Math 493 Lecture 7

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Direct Sums

Let V be a vector space, $W_1, \dots, W_r \subseteq V$ subspaces.

Defn: $W_1, \dots, W_r$ are **independent** if $w_1 = \dots + w_r = 0$, for $w_i \in W_i$, then $w_i = 0$.

Defn: We let $W_1 + \cdots + W_r = \{w_1 = \cdots + w_r \mid w_i \in W_i\}$.

Observations:

1. $W_1 + \cdots + W_r$ is a subspace.
2. Suppose $v_1, \dots, v_r \in V$ are nonzero. Put $W_i = \operatorname{span}(v_i) = \{av_i \mid a \in K\}$. Then $W_1, \dots, W_r$ are independent if and only if $v_1, \dots, v_r$ are linearly independent. Additionally, $W_1 + \cdots + W_r = \operatorname{span}(v_1, \dots, v_r)$.
3. $r = 2$: W_1 and W_2 are independent if and only if $W_1 \cap W_2 = \{0\}$.
   Reason: say W_1 and W_2 are independent, $v \in W_1 \cap W_2$. Then $v + (-v) = 0$, and we have $v \in W_1$, $-v \in W_2$. So $v = 0$.

Defn: V is the (**internal**) **direct sum** of $W_1, \dots, W_r$, written $V = W_1 \oplus \cdots \oplus W_r$ if $W_1, \dots, W_r$ are independent and $W_1 + \cdots + W_r = V$.

Observe: $V = W_1 \oplus \cdots \oplus W_r$ if and only if every $v \in V$ can be written uniquely in the form $w_1 = \cdots + w_r$, with $w_i \in W_i$.
Reason: suppose $v = w_1 + w_2 + \cdots + w_r = w_1' + w_2' + \cdots + w_r'$. Then $0 = (w_1 - w_1') + \cdots + (w_r - w_r')$ (each $w_i - w_i' \in W_i$). Because the W_i are independent, we must have $w_i - w_i' = 0$, so $w_i = w_i'$.

Ex: $K = \mathbb{C}$, $V = M_{n \times n}(\mathbb{C})$. $W_1 = \left\{m \in V \mid {}^T m = m\right\}$, $W_2 = \left\{m \in V \mid {}^T m = -m\right\}$.
Claim: $V = W_1 \oplus W_2$.

- $V = W_1 + W_2$: Given $m \in V$, $m = \left(\frac{m + {}^T m}{2}\right) + \left(\frac{m - {}^T m}{2}\right)$. $\frac{m + {}^T m}{2} \in W_1$ and $\frac{m - {}^T m}{2} \in W_2$.
- $W_1 \cap W_2 = \{0\}$. If $m \in W_1 \cap W_2$, then $m = ^T m = -^T m$. So $m = 0$.

Let V be a vector space, and $U \subseteq V$ a subspace.

Defn: A subspace W of V is called a **complement** to U if $V = U \oplus W$.

Prop: Every subspace U has at least one complement.
Proof: Pick a basis S (possibly infinite) of U. Extend S to a basis T of V. Define $W = \operatorname{span}(T \setminus S)$.
Claim that $V = U \oplus W$.
Well, $V = U + W$. Let $v \in V$. Write $v = a_1x_1 + \cdots + a_nv_n$, $a_i \in K$, $x_i \in T$. Assume $x_i \in S$ for $1 \le i \le k$, $x_i \in T \setminus S$ for $k + 1 \le i \le n$.
Now, independence. Suppose $u + w = 0$, $u \in U$, $w \in W$. Write

$$u = a_1x_1 + \cdots + a_nx_n \quad (a_i \in K, x_i \in S)$$
$$w = b_1y_1 + \cdots + b_my_m \quad (b_i \in K, y_i \in T \setminus S)$$

So

$$u + w = a_1 x_1 = \cdots + a_n x_n + b_1 y_1 = \cdots + b_m y_m$$

T is a basis, and $v_i \in T$, $w_i \in T$, so $a_i = 0$ and $b_j = 0$, $\forall i, j$. So $u = 0$ and $w = 0$. $\square$

Ex: $V = \mathbb{C}^2, U = \operatorname{span}\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) = \{\left[\begin{smallmatrix}a\\0\end{smallmatrix}\right] \mid a \in \mathbb{C}\}$.
Claim: if $w = \left[\begin{smallmatrix}b\\1\end{smallmatrix}\right]$, for any $b \in \mathbb{C}$, then $W = \operatorname{span}(w)$ is a complement of U.
Reason: $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}b\\1\end{smallmatrix}\right]$ are a basis for $\mathbb{C}^2$.

In fact, any line other than the x-axis is a complement to U.

Prop: V is a vector space, $U, W \subseteq V$ subspaces. Let $\pi : V \to V/U$ be the quotient map. Then W is complement to U if $\pi|_W : W \to V/U$ is an isomorphism.
Proof: $\ker(\pi|_W) = \{w \in W \mid \pi(w) = 0\} = \{w \in W \mid w \in \ker(\pi)\} = W \cap \ker(\pi) = W \cap U$.
$\pi|_W$ is injective $\Leftrightarrow W \cap U = \{0\} \Leftrightarrow W, V$ independent.

Suppose $\bar{v} \in \operatorname{im}(\pi|_W)$, $\bar{v} = \bar{w}$ where $w \in W$. So $\overline{v - w} = 0$, thus, $v - w \in U$. $v = w + u$, with $w \in W$ and $u \in U$. Conversely, if $v = w + u$, $w \in W$, $u \in U$, then $\bar{v} = \bar{w}$ because $\bar{u} = 0$.

$\operatorname{im}(\pi|_W) = \{\bar{v} \mid v \in U + W\}$. So $\pi|_W$ is surjective if and only if $U + W = V$. $\square$

Cor: Suppose V is finite dimensional, and U, W are complements. Then $\dim V = \dim U + \dim W$.
Proof: $\dim V = \dim U + \dim V/U = \dim U + \dim W$, because $W \cong V/U$. $\square$

External Direct Sums

Let U and W be vector spaces over K.
Defn: The (**external**) **direct sum** $U \oplus W$ is the set of all ordered pairs (u, w) with $u \in U, w \in W$.

The external direct sum is a vector space:

- $(u, w) + (u', w') = (u + u', w + w')$
- $a(u, v) = (au, av)$

Let $\bar{u} = \{(u, 0) \mid u \in U\} \subseteq U \oplus W$ and $\bar{w} = \{(0, w) \mid w \in W\} \subseteq U \oplus W$.
Then $U \oplus W$ is the internal direct sum of $\bar{u}$ and $\bar{w}$.

Linear Transformations

Let $T : V \to W$ be a linear transformation.

Defn: $\ker(T) = \{v \in V \mid T(v) = 0\}$.

Defn: $\operatorname{im}(T) = \{w \in W \mid \exists v \in V \text{ s.t. } T(v) = w\}$.

Facts:

1. $\ker(T)$ is a subspace of V.
2. $\operatorname{im}(T)$ is a subspace of W.
3. T is injective if and only if $\ker(T) = \{0\}$.
4. First isomorphism theorem holds: T induces an isomorphism $V/\ker(T) \to \operatorname{im}(T)$.

Defn: Suppose V is a finite dimensional vector space. The **rank** of T is $\dim(\mathrm{im}(T))$. The **nullity** of T is $\dim(\ker(T))$.

Thm: (Rank-Nullity) $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim V$.
Proof: $\dim V = \dim V/\ker(T) + \dim(\ker(T)) = \dim(\mathrm{im}(T)) + \dim(\ker(T))$, by the first isomorphism theorem. □

Ex: $V = P_{\leq d} = \{\text{polynomials of degree } \leq d\}$, $K = \mathbb{C}$.

$$\begin{aligned} T = V &\to V \quad \text{is a linear transformation.} \\ f &\mapsto \tfrac{df}{dx} \end{aligned}$$

Then $\dim P_{\leq d} = \mathrm{nullity}(T) + \mathrm{rank}(T) = 1 + d = d + 1$.
Note: if we work over $\mathbb{F}_p$, then $\frac{d}{dx}(x^p) = px^{p-1} = 0$, so nullity can be greater than 1.

Let A be an $n \times m$ matrix (i.e. n rows, m columns) over K. Define a linear transformation $T_A : K^m \to K^n$ by $T_A(v) = Av$.

Prop: Every linear transformation $T : K^m \to K^n$ has the form T_A for a unique matrix A.
Proof: write

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}$$

with $v_i \in K^n$. Then $T_A(e_i) = v_i$.
If $T_A = T_B$, write

$$B = \begin{bmatrix} | & & | \\ w_1 & \cdots & w_m \\ | & & | \end{bmatrix}$$

$T_A(e_i) = T_B(e_i)$, so $v_i = w_i$. Thus, $A = B$.

Given an arbitrary T, put $v_i = T(e_i)$ and

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}$$

Then $T(e_i) = v_i = T_A(e_i)$, so $T = T_A$.
Let $v = \sum a_i e_i$. Then $T(v) = T(\sum a_i e_i) = \sum a_i T(e_i) = \sum a_i T_A(e_i) = T_A(\sum a_i e_i) = T_A(v)$. □