Stokes' Theorem

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If ω is a C^1 $(k-1)$-form on a neighborhood of a compact oriented manifold M, then $\int_M d\omega = \int_{\partial M} \omega$ with the induced orientation on ∂M.

Proof: Focus on the special case where $\operatorname{supp}\omega \subset V \overset{\alpha}{\leftarrow} U$ for orientation-preserving coordinate patch α, we get the general case with finite sums. For

$$\widetilde{\alpha^*\omega} \overset{\text{def}}{=} \begin{cases} \alpha^*\omega & \text{on } U \\ 0 & \text{on } \mathbb{H} \setminus U \end{cases}$$

$$\int\limits_M d\omega = \int\limits_U \alpha^* d\omega = \int\limits_U d(\alpha^*\omega) = \int\limits_{\mathbb{H}^k} d(\widetilde{\alpha^*\omega})$$

Also note: $\displaystyle\int\limits_{\partial M} \omega = \int\limits_{U \cap \partial\mathbb{H}^k} \widetilde{\alpha^*\omega}$, so we can write

$$\widetilde{\alpha^*\omega} = f_1\, dx_2 \wedge \cdots \wedge dx_k + f_2\, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k + \cdots + f_k\, dx_1 \wedge \cdots \wedge dx_{k-1}$$

Thus,

$$d(\widetilde{\alpha^*\omega}) = (D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k) dx_1 \wedge \cdots \wedge dx_k$$

So $\int_{\mathbb{H}^k} d(\widetilde{\alpha^*\omega}) = \int_{\mathbb{H}^k} D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k$.
Replacing $\mathbb{H}^k$ with the box defined by $a_1 \le x_1 \le b_1, \ldots, a_k \le x_k \le b_k$ (where $a_k = 0$), such that every corner of the box is outside of U, yields

$$\int\limits_{\mathbb{H}^k} d(\widetilde{\alpha^*\omega}) = \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k$$

$$(\text{Fubini}) = \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} \int_{a_1}^{b_1} D_1 f_1 - \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \int_{a_2}^{b_2} D_2 f_2 + \cdots + \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \int_{a_{k-1}}^{b_{k-1}} D_{k-1} f_{k-1} + \int\limits_{\mathbb{H}^k} D_k f_k$$

$$(\text{FTC}) = \int_{a_2}^{b_2} \cdots \int_{a_k}^{b_k} (f_1(b_1) - f_1(a_1)) - \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} (f_2(b_2) - f_2(a_2)) + \cdots + \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} (f_{k-1}(b_{k-1}) - f_{k-1}(a_{k-1})) + \int\limits_{\mathbb{H}^k} D_k f_k$$

$$= 0 - 0 + \cdots + 0 + \int\limits_{\mathbb{H}^k} D_k f_k$$

$$(\text{FTC}) = (-1)^{k-1}(-1) \int\limits_{\mathbb{R}^{k-1}} f_k\, dx_1 \wedge \cdots \wedge dx_{k-1}$$

$$= \cancel{(-1)^k (-1)^k} \int\limits_{\partial\mathbb{H}^k} f_k\, dx_1 \wedge \cdots \wedge dx_{k-1} \qquad ((-1)^k \text{ comes from the induced orientation})$$

$$= \int\limits_{\partial\mathbb{H}^k} \alpha^*\omega \ \square$$

Revisiting Examples from Wednesday:

Ex: Sphere S^2

$$\int_{S^2} dx \wedge dy = \int_{\partial S^2} x\,dy = 0$$

$$\int_{S^2} x\,dx \wedge dy = \int_{\partial S^2} \frac{x^2}{2}\,dy = 0$$

$$\int_{S^2} z\,dx \wedge dy = \int_{B^2(1)} d(z\,dx \wedge dy) = \int_{B^2(1)} dz \wedge dx \wedge dy = \frac{4\pi}{3}$$

Ex: $M = \{(x,y,z) \in \mathbb{R}^3 \;:\; x^2+y^2 = 1, 0 \le z \le 1\}$, $\omega = \frac{-y\,dx + x\,dy}{x^2+y^2} =$ "$d\theta$" (so $d\omega = 0$).

Then $\int_M d\omega = \int_{\partial M} \omega = 2\pi - 2\pi = 0$, which is as expected, since $\int_M 0 = 0$.

Long List of Orientation Special Cases

(1) $M^{n\text{-mfd}} \subseteq \mathbb{R}^n$. Use standard orientation of $\mathbb{R}^n$ to get standard orientation of M.

(2) X oriented $(n-1)$-mfd in $\mathbb{R}^n$ (perhaps $X = \partial M$ for some M n-mfd).
Related question: sorting out "inside" vs "outside".
Recall: $\mathcal{T}_{\vec{p}}X$ is the column space of $D\alpha(\vec{q})$ (for $\vec{q} = \alpha^{-1}(\vec{p})$). $\dim \mathcal{T}_{\vec{p}}X = n-1$.

Defn: $N_{\vec{p}}X \overset{\text{def}}{=} (\mathcal{T}_{\vec{p}}X)^{\perp}$ is called the <u>normal space</u>, and has dimension 1.

(2) (continued) Pick $\vec{N}(\vec{p}) \in N_{\vec{p}}X$ s.t. $\left|\left|\vec{N}(\vec{p})\right|\right| = 1$ and $\det(\vec{N}(\vec{p})|D\alpha(\vec{p})) > 0$ $(\star)$.
Convince yourself that this is independent of choice of orientation-preserving coordinate patch α, and that $X \subset C^r \Rightarrow \vec{N}(\vec{p}) \in C^{r-1}(X, \mathbb{R}^n)$. (For the latter, see p315.)
Conversely, given a coice of unit normal vector field for X, use $(\star)$ as a criterion for α to be orientation-preserving, and get an orientation for X.

(2a) $X = \partial M^{n\text{-mfd}}$. Special case: $M = \mathbb{H}^n$. Then $\vec{N}(\vec{p}) = -\vec{e_n}$ (exercise).
General case: $\vec{N}(\vec{p})$ points out of M, i.e., $\vec{p} + \vec{N}(\vec{p}) \notin M$ for $0 < t < \varepsilon$.