Complex Numbers

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Exterior Calculus in $\mathbb{R}^3$	Vector Calculus in $\mathbb{R}^3$
0-form, 3-form	Scalar Function
1-form, 2-form	Vector Field
d	∇
d(0-form)	$\operatorname{grad} f = \nabla f$
d(1-form)	$\operatorname{curl} \vec{F} = \nabla \times \vec{F}$
d(2-form)	$\operatorname{div} \vec{F} = \left\langle \nabla, \vec{F} \right\rangle$

$\left\langle \operatorname{curl} \vec{F}, \vec{N} \right\rangle$ is the rotation of $\vec{F}$ in the plane perpendicular to $\vec{N}$ based on $\int_M d\omega$.

For M oriented 2-manifold in $\mathbb{R}^3$, ω 2-form

$$\int\limits_M \omega = \int\limits_{\vec{p} \in M} \omega(\vec{p})(\vec{v_1}, \vec{v_2})\, ds = \int \vec{F} \cdot \vec{N}\, ds$$

where $\vec{F}$ is the vector field corresponding to ω and $(\vec{v_1}, \vec{v_2}, \vec{N})$ form a positively-oriented orthonormal basis.

We can thus interpret $\vec{F}$ as the velocity of a fluid with unit density, so $\int_M \omega$ is the flux of $\vec{F}$ across M.
$\vec{F} \cdot \vec{N} > 0$ implies a positive flow, $\vec{F} \cdot \vec{N} < 0$ implies a negative flow.

If the fluid has density ρ, then the fluid crosses M at rate (flux of $\rho\vec{F}$).

Ex: For compact 3-manifold $U \subset \mathbb{R}^3$, the flow out of U is equal to the flux of $\rho\vec{F}$ across ∂U, which according to Stokes', is $\int_U \operatorname{div} \vec{F}$.
Additionally, we know the flow out of U is $-\frac{\partial}{\partial t}\int_U \rho = -\int \frac{\partial \rho}{\partial t}$ by Leibniz. Since this is true for all U, we have $\frac{\partial \rho}{\partial t} = -\operatorname{div}(\rho\vec{F})$.

Thus, if $\operatorname{div}(\rho\vec{F}) > 0$, the fluid is expanding, and if $\operatorname{div}(\rho\vec{F}) < 0$, the fuild is contracting. If ρ is constant in space and time, then the fluid is incompressible, $\operatorname{div}(\rho\vec{F}) = 0$.

Now, we will talk about Euler's equations for non-viscous (i.e. no fluid friction), incompressible fluids.
$\operatorname{div} \vec{F} = 0$
$\frac{\partial \operatorname{curl} \vec{F}}{\partial t} + \operatorname{curl}((\operatorname{curl} \vec{F}) \times \vec{F}) = \vec{0}$.

We can upgrade to include viscosity, and we get Navier-Stokes'

Special Case of Euler's equations: $\operatorname{div} \vec{F} = \vec{0}$, $\operatorname{curl} \vec{F} = \vec{0}$ (incompressible and irrotational).
Even more special case: $\vec{F} = \begin{pmatrix} \alpha(x,y) \\ \beta(x,y) \\ 0 \end{pmatrix}$. Get $\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} = 0$ and $\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} = 0$, i.e., $\frac{\partial \alpha}{\partial x} = -\frac{\partial \beta}{\partial y}$, $\frac{\partial \alpha}{\partial y} = \frac{\partial \beta}{\partial x}$.

Complex Integrands

Defn: Consider $f : A \to C$, where $f = u + iv$ for $\mathbb{R}$-valued u and v. We say $u \overset{\text{def}}{=} \operatorname{Re} f$ and $v \overset{\text{def}}{=} \operatorname{Im} f$.

$$\int\limits_A f \overset{\text{def}}{=} \int\limits_A u + i \int\limits_A v$$

Use the extended integral when ext $\int u_+$, ext $\int u_-$, ext $\int v_+$, and ext $\int v_-$ are all finite.

We also have $\int\limits_M f\, dV = \int\limits_M u\, dV + i \int\limits_M v\, dV$.

Exer: $\lambda \in \mathbb{C} \Rightarrow \int\limits_A \lambda f = \lambda \int\limits_A f$

Prop: $\left| \int\limits_A f \right| \leq \int\limits_A |f|$.

Proof: pick θ such that $\left| \left| \int f \right| \right| = e^{i\theta} \int f$.

Then $\left| \int f \right| = \operatorname{Re} \int e^{i\theta} f = \int \operatorname{Re}(e^{i\theta} f) \leq \int \left| \operatorname{Re}(e^{i\theta} f) \right| \leq \int \left| e^{i\theta} f \right| = \int |f|$. $\square$

$\mathbb{C}$-valued k-forms

Recall that a k-form on $A \subseteq \mathbb{R}^n$ is $\omega = \sum\limits_{I \in \{1,\ldots,n\}^k \text{ asc}} b_I(\vec{x}) \Psi_I$ for $b_I \in C(A, \mathbb{R})$.

A $\mathbb{C}$-valued k-form on A is defined exactly the same, except we have each $b_I \in C(A, \mathbb{C})$.

An $\mathbb{R}$-valued k-form maps $A \to \mathcal{A}^k(\mathbb{R}^n) = \{f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R} : f \text{ is } \mathbb{R}\text{-multilinear and alternating}\}$.

Likewise, a $\mathbb{C}$-valued k-form maps $A \to \mathcal{A}^k_{\mathbb{C}}(\mathbb{R}^n) = \{f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{C} : f \text{ is } \mathbb{R}\text{-multilinear and alternating}\}$.

Defn: If $k = n$, $\omega = b_{(1,\ldots,k)} \Psi_{(1,\ldots,k)}$, then $\int\limits_A \omega \overset{\text{def}}{=} \int\limits_A b_{(1,\ldots,k)}$

Rules for $\wedge$, α^*, and d all carry over.
For $\alpha : A^{\text{osso}\mathbb{R}^k} \to Y \subseteq \mathbb{R}^n$, Y_α parameterized k-manifold, ω $\mathbb{C}$-valued k-form, $\int_{Y_\alpha} \omega = \int_A \alpha^* \omega = \cdots$.
We also still have $\int_{Y_\alpha} \lambda \omega = \lambda \int_{Y_\alpha} \omega$.

Consider $k = 1$, $\omega : A \to \mathbb{C}^n_{\text{row}}$.
Exer: We still have the ML estimate: $\left| \int_{Y_\alpha} \omega \right| \leq \text{length}(Y_\alpha) \cdot \sup\limits_{\vec{x} \in Y} ||\omega(\vec{x})||$

Defn: Consider $f = u + iv : A^{\text{osso}\mathbb{R}^k} \to \mathbb{C} \simeq \mathbb{R}^2$. Then $f \in C^r(A, \mathbb{C})$ if and only if $u, v \in C^r(A, \mathbb{R})$.
Also, $D_j f = D_j u + i D_j v$, and $df \overset{\text{def}}{=} du + idv$.

Now, consider $n = 2$, $z = x + iy$. So $\overline{z} = x - iy$, $dz = dx + idy$, and $d\overline{z} = dx - idy$. Thus, we can write

$$dx = \frac{dz + d\overline{z}}{2} \quad \text{and} \quad dy = \frac{dz - d\overline{z}}{2i}$$

Thus, for α, β, f, g $\mathbb{C}$-valued functions, we can correspond $\alpha dx + \beta dy$ with $f dz + g d\overline{z}$.

This leads to another question – given $f = u + iv$, when is fdz closed?
Well, $fdz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy)$.
So both $(udx - vdy)$ and $(vdx + udy)$ must be closed separately.