More Orientation Special Cases
Vector Calculus vs. Exterior Calculus

Professor David Barrett
Transcribed by Thomas Cohn

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Orientation special caes of a k-manifold in $\mathbb{R}^n$:

(1) $k = n$: See notes from 2/11/19.

(2) $k = n - 1$: See notes from 2/11/19.

(3) $k = 1$: Let X be a 1-manifold. Then an orientation on X can be matched with a choice of continuous "forward-pointing" unit tangent. For orientation-preserving coordinate patch α, and $\vec{p} \in X$, $\vec{q} = \alpha^{-1}(\vec{p})$, we have unit tangent $\vec{T}(\vec{p}) = \frac{\alpha'(\vec{p})}{||\alpha'(\vec{p})||}$.

(3a) $X = \partial M^{\text{2-mfd}}$. Then roughly speaking, when looking at the loop from the "outside" of M, the orientation on X is counterclockwise.

(3b) $M^{\text{2-mfd}} \subseteq \mathbb{R}^2$, $X = \partial M$. Then for tangent and normal vectors $\vec{T}(\vec{p})$ and $\vec{N}(\vec{p})$, $\vec{T}(\vec{p})$ is just $\vec{N}(\vec{p})$ rotated $90°$ counterclockwise.

(4) $k = 0$: Recall that a compact 0-manifold is a finite set. A compact, connected 0-manifold is a singleton. Singletons have 2 orientations (denoted ± 1).
So an orientation on $X^{\text{0-mfd}}$ is just a mapping $\varepsilon : X \to \{\pm 1\}$. For compact oriented 0-manifold X and f 0-form,

$$\int\limits_X f \overset{\text{def}}{=} \sum_{\vec{x} \in X} \varepsilon(\vec{x}) f(\vec{x})$$

If $X = \partial M^{\text{1-mfd}}$, then

$$\int\limits_M df = \int\limits_{\partial M} f = f(b) + (-1)f(a) = f(b) - f(a)$$

Building on HW5 #1:
For M oriented k-manifold, orientation-preserving $\underset{\vec{q} \mapsto \vec{p}}{\alpha : U \to V \subset M}$, ω k-form in a neighborhood of M

$$\star \left\{ \begin{array}{l} \alpha^*\omega = f(\vec{x}) \wedge dx_1 \wedge \cdots \wedge dx_k \\ \omega \left\{ \begin{array}{l} \text{positive for } M \text{ at } \vec{p} \\ \text{negative for } M \text{ at } \vec{p} \\ \text{integral for } M \text{ at } \vec{p} \end{array} \right\} \overset{\text{def}}{\Leftrightarrow} \left\{ \begin{array}{l} f(\vec{q}) > 0 \\ f(\vec{q}) < 0 \\ f(\vec{q}) = 0 \end{array} \right\} \end{array} \right.$$

Exer: ω integral at $\vec{p} \Leftrightarrow \omega(\vec{p})(\vec{v_1}, \ldots, \vec{v_k}) = 0$ when $\vec{v_i} \in \mathcal{T}_{\vec{p}}M$.

M is an integral manifold for $\omega \overset{\text{def}}{\Leftrightarrow} \omega$ is integral for M at all $\vec{p} \in M$.

Conversely, given ω nowhere integral on M, we get an orientation on M. Declare ω to be positive, call α orientation-preserving if $\star$ holds. We also get an orientation on each $\mathcal{T}_{\vec{p}}M$: each basis (or <u>frame</u>) $\vec{v_1}, \ldots, \vec{v_k}$

for $\mathcal{T}_{\vec{p}}M$ is positively oriented $\Leftrightarrow \omega(\vec{p})(\vec{v_1},\ldots,\vec{v_k}) > 0$.

Thm: 36.2 For ω k-form on a neighborhoold of M, a compact oriented k-manifold,

$$\int\limits_M f = \int\limits_M \lambda\, dV \text{ where } \begin{array}{l} \lambda : M \to \mathbb{R} \\ \vec{p} \mapsto \omega(\vec{p})(\vec{v_1},\ldots,\vec{v_k}) \end{array} \quad \text{for any positively-oriented orthonormal basis for } \mathcal{T}_{\vec{p}}M$$

Rceall that we did $k = 1$ on November 28: $\in_M \omega = \int\limits_M \omega \cdot \vec{T}\, ds = \int\limits_M \omega(\vec{T})\, ds$.

We would like to be able to do extended integrals over manifolds. What is $\int_M \omega$ for M some non-compact oriented manifold?

Defn: $\text{ext} \int\limits_M \omega \stackrel{\text{def}}{=} \text{ext} \int\limits_M \lambda\, dV = \underbrace{\text{ext} \int\limits_M \lambda_+\, dV}_{=\sup\left\{\int_N \lambda_+\, dV \,:\, N^{\text{cpt } k\text{-mfd}} \subseteq M\right\}} - \text{ext} \int\limits_M \lambda_-\, dV$

(Note that we allow N to inherit its orientation from M.)

Exterior Calculus in $\mathbb{R}^2$	Vector Calculus in $\mathbb{R}^2$
Diffeomorphisms	Isometries (translations, rotations, and reflections)
0-form f	Scalar Function f
1-form $\omega = \alpha\, dx + \beta\, dy$	Vector field $\vec{F} = (\alpha, \beta) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$
2-form $f\, dx \wedge dy$	Scalar Function f
$df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$	$\text{grad}\, F = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right) f = \nabla f$
$\int\limits_{M^{\text{1-mfd}}} \omega^{\text{1-form}}$	$\int\limits_M \left\langle \vec{F}, d\vec{s} \right\rangle = \int\limits_M \left\langle \vec{F}, \vec{T} \right\rangle ds$ (where $d\vec{s} = \begin{pmatrix} dx & dy \end{pmatrix}$)
$\int\limits_M df = \Delta_M f$	$\int\limits_M \left\langle \nabla f, \vec{T} \right\rangle ds = \Delta_m f$

Standard interpretations: f is potential energy, force is $-\nabla f$, and work is $\int_M \left\langle -\nabla f, \vec{T} \right\rangle ds = -\Delta_m f$.

Suppose $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity of a fluid (which could be time-dependent).