Manifold Orientation

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Defn: An <u>orientation</u> of manifold M is a division of connected coordinate patches into Group A, Group B s.t. $\det D(\alpha_2^{-1} \circ \alpha_1)$ is positive for α_1, α_2 in the same group and negative for α_1, α_2 in different groups.

By convention, Group A patches are "orientation-preserving", and Group B patches are "orientation-reversing".

But how do we specify orientation? It can be awkward. And we want a "default" orientation in certain situations.

Prop: Given M with an orientation, then $\exists \omega$ k-form on a neighborhood of M s.t. $\alpha^* \omega$ is a positive multiple of $dx_1 \wedge \cdots dx_k$ when α is an orientation-preserving coordinate patch, and negative when α is an orientation-reversing coordinate patch.
Proof: HW5

Given ω k-form on a neighborhood of a compact oriented manifold M. To define $\int_M \omega$, use partition of unity to write $\omega = \omega_1 + \cdots + \omega_N$ s.t. $\operatorname{supp} \omega_j \subseteq V_j$ for some orientation-preserving coordinate patch $\alpha_j : U_j \to V_j$.

Claim: This does not depend on choices made.
Proof: same as for $\int_M f\, dV$ on 1/11.

Immediate result: $\int\limits_M c\omega = c \int\limits_M \omega$ and $\int\limits_M \omega_1 + \omega_2 = \int\limits_M \omega_1 + \int\limits_M \omega_2$.
Reverse orientation: replace $\int\limits_M \omega$ by $-\int\limits_M \omega$.

Defn: The inverse of a coordinate patch is called a <u>coordinate chart</u>.

Suppose we're just given $U \xrightarrow{\alpha} V \xrightarrow{\beta} \mathbb{R}^k$.

Check: points at which β fails to be a coordinate patch are points at which $\alpha^*(d\beta_1 \wedge \cdots \wedge d\beta_k) = 0$. Such points contribute nothing to the integral.

Ex: Manifold M in the shape of a chef's hat (open at the bottom). Then

$$\int_M dx \wedge dy = \pi$$
$$\int_M dy \wedge dz = 0$$
$$\int_M dx \wedge dz = 0.$$

Ex: Manifold M is the unit sphere in $\mathbb{R}^3$. Then

$$\int_M dx \wedge dy = 0$$
$$\int_M dx \wedge dz = 0$$
$$\int_M dy \wedge dz = 0$$
$$\int_M x\, dx \wedge dy = 0$$
$$\int_M y\, dx \wedge dy = 0$$
$$\int_M z\, dx \wedge dy = z \int_{\{x^2+y^2<1\}} \sqrt{1-x^2-y^2} = 4\pi \int_0^1 \sqrt{1-r^2} r\, dr = \cdots = \frac{4\pi}{3} \neq 0.$$

Goal: $\int_M d\omega = \int_{\partial M} \omega$. To get this, we need an orientation on M to imply an orientation on ∂M.

For M manifold-with-boundary, $U_1, U_2 \subseteq \mathbb{H}^k$, $\alpha_i : U_i \to M$ orientation-preserving coordinate patches, $V_i \overset{\text{def}}{=} \alpha_i[U_i]$, and $\varphi = \alpha_2^{-1} \circ \alpha_1$ orientation-preserving transition map with $\varphi_k \geq 0$. From 1/9, we know $\varphi_k = 0$ when $x_k = 0$.

Consider patches for ∂M, $\tilde{\alpha}_1 \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \end{pmatrix} = \alpha_j \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ 0 \end{pmatrix}$, $\tilde{\varphi} = \tilde{\alpha_2}^{-1} \circ \tilde{\alpha}_1$.

When $x_k = 0$,

$$D\varphi = \left(\begin{array}{ccc|c} & D\tilde{\varphi} & & \\ \underbrace{\frac{\partial \varphi_k}{\partial x_1} \quad \cdots \quad \frac{\partial \varphi_k}{\partial x_{k-1}}}_{\text{all } 0} & & & \frac{\partial \varphi_k}{\partial x_k} \end{array} \right)$$

$\frac{\partial \varphi_k}{\partial x_k} = \lim_{\substack{h \to 0 \\ (h>0)}} \frac{\overbrace{\varphi_k(\vec{p} + h\vec{e_k})}^{\geq 0} - \cancelto{0}{\varphi_k(\vec{p})}}{h}$. This is non-negative fraction, so it's a non-negative limit, so

$$\underbrace{\deg D\varphi}_{\text{positive}} = \det D\tilde{\varphi} \cdot \frac{\partial \varphi_k}{\partial x_k}$$

Because $\det D\varphi$ is positive, and $\frac{\partial \varphi_k}{\partial x_k} > 0$, we must have $\det D\tilde{\varphi}$ also positive. So we get an orientation on ∂M by declaring $\tilde{\alpha}_j$ to be orientation-preserving.