# Integral Manifolds

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Throwback to 11/21/18...

Recall $\int_{Y_\alpha} f\,dV$, where $Y_\alpha$ is a parameterized $k$-manifold.
We also want to know what $\int_M f\,dV$, where $M$ is a $k$-manifold.

For now, we will focus on the case where $M$ is compact and $f$ is continuous.

Special case: Assume $\operatorname{supp} f \subset V$ with $\alpha : U \to V \subset M$ coordinate patch.
Then define $\int_M f\,dV = \int_{V_\alpha} f\,dV = \int_U (f \circ \alpha) V(D\alpha)$. This is guaranteed to exist "in the ordinary sense".

**Prop:** This does not depend on our choice of coordinate patch.
Proof: Suppose we also have $\tilde{\alpha} : \tilde{U} \to \tilde{V} \subset M$. We can replace $V$ and $\tilde{V}$ with $V \cap \tilde{V}$, so we may assume $V = \tilde{V}$. $\tilde{\alpha} = \alpha \circ (\alpha^{-1} \circ \tilde{\alpha})$, and $\alpha^{-1} \circ \tilde{\alpha}$ is a transition map, so from a result we proved on 11/21/18, $\int_{V_\alpha} f\,dV = \int_{V_{\tilde{\alpha}}} f\,dV$. $\square$

*But*, what if we require multiple cooridnate patches to cover $\operatorname{supp} f$?

Choose coordinate patches $\alpha_j : U_j \to V_j \subset M$ for $j \in \{1, 2, \ldots, N\}$, with $M = V_1 \cup \cdots \cup V_N$ (we can assume there are a finite number of $V_i$ because $M$ is compact). Write $V_j = M \cap E_j$ with $E_j^{\text{open}} \subseteq \mathbb{R}^n$.

We can write $1 = \varphi_1 + \cdots + \varphi_N$ on $E_1 \cup \cdots \cup E_N$ with $\operatorname{supp} \varphi_j \subset E_j$ and $(\operatorname{supp} \varphi_j) \cap M \subset V_j$.
So $f = f\varphi_1 + \cdots + f\varphi_N$. Thus, we define

**Defn:** $$\int\limits_M f\,dV = \int\limits_M f\varphi_1\,dV + \cdots + \int\limits_M f\varphi_N\,dV$$

Of course, we need to check that we get the same result using $1 = \tilde{\varphi}_1 + \cdots + \tilde{\varphi}_n$.

$$\begin{array}{ccc}
\sum_j \int f\varphi_j\,dV & \stackrel{?}{=} & \sum_k \int f\tilde{\varphi}_k\,dV \\
\| & & \| \\
\sum_j \int f\varphi_j \left(\sum_k \tilde{\varphi}_k\right) dV & & \sum_k \int f\tilde{\varphi}_k \left(\sum_j \varphi_j\right) dV \\
\| & & \| \\
\sum_j \sum_k \int f\varphi_j\tilde{\varphi}_k\,dV & = & \sum_j \sum_k \int f\varphi_j\tilde{\varphi}_k\,dV
\end{array}$$

**Integral Manifolds**

Consider $\omega$, a 1-form on $A^{\text{open}} \subseteq \mathbb{R}^n$.
Then $\omega : A \to (\mathbb{R}^n)^* = \text{Hom}(\mathbb{R}^n, \mathbb{R})$, and $\omega(\vec{p}) : \mathbb{R}^n \to \mathbb{R}$ is a linear map.

Usually, $\dim(\ker(\omega(\vec{p}))) = n - 1$, but sometimes it's $n$. Consider $\vec{p} + \ker(\omega(\vec{p}))$, an affine set.

**Exer:** (HW 1 #3) Prove for a $k$-manifold $M \subset A$ that the following are equivalent:

(a) $\mathcal{T}_p M \subset \ker(\omega(\vec{p}))$, $\forall \vec{p} \in M$

(b) $\alpha^* \omega = 0$, $\forall \alpha$ coordinate patch for $M$

(c) $\int_C \omega = 0$, $\forall C^{\text{1-mfd}} \subset M$

**Defn:** If $M$ satisfies these conditions, we say that $M$ is an <u>integral manifold</u> for $\omega$.