The First Fundamental Theorem of Calculus for 1-Forms (Part b)

Thomas Cohn

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Recall from Wednesday:
1-form $\omega = \omega_1 dx_1 + \cdots + \omega_n dx_n$ for ω_i scalar functions. Then
ω is closed $\overset{\text{def}}{\Leftrightarrow} D_k\omega_j = D_j\omega_k$.

$\Leftarrow \omega$ is exact $\overset{\text{def}}{\Leftrightarrow} \omega = df$

$\overset{\text{FTC1a}}{\Leftrightarrow} \displaystyle\int_{Y_\alpha} \omega = 0$ when $\alpha \in C^2_{pw}([a,b], A)$, and $\alpha(a) = \alpha(b)$.

Thm: FTC1b for 1-forms
ω closed 1-form on $A \subseteq \mathbb{R}^n$ open and convex $\Rightarrow \omega$ is exact on A.

Lemma: (1) ω C^1 closed 1-form, α C^1 map $\Rightarrow \alpha^*\omega$ closed.

Lemma: (2) ω C^1 1-form on open set containing $R^{\text{box}} \subseteq \mathbb{R}^2 \Rightarrow \displaystyle\int_{\text{Bd } R(\text{counterclockwise})} \omega = \int_R (D_1\omega_2 - D_2\omega_1)$

Cor: Also assume ω closed. Then $\int_{\text{Bd } R} \omega = 0$.

Ex: $\omega = \frac{-x_2}{x_1^2+x_2^2} dx_1 + \frac{x_1}{x_1^2+x_2^2} dx_2$

Exercise: ω closed on $\mathbb{R}^2 \setminus \left\{\vec{0}\right\}$

Exercise: $\omega = d(\arctan \frac{y}{x})$ on $(0, +\infty) \times \mathbb{R}$

Part for $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$, have $\displaystyle\int_{Y_\alpha} \omega = \int_0^{2\pi} -\sin t d\cos t + \cos t d\sin t = \int_0^{2\pi} 1 dt = 2\pi \neq 0$.

Hence, ω is not exact.

Proof of lemma 2: $\displaystyle\int_{\text{Bd } R} \omega = \int_{a_1}^{b_1} \omega_1(x_1, a_2) dx_1 + \int_{a_2}^{b_2} \omega_2(b_1, x_2) dx_2 - \int_{a_1}^{b_1} \omega_1(x_1, b_2) dx_1 - \int_{a_2}^{b_2} \omega_2(a_1, x_2) dx_2 =$

$\displaystyle = -\int_{a_1}^{b_1}\int_{a_2}^{b_2} D_2\omega_1(x_1, x_2) dx_2 dx_1 + (\text{reverse}) = \int_R D_1\omega_2 - D_2\omega_1.$ ✓

Proof of FTC1b
Check that $\displaystyle\int_{Y_\alpha} \omega = 0$ when $\alpha \in C^2_{pw}([a,b], A)$, $\alpha(a) = \alpha(b)$.

Define $\tilde{\alpha} : [a,b] \times [0,1] \to A$. $\tilde{\alpha}$ is affine on each vertical line segment. $\tilde{\alpha}$ is C^2 on each subbox R_j.
So $\displaystyle\int_{\text{Bd } R_j} \tilde{\alpha}^*\omega = 0$ by lemma 2 corollary. Thus $\displaystyle 0 = \sum \int_{\text{Bd } R_j} \tilde{\alpha}^*\omega = \int_{\text{Bd } R} \tilde{\alpha}^*\omega = \int_{[a,b]} \alpha^*\omega$.

Remark: FTC1b also works for A C^2-diffeomorphic to a convex set.

$$\begin{aligned} \omega \text{ closed on } A &\xRightarrow{\text{Lemma 1}} \gamma^*\omega \text{ closed on } B \\ &\xRightarrow{\text{FTC1b}} \gamma^*\omega = df \text{ on } B \\ &\Rightarrow \beta^*\gamma^*\omega = \beta^* df \\ &\Rightarrow \omega = (\gamma \circ \beta)^*\omega d\beta^* f \end{aligned}$$

Hence, ω closed but *not* exact on $\mathbb{R}^2 \setminus \left\{\vec{0}\right\}$, so $\mathbb{R}^2 \setminus \left\{\vec{0}\right\}$ is not diffeomorphic to a convex set.

Thm: $\exists E \subset [0,1]$ such that

(1) t_1, t_2 distinct rational numbers $\Rightarrow (E + t_1) \cap (E + t_2) = \emptyset$

(2) $\mathbb{R} = \bigcup_{t \in \mathbb{Q}} (E + t)$

Proof: $\mathbb{Q}$ is a subgroup of $\mathbb{R}$. So we get an equivalence relation on $\mathbb{R}$: $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$. Equivalence classes are called cosets. Thus $\mathbb{R}$ is the disjoint union of cosets, where each coset is dense, and each coset C can be written as $C = \mathbb{Q} + x$ for some $x \in C \cap [0,1]$. For each coset, pick such an x (we can do this because of the axiom of choice).
For every $y \in \mathbb{R}$, y has a unique representation $y = x + t$ for $x \in E$, $t \in \mathbb{Q}$.
Therefore, $\mathbb{R} = \bigcup_{t \in \mathbb{Q}} (E + t)$. $\square$

$A^{\text{osso}\mathbb{R}^k} \overset{\alpha}{\to} M \subset \mathbb{R}^n$, $\alpha \in C^r$, and α injective. Then each $D\alpha(\vec{x})$ has maximal rank.