Math 493 Lecture 22

Thomas Cohn

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Spectral Theorem

Let V be a real vector space of dimension n with a positive definite, symmetric bilinear form $\langle\ ,\ \rangle$. Note that positive definite means $\langle v, v\rangle > 0$ if $v \neq 0$, and is true iff the bilinear form is non-degenerate and has signature $(n, 0)$.

Defn: A linear operator $T : V \to V$ is **symmetric** if $\langle Tv, w\rangle = \langle v, Tw\rangle$, $\forall v, w \in V$.

The Spectral Theorem states that a symmetric linear operator is diagonalizable. In particular, all eigenvalues are real.

Let $e_1, \ldots, e_n$ be an orthonormal basis (normal means $\langle e_i, e_i\rangle = 1$). This exists because of our assumption of the signature. Let $\mathcal{A}$ be the matrix for T on this basis. $V \cong \mathbb{R}^n$, $\langle v, w\rangle = v^T \mathcal{A} w$. The symmetric condition on T gives us $\langle Tv, w\rangle = \langle v, Tw\rangle$, so $(\mathcal{A}v)^T w = v^T(\mathcal{A}w)$, so $v^T \mathcal{A}^T w = v^T \mathcal{A} w$. We conclude $\mathcal{A} = \mathcal{A}^T$.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$, $v \in \mathbb{C}^n$ a corresponding eigenvector (so we have $\mathcal{A}v = \lambda v$).
Write $v = \left[\begin{smallmatrix} v_1 & \cdots & v_n \end{smallmatrix}\right]^T$. Define $\bar{v} = \left[\begin{smallmatrix} \bar{v}_1 & \cdots & \bar{v}_n \end{smallmatrix}\right]$ (where $\bar{\cdot}$ denotes complex conjugation). Then

$$\bar{v}^T v = \bar{v}_1 v_1 + \cdots + \bar{v}_n v_n = |v_1|^2 + \cdots + |v_n|^2 \geq 0$$

In fact, $\bar{v}^T v > 0$, because $v \neq 0$, so $\exists i$ s.t. $v_i \neq 0$. Thus,

$$\lambda \bar{v}^T v = \bar{v}^T \mathcal{A} v = \bar{v}^T \mathcal{A}^T v = (\mathcal{A}\bar{v})^T v = \overline{(\mathcal{A}v)}^T v = \overline{(\lambda v)}^T v = \overline{\lambda} \bar{v}^T v$$

$\overline{\lambda} \bar{v}^T v = \lambda \bar{v}^T v$, $\forall v$, so because $\bar{v}^T v \neq 0$, $\overline{\lambda} = \lambda$. Thus, $\lambda \in \mathbb{R}$. T has a real eigenvalue λ, so there is a real eigenvector $v \in \mathbb{R}^n$. $V = \mathrm{span}(v) \perp \mathrm{span}(v)^\perp$, so we claim T maps $\mathrm{span}(v)^\perp$ to itself.

Let $w \in \mathrm{span}(v)^\perp$, i.e., $\langle v, w\rangle = 0$. $\langle v, Tw\rangle = \langle Tv, w\rangle = \langle \lambda v, w\rangle = \lambda v, w = 0$. Thus, $Tw \in \mathrm{span}(v)^\perp$.
Let T' be the restrition of T to $\mathrm{span}(v)^\perp$. T' is symmetric. So by induction on the dimension, spectral theorem applies to T'.

Thus, there exists a basis for $\mathrm{span}(v)^\perp$ consisting of eigenvectors for T'. Throwing in v, we get a basis for V consisting of eigenvectors of T. $\square$

Hermitian Forms

Defn: Let V be a $\mathbb{C}$-vector space. A **Hermitian form** on V is a function $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ such that $\forall v, v', w, w' \in V, \alpha, \beta \in \mathbb{C}$,

1. $\langle\ ,\ \rangle$ is $\mathbb{C}$-linear in its second variable: $\langle v, \alpha w + \beta w'\rangle = \alpha \langle v, w\rangle + \beta \langle v, w'\rangle$.
2. $\langle\ ,\ \rangle$ is $\mathbb{C}$-sesquilinear in its first variable: $\langle \alpha v + \beta v', w'\rangle = \overline{\alpha} \langle v, w\rangle + \overline{\beta} \langle v', w\rangle$.
3. $\langle w, v\rangle = \overline{\langle v, w\rangle}$. (Note: $\langle v, v\rangle \in \mathbb{R}$.)

Defn: We say $\langle\ ,\ \rangle$ is **positive definite** if $\langle v, v\rangle > 0$ for $v \neq 0$.

Ex: $V = \mathbb{C}^n$, $\langle v, w\rangle = \sum_{i=1}^{n} \overline{v_i} w_i$.
This is a positive definite Hermitian form.

Anti-Symmetric Forms

Let $\langle\ ,\ \rangle$ be a non-degenerate anti-symmetric (that is, $\langle v, w\rangle = -\langle w, v\rangle$) form on V. (And, of course, V is an F-vector space, where F is not of characteristic 2.)

Defn: A **symplectic basis** for V is a basis $e_1, f_1, \ldots, e_n, f_n$ where

- $\langle e_i, e_j\rangle = 0$ if $i \neq j$
- $\langle e_i, f_j\rangle = 0$ if $i \neq j$
- $\langle f_i, f_j\rangle = 0$ if $i \neq j$
- $\langle e_i, f_i\rangle = 1$

The matrix of $\langle\ ,\ \rangle$ with respect to a symplectic basis is

$$\begin{array}{c} \\ e_1 \\ f_1 \\ e_2 \\ f_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & f_1 & e_2 & f_2 & \cdots \end{array} \\ \left[\begin{array}{cc|cc|c} 0 & -1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] \end{array}$$

Prop: $\langle\ ,\ \rangle$ is a non-degenerate, anti-symmetric form on V. Then there is a symplectic basis.
Proof: pick $e_1 \neq 0$ in V. By non-degeneracy, $\exists f_1'$ s.t. $\langle e_1, f_1'\rangle \neq 0$. So we can scale f_1' to f_1 s.t. $\langle e_1, f_1\rangle = 1$. Thus, $V = \mathrm{span}(e_1, f_1) \perp \mathrm{span}(e_1, f_1)^{\perp}$, and continue by induction. $\square$

Cor: First Corollary: If $\langle\ ,\ \rangle$ is a non-degenerate, anti-symmetric form on V, then $\dim V$ is even.

Defn: A **symplectic space** is a pair $(V, \langle\ ,\ \rangle)$, where $\langle\ ,\ \rangle$ is a non-degenerate anti-symmetric form.

Cor: Second Corollary: Any two symplectic spaces of the same dimension are isomorphic.

Group Representations

Let G be a group, V a vector space over a field F.

Defn: A **linear action** (or **representation**) of G on V is an action $G \times V \to V$ such that
$g \cdot (\alpha v + \beta w) = \alpha(g \cdot v) + \beta(g \cdot w)$, $\forall \alpha, \beta \in F, v, w \in V$.

So, given a representation of G on V, for each $g \in G$, we get an invertible linear map $\begin{array}{l} \rho(g) : V \to V \\ \qquad\quad v \mapsto g \cdot v \end{array}$

This is an element $\rho(g) \in \mathrm{GL}(V)$, so $\begin{array}{l} \rho : G \to \mathrm{GL}(V) \\ \quad\ \ g \mapsto \rho(g) \end{array}$ is a group homomorphism.

Conversely, if $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism, then defining $g \cdot v = \rho(g)v$ gives a representation of G on V.

So, we have a correspondence between representations on V and group homomorphisms $\rho : G \to \mathrm{GL}(V)$.

Defn: A **representation** of G is a pair (V, ρ) where V is a vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. We often omit ρ from the notation.

Ex:

1. $G = \mathrm{GL}(V)$, $\rho : G \times \mathrm{GL}(V) \to \mathrm{GL}(V)$ is the identity map.
   This is a representation of G on V, called the **standard** representation of G.
2. $G = S_n$, $V = F^n$ with basis $(e_1, \ldots, e_n)$. $\sigma(e_i) = e_{\sigma(i)}$. $\rho(\sigma)$ is the permutation matrix of σ.
3. In general, say a group G acts on a set X. Let $F[X]$ be the vector space with basis symbols $[x]$ for $x \in X$. An element of $F[X]$ is a formal sum $\sum_{x \in X} c_x[x]$ where $c_x \in F$ and only finitely many c_x's are nonzero. $F[X]$ is naturally a representation of G by $g \cdot [x] = [gx]$. These are called **permutation representations**.
4. $G = D_n = \langle a, b \mid a^2, b^n, (ab)^2 \rangle$. Define $\rho : G \to \mathrm{GL}_2(\mathbb{R})$ by $a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $b \mapsto \begin{pmatrix} \cos \frac{2\pi}{n} & -\sin \frac{2\pi}{n} \\ \sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} \end{pmatrix}$.
   This is a two-dimensional representation of D_n.
5. $G = \{1, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$. $V = \mathbb{C}[x, y]$, the set of all polynomials in two variables. $\sigma \cdot f(x, y) = f(y, x)$.
   This is an infinite dimensional representation of G.