The Wedge Product

Thomas Cohn

1/25/19

Recall: $\mathscr{L}^k(V) = \{f : V^k \to \mathbb{R} \mid f \text{ multilinear}\}$.
$f \in \mathscr{L}^k(V)$ and $g \in \mathscr{L}^\ell(V)$ yields $f \otimes g \in \mathscr{L}^{k+\ell}(V)$.
Rules:

- $f \otimes g$ is linear w.r.t f and g
- $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
- $T^*(f \otimes g) = T^* f \otimes T^* g$
- For $I = (i_1, \ldots, i_k)$, $\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$

$\mathcal{A}^k(V) = \Big\{f \in \mathscr{L}^k(V) \mid f \text{ alternating}\Big\}$.

Given $f \in \mathcal{A}^k(V), g \in \mathcal{A}^\ell(V)$, we don't necessarily have $f \otimes g \in \mathcal{A}^{k+\ell}(V)$.

Thm: There is some map $\begin{aligned} \wedge : \mathcal{A}^k \times \mathcal{A}^k &\to \mathcal{A}^{k+\ell} \\ (f,g) &\mapsto f \wedge g \end{aligned}$ which satisfies

(a) $f \wedge g$ is linear in f and linear in g
(b) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$
(c) $g \wedge f = (-1)^{k\ell} f \wedge g$
(d) $\psi_I = \psi_{i_1} \wedge \cdots \wedge \psi_{i_k}$
(e) $T^*(f \wedge g) = T^* f \wedge T^* g$

From last time, we have a basis for $\mathrm{Alt}^k(V)$ $\psi_I = \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \cdot \psi_{I_\sigma}$ where $I_\sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.
Note that $\psi_i = \phi_i$.

Rules determine the operations:

$$f = \sum_{\substack{I \text{ asc } k\text{-tuple} \\ \text{entries} \in \{1,\ldots,n\}}} \alpha_I \psi_I \qquad\qquad g = \sum_{\substack{J \text{ asc } k\text{-tuple} \\ \text{entries} \in \{1,\ldots,n\}}} \beta_J \psi_J$$

$$f \wedge g = \sum_{\substack{I \text{ asc} \\ J \text{ asc}}} \alpha_I \beta_J \psi_I \wedge \psi_J = \sum_{\substack{I \text{ asc} \\ J \text{ asc} \\ \text{no duplicates} \\ I_{\text{set}} \cap J_{\text{set}} = \emptyset}} \alpha_I \beta_J \operatorname{sgn}(I,J) \psi_{\text{sort}(I,J)}$$

Where $\operatorname{sgn}(I,J) = (-1)^{\#\text{ transpositions to set } (I,J)}$.

Claim: $\wedge$ defined by this formula satisfies conditions (a) through (e).