# Beginning Integration

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On Friday, we proved that given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$, $Hf(\vec{x}) \geq 0\ \forall \vec{x} \in \Omega$, and $Df(\vec{x_0}) = 0$, then $f(\vec{x}) \geq f(\vec{x_0})$ for all $\vec{x} \in \Omega$.

**Cor:** Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$ and $Hf \geq 0$ on $\Omega$, then $f(\vec{x}) \geq f(\vec{x_0}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) - f(\vec{x_0})$.
This is a strict inequality for $\vec{x} \neq \vec{x_0}$ if $Hf > 0$ on $\Omega$.
Proof: Let $g(\vec{x}) = f(\vec{x}) - Df(\vec{x_0})(\vec{x} - \vec{x_0}) - f(\vec{x_0})$.
Then $Dg(\vec{x}) = Df(\vec{x}) - Df(\vec{x_0})$ and $Hg(\vec{x}) = Hf(\vec{x})$.
Notice that $Dg(\vec{x_0}) = \vec{0}$. So $g(\vec{x}) \geq g(\vec{x_0})$. □

**Defn:** For $\psi : \Omega \to \mathbb{R}$, the epigraph of $\psi$ is $\{(\vec{x}, y) \in \Omega \times \mathbb{R} : y \geq \psi(\vec{x})\}$.

**Defn:** The opposite of the epigraph is the hypograph.

**Cor:** Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$ and $Hf \geq 0$, then
$\text{epi}(f) = \bigcap_{\vec{x_0} \in \Omega} \{(\vec{x}, y) \in \Omega \times \mathbb{R} : y \geq f(\vec{x_0}) + Df(\vec{x_0})(\vec{x} - \vec{x_0})\}$
Proof: $(\vec{x}, y) \in \text{epi}(f) \Rightarrow (\vec{x}, y) \in \text{RHS}$ by previous result. So assume $(\vec{x}, y) \in \text{RHS}$ Then let $\vec{x_0} = \vec{x}$; then $y \geq f(\vec{x}) \Rightarrow (\vec{x}, y) \in \text{epi}(f)$. □

**Cor:** Same hypothesis as above $\Rightarrow$ the epigraph is convex.

**Defn:** For $\Omega^{\text{convex}} \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$, $f$ is convex $\overset{\text{def}}{\Leftrightarrow}$ epi$(f)$ is convex.
$\overset{\text{HW8}}{\Leftrightarrow} f((1-t)\vec{x_0} + t\vec{x_1}) \leq (1-t)f(\vec{x_0}) + tf(\vec{x_1})$
for $\vec{x_0}, \vec{x_1} \in \Omega$ and $0 \leq t \leq 1$.

Assume $f \in C^2(\Omega, \mathbb{R})$.

$$\begin{array}{rcl} Hf(\vec{x_0}) \not\geq 0 & \Leftrightarrow & \vec{a}^T Hf(\vec{x_0})\vec{a} < 0 \text{ for some } \vec{a}. \\ \text{(Friday)} & \Rightarrow & (f \circ \varphi)''(0) < 0 \text{ for } \varphi(t) = \vec{x_0} + t\vec{a} \\ & \Rightarrow & \text{epi}(f) \cap \{(\vec{x_0} + t\vec{a}, y) : t, y \in \mathbb{R}\} \text{ is affine, and hence convex.} \\ & \Rightarrow & \text{epi}(f) \text{ is } \textit{not} \text{ convex.} \end{array}$$

**Cor:** Given $f \in C^2(\Omega^{\text{convex osso}\mathbb{R}^n}, \mathbb{R})$, then $f$ is convex if and only if $Hf(\vec{x}) \geq 0$ for all $\vec{x} \in \Omega$.

| | $\mathbb{R}$ | $\mathbb{R}^n$ |
|---|---|---|
| Reimann/Darboux | 295/297 | Munkres/Lecture |
| Lebesgue | IBL | Later |

Lebesgue integration is more robust and coherent.

$Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Munkres calls this a rectangle. We'll call it a box.

**Defn:** $V(Q) \overset{\text{def}}{=} (b_1 - a_1) \cdots (b_n - a_n) = \displaystyle\prod_{i=1}^{n} (b_i - a_i)$. In IBL, we may say $m(Q)$.

We want to define $\int_Q f$ for $f : Q \to \mathbb{R}$ (assume $Q$ is bounded).

We hope to have $\int_Q c = c \cdot V(Q)$, and $f \le g \to \int_Q f \le \int_Q g$.

Subdivide each $[a_j, b_j]$ with finitely many partition points. We want $\int_Q f = \sum \int_R f$ for $R$ subbox of $Q$. So...

Set $m_R(f) = \inf\limits_R f = \inf \{f(\vec{x}) : \vec{x} \in R\}$
$M_R(f) = \sup\limits_R f$
$L(f, P) \overset{\text{def}}{=} \sum_R m_R(f) \cdot V(R)$
$U(f, P) \overset{\text{def}}{=} \sum_R M_R(f) \cdot V(R)$

Then we have $L(f, P) \le U(f, P)$.

**Defn:** $P'$ <u>refines</u> $P$ if and only if $P'$ is obtained from $P$ by adding more partition points.

Then $L(f, P) \le L(f, P') \le U(f, P') \le U(f, P)$.

**Lemma:** $P, P'$ arbitrary partitions of $Q$. Then $L(f, P) \le U(f, P')$.
Proof: Let $P''$ use all partition points in $P$ and $P'$. Then it refines both, so $L(f, P) \le L(f, P'') \le U(f, P'') \le U(f, P')$. $\square$

**Defn:** $\displaystyle\underline{\int_Q} f \overset{\text{def}}{=} \sup_P L(f, P)$

**Defn:** $\displaystyle\overline{\int_Q} f \overset{\text{def}}{=} \inf_P U(f, P)$

**Defn:** Lemma + ${}^{295\#11}/_{297\#12} \Rightarrow \displaystyle\underline{\int_Q} f = \overline{\int_Q} f \overset{\text{def}}{=} \int_Q f$ (if they match).

**Thm:** ("Riemann Criterion" or "Cauchy Criterion" for Integrability)
$f$ is (Riemann)-integrable on $Q \Leftrightarrow \forall \varepsilon > 0, \exists P$ partition s.t. $U(f, P) - L(f, P) < \varepsilon$.