

Complex Numbers

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Exterior Calculus in \mathbb{R}^3	Vector Calculus in \mathbb{R}^3
0-form	Scalar Function
3-form	
1-form	Vector Field
2-form	
d	∇
$d(0\text{-form})$	$\text{grad } f = \nabla f$
$d(1\text{-form})$	$\text{curl } \vec{F} = \nabla \times \vec{F}$
$d(2\text{-form})$	$\text{div } \vec{F} = \langle \nabla, \vec{F} \rangle$

$\langle \text{curl } \vec{F}, \vec{N} \rangle$ is the rotation of \vec{F} in the plane perpendicular to \vec{N} based on $\int_M d\omega$.

For M oriented 2-manifold in \mathbb{R}^3 , ω 2-form

$$\int_M \omega = \int_{\vec{p} \in M} \omega(\vec{p})(\vec{v}_1, \vec{v}_2) ds = \int \vec{F} \cdot \vec{N} ds$$

where \vec{F} is the vector field corresponding to ω and $(\vec{v}_1, \vec{v}_2, \vec{N})$ form a positively-oriented orthonormal basis.

We can thus interpret \vec{F} as the velocity of a fluid with unit density, so $\int_M \omega$ is the flux of \vec{F} across M . $\vec{F} \cdot \vec{N} > 0$ implies a positive flow, $\vec{F} \cdot \vec{N} < 0$ implies a negative flow.

If the fluid has density ρ , then the fluid crosses M at rate (flux of $\rho\vec{F}$).

Ex: For compact 3-manifold $U \subset \mathbb{R}^3$, the flow out of U is equal to the flux of $\rho\vec{F}$ across ∂U , which according to Stokes', is $\int_U \text{div } \vec{F}$.

Additionally, we know the flow out of U is $-\frac{\partial}{\partial t} \int_U \rho = -\int \frac{\partial \rho}{\partial t}$ by Leibniz. Since this is true for all U , we have $\frac{\partial \rho}{\partial t} = -\text{div}(\rho\vec{F})$.

Thus, if $\text{div}(\rho\vec{F}) > 0$, the fluid is expanding, and if $\text{div}(\rho\vec{F}) < 0$, the fluid is contracting. If ρ is constant in space and time, then the fluid is incompressible, $\text{div}(\rho\vec{F}) = 0$.

Now, we will talk about Euler's equations for non-viscous (i.e. no fluid friction), incompressible fluids.

$$\begin{aligned} \text{div } \vec{F} &= 0 \\ \frac{\partial \text{curl } \vec{F}}{\partial t} + \text{curl}((\text{curl } \vec{F}) \times \vec{F}) &= \vec{0}. \end{aligned}$$

We can upgrade to include viscosity, and we get Navier-Stokes'

Special Case of Euler's equations: $\operatorname{div} \vec{F} = \vec{0}$, $\operatorname{curl} \vec{F} = \vec{0}$ (incompressible and irrotational).

Even more special case: $\vec{F} = \begin{pmatrix} \alpha(x, y) \\ \beta(x, y) \\ 0 \end{pmatrix}$. Get $\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = 0$ and $\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} = 0$, i.e., $\frac{\partial \alpha}{\partial x} = -\frac{\partial \beta}{\partial y}$, $\frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x}$.

Complex Integrands

Defn: Consider $f : A \rightarrow \mathbb{C}$, where $f = u + iv$ for \mathbb{R} -valued u and v . We say $u \stackrel{\text{def}}{=} \operatorname{Re} f$ and $v \stackrel{\text{def}}{=} \operatorname{Im} f$.

$$\int_A f \stackrel{\text{def}}{=} \int_A u + i \int_A v$$

Use the extended integral when $\operatorname{ext} \int u_+$, $\operatorname{ext} \int u_-$, $\operatorname{ext} \int v_+$, and $\operatorname{ext} \int v_-$ are all finite.

We also have $\int_M f dV = \int_M u dV + i \int_M v dV$.

Exer: $\lambda \in \mathbb{C} \Rightarrow \int_A \lambda f = \lambda \int_A f$

Prop: $\left| \int_A f \right| \leq \int_A |f|$.

Proof: pick θ such that $||f f|| = e^{i\theta} \int f$.

Then $|\int f| = \operatorname{Re} \int e^{i\theta} f = \int \operatorname{Re}(e^{i\theta} f) \leq \int |\operatorname{Re}(e^{i\theta} f)| \leq \int |e^{i\theta} f| = \int |f|$. \square

\mathbb{C} -valued k -forms

Recall that a k -form on $A \subseteq \mathbb{R}^n$ is $\omega = \sum_{I \in \{1, \dots, n\}^k \text{ asc}} b_I(\vec{x}) \Psi_I$ for $b_I \in C(A, \mathbb{R})$.

A \mathbb{C} -valued k -form on A is defined exactly the same, except we have each $b_I \in C(A, \mathbb{C})$.

An \mathbb{R} -valued k -form maps $A \rightarrow \mathcal{A}^k(\mathbb{R}^n) = \{f : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_k \rightarrow \mathbb{R} : f \text{ is } \mathbb{R}\text{-multilinear and alternating}\}$.

Likewise, a \mathbb{C} -valued k -form maps $A \rightarrow \mathcal{A}_{\mathbb{C}}^k(\mathbb{R}^n) = \{f : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_k \rightarrow \mathbb{C} : f \text{ is } \mathbb{R}\text{-multilinear and alternating}\}$.

Defn: If $k = n$, $\omega = b_{(1, \dots, k)} \Psi_{(1, \dots, k)}$, then $\int_A \omega \stackrel{\text{def}}{=} \int_A b_{(1, \dots, k)}$

Rules for \wedge , α^* , and d all carry over.

For $\alpha : A^{\operatorname{osso} \mathbb{R}^k} \rightarrow Y \subseteq \mathbb{R}^n$, Y_α parameterized k -manifold, ω \mathbb{C} -valued k -form, $\int_{Y_\alpha} \omega = \int_A \alpha^* \omega = \dots$.

We also still have $\int_{Y_\alpha} \lambda \omega = \lambda \int_{Y_\alpha} \omega$.

Consider $k = 1$, $\omega : A \rightarrow \mathbb{C}_{\text{row}}^n$.

Exer: We still have the ML estimate: $\left| \int_{Y_\alpha} \omega \right| \leq \operatorname{length}(Y_\alpha) \cdot \sup_{\vec{x} \in Y} ||\omega(\vec{x})||$

Defn: Consider $f = u + iv : A^{\operatorname{osso} \mathbb{R}^k} \rightarrow \mathbb{C} \simeq \mathbb{R}^2$. Then $f \in C^r(A, \mathbb{C})$ if and only if $u, v \in C^r(A, \mathbb{R})$.

Also, $D_j f = D_j u + i D_j v$, and $df \stackrel{\text{def}}{=} du + i dv$.

Now, consider $n = 2$, $z = x + iy$. So $\bar{z} = x - iy$, $dz = dx + idy$, and $d\bar{z} = dx - idy$. Thus, we can write

$$dx = \frac{dz + d\bar{z}}{2} \quad \text{and} \quad dy = \frac{dz - d\bar{z}}{2i}$$

Thus, for α, β, f, g \mathbb{C} -valued functions, we can correspond $\alpha dx + \beta dy$ with $f dz + g d\bar{z}$.

This leads to another question – given $f = u + iv$, when is $f dz$ closed?

Well, $f dz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy)$.

So both $(udx - vdy)$ and $(vdx + udy)$ must be closed separately.