Integrating Factor Examples

Thomas Cohn

1/16/19

Recall: For ω non-zero 1-form on an open susbet of $\mathbb{R}^n$, if there exists some g such that $B\omega = dg$ where B is a continuous non-vanishing "integrating factor", then the level sets $g^{-1}(c)$ of g are integral $(n-1)$-manifolds for g.

Special case: $\omega = u(x,y)dx + v(x,y)dy$ (i.e. $n = 2$).
Then the integral curves for ω are graphs of solutions of $\frac{dy}{dx} = \frac{-u(x,y)}{v(x,y)}$, i.e., $f'(x) = \frac{-u(x,f(x))}{v(x,f(x))}$.

<u>Two Classes of Examples</u>

1) $f'(x) = \beta(f(x))$ $(\star\star)$. Solutions satisfy $\int \frac{dy}{\beta(y)} = x + C$.
We call points where $\beta(y) = 0$ "equilibrium points".

Consider a path from y_0 to y_1 taken from time x_0 to x_1. Then $x_1 - x_0 = \int_M dx$.
$\omega = -\beta(y)dx + dy$ or $\omega = -dx + \frac{dy}{\beta(y)}$.

$$\text{So } x_1 - x_0 = \int\limits_M dx = \int\limits_M dx + \underbrace{\int\limits_M \left(-dx + \frac{dy}{\beta(y)}\right)}_{0} = \int\limits_M \frac{dy}{\beta(y)} = \int\limits_{y_0}^{y_1} \frac{dy}{\beta(y)}$$

Followup: This last integral diverges (in the extended sense) <u>if</u> β is Lipschitz and β vanishes somewhere in the interval $[y_0, y_1)$. So the integral is finite if and only if it's the "non-deterministic" case. Compare this with 395 HW 8 #3 — $\beta(y) = \sqrt[3]{y}$.

2) $f''(x) = \beta(f(x))$ $(\star\star\star)$. This is a particle subject to a force field.
Let $h(x) = f'(x)$. Get

$$\begin{cases} f'(x) = h(x) \\ h'(x) = \beta(f(x)) \end{cases} \quad \text{i.e.} \quad \begin{pmatrix} f \\ h \end{pmatrix}'(x) = \begin{pmatrix} h(x) \\ \beta(f(x)) \end{pmatrix} = \Psi \begin{pmatrix} f(x) \\ h(x) \end{pmatrix} \quad \text{where} \quad \Psi \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} q \\ \beta(p) \end{pmatrix}$$

Let $\alpha : x \mapsto \begin{pmatrix} x \\ f(x) \\ h(x) \end{pmatrix}$ graph parameterization. For $\begin{pmatrix} x \\ y \\ v \end{pmatrix}$ coords in $\mathbb{R}^3$,
$y = f(x)$ and $v = h(x) = f'(x)$.

Thus, Y_α integral $\omega_1 = dy - v\,dx$ and $\omega_2 = dv - \beta(y)\,dx$.
From HW 2: $\omega_1 = x_1\,dx_2 + dx_3$ has no integral 2-manifolds.

Exer: M integral for ω_1 and for $\omega_2 \Rightarrow M$ integral for $f_1\omega_1 + f_2\omega_2$.

Apply to the specific situation $\omega_3 \overset{\text{def}}{=} -\beta(y)\omega_1 + v\omega_2 = \cdots = v\,dv - \beta(y)\,dy = d\left(\frac{v^2}{2} - \int \beta(y)\,dy\right)$.

$$\text{So we can write } \underbrace{\frac{v^2}{2}}_{\text{kinetic}} - \underbrace{\int \beta(y)\,dy}_{\text{potential}} = \underbrace{E}_{\text{energy}}, \text{ i.e., } \beta = \frac{F}{m}.$$

$f'(x) = V = \sqrt{2(E + \int \beta(y)\,dy)}$ "type 1 autonomous"

Use the method for type 1 autonomous equations, get $x + C = \pm \int \frac{dy}{\sqrt{2(E + \int \beta(y)\,dy)}}$.

We still have $y = f(x)$ – try to solve for f.

Ex: $\beta(y) = -y$, $E = \frac{v^2+y^2}{2}$

$$x + C = \pm \int \frac{dy}{\sqrt{2E - y^2}} = \pm \arcsin \frac{y}{\sqrt{2E}}.$$

So $y = \sqrt{2E}\sin(x + C)$ "Simple Harmonic Motion"

Ex: $\beta(y) = -\sin y$, $E = \frac{v^2}{2} - \cos y$ "fritionless pendulum"

$$v^2 = 2(E + \cos y) \to v = \pm\sqrt{2(E + \cos(y))}$$