Connectedness

Thomas Cohn

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Prop: The Following are Equivalent (TFAE):
(1) There exists $f : X \to \{0, 1\}$ continuous and surjective.
(2) There exists $A \subset X$ open and closed in X with $\emptyset \neq A \neq X$.

Proof:
(1) $\to$ (2) $A = f^{-1}[\{1\}]$. Clearly, $\emptyset \neq A \neq X$. A is closed because $\{1\}$ is closed and f is continuous. A is open because $A^{\complement} = X \setminus A = f^{-1}[\{0\}]$ is closed.
(2) $\to$ (1) Let $f = \mathbb{I}_A$ (the indicator function for A). $A \neq \emptyset$, and $A \neq X$, so f is surjective. $f^{-1}[\{1\}]$ is open, $f^{-1}[\{0\}]$ is open, so f is continuous.
$\square$

Defn: If this holds, we say that X is <u>disconnected</u>. We say X is <u>connected</u> if it is not disconnected.

Prop: $[0, 1]$ is connected. Proof: Let $f : [0, 1] \to \{0, 1\}$ be continuous. It's enough to show that f is not surjective. Use the intermediate value theorem. $\square$

Defn: A topological space X is said to be <u>path-connected</u> $\leftrightarrow$ $\forall \alpha, \beta \in X$, there is a continuous map $\varphi : [0, 1] \to X$ with $\varphi(0) = \alpha$ and $\varphi(1) = \beta$.

Prop: X is path connected $\to$ X is connected.
Proof: Suppose to the contrary X is path-connected, but $\exists f : X \to \{0, 1\}$ continuous and surjective. Pick $\alpha, \beta \in X$ with $f(\alpha) = 0$, $f(\beta) = 1$, and φ as above. Then $(f \circ \varphi) : [0, 1] \to \{0, 1\}$ is also continuous and surjective. So then $[0, 1]$ is disconnected. Oops! $\square$

Examples and Special Cases:

1. $X \subset \mathbb{R}^n$ is convex $\to$ X is path-connected $\to$ X is connected.
2. $X = \left\{(x, y) \in \mathbb{R}^2 : x > 0, y = \sin\left(\frac{1}{x}\right)\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : x = 0, -1 \leq y \leq 1\right\}$ is connected, but *not* path-connected (proven in a supplement, to be given later).
3. $X \subset \mathbb{R}^n$ is open, connected $\to$ X is path-connected (proved in HW 3).

Given $x \in X$ metric space, $B \subset X$.
Set $d(x, B) = \inf \{d(x, b) : b \in B\}$.
$d(x, B) > 0 \leftrightarrow x \in \text{Ext}\, B$.
$d(x, B) = 0 \leftrightarrow x \notin \text{Ext}\, B \leftrightarrow x \in \text{Int}\, B \cup \text{Bd}\, B \leftrightarrow x \in \bar{B}$.

Fact: $X \to \mathbb{R}$, $x \mapsto d(x, B)$ is Lipschitz, and hence continuous (proved in HW 3).

Defn: Given $A, B \subset X$, $d(A,B) = \inf\{d(a,B) : a \in A\} = \inf\{d(a,b) : a \in A, b \in B\}$.

$d(B,A) = d(A,B)$ and $d(A,B) \geq 0$. But there's no triangle inequality!