ with $\vec{x_0}$ in the domain of f and $Df(\vec{x_0}) = \vec{0}$ (that is, $\vec{x_0}$ is a critical point for f).

$Hf(\vec{x_0}) > 0 \Rightarrow Hf(\vec{x}) > 0$ for $\vec{x} \in \mathcal{U}(\vec{x_0}, \delta) \Rightarrow f$ has a strict local minimum at $\vec{x_0}$.
$Hf(\vec{x_0}) \ngeq 0 \Rightarrow f$ has a strict local max along some line through $\vec{x_0}$. f does <u>not</u> have a local max at $\vec{x_0}$.
$Hf(\vec{x_0}) < 0 \Rightarrow f$ has a strict local max at $\vec{x_0}$.
$Hf(\vec{x_0}) \nleq 0 \Rightarrow f$ does not have a local max at $\vec{x_0}$.
$Hf(\vec{x_0}) \begin{smallmatrix}\ngeq \\ \nleq\end{smallmatrix} 0 \Rightarrow f$ does not have a local max or min at $\vec{x_0}$.

Ex: Consider $f(x) = x^3$, $f_1(x) = x^3 + \frac{x}{10}$, and $f_2(x) = x^3 - \frac{x}{10}$. They're all different – f is delicate.

Ex: $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$

$Df\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ critical point

$Hf\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

This gives us a "saddle point".