Proving Facts about the Wedge Product

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Recall:
$I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ k-forms. $\phi_I(\vec{a_J}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$ for $\vec{a_J} = (\vec{a_{j_1}}, \ldots, \vec{a_{j_k}})$, where the $\vec{a_j}$ are a basis for V.
$\phi_I \in \mathscr{L}^k(V)$.
$\operatorname{sgn}(I, J) \stackrel{\text{def}}{=} (-1)^{\#\text{ of transpositions to sort }(I,J)}$.

$\psi_i = \phi_i \in \mathscr{L}^1(V) = \mathcal{A}^1(V) = V^*$. For I asc, $\psi_I = \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot \phi_{I_\sigma} \in \mathcal{A}^k$.

Wedge product $\begin{array}{l} \mathcal{A}^k(V) \times \mathcal{A}^k(V) \to \mathcal{A}^{k+\ell}(V) \\ \qquad (f, g) \mapsto f \wedge g \end{array}$ where

$$\left(\sum_{I\text{ asc}} \alpha_I \psi_I\right) \wedge \left(\sum_{J\text{ asc}} \beta_J \psi_J\right) = \sum_{\substack{I\text{ asc} \\ J\text{ asc} \\ \text{no duplications}}} \alpha_I \beta_J \operatorname{sgn}(I, J)$$

Thm:

(a) $f \wedge g$ linear in f, linear in g.
Proof: clear from definition.

(b) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$.

(c) $g \wedge f = (-1)^{k\ell} f \wedge g$.

(d) $\psi_I = \psi_{i_1} \wedge \cdots \wedge \psi_{i_k}$.
Proof: clear from definition.

(e) $T^*(f \wedge g) = T^* f \wedge T^* g$.

Special case: $\dim V = 3$.

$(\alpha\psi_1 + \beta\psi_2 + \gamma\psi_3) \wedge (\tilde{\alpha}\psi_4 + \tilde{\beta}\psi_5 + \tilde{\gamma}\psi_6) \wedge (\hat{\alpha}\psi_7 + \hat{\beta}\psi_8 + \hat{\gamma}\psi_9) = \cdots = \det \begin{pmatrix} \alpha & \beta & \gamma \\ \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\ \hat{\alpha} & \hat{\beta} & \hat{\gamma} \end{pmatrix}$.
Wedge of 3 $(n-1)$-tensors yields the determinant.

$(\alpha\psi_1 + \beta\psi_2 + \gamma\psi_3) \wedge (A\psi_{(2,3)} \pm B\psi_{(1,3)} + C\psi_{(1,2)}) = \cdots = (\alpha A \pm \beta B + \gamma C)\psi_{(1,2,3)}$.
Wedge of 1-tensor and alternating $(n-1)$-tensor yields the dot product (with minus signs).

$(\alpha\psi_1 + \beta\psi_2 + \gamma\psi_3) \wedge (\tilde{\alpha}\psi_4 + \tilde{\beta}\psi_5 + \tilde{\gamma}\psi_6) = (\alpha\tilde{\beta} - \beta\tilde{\alpha})\psi_{(1,2)} + (\beta\tilde{\gamma} - \gamma\tilde{\beta})\psi_{(2,3)} + (\alpha\tilde{\gamma} - \gamma\tilde{\alpha})\psi_{(1,3)}$.
Wedge of 2 $(n-1)$-tensors yields the cross product.

Proof of (c): Examine the key special case where $f = \psi_I$ and $g = \psi_J$ are basis elements.
Then $\operatorname{sgn}(J, I) = (-1)^{k\ell} \operatorname{sgn}(I, J)$, so $\psi_J \wedge \psi_I = (-1)^{k\ell} \psi_I \wedge \psi_J$. The general case is left as an exercise. $\square$

Proof of (b): Examine the key special case where $f = \psi_I$, $g = \psi_J$, and $h = \psi_K$ are basis elements.
Then $\operatorname{sgn}(I,J)\operatorname{sgn}(\operatorname{sort}(I,J),K) = \operatorname{sgn}(I,\operatorname{sort}(J,K))\operatorname{sgn}(J,K) = \operatorname{sgn}(I,J,K)$.
So $(\psi_I \wedge \psi_J) \wedge \psi_K = \psi_I \wedge (\psi_J \wedge \psi_K)$. The general case is left as an exercise. $\square$

Proof of (e): We need a basis for V, W. Related questions: can we provide a "basis-free" definition of $\wedge$? Does $\wedge$ depend on the choice of basis?

Defn: $\begin{array}{rl} \mathring{A} : \mathscr{L}^k(V) \to \mathcal{A}^k(V) \\ f \mapsto \sum_\sigma \operatorname{sgn}(\sigma) f^\sigma \end{array}$.

Exer: $\mathring{A}\, f \in \mathcal{A}^k(V)$ (see p238)

Note:

a) $\mathring{A}$ doesn't use a basis
b) I asc $\Rightarrow \mathring{A}\,\phi_I = \psi_I$
c) $\#(I_{\text{set}}) < k \Rightarrow \mathring{A}\,\phi_I = 0$
d) $f \in \mathcal{A}^k(V) \Rightarrow \mathring{A}\, f = k! f$

Prop: $f \wedge g = \frac{1}{k!\ell!}\mathring{A}(f \otimes g)$

Cor: Resultant definition of $\wedge$ does not depend on the choice of basis.

Proof of (e), assuming Prop:

$$\begin{aligned} T^*(f \wedge g) &= \frac{1}{k!\ell!} T^* \left(\sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(f \otimes g)^\sigma \right) \\ &= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(T^*(f \otimes g))^\sigma \\ &= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)(T^* f \otimes T^* g)^\sigma \\ &= \frac{1}{k!\ell!} \mathring{A}(T^* f \otimes T^* g) \\ &= T^* f \wedge T^* g \qquad \square \end{aligned}$$

Proof of Prop: Examine the key special case where $f = \psi_I$ and $g = \psi_J$ with I, J asc.
It is enough to show $(\psi_I \wedge \psi_J)(\vec{a}_S) = \frac{1}{k!\ell!}\mathring{A}(\psi_I \otimes \psi_J)(\vec{a}_S)$ for S asc $(k+\ell)$-tuple.

$$LHS = \begin{cases} 0 & I_{\text{set}} \cup J_{\text{set}} \neq S_{\text{set}} \\ \operatorname{sgn}(I,J) & \text{o/w} \end{cases}$$

$$RHS = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \underbrace{\operatorname{sgn}(\sigma)(\psi_I \otimes \psi_J)(\vec{a}_{S_\sigma})}_{*}.$$

$$* = \begin{cases} 0 & I_{\text{set}} \neq (S'_\sigma)_{\text{set}} \vee J_{\text{set}} \neq (S''_\sigma)_{\text{set}} \\ \operatorname{sgn}\sigma \cdot \operatorname{sgn}\sigma' \cdot \operatorname{sgn}\sigma'' \cdots \stackrel{\dagger}{=} \operatorname{sgn}(I,J) & \text{o/w} \end{cases}$$

† because it happens $k!\ell!$ times.

Thus, $RHS = LHS$. $\square$

We've now learned the basics of exterior algebra. It was developed by Grassman in the mid 1800s, with the goal of studying subspaces of vector spaces.

We need to move on to exterior calculus. It was developed by Elie Catan from 1869-1951.