Fluids in $\mathbb{R}^2$

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$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ (or perhaps $A^{\mathrm{osso}\mathbb{R}^2} \to \mathbb{R}^2$) velocity of a fluid (could be time dependent).
$\left\langle \vec{F}, \vec{T} \right\rangle$ is the tangential velocity. $\int\limits_X \left\langle \vec{F}, \vec{T} \right\rangle ds$ is the circulation of $\vec{F}$ along X.
The average tangential velocity is $\frac{\int_X \langle \vec{F}, \vec{T} \rangle \, ds}{\int_X ds}$.

Suppose X is a circle of radius r. Then the average angular velocity is equal to the average tangential velocity times 1 radian per r units of distance. So it's equal to the circulation divided by $2\pi r^2$.

From Wednesday, for $\vec{F} = (\alpha, \beta) \leftrightarrow \omega = \alpha dx + \beta dy$, we have circulation

$$\int\limits_X \alpha \, dx + \beta \, dy = \int\limits_{\text{disc}} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dx \wedge dy$$

and average angular velocity

$$\frac{\int\limits_{\text{disk}} \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}}{2\pi r^2}$$

As r decreases to 0, we have $\frac{1}{2} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right)$. This is the two-dimensional version of the curl.

Now, let's return to $\mathbb{R}^3$...

Exer: For $\vec{p} \in \mathbb{R}^3$, $\vec{v_1}, \vec{v_2} \in \mathbb{R}^3$ orthonormal, M_r parameterized by $\theta \mapsto \vec{p} + r\cos\theta \vec{v_1} + r \sin\theta \vec{v_2}$, $\vec{F} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}^T$ vector field, $\omega = \alpha dx + \beta dy + \gamma dz$, then the average angular velocity is $\frac{\text{circulation of } \vec{F} \text{ along } M_r}{2\pi r^2}$, and as r decreases to 0, this tends to

$$\frac{1}{2} d\omega(\vec{p})(\vec{v_1}, \vec{v_2}) = \frac{1}{2} \left\langle \operatorname{curl} \vec{F}(\vec{p}), \vec{v_1} \times \vec{v_2} \right\rangle = \frac{1}{2} \det \left(\begin{array}{c:c:c} | & | & | \\ \operatorname{curl} \vec{F}(\vec{p}) & \vec{v_1} & \vec{v_2} \\ | & | & | \end{array} \right)$$

In the general case...

For M compact oriented 2-manifold in $\mathbb{R}^3$, the circulation of $\vec{F}$ along ∂M is

$$\int\limits_{\partial M} \omega = \int\limits_M d\omega = \int\limits_M d\omega(\overbrace{\vec{v_1}, \vec{v_2}}^{\text{positively-oriented basis for } \mathcal{T}_{\vec{p}} M}) \, ds = \int\limits_M \left\langle \operatorname{curl} \vec{F}, \vec{v_1} \times \vec{v_2} \right\rangle ds = \int\limits_M \left\langle \operatorname{curl} \vec{F}, \vec{N} \right\rangle ds$$