Beginning Integration

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On Friday, we proved that given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$, $Hf(\vec{x}) \geq 0\ \forall \vec{x} \in \Omega$, and $Df(\vec{x_0}) = 0$, then $f(\vec{x}) \geq f(\vec{x_0})$ for all $\vec{x} \in \Omega$.

Cor: Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$ and $Hf \geq 0$ on Ω, then $f(\vec{x}) \geq f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) - f(\vec{x_0})$.
This is a strict inequality for $\vec{x} \neq \vec{x_0}$ if $Hf > 0$ on Ω.
Proof: Let $g(\vec{x}) = f(\vec{x}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) - f(\vec{x_0})$.
Then $Dg(\vec{x}) = Df(\vec{x}) - Df(\vec{x_0})$ and $Hg(\vec{x}) = Hf(\vec{x})$.
Notice that $Dg(\vec{x_0}) = \vec{0}$. So $g(\vec{x}) \geq g(\vec{x_0})$. $\square$

Defn: For $\psi : \Omega \to \mathbb{R}$, the <u>epigraph</u> of ψ is $\{(\vec{x}, y) \in \Omega \times \mathbb{R} : y \geq \psi(\vec{x})\}$.

Defn: The opposite of the epigraph is the <u>hypograph</u>.

Cor: Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$ and $Hf \geq 0$, then
$\text{epi}(f) = \bigcap_{\vec{x_0} \in \Omega} \{(\vec{x}, y) \in \Omega \times \mathbb{R} : y \geq f(\vec{x_0}) + Df(\vec{x_0})(\vec{x} - \vec{x_0})\}$
Proof: $(\vec{x}, y) \in \text{epi}(f) \Rightarrow (\vec{x}, y) \in \text{RHS}$ by previous result. So assume $(\vec{x}, y) \in \text{RHS}$ Then let $\vec{x_0} = \vec{x}$; then $y \geq f(\vec{x}) \Rightarrow (\vec{x}, y) \in \text{epi}(f)$. $\square$

Cor: Same hypothesis as above $\Rightarrow$ the epigraph is convex.

Defn: For $\Omega^{\text{convex}} \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$, f is convex $\overset{\text{def}}{\Leftrightarrow}$ $\text{epi}(f)$ is convex.

$$\overset{\text{HW8}}{\Leftrightarrow} f((1-t)\vec{x_0} + t\vec{x_1}) \leq (1-t)f(\vec{x_0}) + tf(\vec{x_1})$$

for $\vec{x_0}, \vec{x_1} \in \Omega$ and $0 \leq t \leq 1$.

Assume $f \in C^2(\Omega, \mathbb{R})$.

$$\begin{aligned}
Hf(\vec{x_0}) \not\geq 0 \quad &\Leftrightarrow \quad \vec{a}^T Hf(\vec{x_0})\vec{a} < 0 \text{ for some } \vec{a}. \\
\text{(Friday)} \quad &\Rightarrow \quad (f \circ \varphi)''(0) < 0 \text{ for } \varphi(t) = \vec{x_0} + t\vec{a} \\
&\Rightarrow \quad \text{epi}(f) \cap \{(\vec{x_0} + t\vec{a}, y) : t, y \in \mathbb{R}\} \text{ is affine, and hence convex.} \\
&\Rightarrow \quad \text{epi}(f) \text{ is } \textit{not} \text{ convex.}
\end{aligned}$$

Cor: Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$, then f is convex if and only if $Hf(\vec{x}) \geq 0$ for all $\vec{x} \in \Omega$.

	$\mathbb{R}$	$\mathbb{R}^n$
Reimann/Darboux	295/297	Munkres/Lecture
Lebesgue	IBL	Later

Lebesgue integration is more robust and coherent.

$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Munkres calls this a rectangle. We'll call it a box.

Defn: $V(Q) \stackrel{\text{def}}{=} (b_1 - a_1) \cdots (b_n - a_n) = \prod_{i=1}^{n} (b_i - a_i)$. In IBL, we may say $m(Q)$.

We want to define $\int_Q f$ for $f : Q \to \mathbb{R}$ (assume Q is bounded).

We hope to have $\int_Q c = c \cdot V(Q)$, and $f \le g \to \int_Q f \le \int_Q g$.

Subdivide each $[a_j, b_j]$ with finitely many partition points. We want $\int_Q f = \sum \int_R f$ for R subbox of Q. So...

Set $m_R(f) = \inf_R f = \inf \{ f(\vec{x}) : \vec{x} \in R \}$
$M_R(f) = \sup_R f$
$L(f, P) \stackrel{\text{def}}{=} \sum_R m_R(f) \cdot V(R)$
$U(f, P) \stackrel{\text{def}}{=} \sum_R M_R(f) \cdot V(R)$

Then we have $L(f, P) \le U(f, P)$.

Defn: P' refines P if and only if P' is obtained from P by adding more partition points.

Then $L(f, P) \le L(f, P') \le U(f, P') \le U(f, P)$.

Lemma: P, P' arbitrary partitions of Q. Then $L(f, P) \le U(f, P')$.
Proof: Let P'' use all partition points in P and P'. Then it refines both, so
$L(f, P) \le L(f, P'') \le U(f, P'') \le U(f, P')$. $\square$

Defn: $\underline{\int_Q} f \stackrel{\text{def}}{=} \sup_P L(f, P)$

Defn: $\overline{\int_Q} f \stackrel{\text{def}}{=} \inf_P U(f, P)$

Defn: Lemma + ${}^{295\#11}/_{297\#12} \Rightarrow \underline{\int_Q} f = \overline{\int_Q} f \stackrel{\text{def}}{=} \int_Q f$ (if they match).

Thm: ("Riemann Criterion" or "Cauchy Criterion" for Integrability)
f is (Riemann)-integrable on $Q \Leftrightarrow \forall \varepsilon > 0, \exists P$ partition s.t. $U(f, P) - L(f, P) < \varepsilon$.