Residue Theorem

Professor David Barrett
Transcribed by Thomas Cohn

2/22/19

Recall: $A^{\text{osso}\mathbb{C}} \xrightarrow{f=u+iv} \mathbb{C}$. f is <u>holomorphic</u> if and only if f satisfies one/all of the following equivalent conditions:

1) $f \in C^1$ and $f\,dz$ is closed.
2) $u, v \in C^1(A, \mathbb{R})$ satisfy the CR equations.
3) f is $\mathbb{C}$-differentiable at each point of A and $f'_{\mathbb{C}}$ is continuous on A.

Thm: If $f : A \to B$ is a holomorphic diffeomorphism, then $f^{-1} : B \to A$ is a holomorphism.
Proof: Write $f = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$, $f^{-1} = \left(\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right)$. $D\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)(z_0) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.
By Thm 7.4, $D\left(\begin{smallmatrix} \tilde{u} \\ \tilde{v} \end{smallmatrix}\right)(f(z_0)) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\alpha}{\alpha^2+\beta^2} & \frac{\beta}{\alpha^2+\beta^2} \\ \frac{-\beta}{\alpha^2+\beta^2} & \frac{\alpha}{\alpha^2+\beta^2} \end{pmatrix} = \frac{1}{\alpha^2+\beta^2}\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.
This satisfies the CR equations, so f^{-1} is holomorphic. $\square$

Thm: (Cauchy Integral Theorem V.2) Given $M^{\text{cpt 2-mfd}} \subset \mathbb{C}$, f holomorphic on some neighborhood of M, then

$$\int_{\partial M} f\,dz \overset{\text{Stokes'}}{=} \int_M d(f\,dz) = \int_M 0 = 0$$

Important special case: when ∂M is the union of 2 connected curves C_1 and C_2, we have

$$0 = \int_{\partial M} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz$$

So $\displaystyle\int_{C_1} f\,dz = \int_{C_2} f\,dz$.

Cor: Given $z_0 \in U^{\text{osso}\mathbb{C}}$, f holomorphic on $U \setminus \{z_0\}$, then $\displaystyle\int_{\{z:|z-z_0|=r\}} f\,dz$ is independent of $r \in (0, d(z_0, \operatorname{Bd} U))$.
(Note that this set is a circle; we're traveling counterclockwise.)

Defn: This integral is equal to $2\pi i \operatorname{Res}(f\,dz, z_0)$, where $\operatorname{Res}(f\,dz, z_0)$ is the <u>residue</u> of $f\,dz$ at z_0.

Cor: (Residue Theorem) Given $M^{\text{cpt 2-mfd}} \subset U^{\text{open}} \subset \mathbb{C}$, $z_1, \ldots, z_k$ distinct in $M \setminus \partial M$, and f holomorphic on $U \setminus \{z_1, \ldots, z_k\}$, then

$$\int_{\partial M} f\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f\,dz, z_j)$$

Proof: Draw small ε-balls around each z_j which don't intersect with ∂M. Then apply V.2 of the Cauchy Integral Theorem. $\square$

The ***most*** important special case: consider g holomorphic on $U \ni z_0$, $f(z) = \frac{g(z)}{z-z_0}$. Then

$$\int\limits_{\{z:|z-z_0|=r\}} f(z)\,dz = \int\limits_0^{2\pi} \frac{g(z_0+re^{it})}{re^{it}} ire^{it}\,dt = i\int\limits_0^{2\pi} g(z_0+re^{it})\,dt$$

This is independent of r, so

$$\lim_{r\to 0} i\int\limits_0^{2\pi} g(z_0+re^{it})\,dt = i\int\limits_0^{2\pi} g(z_0)\,dt = 2\pi i g(z_0)$$

because the limit converges uniformly. Thus, $\operatorname{Res}\left(\frac{g(z)}{z-z_0}, z_0\right) = g(z_0)$.

Cor: (Cauchy Integral Formula) Given $M^{\text{cpt 2-mfd}} \subset \mathbb{C}$, g holomorphic on $U^{\text{osso } \mathbb{C}} \supset M$, then

$$\int\limits_{\partial M} \frac{g(z)}{z-z_0}\,dz = 2\pi i g(z_0)$$

Unfortunately, this doesn't work in $\mathbb{R}$.