Math 493 Lecture 12

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Defn: Let G be a group, and X be a set. An **action** of G on X is a function $\begin{matrix} G \times X \to X \\ (g,x) \mapsto g \cdot x \end{matrix}$.

s.t.

1. $1 \cdot x = x$, $\forall x \in X$
2. $(gh)x = g(hx)$, $\forall g, h \in G$, $\forall x \in X$

We may also say X is a G**-set**.

Ex:

1. M, the group of rigid motions of the plane $P = \mathbb{R}^2$. M acts on P.
2. G, any group, acts on $X = G$ by
   (a) $g \cdot h = gh$
   (b) $g \cdot h = hg^{-1}$. Check:
   $\forall g, h, x \in G$, $(gh) \cdot x = x(gh)^{-1} = xh^{-1}g^{-1} = (xh^{-1})g^{-1} = g \cdot (xh^{-1}) = g \cdot (h \cdot x)$
   (c) $g \cdot x = gxg^{-1}$
3. $G = S_n$, $X = \{1, 2, \ldots, n\}$. $\sigma \in G$, $i \in X$, $\sigma \cdot i = \sigma(i)$.
4. $G = \mathrm{GL}_n(F)$, (F is a field). $X = F^n$. $g \cdot x$ is just matrix-vector multiplication.

Defn: X and Y are G-sets. A **homomorphism** from X to Y is a function $f : X \to Y$ such that $f(gx) = gf(x)$, $\forall x \in X, g \in G$. We say that f is a G**-map**.

Defn: An **isomorphism** of G-sets is a bijective homomorphism.

Defn: If G is any group, X is any set, we have the **trivial action** of G on X by $g \cdot x = x$, $\forall g \in G, x \in X$.

Defn: Let X be a G-set. The **orbit** of $x \in X$ (under G) is $O_x = \{g \cdot x \mid g \in G\}$.

Note: we can define an equivalence relation on X by $x \sim y$ if $\exists g \in G$ s.t. $x = gy$. O_x is the equivalence class.
Equivalence classes partition their set, so the orbits partition X.

Observation: given $y \in O_x g \in G$, $gy \in O_x$, because $y = hx$ for some $h \in O_x$, so $gy = ghx = (gh)x \in O_x$.

Defn: We say G acts **transitively** on X if there is only one orbit.

Ex:

1. $S_n \circlearrowright \{1, \ldots, n\}$. This action is transitive.
2. $\mathrm{GL}_n(F) \circlearrowright F^n$. There are two orbits, $\{0\}$ and $F^n \setminus \{0\}$.
3. $G \circlearrowright G$ by left multiplication. This is transitive, because $\forall g \in G$, $g = g \cdot 1$, so $g \in O_1$.
4. $G \circlearrowright G$ by conjugation. Given $g \in G$, $O_g = \left\{hgh^{-1} \mid h \in G\right\}$, the set of elements conjugate to g.

Defn: The set of elements conjugate to g is called the **conjugacy class** of g, and often denoted C_g.

Defn: Given G-sets X and Y, we can define a new G-set $X \amalg Y$ called the **disjoint union**, by putting together X and Y. X and Y are isomorphic to G-subsets of $X \amalg Y$ and $X \amalg Y = X \cup Y$ and $X \cap Y = \emptyset$.

Prop: Every G-set is isomorphic to a disjoint union of transitive G-sets.
Proof: X is the disjoint union of its orbits. $\square$

Defn: Let X be a G-set. A subset Y of X is **G-stable** if $y \in Y, g \in G \Rightarrow gy \in Y$. Y is also called a **G-subset**.

Defn: Let X be a G-set, $x \in X$. The **stabilizer** of X is $G_x = \{g \in G \mid gx = x\}$.

Observe: G_x is a subgroup of G.
Proof:

- $1 \in G_x$ because $1 \cdot x = x$.
- If $g \in G_x$, then $gx = x$, so $g^{-1}gx = g^{-1}x$. Thus, $x = g^{-1}x$, so $g^{-1} \in G_x$.
- If $g, h \in G_x$, then $(gh)x = g(hx) = gx = x$, so $gh \in G_x$.

Thus, G_x is a subgroup of G. $\square$

Prop: Let $x \in X, g \in G$. $G_{gx} = gG_xg^{-1}$.
Proof: Suppose $h \in G_x$. Then $(ghg^{-1}) \cdot gx = ghx = gx$, so $ghg^{-1} \in G_{gx}$. Thus, $G_{gx} \supseteq gG_xg^{-1}$.
The other direction is similar. $\square$

Cor: Any 2 elements in the same orbit have conjugate stabilizers.

Given $H \subseteq G$ a subgroup, consider the set of all left cosets $G/H = \{gH \mid g \in G\}$. This is naturally a G-set by $g \cdot (g'H) = gg'H$.

Some observations:

- G/H is a transitive G-set $gH = g \cdot H$. Thus, every coset is in the orbit of H.
- The stabilizer of H under this action is H, because $g \cdot H = gH$; if $g \in H$, then $g \cdot H = H$. Conversely, if $g \cdot H = H$, then $g \in H$.

Thm: Every transitive G-set is isomorphic to G/H for some subgroup H. More precisely, if $x \in X$, $H = G_x$, then the function $\begin{aligned} f : G/H &\to X \\ gH &\mapsto gx \end{aligned}$ is well defined and an isomorphism of G-sets.
Proof: First, check that f is well defined. Suppose $gH = g'H$. It's enough to show $gx = g'x$. We know $g = g'h$ for some $h \in H$. So $gx = g'hx = g'x$, because h stabilizes x.
Map of G-sets: Let $g, g' \in G$. Then $f(g \cdot (g'H)) = f(gg'H0 = gg'x = g(g'x) = gf(g'H)$.
Surjectivity: Given $y \in X$, we can write $y = gx$ for some $g \in G_x$ because X is transitive. Thus, $y = f(gH)$, so $y \in \mathrm{im} f$.
Injectivity: Suppose $f(gH) = gx = f(g'H) = g'x$. Then $x = g^{-1}g'x$, so $g^{-1}g' \in G_x = H$, so $gH = g'H$. $\square$

Prop: Given subgroups $H, H' \subseteq G$, G/H and G/H' are isomorphic as G-sets iff H and H' are conjugates.
Proof: this is left as an exercise to the reader.

Cor: (Counting Formula) Suppose f is finite, X is a G-set, and $x \in X$. Then $|G| = |O_x| \cdot |G_x|$.
Proof: O_x is a transitive G-set, and G_x is the stabilizer of $x \in O_x$. So $O_x \cong G/G_x$, so $|O_x| = [G : G_x]$. $\square$