Integrating 1-Forms

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In $\mathbb{R}^2$, $d(\alpha\,dx + \beta\,dy) = \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx \wedge dy$.

Thm: Green's Thm (Rectangle Version – Lemma J.7)

$$\int\limits_{\partial R^{\text{box}}} (\alpha\,dx + \beta\,dy) = \int\limits_{R} \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right)$$

Alternatively, $\displaystyle\int\limits_{\partial} \omega = \int\limits_{R} \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) dx \wedge dy = \int\limits_{R} d\omega$.

Defn: For C^r k-form $\omega = \displaystyle\sum_{\substack{I \text{ asc} \\ k\text{-tuple}}} b_I(\vec{x})\,dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $d\omega \overset{\text{def}}{=} \displaystyle\sum_{\substack{I \text{ asc} \\ k\text{-tuple}}} db_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. $d\omega$ is at least C^{r-1}.

1-forms on $\mathbb{R}^n$

We still have ω closed $\Leftrightarrow d\omega = 0$.

Prop: (15) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ (for $\deg(\omega_1) = \deg(\omega_2)$).

Prop: $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg\omega_1}\omega_1 \wedge d\omega_2$ (note that $\deg(\omega_1)$ is not necessarily equal to $\deg(\omega_2)$).
Proof: Note that $d(fg) = f\,dg + g\,df$ for scalar functions f and g (395 rule (14)).
$d(\omega_1 \wedge \omega_2)$ is gross, so we'll go term by term.

$$\begin{aligned} d(\alpha_I\Psi_I \wedge \beta_J\Psi_J) &= d(\alpha_I\beta_J) \wedge \Psi_I \wedge \Psi_J \\ &= d(\alpha_I\beta_J) \wedge \Psi_I \wedge \Psi_J \\ &= d\alpha_I\beta_J \wedge \Psi_I \wedge \Psi_J + \alpha I\,d\beta_J \wedge \Psi_I \wedge \Psi_J \\ &= d(\alpha_I\Psi_I) \wedge (\beta_J\Psi_J) + (-1)^{\deg\omega}(\alpha_I\Psi_I)\,d(\beta_J\Psi_J) \end{aligned}$$

$\square$

Prop: (17) $dd\omega = 0$ (assuming ω is C^2).
Proof: If $\deg\omega = 0$, then we're done because exact 1-forms are closed.
In general, $dd\left(\sum \alpha_I\Psi_I\right) = d\left(\sum d\alpha_I \wedge dx_I\right) = \cancel{\sum dd\alpha_I \wedge dx_I} \pm \cancel{\sum d\alpha_I \wedge ddx_I} = 0$. $\square$

Prop: $d(\Phi^*\omega) = \Phi^* d\omega$.

Proof of prop: We already know this to be true for $\deg\omega = 0$.

In general:

$$\begin{aligned}
d(\Phi^*\omega) &= d\left(\Phi^*\left(\sum b_I dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)\right)\\
&= \sum d\left(\Phi^* b_I\cdot\Phi^*(dx_{i_1})\wedge\cdots\wedge\Phi^*(dx_{i_k})\right)\\
&= \sum d\left(\Phi^* b_I\cdot d(\Phi^* x_{i_1})\wedge\cdots\wedge d(\Phi^*(x_{i_k}))\right)\\
&= \sum \Phi^*(db_I)\cdot\Phi^*(dx_{i_1})\wedge\cdots\wedge\Phi^*(dx_{i_k})\\
&= \Phi^*\left(\sum db_I dx_I\right)\\
&= \Phi^* d\omega
\end{aligned}$$

□

Integration

0.1 Integrating k-forms over Open Subsets of $\mathbb{R}^k$

Let $U^{\text{open}}\subset\mathbb{R}^k$ (or $\mathbb{H}^k$).
Defn: $\omega = f\,dx_1\wedge\cdots\wedge dx_k$. $\int\omega\overset{\text{def}}{=}\int_U f$.

Existence is guaranteed if $\operatorname{supp} f$ is compact (because then we can cover $\operatorname{supp} f$ with finitely many closed boxes contained in U).

Consider $\Phi^{\text{diffeo}}U^{\text{osso}\mathbb{R}^k\text{ or }\mathbb{H}^k}\to V^{\text{osso}\mathbb{R}^k\text{ or }\mathbb{H}^k}$, $\omega = f\,dx_1\wedge\cdots\wedge dx_k$ k-form on V. Then

$$\begin{aligned}
\int\limits_U \Phi^*\omega &= \int\limits_U (\Phi^* f)\Phi^* dx_1\wedge\cdots\wedge\Phi^* dx_k\\
&= \int\limits_U (\Phi^* f)\,d(\Phi_1)\wedge\cdots\wedge d(\Phi_k)\\
&= \int\limits_U f\circ\Phi h(D\Phi)\,dx_1\wedge\cdots\wedge dx_k\\
&= I\int\limits_V f
\end{aligned}$$

Note that $h(D\Phi)$ is an alternating multilinear function of the rows of $D\Phi$. $h(I)=1$, and $h(D\Phi)=\det D\Phi$. So $\Phi^*(dx_1\wedge\cdots\wedge dx_k) = (\det D\Phi)dx_1\wedge\cdots\wedge dx_k$.

Also, I is positive if $\det D\Phi > 0$, and $-$ if $\det D\Phi < 0$. Split it into integrals on the connected components if U is disconnected.

0.2 Integrating k-forms over Parameterized Manifolds

Now, consider paramterized manifolds. Let $\alpha : U^{\text{osso}\mathbb{R}^k} \to Y \overset{\text{def}}{=} \alpha(U) \subset \mathbb{R}^n$, and let ω be a k-form on a neighborhood of Y. Then $\int_{Y_\alpha} \omega = \int_U \alpha^* \omega$.

What if we reparameterize with $\tilde{\alpha} : V^{\text{osso}\mathbb{R}^k} \to Y$, with Φ a diffeomorphic transition map. Then

$$\int\limits_{Y_{\tilde{\alpha}}} \omega = \int\limits_{V} \tilde{\alpha}^* \omega = \pm \int\limits_{U} \Phi^* \tilde{\alpha}^* \omega = I \int\limits_{Y_\alpha} \omega$$

Where I is positive if $\deg D\Phi > 0$, and negative if $\det D\Phi < 0$.

0.3 Integrating k-forms over Manifolds

Let M be a compact k-manifold. We want to find $\int_M \omega$.

Strategy: use partitions of unity to write $\omega = \omega_1 + \cdots + \omega_N$ s.t. $\operatorname{supp} \omega_j \subseteq V_j$ with $\alpha_j : U_j \to V_j$ coordinate patch.

Then set $\int\limits_M \omega = \int\limits_{(V_1)_{\alpha_1}} \omega + \cdots + \int\limits_{(V_N)_{\alpha_N}} \omega_N$.