Math 493 Lecture 8

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Prop: Suppose V is a vector space on field K. We have a bijection

$$\begin{aligned}\{(v_1,\ldots,v_n \in V^n)\} &\leftrightarrow \{\text{linear transformation } K^n \to V\}\\ (v_1,\ldots,v_n) &\mapsto \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \to \textstyle\sum_{i=1}^n a_i v_i\right)\\ \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix} &\leftarrow\!\shortmid T\end{aligned}$$

We write $i_{(v_1,\ldots,v_n)}$ for the linear transformation $K^n \to V$ corresponding to $(v_1,\ldots,v_n)$.

$$\operatorname{im}(i_{(v_1,\ldots,v_n)}) = \operatorname{span}\{v_1,\ldots,v_n\}$$

$$\ker(i_{(v_1,\ldots,v_n)}) = \{\text{linear relations between } v_1,\ldots,v_n\} = \left\{(a_1,\ldots,a_n) \in K^n \;\middle|\; \sum_{i=1}^n a_i v_i = 0\right\}$$

$i_{(v_1,\ldots,v_n)}$ is surjective if and only if $v_1,\ldots,v_n$ span, and injective if and only if $v_1,\ldots,v_n$ are linearly independent. Hence, it's a bijection if and only if $v_1,\ldots,v_n$ forms a basis.

Conclusion: ordered bases of V correspond with isomorphisms $K^n \to V$.

Let $T : V \to W$ be a linear transformation. Let $B = (v_1,\ldots,v_m)$ and $C = (w_1,\ldots,w_n)$ be bases of V and W, respectively.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\wr" left, "i_B"] & W \arrow[d, "\wr" left, "i_C"] \\ K^m \arrow[r, "T'"] & K^n \end{tikzcd}$$

where $T' = (i_C^{-1}) \circ (T) \circ (i_B)$.

T' is left multiplication by some $n \times n$ matrix, denoted A_p.

Defn: A_p is the **matrix of** T with respect to B and C.

Ex: Let $T : \begin{aligned} P_{\le 2}(x) &\to P_{\le 2}(x) \\ f &\mapsto \tfrac{df}{dx} \end{aligned}$. Let $B = C = (1, x, x^2)$. Want to find T'.

$$T'(e_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad T'(e_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad T'(e_3) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

So

$$A = \begin{bmatrix} | & | & | \\ T'(e_1) & T'(e_2) & T'(e_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now let $B = (2, x - x^2, -x)$ and $C = (1, x, x^2)$. Then

$$T'(e_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad T'(e_2) = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \qquad T'(e_3) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

So

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now suppose $B = (v_1, \ldots, v_m)$ and $B' = (v_1', \ldots, v_m')$ are bases of V. Then

$$\begin{tikzcd} & V \arrow[dl, "\sim"', "i_B"] \arrow[dr, "\sim", "i_{B'}"'] & \\ K^m \arrow[rr, "T_X"] & & K^m \end{tikzcd}$$

with $T_X = (i_{B'}^{-1}) \circ (i_B)$. X is the matrix of T_X, associated with the standard basis of K^m.

$$T_X(e_1) = ((i_{B'}^{-1}) \circ (i_B))(e_1) = (i_{B'}^{-1})(v_1) = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \text{ s.t. } v_1 = \sum_{i=1}^{m} a_i v_i'$$

Now also suppose $C = (w_1, \ldots, w_n)$, $C' = (w_1', \ldots, w_n')$ are bases for W. Then we have

$$\begin{tikzcd} V \arrow[r, "\mathrm{Id}"] \arrow[d, "\wr"', "i_{B'}"] & V \arrow[r, "T"] \arrow[d, "\wr"', "i_B"] & W \arrow[r, "\mathrm{Id}"] \arrow[d, "\wr"', "i_C"] & W \arrow[d, "\wr"', "i_{C'}"] \\ K^m \arrow[rrr, bend right, "T''"'] & K^m \arrow[l, "T_X"'] \arrow[r, "T'=A"] & K^n \arrow[r, "T_Y"] & K^n \end{tikzcd}$$

$T'' = T_Y \circ T' \circ T_X^{-1}$. So the matrix for T'' is $M = YAX^{-1}$.

Conclusion: If A is the matrix for T w.r.t. B, C, then YAX^{-1} is the matrix for T w.r.t. B', C'.

Defn: An **endomorphism** of V (or a **linear operator** on V) is a linear transformation on V.

Let T be an endomorphism of V, and B and ordered basis of V. We get a matrix T w.r.t. B. Call it A. If B' is a different basis of V, XAX^{-1} is the matrix of T with respect to B'.

Defn: A, A' are two $n \times n$ matrices. We say they are **similar** if $\exists X \in \mathrm{GL}_n(K)$ s.t. $A' = XAX^{-1}$.

Note: this implies that if A and A' are matrices of T w.r.t. two bases, then A and A' are similar. Using this, we can define some numerical invariants of a linear transformation.

Defn: $\det(T) = \det(A)$, where A is a matrix of T. This is well defined because similar matrices have equal determinants.

Defn: $\mathrm{tr}(T) = \mathrm{tr}(A)$, where A is a matrix of T. This is well defined because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Ex: $\begin{aligned} T : P_{\leq 2} &\to P_{\leq 2} \\ f &\mapsto \tfrac{df}{dx} \end{aligned}$. Consider basis $(1, x, x^2)$.

Then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{tr}(T) = 0, \det(T) = 0$$

Eigenvalues and Eigenvectors

Let T be a linear operator on V.

Defn: An **eigenvector** for T is a nonzero $v \in V$ s.t. $T(v) = \lambda v$, for some $\lambda \in K$. We say λ is an **eigenvalue** for T.

λ is an eigenvalue for T

$\Leftrightarrow$ $T - \lambda \mathrm{Id}$ has a nontrivial kernel.
$\Leftrightarrow$ $T - \lambda \mathrm{Id}$ is not invertible.
$\Leftrightarrow$ $\det(T - \lambda \mathrm{Id}) = 0$.

Defn: The **characteristic polynomial** of T is

$$\mathcal{X}_T(t) = \det(T - t\mathrm{Id}) = (-t)^m \pm \mathrm{tr}(T)t^{m-1} + \cdots + \det(T)$$

The eigenvalues of T are the roots of $\mathcal{X}$.

How to find eigenvectors of T:

1. Compute $\mathcal{X}_T$.
2. Find the roots of $\mathcal{X}_T$ (eigenvalues).
3. For each eigenvalue λ_i, compute $\ker(T - \lambda_i \mathrm{Id})$.

Defn: Let A be an $m \times m$ matrix. We can say A is **diagonalizable** if it is similar to a diagonal matrix.

Let $T : V \to V$ be an endomorphism. Pick a basis and supposed the matrix of T is diagonalizable. So $\exists X \in \mathrm{GL}_m(K)$ s.t. XAX' is diagonal.
Let $B' = (v_1', \ldots, v_m')$ be the basis with change of basis matrix X.
Then the matrix of T with respect to B' is

$$A' = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}$$

According to

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "i_B", "\wr"'] & V \arrow[d, "i_{B'}", "\wr"'] \\ K^m \arrow[r, "T_A'"] & K^m \end{tikzcd}$$

So $T_A'(e_1) = A'e_1 = \lambda e_1$, thus $T(v_1') = \lambda_1 v_1'$.

Conclusion: Let $T : V \to V$ be an endomorphism with matrix A w.r.t. some basis. The following are equivalent:

1. A is diagonalizable.
2. There is a basis of V consisting of eigenvectors of T.