Beginnings of Complex Analysis

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Recall, $\begin{array}{l} dz = dx + idy \\ d\overline{z} = dx - idy \end{array} \quad \Rightarrow \quad \begin{array}{l} dx = \frac{dz+d\overline{z}}{2} \\ dy = \frac{dz-d\overline{z}}{2i} \end{array}$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[dr] & \mathbb{C} \arrow[d, leftrightarrow] \\ \begin{pmatrix} u \\ v \end{pmatrix} & \mathbb{R}^2 \end{tikzcd}$$

So we have $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, with $\begin{pmatrix} x \\ y \end{pmatrix}$ mapping to $x + iy$.

When is fdz closed? Well, $fdz = (u+iv)(dx+idy) = (u+iv)dx + (-v+iu)dy$.
So we need $D_1(-v+iu) = D_2(u+iv)$, i.e., $D_1u = D_2v$ and $D_2u = -D_1v$.
This is equivalent to $\begin{pmatrix} u \\ -v \end{pmatrix}$ is incompressible and irrotational.
This is also equivalent to $(u+iv)d\overline{z}$ is closed.

From §J, we have
Thm: Given $f \in C^1(A^{\text{osso}\mathbb{R}^2}, \mathbb{C})$, A diffeomorphic to a convex set, then $f(z)dz$ is exact if and only if f satisfies the CR equations.

Ex: $\frac{dz}{z} = \left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right)dz$. Check that the CR equations hold.
$\frac{dz}{z} = \left(\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy\right) + i\left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) = d\ln(\sqrt{x^2+y^2}) +$ "$id\theta$".
Choose a ray X starting at 0. Then θ_X is the anti-derivative for the above on $\mathbb{R}^2 \setminus X$.
We get $\frac{dz}{z} = d(\ln(r) + i\theta_X)$ on $\mathbb{R}^2 \setminus X$.
This suggests $\ln_X(z) \overset{\text{def}}{=} \ln(r) + i\theta_X$.

Consider V, W finite-dimensional $\mathbb{R}$-v.s., $f : A^{\text{osso}V} \to W$ diffeomorphic at $\vec{x}_0 \in A$.
Then we have the affine approximation $\star$ $f(\vec{x}) \approx f(\vec{x}_0) + Df(\vec{x}_0)(\vec{x} - \vec{x}_0)$.

Defn: For V, W $\mathbb{C}$-v.s., $f : V \to W$ is $\mathbb{C}$-linear $\overset{\text{def}}{\Leftrightarrow}$ $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$ and $f(\lambda\vec{v}) = \lambda f(\vec{v})$ for $\lambda \in \mathbb{C}$.

More formally, $Df(\vec{x_0}) = T \in \text{Hom}_{\mathbb{R}}(V, W)$ means $\lim_{\vec{x} \to \vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0) - T(\vec{x} - \vec{x}_0)}{||\vec{x} - \vec{x}_0||} = 0$. The definition of $\mathbb{C}$-differentiation is exactly the same, except for $T \in \text{Hom}_{\mathbb{C}}(V, W)$. So f is $\mathbb{C}$-differentiable at $\vec{p}$ if and only if f is $\mathbb{R}$-differentiable at $\vec{p}$ and $Df(\vec{p})$ is $\mathbb{C}$-linear.

Now, let $V = W = \mathbb{R}^2 \simeq \mathbb{C}$. Then $Df\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} D_1u & D_2v \\ D_1v & D_2v \end{bmatrix}$. Multiplying by $\alpha + i\beta$ is equivalent with multiplying by $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.

f is $\mathbb{C}$-differentiable at $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Leftrightarrow f$ is $\mathbb{R}$-differentiable and the CR equations hold.

If f is $\mathbb{C}$-differentiable at $\vec{p}$, let $f'_{\mathbb{C}}(z_0) = (D_1 u + iD_1 v)(z_0)$.
So we have the $\mathbb{C}$-affine approximation $f(z) \approx f(z_0) + f'_{\mathbb{C}}(z_0)(z - z_0)$.

Exer: For $f : A^{\text{osso}\mathbb{C}} \to \mathbb{C}$, we have f $\mathbb{C}$-differentiable at $z_0 \in A$ iff $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists (is finite). In this case, we call that limit $f'_{\mathbb{C}}(z_0)$.

Thm: For $f = u + iv : A^{\text{osso}\mathbb{C}} \to \mathbb{C}$, the following are equivalent:

1) f is C^1 (in the real sense) and $f dz$ is closed.
2) $u, v \in C^1(A, \mathbb{R})$ satisfies the CR equations.
3) f is $\mathbb{C}$-differentiable at each point of A and $f'_{\mathbb{C}} : A \to \mathbb{C}$ is continuous.

Defn: If the above holds, we say f is holomorphic (or analytic, complex-analytic) on A.

Cauchy's Integral Theorem (v.1)

Given $A^{\text{osso}\mathbb{C}}$ diffeomorphic to a convex set, f holomorphic on A, $\alpha : [a,b] \to A$ piecewise C^1, and $\alpha(a) = \alpha(b)$, then $\int_{Y_\alpha} f\, dz = 0$.
Proof: Hyp $\overset{J.8}{\Rightarrow}$ $f dz$ exact $\overset{J.1}{\Rightarrow}$ $\int_{Y_\alpha} f\, dz = 0$. $\square$

Ex: Some holomorphic functions:
(1) $f(z) = C$ (for constant C).
(2) $f(z) = z = x + iy$.
(3) $f(z) = z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.
(4) $f(z) = \log_X(z) = \log|z| + i\theta_X(z)$. Note that the inverse of $\log_X$ is $\exp : x + iy \mapsto \cos y + ie^x \sin y$, and does not depend on X.
(5) $f(z) = e^z$ (verify using method 1 or method 2).

Ex: Some not holomorphic functions:
(1) $f(z) = \overline{z}$.
(2) $f(z) = \text{Re}(z)$.
(3) $f(z) = \text{Im}(z)$.

Thm: If f, g holomorphic, then the following are holomorphic wherever defined:
(a) $f + g$
(b) $f - g$
(c) fg
(d) f/g
(e) $f \circ g$
Proof: Method 1

Alternative methods: (a)(b) using CR equations, and (e) using the fact that the $\mathbb{R}$-affine approximation for $f \circ g$ is the composition of the corresponding $\mathbb{R}$-affine approximations. If these are, in fact, $\mathbb{C}$-affine, then so is the first one.

Cor: From (e), g holomorphic $\Rightarrow 1/g$ holomorphic where defined.

Lemma: $f(z) = z^2 = (x^2 - y^2) + i2xy$ is holomorphic.
Proof: Use CR equations.

Cor: f, g holomorphic $\Rightarrow fg = \frac{(f+g)^2 - (f-g)^2}{2}$ holomorphic.

Cor: f, g holomorphic $\Rightarrow f/g$ holomorphic where defined.