

Manifold Boundary is a Manifold-Without-Boundary

Professor David Barrett

Transcribed by Thomas Cohn

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Recall that $\mathbb{H}^k \stackrel{\text{def}}{=} \{(x_1, \dots, x_k) : x_k \geq 0\}$ and $\mathbb{H}_+^k \stackrel{\text{def}}{=} \{(x_1, \dots, x_k) : x_k > 0\} = \text{Int } \mathbb{H}^k$.

Consider U, W (relatively) open subsets of \mathbb{H}^k , and $\gamma : U \rightarrow W$ a diffeomorphism.

Then $U \cap \mathbb{H}_+^k$ is open in \mathbb{R}^k , and $D\gamma(\vec{x})$ is invertible for $\vec{x} \in U \cap \mathbb{H}_+^k$.

So by the inverse function theorem, $\gamma[U \cap \mathbb{H}_+^k]$ is open in \mathbb{H}_+^k . Hence, $\gamma[U \cap \mathbb{H}_+^k] \subset \mathbb{H}_+^k$.

We can apply the same argument to γ^{-1} , and we get $\gamma[U \cap \mathbb{H}_+^k] = W \cap \mathbb{H}_+^k$. So because γ is bijective, and $\gamma[U \cap \mathbb{H}_+^k] = W \cap \mathbb{H}_+^k$, we have

$$\gamma \left[U \cap \underbrace{(\mathbb{R}^{k-1} \times \{0\})}_{\text{Bd } \mathbb{H}^k} \right] = W \cap (\mathbb{R}^{k-1} \cap \{0\})$$

This is used in the proof that ∂M (for manifold M) is a $(k-1)$ -manifold-without-boundary.

Consider two subsets U_1, U_2 of \mathbb{H}^k , where α_1 and α_2 map them onto M . Then $\alpha_1^{-1} \circ \alpha_2$ is a diffeomorphism that “takes boundary to boundary”. We can use $\alpha_1|_{\text{Bd } U_1}$ and $\alpha_2|_{\text{Bd } U_2}$ as coordinate patches for ∂M .

We previously stated the following theorem:

Thm: Every connected C^r 1-manifold is C^r -diffeomorphic to an interval in \mathbb{R} or to S^1 .

Cor: Every connected C^r 1-manifold is C^r -diffeomorphic to exactly one of the following:

- $(0, 1)$
- $(0, 1]$
- $[0, 1]$
- S^1

Proof: Let M be a connected 1-manifold, and $x_0 \in M \setminus \partial M$.

Exer: For $x_1 \in M \setminus \{x_0\}$, $\exists I \subseteq M$ such that I is homeomorphic to a closed interval and $\partial I = \{x_0, x_1\}$.

Hint: use path-connectedness, mimic proof of 395 HW3 #4.

Case 1: There’s exactly one such I_{x_0, x_1} for each x_1 . Then partition $M \setminus \{x_0\}$ into two subsets according to whether I_{x_0, x_1} lies to the “left” or “right” of x_0 .

Let $f : M \rightarrow \mathbb{R}$ be defined by

$$f(x_1) = \begin{cases} 0 & x_1 = x_0 \\ \text{length}(I_{x_0, x_1}) & x_1 \text{ is to the right of } x_0 \\ -\text{length}(I_{x_0, x_1}) & x_1 \text{ is to the left of } x_0 \end{cases}$$

Check that f is continuous.

So $f[M]$ is connected, and connected subsets of \mathbb{R} are intervals.

Exer: $\{y \in f[M] : \#(f^{-1}(y)) = 1\}$ is open in $f[M]$, closed in $f[M]$, and nonempty.
So this set is equal to $f[M]$, and thus f is a bijection.

Consider a coordinate patch $\alpha : U \rightarrow V^{\text{osso}M}$, and $[t_1, t_2] \subset U$ (with $t_1 \neq t_2$).

$$\int_{[t_1, t_2]} \|D\alpha\| = \text{length}(\alpha[[t_1, t_2]]) = f(\alpha(t_2)) - f(\alpha(t_1))$$

So by the fundamental theorem of calculus, $D(f \circ \alpha) = \|D\alpha\| \geq 0$. In fact, because $t_1 \neq t_2$, $D(f \circ \alpha) = \|D\alpha\| > 0$.

$D\alpha$ is C^{r-1} . $\|\cdot\|$ is C^∞ everywhere except $\vec{0}$. Because $D\alpha$ never reaches $\vec{0}$, we can treat $\|\cdot\|$ as C^∞ . So $\|D\alpha\|$ is C^{r-1} .

$f \circ \alpha$ is C^r , and $D(f \circ \alpha) \neq 0$, so by the inverse function theorem, $(f \circ \alpha)^{-1}$ is C^{r-1} .

$f \circ \alpha$ is C^r , and α^{-1} is C^r , so $(f \circ \alpha) \circ \alpha^{-1} = f$ is C^r . And $\alpha \circ (f \circ \alpha)^{-1} = f$ is C^r . So case 1 works.

Case 2: There are $I_1, I_2 \subseteq M$ homeomorphic to closed intervals where $\partial I_1 = \partial I_2 = \{x_0, x_1\}$ and $I_1 \neq I_2$.
WOLOG assume $I_1 \not\subseteq I_2$.

Exer: $I_1 \setminus I_2$ is open in $I_1 \setminus \partial I_1$ (relatively open), closed in $I_1 \setminus \partial I_1$, and nonempty.
This implies $I_1 \setminus I_2 = I_1 \setminus \partial I_1$, i.e., $I_1 \cap I_2 = \{x_0, x_1\}$.

Exer: $I_1 \cup I_2$ is open in M , closed in M , and nonempty.
This implies $I_1 \cup I_2 = M$.

Using the same f as above, $f[M] = [-\text{length}(I_2), \text{length}(I_1)]$. So we have “competing values” for $f(x_1)$.
Let

$$t_1 \xrightarrow{“g”} \left(\cos \frac{2\pi t}{\ell_1 + \ell_2}, \sin \frac{2\pi t}{\ell_1 + \ell_2} \right)$$

Exer: This composition $g \circ f$ is a diffeomorphism.