# Multilinear Algebra

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1/23/19

Recall: a $k$-tensor on vector space $V$ is a multilinear map $f : \underbrace{V \times \cdots \times V}_{V^k} \to \mathbb{R}$.

**Defn:** $\mathscr{L}^k(V)$ is defined to be the set of all $k$-tensors on $V$.

**Defn:** $\mathrm{Sym}^k(V) \overset{\text{def}}{=} \left\{ f \in \mathscr{L}^k(V) : f \text{ is symmetric} \right\}$

**Defn:** $\mathcal{A}^k(V) \overset{\text{def}}{=} \left\{ f \in \mathscr{L}^k(V) : f \text{ is alternating} \right\}$. Sometimes written as $\mathrm{Alt}^k(V)$.

Recall: $\mathscr{L}^1(V) = \mathrm{Sym}^1(V) = A^1 = V^*$.

Suppose $\vec{a_1}, \ldots, \vec{a_n}$ are a basis for $V$. We can write $\vec{a} \in V$ as $\vec{a} = \sum_{j=1}^n c_j \vec{a_j}$. So for $f \in \mathscr{L}^k$,

$$\begin{aligned}
f(\vec{v_1}, \ldots, \vec{v_k}) &= f\left( \sum_{j_1=1}^n c_{1,j_1} \vec{a_{j_1}}, \ldots, \sum_{j_k=1}^n c_{k,j_k} \vec{a_{j_k}} \right) \\
&= c_{1,1} f\left( \vec{a_1}, \sum_{j_2=1}^n c_{2,j_2} \vec{a_{j_2}}, \ldots, \sum_{j_k=1}^n c_{n,j_k} \vec{a_{j_k}} \right) \\
&\quad + c_{1,2} f\left( \vec{a_2}, \sum_{j_2=1}^n c_{2,j_2} \vec{a_{j_2}}, \ldots, \sum_{j_k=1}^n c_{n,j_k} \vec{a_{j_k}} \right) \\
&\quad + \quad \vdots \\
&\quad + c_{1,n} f\left( \vec{a_n}, \sum_{j_2=1}^n c_{2,j_2} \vec{a_{j_2}}, \ldots, \sum_{j_k=1}^n c_{n,j_k} \vec{a_{j_k}} \right) \\
\cdots &= \sum_{j_1, j_2, \ldots, j_k \in \{1, \ldots, n\}} c_{1,j_1} c_{2,j_2} \cdots c_{k,j_k} f(\vec{a_{j_1}}, \ldots, \vec{a_{j_k}})
\end{aligned}$$

Where the final sum is over every possible combination of one coefficient from every $j_\ell$.
So $f$ is determined by $f(\vec{a_{j_1}}, \ldots, \vec{a_{j_k}})$.

Let $I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k = \underbrace{\{1, \ldots, n\} \times \cdots \times \{1, \ldots, n\}}_{k}$.

**Defn:** $\phi_I : \left( \sum_{j_1=1}^n c_{1,j_1} \vec{a_{j_1}}, \ldots, \sum_{j_k=1}^n C_{k,j_k} \vec{a_{j_k}} \right) \mapsto c_{1,j_1} \cdots c_{k,j_k}$.

**Exer:** $\phi_I \in \mathscr{L}^k(V)$

**Exer:** $\phi_I(\vec{a_J}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$

**Exer:** $\left\{\phi_I : I \in \{1, \ldots, n\}^k\right\}$ is linearly independent.

**Prop:** $f \in \mathscr{L}^k(V) \Rightarrow f = \sum_I f(\vec{a_I})\phi_I$

Proof follows from the uniqueness result before.

Thus, $\{\phi_I\}_{I\in\{1,\ldots,n\}^k}$ is a <u>basis</u> for $\mathscr{L}^k(V)$. $\dim(\mathscr{L}^k(V)) = n^k = (\dim V)^k$.

We can pick any constants $C_I$, and get $f = \sum c_I \phi_I$ with $f(\vec{a_I}) = c_I$.

**Defn:** Given $f \in \mathscr{L}^k(V), g \in \mathscr{L}^\ell(V)$, we say $f \otimes g \in \mathscr{L}^{k+\ell}(V)$, where $(f \otimes g)(\vec{v_1}, \ldots, \vec{v_{k+\ell}}) = f(\vec{v_1}, \ldots, \vec{v_k})g(\vec{v_{k+1}}, \ldots, \vec{v_{k+\ell}})$.

Some rules:

- $f \otimes (g \otimes h) = (f \otimes g) \otimes h$
- $(cf) \otimes g = c(f \otimes g) = f \otimes (cg)$
- $(f + g) \otimes h = (f \otimes h) + (g \otimes h)$
- $f \otimes (g + h) = (f \otimes g) + (f \otimes h)$

Also, $\phi_I = \phi_{i_1} \otimes \cdots \otimes \phi_{i_k}$ with each $\phi_{i_\ell} \in \mathscr{L}^1(V) = V^*$.

So $T : V \to W$ linear induces $T^* : \mathscr{L}^k(W) \to \mathscr{L}^k(V)$ by $(T^*f)(\vec{v_1}, \ldots, \vec{v_k}) = f(T(\vec{v_1}), \ldots, T(\vec{v_k}))$. Rules:

- $T^*$ is linear
- $T^*(f \otimes g) = T^*f \otimes T^*g$
- $(S \circ T)^*f = T^*(S^*f)$

Let $f \in \mathrm{Alt}^k(V)$ and $|\{i_1, \ldots, i_k\}| < k$ (i.e., there's a repitition). Then $f(a_I) = 0$.

**Defn:** $I \in \{1, \ldots, n\}^k$ is <u>ascending</u> $\overset{\text{def}}{\Leftrightarrow} 1 \le i_1 < i_2 < \cdots < i_k$

**Defn:** $J = \{1, \ldots, n\}^k$ and $|\{j_1, \ldots, j_k\}| = k \Rightarrow \exists!$ ascending $\sigma \in S_k$ s.t. $J = I_\sigma \overset{\text{def}}{=} (i_{\sigma(1)}, \ldots, i_{\sigma(k)})$.

$$
\begin{aligned}
f &= \sum_{I\in\{1,\ldots,n\}^k} f(a_I)\phi_I \\
&= \sum_{I\in\{1,\ldots,n\}^k \text{ asc.}} \left[\sum_{\sigma\in S_k} f(\vec{a_{I_\sigma}})\phi_{I_\sigma}\right] \\
&= \sum_{I \text{ asc.}} \left[\sum_{\sigma\in S_k} \mathrm{sgn}_\sigma\, f(\vec{a_I})\phi_{I_\sigma}\right] \\
&= \sum_{I \text{ asc.}} \left[ f(\vec{a_I}) \underbrace{\sum_{\sigma\in S_k} \mathrm{sgn}_\sigma\, \phi_{I_\sigma}}_{\overset{\text{def}}{=}\psi_I} \right] \\
&= \sum_{I \text{ asc.}} f(\vec{a_I})\psi_I
\end{aligned}
$$

Thus, $\{\psi_I\}_{I \text{ asc.}}$ forms a basis for $\mathcal{A}^k(V)$.
$k > n \Rightarrow \mathcal{A}^k(V) = \{0\}$.
$1 \le k \le n \Rightarrow \dim \mathcal{A}^k(V)$ is the number of ascending $k$-tuples in $\{1, \ldots, n\}$.

Specialize to $V = \mathbb{R}^n$.
For $M = (\vec{x_1}, \dots, \vec{x_n}) \in \mathrm{Mat}(n,n)$, we have $\det(M) = \psi_{(1,\dots,n)}(\vec{x_1}, \dots, \vec{x_n})$.

Check: $\det I_n = \psi_{(1,\dots,n)}(\vec{e_1}, \dots, \vec{e_n}) = 1$.

We want some sort of product $\mathcal{A}^k \wedge \mathcal{A}^\ell \to \mathcal{A}^{k+\ell}$.