Riemann Integrability

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f is Riemann integrable on Q if and only if $\forall\varepsilon > 0$, $\exists P$ partition of Q such that $U(f,P) - L(f,P) < \varepsilon$.

Proof $\Leftarrow$: $\forall\varepsilon > 0$, we have $0 \le \overline{\int_Q} f - \underline{\int_Q} f \le U(f,P) - L(f,P) < \varepsilon$. So $\overline{\int_Q} f = \underline{\int_Q} f$.

Proof $\Rightarrow$: For ε, we have partitions P, P' with $U(f,P') < L(f,P) + \varepsilon$. Choose P'' refining P and P'. Then $U(f,P'') \le U(f,P') < L(f,P) + \varepsilon \le L(f,P'') + \varepsilon$.

$\square$

Defn: $\mathrm{OSC}(f,\vec{a}) \overset{\text{def}}{=} \inf\limits_{\delta>0} \left\{ \sup\limits_{U(\vec{a},\delta)\cap Q} f - \inf\limits_{U(\vec{a},\delta)\cap Q} f \right\}$

$\mathrm{OSC}(f,\vec{a}) < \varepsilon \Leftrightarrow \exists U^{\text{open}} \ni \vec{a}$ s.t. $\sup\limits_{U\cap Q} f - \inf\limits_{U\cap Q} f < \varepsilon$.

Note: $\{\vec{a} : \mathrm{OSC}(f,\vec{a}) < \varepsilon\}$ is open. OSC is upper semi-continuous.

Ex: $f(x) = \begin{cases} \sin\frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$ $\qquad \mathrm{OSC}(f,a) = \begin{cases} 0 & a \ne 0 \\ 2 & a = 0 \end{cases}$

Defn: Let $\mathcal{D}_k = \left\{\vec{a} \in Q : \mathrm{OSC}(f,\vec{a}) \ge \frac{1}{k}\right\}$ closed. $\mathcal{D} \overset{\text{def}}{=} \bigcup_{k=1}^{\infty} \mathcal{D}_k = \{\vec{a} \in Q : f \text{ is not cts at } \vec{a}\}$. This might not be a closed set.

Thm: The following are equivalent:

1. f is Riemann-integrable on Q
2. For $\varepsilon > 0$, $\exists P$ partition of Q with $U(f,P) < L(f,P) + \varepsilon$
3. For $\varepsilon > 0$, $k \in \mathbb{N}$, we can write $\mathcal{D}_k \subset R_1 \cup \cdots \cup R_j$ boxes with $\sum_{\ell=1}^{j} V(R_\ell) < \varepsilon$
4. For $\varepsilon > 0$, we can write $\mathcal{D} \subset \bigcup_{p=1}^{\infty} R_p^{\text{box}}$ with $\sum_{p=1}^{\infty} V(R_p) < \varepsilon$
5. For $\varepsilon > 0$, we can write $\mathcal{D} \subset \bigcup_{p=1}^{\infty} \mathrm{rInt}\,(R_p^{\text{box}})$ with $\sum_{p=1}^{\infty} V(R_p) < \varepsilon$

Proof 2 $\Rightarrow$ 3: Pick P s.t. $U(f,P) - L(f,P) = \sum\limits_{R \text{ det'd by } P} \left(\sup\limits_P f - \inf\limits_P f\right) V(R) < \dfrac{\varepsilon}{k}$. Let $R_1, \dots, R_\ell$ be the boxes determined by P whose interior meets $\mathcal{D}_k$.

Then $\dfrac{1}{k} \sum\limits_{p=1}^{\ell} V(R_p) \le \sum\limits_{p=1}^{\ell} \left(\sup\limits_{R_p} f - \inf\limits_{R_p} f\right) V(R_p) \le \dfrac{\varepsilon}{k}$. So $\sum\limits_{p=1}^{\ell} V(R_p) < \varepsilon$.

Note: $\mathcal{D}_k \overset{?}{\subset} R_1 \cup \cdots \cup R_\ell$? Maybe not. But $\mathcal{D}_k \subset R_1 \cup \cdots \cup R_\ell \cup \mathrm{Bd}\,\widetilde{R_1} \cup \cdots \cup \mathrm{Bd}\,\widetilde{R_\ell}$, and the sum of the volumes is less than ε.

Proof 3 $\Rightarrow$ 4: We can cover $\mathcal{D}_k$ with finitely many boxes with volume sum less than $\frac{\varepsilon}{2k}$. Combined them – the new volume sum is less than $\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots = \varepsilon$. Given $R^{\text{box}} \subset Q$, $y > V(R)$, then $\exists \widetilde{R}^{\text{box}}$ with $R \subset \mathrm{rInt}\,(\widetilde{R}) \subset \widetilde{R} \subset Q$, and $V(\widetilde{R}) < y$.

Proof 4 $\Rightarrow$ 5: Pick $\mathcal{D} \subset \bigcup_{p=1}^{\infty} R_p$ with $\sum_{p=1}^{\infty} V(R_p) < \dfrac{\varepsilon}{4}$. Pick $R_p \subset \operatorname{rInt}(\widetilde{R_p}) \subset \widetilde{R_p} \subset Q$ with $V(\widetilde{R_p}) < 2V(R_p)$ if $v(R_p) > 0$ and $V(\widetilde{R_p}) < \frac{\varepsilon}{2^{p+1}}$ if $V(R_p) = 0$. Then the new volume sum is at most $2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon$.