Fluids in $\mathbb{R}^2$

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$\vec{F}:\mathbb{R}^2\to\mathbb{R}^2$ (or perhaps $A^{\text{osso}\mathbb{R}^2}\to\mathbb{R}^2$) velocity of a fluid (could be time dependent).
$\left\langle \vec{F},\vec{T}\right\rangle$ is the tangential velocity. $\int\limits_X \left\langle \vec{F},\vec{T}\right\rangle ds$ is the circulation of $\vec{F}$ along X.

The average tangential velocity is $\frac{\int_X \langle \vec{F},\vec{T}\rangle\, ds}{\int_X ds}$.

Suppose X is a circle of radius r. Then the average angular velocity is equal to the average tangential velocity times 1 radian per r units of distance. So it's equal to the circulation divided by $2\pi r^2$.

From Wednesday, for $\vec{F}=(\alpha,\beta)\leftrightarrow\omega=\alpha dx+\beta dy$, we have circulation

$$\int\limits_X \alpha\, dx+\beta\, dy=\int\limits_{\text{disc}}\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)dx\wedge dy$$

and average angular velocity

$$\frac{\int\limits_{\text{disk}}\dfrac{\partial\beta}{\partial x}-\dfrac{\partial\alpha}{\partial y}}{2\pi r^2}$$

As r decreases to 0, we have $\frac{1}{2}\left(\frac{\partial\beta}{\partial x}-\frac{\partial\alpha}{\partial y}\right)$. This is the two-dimensional version of the curl.

Now, let's return to $\mathbb{R}^3$...

Exer: For $\vec{p}\in\mathbb{R}^3$, $\vec{v_1},\vec{v_2}\in\mathbb{R}^3$ orthonormal, M_r parameterized by $\theta\mapsto\vec{p}+r\cos\theta\vec{v_1}+r\sin\theta\vec{v_2}$, $\vec{F}=\begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}^T$ vector field, $\omega=\alpha dx+\beta dy+\gamma dz$, then the average angular velocity is $\frac{\text{circulation of }\vec{F}\text{ along }M_r}{2\pi r^2}$, and as r decreases to 0, this tends to

$$\frac{1}{2}d\omega(\vec{p})(\vec{v_1},\vec{v_2})=\frac{1}{2}\left\langle \operatorname{curl}\vec{F}(\vec{p}),\vec{v_1}\times\vec{v_2}\right\rangle=\frac{1}{2}\det\left(\begin{array}{c:c:c} | & | & | \\ \operatorname{curl}\vec{F}(\vec{p}) & \vec{v_1} & \vec{v_2} \\ | & | & | \end{array}\right)$$

In the general case...

For M compact oriented 2-manifold in $\mathbb{R}^3$, the circulation of $\vec{F}$ along ∂M is

$$\int\limits_{\partial M}\omega=\int\limits_M d\omega=\int\limits_M d\omega(\overbrace{\vec{v_1},\vec{v_2}}^{\text{positively-oriented basis for }\mathcal{T}_{\vec{p}}M})\,ds=\int\limits_M\left\langle \operatorname{curl}\vec{F},\vec{v_1}\times\vec{v_2}\right\rangle ds=\int\limits_M\left\langle \operatorname{curl}\vec{F},\vec{N}\right\rangle ds$$