Metric Spaces

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Exercises to study: Munkres §3 #1,3,5

Defn: A metric on set X is a function $d : X \times X \to \mathbb{R}$ s.t.
(1) $d(x, y) = d(y, x)$
(2) a) $d(x, y) \geq 0$
b) $d(x, y) = 0 \leftrightarrow x = y$
c) $d(x, z) \leq d(x, y) + d(y, z)$.

Defn: A metric space is a set X equipped with a metric d.

Defn: With $Y \subset X$, $d|_{Y \times Y}$ is a metric on Y, called the induced metric.

The most important exapmles for the Munkres material:

1. $X = \mathbb{R}^n$, $d_{\text{eucl}}(\vec{x}, \vec{y}) = \sqrt{\sum_{j=1}^{n} (y_j - x_j)^2}$.
2. $Y \subset X$ with the induced metric.

Defn: For $x_0 \in X$, $\varepsilon > 0$, the set $\mathcal{U}(x_0, \varepsilon) = \{x \in X : d(x_0, x) < \varepsilon\}$. This is the ε-neighborhood of x_0, or the ε-ball centered at x_0.

Consider $A \subset X$.

Defn: $x_0 \in X$ is interior to $A \leftrightarrow \exists \varepsilon > 0$ s.t. $\mathcal{U}(x_0, \varepsilon) \subset A$.
Int A is the set of interior points to A.

Defn: $x_0 \in X$ is exterior to $A \leftrightarrow \exists \varepsilon > 0$ s.t. $\mathcal{U}(x_0, \varepsilon) \cap A = \emptyset$.
Ext A is the set of exterior points to A.

Defn: $x_0 \in X$ is a boundary point of $A \leftrightarrow x_0$ is neither interior nor exterior to A.
$\leftrightarrow$ each $\mathcal{U}(x_0, \varepsilon)$ intersects A and $X \setminus A$.
Bd A is the set of boundary points of A.

Note that we have $X = \text{Int } A \sqcup \text{Ext } A \sqcup \text{Bd } A$. Note that $\sqcup$ can have multiple meanings (but it's ok for us to use it this way). For more on the different meanings of "$\sqcup$", look up "disjoint union" on Wikipedia.

Note that Ext $A = \text{Int } (X \setminus A)$.

Defn: A is open $\leftrightarrow A = \text{Int } A$.

Prop: This defines a topology on X.
Proof: See Munkres §3.

Some facts:

1. Each $\mathcal{U}(x_0, \varepsilon)$ is open (Munkres §3 #1).
2. Int A is the largest open subset of A.
3. A is closed $\leftrightarrow A = \text{Int } A \cup \text{Bd } A$.
4. $\bar{A}$ is defined as $\text{Int } A \cup \text{Bd } A$.
5. $\bar{A}$ is the smallest closed superset of A.
6. $\text{Bd } (X \setminus A) = \text{Bd } A$.
7. Bd A is closed.

Defn: Given (x_n) sequence in X $(x_n) : \mathbb{N} \to X$, $n \mapsto x_n$, (X, d) metric
$x_n \to x \leftrightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon$ when $n > N$.
(x_n) converges if $\exists x$ s.t. $x_n \to x$.

If $x_n \to x$ and $x_n \to y$, then $x = y$.

Defn: $f : (X, d_x) \to (Y, d_y)$ is sequentially continuous $\leftrightarrow x_n \to x$ implies $f(x_n) \to f(x)$.

Thm: f is sequentially continuous $\leftrightarrow f$ is continuous.
Proof: We basically did this last year.

Defn: (x_n) is Cauchy $\leftrightarrow \forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon \; \forall n, m > N$.

Defn: A metric space is complete iff all Cauchy sequences converge.

Some facts:

1. $(\mathbb{R}^n, d_{\text{eucl}})$ is complete.
2. For $Y \subset \mathbb{R}^n$, (Y, d_{eucl}) is complete iff Y is closed.

$Z \subset X$ is closed $\leftrightarrow Z$ is sequentially closed.

Weird example of a metric: We can map between $\mathbb{R}$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with tan and arctan. Let $\widetilde{d}(x, y) = |\arctan x - \arctan y|$.

Defn: A topological space X is compact $\leftrightarrow$ every open cover of X has a finite subcover.
$\leftrightarrow$ if $X = \bigcup_{\alpha \in A} X_\alpha$ with X_α open,
then $\exists \alpha_1, \ldots, \alpha_k \in A$ s.t. $X = X_{\alpha_1} \cup \cdots \cup X_{\alpha_k}$.

Thm: (Bolzano-Weierstrass) (X, d) is compact if and only if every sequence in X admits a convergent subsequence ("sequential compactness").
Proof: We will do this on Wednesday.

Thm: (Heine-Borel) If $Y \subset \mathbb{R}^n$ then (Y, d_{eucl}) is compact $\leftrightarrow Y$ is closed and bounded.
Proof: Math 296 #158 or Math 297 #100 or Munkres Thms 4.3, 4.9.

Ex: Set S, with $\mathcal{P}^{\text{finite}}(test)$.
$$d_1(A, B) = \begin{cases} 0 & A = B \\ 1 & A \neq B \end{cases} \text{ gives us the discrete topology.}$$

$d_2(A, B) = |A \triangle B|$ *also* gives us the discrete topology.