The Extended Reimann Integral

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Consider $f \in C(A^{\text{osso}\mathbb{R}^n}, \mathbb{R})$ (with f and/or A possibly unbounded). For now, assume $f \geq 0$.

Defn: We define the <u>Extended Reimann Integral</u> $\text{ext}\int_A f \overset{\text{def}}{=} \sup\left\{\int_E f : E^{\text{cpt,rect}} \subset A\right\}$.

Lemma: $B^{\text{open}} \subset A^{\text{open}} \Rightarrow \text{ext}\int_B f \leq \text{ext}\int_A f$

What if A and f are bounded?

(i) The old $\int_A f$ may not exist. (It exists if A is rectifiable.)

(ii) $\int_E f = \underline{\int_E} f \leq \underline{\int_A} f \overset{\text{def}}{=} \underline{\int_Q} f_A$. So $\text{ext}\int_A f \leq \underline{\int_A} f$

(iii) Let P be a partition of $Q^{\text{box}} \supset A$. Then $L(f_A, P) \leq \int_{\text{union of } P\text{-boxes}} f \leq \text{ext}\int_A f$

(iv) Thus, $\text{ext}\int_A f = \underline{\int_A} f$. So $\underline{\int_A} f \leq \text{ext}\int_A f$. So $\text{ext}\int_A f = \text{old}\int_A f$ if old $\int_A f$ exists.

How do we compute?

Suppose we have an infinite sequence of compact rectifiable sets $E_1 \subset E_2 \subset E_3 \subset \cdots \subset A$, and $\bigcup_{j=1}^{\infty} \text{Int}\, E_j = A$. Then we claim $\text{ext}\int_A f = \lim_{j\to\infty}\int_{E_j} f$

Ex: $E_j = [-j, 0] \cup [\frac{1}{j}, j]$. Then $\bigcup_{j=1}^{\infty} E_j = \mathbb{R}$, and $\bigcup_{j=1}^{\infty} \text{Int}\, E_j = \mathbb{R} \setminus \{0\}$.

Proof of claim: $\int_{E_j} \leq \text{ext}\int_A f$, so $\lim_{j\to\infty}\int_{E_j} f \leq \text{ext}\int_A f$.

If $E \subset A$ is compact and rectifiable, then $E \subset E_j$ for some j. Thus, $\int_E f \leq \int_{E_j} f \leq \lim_{j\to\infty}\int_{E_j} f$.

Therefore, $\text{ext}\int_E f \leq \lim_{j\to\infty}\int_{E_j} f$. $\square$

Alternate proof (outline): Let E_j be the union of all closed (hyper) cubes with side length $\frac{1}{2^j}$, subsets of A, with each vertex having coords in $\mathbb{Z}/2^j \cap [-j, j]$.

Ex: $\displaystyle\int\limits_{\mathbb{R}} \frac{1}{1+x^2}$

Let $E_j = [-j, j]$. Then $\displaystyle\int\limits_{\mathbb{R}} \frac{1}{1+x^2} = \lim_{j\to\infty} \int\limits_{-j}^{j} \frac{1}{1+x^2} = \lim_{j\to\infty} [\arctan x]_{x=-j}^{x=j} = \lim_{j\to\infty} 2\arctan j = \pi$

Ex: $\displaystyle\int\limits_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} = \lim_{j\to\infty} \int\limits_{-j}^{j}\int\limits_{-j}^{j} \frac{1}{1+x^2+y^2} dxdy = \cdots = \lim_{j\to\infty} \int\limits_{-j}^{j} \frac{2\arctan\frac{j}{\sqrt{1+y^2}}}{\sqrt{1+y^2}} dy$. Ew.

$$\int\limits_{\mathbb{R}^2} \frac{1}{1+x^2+y^2} = \lim_{j\to\infty}\lim_{k\to\infty} \int\limits_{\substack{-k\le x\le k\\ -j\le y\le j}} \frac{1}{1+x^2+y^2} = \lim_{j\to\infty}\lim_{k\to\infty} \int\limits_{-j}^{j}\int\limits_{-k}^{k} \frac{1}{1+x^2+y^2} dxdy =$$

$$= \lim_{j\to\infty}\lim_{k\to\infty} \int\limits_{-j}^{j} \frac{2\arctan\frac{k}{\sqrt{1+y^2}}}{\sqrt{1+y^2}} = \lim_{j\to\infty} \int\limits_{-j}^{j} \frac{\pi}{\sqrt{1+y^2}} dy = \pi \lim_{j\to\infty} \int\limits_{-j}^{j} \frac{1}{\sqrt{1+y^2}} dy = \overset{\text{use } y=\sinh u}{\cdots} =$$

$$= \pi \lim_{j\to\infty} [\operatorname{arcsinh} y]_{y=-j}^{y=j} = +\infty - (-\infty) = +\infty$$

We “computed” this extended integral.