Introduction to Exterior Calculus

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The previous 3 lectures have covered exterior algebra. Our key object is $\mathcal{A}^k(V) = \mathrm{Alt}^k(V)$. If $\dim V = n$, then $\dim \mathcal{A}^k(V) = \binom{n}{k}$. We can bijectively map a basis of $\mathcal{A}^k(V)$ with ascending k-tuples in $\{1, \dots, n\}$, and with size-k subsets of $\{1, \dots, n\}$. Now, we are ready to move on to exterior calculus.

Defn: Let $U \subseteq \mathbb{R}^n$ be open. (Later, we will consider an n-manifold.)
A $\underline{k\text{-form}}$ on U is a continuous map $\omega : U \to \mathcal{A}^k(\mathbb{R}^n)$.

$$\omega(\vec{x}) = \sum_{\substack{I \text{ asc} \\ k\text{-tuple}}} b_I(\vec{x})\Psi_I = \sum_{\substack{I \text{ asc} \\ k\text{-tuple}}} b_I(\vec{x})\Psi_{i_1} \wedge \cdots \wedge \Psi_{i_k} = \sum_{\substack{I \text{ asc} \\ k\text{-tuple}}} b_I(\vec{x})\, dx_{i_1} \wedge \cdots dx_{i_k}$$

Defn: $\underline{\omega \in C^r} \overset{\text{def}}{\Leftrightarrow} b_I \in C^r$ for all b_I.

Let ω, ν be k-forms on U, ϖ be an ℓ-form on U, and g a scalar function on U. Then

- $\omega + \nu$ is a k-form on U,
- $g\omega$ is a k-form on U, and
- $\omega \wedge \varpi$ is a $(k+\ell)$-form on U.

Given $\Phi \in C^1(U, V)$, ω k-form on V, we get $\Phi^*\omega$ k-form on U defined by $\Phi^*\omega(\vec{x}) = (D\Phi(\vec{x}))^*\omega(\Phi(\vec{x}))$.

Consider the case where $k = 1$. Then $\mathcal{A}^1(V) = \mathscr{L}^1(V) = V^* = \mathbb{R}^n_{\text{row}}$.
We have the basis $\Psi_j = \phi_j : \sum c_k \vec{a_k} \mapsto c_j$, where $\vec{a_j} = \vec{e_j}^T = dx_j$.
Thus, we can define the exterior derivative d, which maps a k-form on U to a $(k+1)$-basis on U.

Our goal is rule 19: $\displaystyle\int_M d\omega = \int_{\partial M} \omega$.

Some words about $k = 0$: our original definition for $\mathcal{A}^0(V)$ doesn't make much sense, but reverse engineering from last week, we find $\dim(\mathcal{A}^0(\mathbb{R}^n)) = 1$. $\mathcal{A}^0(\mathbb{R}^n)$ has basis $\Psi_\emptyset$.

Defn: $\mathcal{A}^0(\mathbb{R}^n) = \mathbb{R}$, $\Psi_\emptyset = 1$.

From the last 395 lecture, rule 19 plus the fact that $\partial\partial M = \emptyset$ leads us to expect $dd\omega = 0$ (rule 17).
We also expect some sort of product rule. This suggests we define

$$d\left(\sum_{I \text{ asc}} b_I(\vec{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = \sum_{I \text{ asc}} db_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

In $\mathbb{R}^2$, this gives us

$$\begin{aligned}
d(\alpha\,dx + \beta\,dy) &= d(\alpha\,dx) + d(\beta\,dy) \\
&= d\alpha \wedge dx + d\beta \wedge dy \\
&= \left(\frac{\partial\alpha}{\partial x}dx + \frac{\partial\alpha}{\partial y}dy\right) \wedge dx + \left(\frac{\partial\beta}{\partial x}dx + \frac{\partial\beta}{\partial y}dy\right) \wedge dy \\
&= \frac{\partial\alpha}{\partial y}dy \wedge dx + \frac{\partial\beta}{\partial x}dx \wedge dy \\
&= -\frac{\partial\alpha}{\partial y}dx \wedge dy + \frac{\partial\beta}{\partial x}dx \wedge dy \\
&= \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx \wedge dy
\end{aligned}$$