TITLE

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Cor: For M as in the Cauchy Integral Theorem,

$$\int\limits_{\partial M} \frac{dz}{z-z_0} = \begin{cases} 0 & z_0 \notin M \\ 2\pi i & z_0 \in M \setminus \partial M \\ \text{Diverges} & z_0 \in \partial M \end{cases}$$

Cor: (Once Differentiated CIF)
For the same setup as above,

$$g'_{\mathbb{C}}(z_0) = \frac{1}{2\pi i} \int\limits_{\partial M} \frac{g(z)}{(z-z_0)^2}\, dz$$

Proof: We need $\dfrac{g(z_0+h)-g(z_0)}{h} \to \dfrac{1}{2\pi i} \displaystyle\int\limits_{\partial M} \frac{g(z)}{z-z_0}\, dz$ as $h \to 0$.
Well, $\dfrac{g(z_0+h)-g(z_0)}{h} = \dfrac{1}{2\pi i h} \displaystyle\int\limits_{\partial M} g(z)\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) dz$. And

$$\begin{aligned} |LHS - RHS| &= \left| \frac{1}{2\pi i} \int\limits_{\partial M} g(z)\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) dz \right| \\ &= \left| \frac{1}{2\pi i} \int\limits_{\partial M} g(z)\left(\frac{1}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) - \frac{1}{(z-z_0)^2}\right) dz \right| \\ &= \left| \frac{1}{2\pi i} \int\limits_{\partial M} g(z)\left(\frac{h}{(z-z_0-h)(z-z_0)^2}\right) dz \right| \\ &\overset{\text{ML}}{\leq} \frac{\mathscr{C}(\partial M)}{2\pi} |h| \sup_{\partial M} |g| \frac{1}{(d(z_0,\partial M)-h)(d(z_0,\partial M)^2)} \end{aligned}$$

Which goes to 0 as $h \to 0$ as required. □

Additionally,

1) $g''_{\mathbb{C}}(z_0) = \dfrac{1}{2\pi i} \displaystyle\int\limits_{\partial M} \frac{zg(z)}{(z-z_0)^3}\, dz$

2) In particular, we know that $g''_{\mathbb{C}}(z_0)$

Cor: g holomorphic $\Rightarrow$ g infinitely $\mathbb{C}$-differentiable ($\Rightarrow$ g infinitely $\mathbb{R}$-differentiable)
Proof 1: Induction
Proof 2: $g''^{(m)}_{\mathbb{C}}(z_0) = \dfrac{m!}{2\pi i} \displaystyle\int\limits_{\partial M} \frac{g(z)}{(z-z_0)^m}\, \mathrm{dz}$

Thm: (Taylor's Theorem) $f(z)$ holomorphic at z_0, $|z_0 - z| < \delta$. Then $f(z) = \displaystyle\sum_{k=0}^{\infty} \frac{f_{\mathbb{C}}^{(k)}(z_0)}{k!}(z-z_0)^k \quad \star$

At this point, we need to mention that for $0 < r < p$, we have $\star$ converges uniformly on $|z - z_0| \leq r$ if and only if $\star$ converges uniformly on each $K^{\text{cpt}} \subseteq U(z_0, \delta)$ if and only if $\star$ converges almost uniformly on $U(z_0, \delta)$.

1. Series could converge but not to f
2. Series might not converge (except at z_0)
3. In $\mathbb{R}^m$ we have Taylor's Theorem with Remainder

Proof of Taylor's Theorem: Pick $0 < r < \tilde{r} < \delta$. Then because $|z - z_0| \leq r$,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int\limits_{|\mathscr{S} - z_0| = \tilde{r}} \frac{f(\mathscr{S})}{\mathscr{S} - z_0}\, d\mathscr{S} \\
&= \frac{1}{2\pi i} \int\limits_{|\mathscr{S} - z_0| = \tilde{r}} \frac{1}{\mathscr{S} - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\mathscr{S} - z_0}} f(\mathscr{S})\, d\mathscr{S} \\
&\overset{\star}{=} \frac{1}{2\pi i} \int\limits_{|\mathscr{S} - z_0| = \tilde{r}} \frac{1}{\mathscr{S} - z_0} \left(\sum_{k=0}^{\infty} \left(\frac{z - z_0}{\mathscr{S} - z_0} \right)^k f(\mathscr{S}) \right) d\mathscr{S} \\
&= \sum_{k=0}^{\infty} (z - z_0)^k \frac{1}{2\pi i} \int\limits_{|\mathscr{S} - z_0| = \tilde{r}} \frac{f(\mathscr{S})}{(\mathscr{S} - z_0)^{k+1}}\, dy \\
&= \sum_{k=0}^{\infty} (z - z_0)^k \frac{f_{\mathbb{C}}^{(k)}(z_0)}{k!}
\end{aligned}$$