Integral Manifolds and Differential Equations

Thomas Cohn

1/14/19

Recall from Friday/HW 1 #3:

Defn: $M \subset A$ is an integral manifold for ω 1-form on $A^{\text{open}} \subset \mathbb{R}^n$ when any of the following conditions are true (TFAE):

(a) $\mathcal{T}_{\vec{p}}M \subset \ker \omega(\vec{p})$, $\forall \vec{p} \in M$

(b) $\alpha^*\omega = 0$, $\forall \alpha$ coordinate patch for M

(c) $\int_C \omega = 0$, $\forall C^{\text{1-mfd}} \subset M$

Ex: Consider $n = 2$...
$f \in C^1(A, \mathbb{R})$, $df \neq 0$ on A.
Then each level set $f^{-1}(c)$ is a 1-mfd-wob.
Then each level set of f is an integral manifold for df.
Proof: $\alpha^*(df) \underset{15}{=} d(\alpha^* f) \underset{4}{=} d(f \circ \alpha) = d(c) \underset{12}{=} 0$

Suppose $\omega = u(x,y)dx + v(x,y)dy$ on $A^{\text{open}} \subset \mathbb{R}^2$ with v non-vanishing.

Given M is a 1-mfd, use a graph paramterization $\begin{array}{l} \alpha : (a,b) \to M \\ \quad x \mapsto (x, f(x)) \end{array}$

All other coord patches β satisfy $\beta = \alpha \circ \underbrace{(\alpha^{-1} \circ \beta)}_{\text{transition map}} = \alpha \circ \gamma$ for γ C^1 diffeomorphism.

That is, $\beta^*\omega \overset{\gamma}{=} \gamma^*(\alpha^*\omega)$.

Thus, M is an integral manifold for ω if and only if $\alpha^*\omega = 0$.
$\alpha^*\omega = u(x, f(x))dx + v(x, f(x))d(f(x)) = u(x, f(x))d + v(x, f(x))f'(x)dx$.
This is 0 if and only if $f'(x) = \frac{-u(x,f(x))}{v(x,f(x))}$. This is a differential equation for f.

Suppose further that $u, v \in C^1$, so $-\frac{u(x,y)}{v(x,y)}$ is C^1. Consider $(0, y_0) \in A$.
Claim: $\exists \Phi \in C^1(\mathbb{R}^2, \mathbb{R})$ with $D\Phi : \mathbb{R}^2 \to \mathbb{R}^2_{\text{row}} = \text{Hom}(\mathbb{R}^2, \mathbb{R})$ bounded and $\Phi(x,y) = -\frac{u(x,y)}{v(x,y)}$ for (x,y) in a neighborhood of $(0, y_0)$.
Proof: Pick a $\Psi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with $\Psi \equiv 1$ in a neighborhood around $(0, y_0)$, and $\operatorname{supp} \Psi$ is a compact subset of A (see notes 11/21 for why we can do this).
Set $\Phi(x,y) = \begin{cases} -\frac{u(x,y)}{v(x,y)} \cdot \Psi(x,y) & (x,y) \in A \\ 0 & (x,y) \notin A \end{cases}$

Exer: Check that this works.

395 HW 3 #2 $\Rightarrow$ Φ is Lipschitz on $\mathbb{R}^2$ $\Rightarrow$ Φ is partial-Lipschitz on $\mathbb{R}^2$
395 HW 10 #6 $\Rightarrow \exists \tilde{\varepsilon} > 0$ s.t. $f'(x) = \Phi(x, f(x))$ with $f(0) = y_0$ has a unique solution for $x \in (-\tilde{\varepsilon}, \tilde{\varepsilon})$ (perhaps shrinking $\tilde{\varepsilon}$ if necessary).

Result: $\exists \varepsilon > 0$ such that $f'(x) = -\frac{u(x,f(x))}{v(x,f(x))}$ has a unique solution for $x \in (-\varepsilon, \varepsilon)$. So $\exists!$ local integral curve for ω passing through $(0, y_0)$.

Claim: Get the same result based at (x_0, y_0).
Proof: Use a translation in the x-direction.

395 HW 11 #6: $\omega = x^{1/3}dx - dy$ is a C^0 but not C^1 1-form. ω does not have unique solutions.

Conversely, given a differential equation $f'(x) = \Phi(x, f(x))$ $\star$
Then graphs of solutions of $(\star)$ are integral curves for $\omega = -B(x,y)\Phi(x,y)dx + B(x,y)dy$.
I.e. "$0 = -B(x,y)\Phi(x,y) + B(x,y)\frac{dy}{dx}$".
Suppose we can choose non-zero B such that $B\omega$ is exact, i.e., $B\omega = dg$ for some g.

Defn: Then B is an <u>integrating factor</u> for $-\Phi(x,y)dx + dy$.

From earlier in lecture, the level curves $g^{-1}(c)$ are integral manifolds for dg, which are integral manifolds for $-\Phi(x,y)dx + dy$, which are graphs of solutions of $(\star)$.

Conversely, f solves $(\star) \Rightarrow \frac{d}{dx}g(x, f(x)) = \begin{pmatrix} -B(x,f(x)) & \Phi(x,f(x)) \end{pmatrix} \begin{pmatrix} 1 \\ \Phi(x,f(x)) \end{pmatrix} = 0.$
So $g(x, f(x))$ is (locally) constant!

Good News 1: Such a B always exists!
Good News 2: Looking for such a B is often a useful approach to solving $(\star)$!
Bad News: It's not always easier to find B than to solve $(\star)$.

Two important classes of examples:

1 $f'(x) = \beta(f(x))$ $(\star\star)$, an "autonomous differential equation". Solutions are integral curves for $-\beta(y)dx - dy$. Use integrating factor $B(x,y) = \frac{1}{\beta(y)}$ (assume for now that β doesn't vanish).
Solutions: integral cuves for $-dx + \frac{dy}{\beta(y)} = d(-x - \int \frac{dy}{\beta(y)})$.
Solutions of $(\star\star)$ satisfy $-x + \int \frac{dy}{\beta(y)} = C$, i.e., $\int \frac{dy}{\beta(y)} = x + C$.

Exer: Try to solve for y using implicit function theorem if there's no nice closed form solution as a function of y.