Manifold Boundary is a Manifold-Without-Boundary

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Recall that $\mathbb{H}^k \overset{\text{def}}{=} \{(x_1, \dots, x_k) : x_k \geq 0\}$ and $\mathbb{H}^k_+ \overset{\text{def}}{=} \{(x_1, \dots, x_k) : x_k > 0\} = \operatorname{Int} \mathbb{H}^k$.

Consider U, W (relatively) open susbets of $\mathbb{H}^k$, and $\gamma : U \to W$ a diffeomorphism.
Then $U \cap \mathbb{H}^k_+$ is open in $\mathbb{R}^k$, and $D\gamma(\vec{x})$ is invertible for $\vec{x} \in U \cap \mathbb{H}^k_+$.
So by the inverse function theorem, $\gamma[U \cap \mathbb{H}^k_+]$ is open in $\mathbb{H}^k_+$. Hence, $\gamma[U \cap \mathbb{H}^k_+] \subset \mathbb{H}^k_+$.

We can apply the same argument to γ^{-1}, and we get $\gamma[U \cap \mathbb{H}^k_+] = W \cap \mathbb{H}^k_+$. So because γ is bijective, and $\gamma[U \cap \mathbb{H}^k_+] = W \cap \mathbb{H}^k_+$, we have

$$\gamma\left[U \cap \underbrace{\left(\mathbb{R}^{k-1} \times \{0\}\right)}_{\operatorname{Bd} \mathbb{H}^k}\right] = W \cap \left(\mathbb{R}^{k-1} \cap \{0\}\right)$$

This is used in the proof that ∂M (for manifold M) is a $(k-1)$-manifold-without-boundary.

Consider two subsets U_1, U_2 of $\mathbb{H}^k$, where α_1 and α_2 map them onto M. Then $\alpha_1^{-1} \circ \alpha_2$ is a diffeomorphism that "takes boundary to boundary". We can use $\alpha_1|_{\operatorname{Bd} U_1}$ and $\alpha_2|_{\operatorname{Bd} U_2}$ as coordinate patches for ∂M.

We previously stated the following theorem:

Thm: Every connected C^r 1-manifold is C^r-diffeomorphic to an interval in $\mathbb{R}$ or to S^1.

Cor: Every connected C^r 1-manifold is C^r-diffeomorphic to exactly one of the following:

- $(0, 1)$
- $(0, 1]$
- $[0, 1]$
- S^1

Proof: Let M be a connected 1-manifold, and $x_0 \in M \setminus \partial M$.

Exer: For $x_1 \in M \setminus \{x_0\}$, $\exists I \subseteq M$ such that I is homeomorphic to a closed interval and $\partial I = \{x_0, x_1\}$.
Hint: use path-connectedness, mimic proof of 395 HW3 #4.

Case 1: There's exactly one such I_{x_0,x_1} for each x_1. Then partition $M \setminus \{x_0\}$ into two subsets according to whether I_{x_0,x_1} lies to the "left" or "right" of x_0.

Let $f : M \to \mathbb{R}$ be defined by

$$f(x_1) = \begin{cases} 0 & x_1 = x_0 \\ \operatorname{length}(I_{x_0,x_1}) & x_1 \text{ is to the right of } x_0 \\ -\operatorname{length}(I_{x_0,x_1}) & x_1 \text{ is to the left of } x_0 \end{cases}$$

Check that f is continuous.

So $f[M]$ is connected, and connected subsets of $\mathbb{R}$ are intervals.

Exer: $\left\{y \in f[M] : \#(f^{-1}(y)) = 1\right\}$ is open in $f[M]$, closed in $f[M]$, and nonempty.
So this set is equal to $f[M]$, and thus f is a bijection.

Consider a coordinate patch $\alpha : U \to V^{\text{osso}M}$, and $[t_1, t_2] \subset U$ (with $t_1 \neq t_2$).

$$\int_{[t_1,t_2]} ||D\alpha|| = \text{length}(\alpha[[t_1, t_2]]) = f(\alpha(t_2)) - f(\alpha(t_1))$$

So by the fundamental theorem of calculus, $D(f \circ \alpha) = ||D\alpha|| \geq 0$. In fact, because $t_1 \neq t_2$, $D(f \circ \alpha) = ||D\alpha|| > 0$.

$D\alpha$ is C^{r-1}. $|| \cdot ||$ is C^∞ everywhere except $\vec{0}$. Because $D\alpha$ never reaches $\vec{0}$, we can treat $|| \cdot ||$ as C^∞. So $||D\alpha||$ is C^{r-1}.

$f \circ \alpha$ is C^r, and $D(f \circ \alpha) \neq 0$, so by the inverse function theorem, $(f \circ \alpha)^{-1}$ is C^{r-1}.

$f \circ \alpha$ is C^r, and α^{-1} is C^r, so $(f \circ \alpha) \circ \alpha^{-1} = f$ is C^r. And $\alpha \circ (f \circ \alpha)^{-1} = f$ is C^r. So case 1 works.

Case 2: There are $I_1, I_2 \subseteq M$ homeomorphic to closed intervals where $\partial I_1 = \partial I_2 = \{x_0, x_1\}$ and $I_1 \neq I_2$. WOLOG assume $I_1 \not\subseteq I_2$.

Exer: $I_1 \setminus I_2$ is open in $I_1 \setminus \partial I_1$ (relatively open), closed in $I_1 \setminus \partial I_1$, and nonempty.
This implies $I_1 \setminus I_2 = I_1 \setminus \partial I_1$, i.e., $I_1 \cap I_2 = \{x_0, x_1\}$.

Exer: $I_1 \cup I_2$ is open in M, closed in M, and nonempty.
This implies $I_1 \cup I_2 = M$.

Using the same f as above, $f[M] = [-\text{length}(I_2), \text{length}(I_1)]$. So we have "competing values" for $f(x_1)$. Let

$$t_1 \overset{\text{“}g\text{”}}{\mapsto} \left(\cos \frac{2\pi t}{\ell_1 + \ell_2}, \sin \frac{2\pi t}{\ell_1 + \ell_2}\right)$$

Exer: This composition $g \circ f$ is a diffeomorphism.