Multilinear Algebra

Thomas Cohn

1/18/19

Recall from last time, when $f''(x) = \beta(f(x))$, we then have $E = \frac{(f'(x))^2}{2} - \int \beta(y)\,dy\big|_{y=f(x)}$ constant.

We get the solution $x + C = \pm \int \left.\frac{dy}{\sqrt{2(E + \int \beta(y)\,dy)}}\right|_{y=f(x)}$.

Special Case: Frictionless Pendulum
$f''(x) = -\sin(f(x))$.
Then $E = \frac{v^2}{2} - \cos y$, so

$$x + C = \pm \int \frac{dy}{\sqrt{2(E + \cos y)}} = \pm \frac{\sqrt{2}\,\text{EllipticF}\left[\frac{y}{2}, \frac{2}{1+E}\right]}{\sqrt{1+E}}$$

Variant: $f''(x) = \beta(f(x)) - \gamma \cdot f'(x)$. Let $E = \frac{v^2}{2} - \cos y$.

Exer: $E'(x) \leq 0$

Multilinear Algebra

Consider an n-dimensional parallelopiped $P \subseteq \mathbb{R}^n$. It's the image of $[0,1]^n$ under $\vec{x} \mapsto M\vec{x} + \vec{b}$, where $M = (\vec{x_1}, \ldots, \vec{x_n})$ linearly independent. Then the volume of P, $V(\vec{x_1}, \ldots, \vec{x_n}) \stackrel{\text{def}}{=} |\det M|$.

Contrast this with $T(\vec{x_1}, \ldots, \vec{x_n}) = \det M = \pm V(\vec{x_1}, \ldots, \vec{x_n})$. This will be positive if and only if the basis belongs to the standard orientation on $\mathbb{R}^n$.

See 296 HW #168,221 and/or 297 HW #136,139.

Let V be a vector space.

Defn: A <u>k-tensor</u> on V is a map $\underbrace{V \times \cdots \times V}_{V^k} \xrightarrow{f} \mathbb{R}$ s.t. $f(\vec{x_1}, \ldots, \vec{x_n})$ is linear with respect to each $\vec{x_j}$.

(We could use fields other than $\mathbb{R}$, but the field of characteristic 2 is problematic.)

Ex: $f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = x_1y_2 + x_2y_1$ is a 2-tensor on $\mathbb{R}^2$.

Ex: $f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = x_1y_1 + x_2y_2$ is **not** a 2-tensor on $\mathbb{R}^2$.

Defn: A k-tensor is <u>symmetric</u> $\stackrel{\text{def}}{\Leftrightarrow}$ $f(\vec{x_1}, \ldots, \vec{x_j}, \ldots, \vec{x_k}, \ldots, \vec{x_n}) = f(\vec{x_1}, \ldots, \vec{x_k}, \ldots, \vec{x_j}, \ldots, \vec{x_n})$.

Defn: A k-tensor is <u>alternating</u> $\stackrel{\text{def}}{\Leftrightarrow}$ $f(\vec{x_1}, \ldots, \vec{x_j}, \ldots, \vec{x_k}, \ldots, \vec{x_n}) = -f(\vec{x_1}, \ldots, \vec{x_k}, \ldots, \vec{x_j}, \ldots, \vec{x_n})$.

Exer: A k-tensor is alternating iff $f(\vec{x_1}, \ldots, \vec{x_n}) = 0$ when there are $j \neq \ell$ s.t. $\vec{x_j} = \vec{x_\ell}$.

A 1-tensor is just a linear map $f : V \to \mathbb{R}$.

Defn: A <u>k-form</u> on $A^{\text{open}} \subseteq V$ is a continuous $\omega : A \to \{\text{alternating } k\text{-tensors on } V\}$.

Note that the set of alternating k-tensors on V is a finite dimensional vector space.