The First Fundamental Theorem of Calculus for 1-Forms

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Let $f : [a, b] \to \mathbb{R}^n$

Prop: TFAE:

(1) f extends to a function in $C^k(\mathbb{R}, \mathbb{R}^n)$

(2) $f \in C([a, b])$ and $f|_{(a,b)}$ is C^k and $\lim_{t \searrow a} f^{(j)}(t), \lim_{t \nearrow b} f^{(j)}(t)$ exist and are finite for $j = 1, \ldots, k$.

Proof (1)$\Rightarrow$(2): Trivial
Proof (2)$\Rightarrow$(1): Use Taylor polynomials to extend

Defn: $f \in C^k_{pw} \overset{\text{def}}{\Leftrightarrow} f$ cts on $[a, b]$ and $f|_{[t_{j-1}, t_j]} \in C^k$ for each j (where the t_j's partition $[a, b]$). We say that f is <u>piecewise C^k</u>.

Defn: Let ω be a 1-form on $A^{\text{open}} \subseteq \mathbb{R}^n$, $I = [a, b] \subset \mathbb{R}$, and $\alpha \in C^1_{pw}(I, A)$ (a "path in A").
Then $\displaystyle\int\limits_{Y_\alpha} \omega \overset{\text{def}}{=} \int\limits_I \alpha^* \omega = \int\limits_I (\omega \circ \alpha) D\alpha$.

Rewrite: $\displaystyle\int\limits_{Y_\alpha} \sum_{i=1}^n \omega_i dx_i = \int\limits_I \sum_{i=1}^n \omega_i(\alpha(t)) \frac{dx_i}{dt}$

The idea is that $\vec{x}$ is the position at time t given by $\alpha(t)$, i.e., $x_j = \alpha_j(t)$.

From Monday: For $\alpha \in C^1$, we have $\displaystyle\int\limits_{Y_\alpha} df = \triangle_{Y_\alpha} f \overset{\text{def}}{=} f(\alpha(b)) - f(\alpha(a))$.

Exercise: show this still works for $\alpha \in C^1_{pw}$.

Thm: Given $A^{\text{conn,open}} \subseteq \mathbb{R}^n$, $f \in C^1(A, \mathbb{R})$. Then $df = 0$ on A if and only if f is constant.

HW3#4: Choose $\alpha \in C([0, 1], A)$ such that $\alpha(0) = a$, and $\alpha(1) = b$.
Proof $\Leftarrow$: trivial
Proof $\Rightarrow$: Use $\triangle_{Y_\alpha} f = \sum_j \triangle_{j^{\text{th}} \text{ piece}} f = 0$.

Note: for A open, *disconnected*, $df = 0 \Leftrightarrow f$ is constant on each connected component of A.

Problem: Can we rewrite a 1-form integral $\int_{Y_\alpha} \omega$ as a scalar integral $\int_{Y_\alpha} g ds$?

Recall: $\displaystyle\int\limits_{Y_\alpha} g ds = \int\limits_I (g \circ \alpha) V(D\alpha) = \int\limits_I (g \circ \alpha) \sqrt{\det D\alpha^T D\alpha} = \int\limits_I (g \circ \alpha) \, ||D_\alpha|| = \int\limits_I (g \circ \alpha) \, ||\alpha'||$

$$\int\limits_{Y_\alpha} \omega = \int\limits_I (\omega \circ \alpha) D\alpha = \int\limits_I (\omega \circ \alpha) a'.$$

So match if $(g \circ \alpha)\,||a'|| = (\omega \circ \alpha)\alpha'$, i.e., if for any point $\vec{p} = \alpha(t) \in Y$, we have $g(\vec{p})\,||\alpha'(t)|| = \omega(\vec{p})\alpha'(t)$ – $g(\vec{p}) = \omega(\vec{p}) \cdot \frac{\alpha'(t)}{||\alpha'(t)||}$.

Trouble if $\alpha'(t) = 0$. But if $\alpha \in C^1$, α injective, then $\alpha' \neq 0$.

Set $T : Y \to \mathbb{R}^n$, with $\alpha(t) \mapsto \frac{\alpha'(t)}{||\alpha'(t)||}$. T is the unit tangent vector function.

We get $\int\limits_{Y_\alpha} (\omega \cdot T) ds = \int\limits_{Y_\alpha} \omega$. Reverse: $\int g ds = \int g T^T$.

Thm: FTC1a for 1-forms
Given ω 1-form on $A^{\text{open,conn}} \subset \mathbb{R}^n$.
Then TFAE:

(1) $\omega = df$ for some $f \in C^1(A, \mathbb{R}) \overset{\text{def}}{\Leftrightarrow} \omega$ is <u>exact</u> on A.

(2) $\int_{Y_\alpha} \omega = 0$ when $\alpha \in C^1_{pw}([a,b], A)$ with $\alpha(a) = \alpha(b)$.

(3) $\int_{Y_{\alpha_1}} \omega = \int_{Y_{\alpha_2}} \omega$ when $\alpha_j \in C^1_{pw}([a_j, b_j], A)$ with $\alpha_1(a_1) = \alpha_2(a_2)$ and $\alpha_1(b_1) = \alpha_2(b_2)$.
"Path Independence"

Proof (1)⇒(2): $\int_{Y_\alpha} df \overset{ptxtFTC2}{=} f(\alpha(b)) - f(\alpha(a)) = 0$.

Proof (2)⇒(3): Form a single path α from α_1 and the reverse of α_2.
$\int\limits_{Y_{\alpha_1}} \omega - \int\limits_{Y_{\alpha_2}} \omega = \int\limits_{Y_\alpha} \omega = 0$. So $\int\limits_{Y_{\alpha_1}} \omega = \int\limits_{Y_{\alpha_2}} \omega$

Proof (3)⇒(1): We can define $\int_x^y \omega$ for $x, y \in A$ with $\int_x^y \omega + \int_y^z \omega = \int_x^z \omega$. Fix $x_0 \in A$, define $f(x) = \int_{x_0}^x \omega$.
Claim: $df = \omega$, i.e., $\star = \frac{f(x+h) - f(x) - \omega(x) \cdot h}{||h||} \to 0$ as $h \to 0$.
But $f(x+h) - f(x) = \int_x^{x+h} \omega = \int_0^1 \omega(x + th) \cdot h$. So $\star = \frac{\int_0^1 (\omega(x+th) - \omega(x)) \cdot h dt}{||h||}$
Thus, $||\star|| \leq \max\limits_{0 \leq t \leq h} ||\omega(x + th) - \omega(x)|| \to 0$ as $h \to 0$.

Exercise: this still works for C^1_{pw}, C^k_{pw}, and C^∞_{pw}.

An alternate approach: Need $D_j f = \omega_j$ (a system of partial differential equations).

Recall: Thm 6.3 gives us $f \in C^2 \Rightarrow D_k D_j f = D_j D_k f$. $D_j f = \omega_j$ and $D_k f = \omega_k$. So $D_k \omega_j = D_j \omega_k$.
So ω is C^1 and exact on A, and thus $D_k \omega_j = D_j \omega_k \overset{\text{def}}{\Leftrightarrow} \omega$ is closed on A.

Thm: FTC1b for 1-forms
Given ω C^1 closed 1-form, $\alpha \in C^2$, then $\alpha^* \omega$ closed.
Pf1: Wait for Thm 32.3
Pf2: Read 4-line computation in Lemma J.6