Comparison of Exterior and Vector Calculus

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From Wednesday, we have an orientation of manifold M induces an orietnation of ∂M.

$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$ orientation-preserving on $\mathbb{H}^k$.

$(x_1, \ldots, x_{k-1}) \mapsto (x_1, \ldots, x_{k-1}, 0)$ orientation-preserving on $\partial\mathbb{H}^k$.

Exterior Calculus in $\mathbb{R}^3$	Vector Calculus in $\mathbb{R}^3$
Works well with diffeomorphisms	Works well with isometries (translations and rotations)
0-form f	Scalar Field f
1-form $\alpha\,dx + \beta\,dy + \gamma\,dz$	Vector Field $\vec{F} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$
2-form $\alpha(dy \wedge dz) + \beta(dx \wedge dz) + \gamma(dx \wedge dy)$	Vector Field $\vec{F} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$
3-form $f(dx \wedge dy \wedge dz)$	Scalar Field f
1-form $\wedge$ 2-form	$\left\langle \vec{F_1}, \vec{F_2} \right\rangle$ inner product
1-form $\wedge$ 1-form	$\vec{F_1} \times \vec{F_2}$ cross product
1-form $\wedge$ 1-form $\wedge$ 1-form	$\det(\vec{F_1}, \vec{F_2}, \vec{F_3})$ scalar field
d	$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$
df	$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$
$d(\alpha\,dx + \beta\,dy + \gamma\,dz)$	$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$
$d(\alpha\,dy \wedge dz + \beta\,dx \wedge dz + \gamma\,dx \wedge dy)$	$\operatorname{div} \vec{F} = \left\langle \nabla, \vec{F} \right\rangle$
$\displaystyle\int_M \omega$	$k = 1$ work, circulation $k = 2$ flux across M

Notation clarification: vector field $V : A \to \mathbb{R}^n$.
For a local diffeomorphism, $\alpha^* V(\vec{x}) \stackrel{\text{def}}{=} (D\alpha(\vec{x}))^* V(\alpha(\vec{x}))$

Vector field $\leftrightarrow$ 1-form

$$\begin{array}{ccc} \alpha^*(V^T)(\vec{x}) & \stackrel{?}{=} & (\alpha^* V)^T(\vec{x}) \\ \| & & \| \\ V^*(\alpha(\vec{x}))D\alpha(\vec{x}) & & V^T(\alpha(\vec{x}))(D\alpha(\vec{x})^{-1})^T \end{array}$$

This only works if $D\alpha = ((D\alpha)^{-1})^T \Leftrightarrow D\alpha$ orthonormal $\Leftrightarrow D\alpha$ is an isometry.

So if the coordinate patches are isometries, the distinction between vector fields and 1-forms (exterior and vector calculus) collapses since the pullback rules are the same.