Math 493 Lecture 19

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Bilinear Forms

Recall the dot product.
Given $v, w \in \mathbb{R}^n$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then their dot product is $v \cdot w = v_1 w_1 + \cdots + v_n w_n$.

The dot product has three important properties:

1. Bilinearity: $(\alpha v + \beta v') \cdot w = \alpha(v \cdot w) + \beta(v' \cdot w)$ for $\alpha, \beta \in \mathbb{R}, v, v', w \in \mathbb{R}^n$ (and similarly in the second argument)
2. Symmetry: $v \cdot w = w \cdot v$
3. Positive Definite: $v \neq 0 \Rightarrow v \cdot v > 0$

We want to generalize the dot product to arbitrary vector spaces, and understand what that looks like.

Some setup: Let F be a field with $\text{char}(F) \neq 2$ (i.e. $2 \neq 0$, $1 \neq -1$). Let V be a F-vector space with finite dimension.

Defn: A **bilinear form** on V is a function $\langle\ ,\ \rangle : V \times V \to F$ s.t.
- $\langle \alpha v + \beta v', w \rangle = \alpha \langle v, w \rangle + \beta \langle v', w \rangle$, $\forall \alpha, \beta \in F, v, v', w \in V$
- $\langle v, \alpha w + \beta w' \rangle = \alpha \langle v, w \rangle + \beta \langle v, w' \rangle$, $\forall \alpha, \beta \in F, v, w, w' \in V$

Defn: A bilinear form $\langle\ ,\ \rangle$ is said to be
- **symmetric** if $\langle v, w \rangle = \langle w, v \rangle$, $\forall v, w \in V$
- **antisymmetric** if $\langle v, w \rangle = -\langle w, v \rangle$, $\forall v, w \in V$

Ex:
1. $V = \mathbb{R}^n$, $\langle v, w \rangle = v \cdot w$ is symmetric.
2. $V =$ anything, $\langle v, w \rangle = 0$, $\forall v, w \in V$ is symmetric and antisymmetric.
3. $V = F^n$, $\langle v, w \rangle = v_1 w_1 + \cdots + v_n w_n$ is symmetric.
4. $V = F^n$, pick $\alpha_1, \ldots, \alpha_n \in F^n$, $\langle v, w \rangle = \alpha_1 v_1 w_1 + \cdots + \alpha_n v_n w_n$ is symmetric.
5. $V = F^2$, $\langle v, w \rangle = \det \left[\begin{smallmatrix} v_1 & w_1 \\ v_2 & w_n \end{smallmatrix}\right]$ is antisymmetric.

Let $\langle\ ,\ \rangle$ be a bilinear form on V, $e_1, \ldots, e_n$ be a basis of V. Let $a_{i,j} = \langle e_i, e_j \rangle$.
Given $v = v_1 e_1 + \cdots + v_n e_n, w = w_1 e_1 + \cdots + w_n e_n$ (for $v_i, w_i \in F$), we have

$$\langle v, w \rangle = \sum_{i,j=1}^{n} v_i w_j \langle e_i, e_j \rangle = \sum_{i,j=1}^{n} a_{i,j} v_i w_j$$

If $n = 2$, then this becomes $\left[\begin{smallmatrix} v_1 & v_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{smallmatrix}\right] \left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right]$.

This works in general. Let $A = [a_{i,j}]_{1 \leq i,j \leq n}$, an $n \times n$ matrix. Then $\langle v, w \rangle = [v]_{\mathfrak{B}}^T A [w]_{\mathfrak{B}}$ (where $[v]_{\mathfrak{B}}$ is the vector v in the $\mathfrak{B} = (e_1, \ldots, e_n)$ basis).

Note: $\langle\ ,\ \rangle$ is symmetric iff A is symmetric, i.e., $A^T = A$. $\langle\ ,\ \rangle$ is antisymmetric iff $A^T = -A$.

Change of basis: Say $\mathfrak{C} = (f_1, \ldots, f_n)$ is a second basis. Let $Q \in \mathrm{GL}_n(F)$ be the change of basis matrix, so $[v]_{\mathfrak{B}} = Q[v]_{\mathfrak{C}}$.

Let A be the matrix for $\langle\ ,\ \rangle$ with respect to the $\mathfrak{B}$ basis. $A_{i,j} = \langle e_i, e_j\rangle$.
Let A' be the matrix for $\langle\ ,\ \rangle$ with respect to the $\mathfrak{C}$ basis. $A'_{i,j} = \langle f_i, f_j\rangle$.
Then

$$\langle v, w\rangle = [v]_{\mathfrak{B}}^T A[w]_{\mathfrak{B}} = [v]_{\mathfrak{C}}^T A'[w]_{\mathfrak{C}} = (Q[v]_{\mathfrak{B}})^T A'(Q[w]_{\mathfrak{B}}) = [v]_{\mathfrak{B}}^T (Q^T A' Q)[w]_{\mathfrak{B}}$$

So $[v]_{\mathfrak{B}}^T A[w]_{\mathfrak{B}} = [v]_{\mathfrak{B}}^T (Q^T A' Q)[w]_{\mathfrak{B}}$, $\forall v, w \in V$. So we have $a^T A b = a^T (Q^T A' A) b$, $\forall a, b \in F^n$, and thus, $A = Q^T A' Q$ (we can take a and b to be the standard basis vectors e_i, e_j, then $a^T A b = A_{i,j}$). We've just proved:

Prop: Let $\langle\ ,\ \rangle$ be a bilinear form on V. Let A be the matrix of $\langle\ ,\ \rangle$ in some basis. Then the matrix in an arbitrary basis has the form $Q^T A Q$ for $Q \in \mathrm{GL}_n(F)$.

From now on, focus on the case where the bilinear form is symmetric.

Defn: A **quadratic space** is a pair $(V, \langle\ ,\ \rangle)$ where V is a finite dimensional F-vector space and $\langle\ ,\ \rangle$ is a symmetric bilinear form on V.

Defn: Let V, W be quadratic spaces. An **isometry** from V to W is a linear isomorphism $T : V \to W$ such that $\langle Tv, Tv'\rangle = \langle v, v'\rangle$, $\forall v, v' \in V$.

Problem: classify quadratic spaces up to isometry.

$$\{n\text{-dimensional quadratic spaces}\} / \text{isometry} \cong \{n \times n \text{ symmetric matrices}\} / \sim$$

where $A \sim B$ if $A = Q^T B Q$ for some $Q \in \mathrm{GL}_n(F)$. Reason:
- Given a quadratic spaces V, we get the elements of $M_n(F)/\sim$ by taking matrix of form (?)
- If $T : V \to W$ is an isometry, let $e_1, \ldots, e_n$ be a basis for V. Then $Te_1, \ldots, Te_n$ is a basis for W. Because T is an isometry, $\langle Te_i, Te_j\rangle = \langle e_i, e_j\rangle$, which tells us that the matrices are the same.

Invariants of Quadratic Spaces

- Dimension of V – not a complete invariant. Consider I_n and 0_n (matrix of zeros). These are not equivalent.
- Discriminant: Let $A, B \in M_n(F)$ s.t. $A \sim B$. Then $\exists Q \in \mathrm{GL}_n(F)$ s.t. $A = Q^T B Q$, so $\det A = (\det Q)^2 \det B$, so $\det(A) = \det(B)$ in $F/(F^\times)^2 = \{0\} \cup F^\times/(F^\times)^2$.

Defn: The **discriminant** of a bilinear form or a quadratic spaces is the determinant of the matrix as an element of $F/(F^\times)^2$.

Note: It's well-defined!

Ex: $F = \mathbb{Q}$, $V = \mathbb{Q}^2$, p is a prime.

1. $\langle v, w\rangle = v_1 w_1 + v_2 w_2$ has matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, so the discriminant is 1.
2. $(v, w) = v_1 w_1 + p v_2 w_2$ has matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right]$, so the discriminant is p.

$p \neq 1$ on $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$, so $(V, \langle\ ,\ \rangle)$ and $(V, (\ ,\ ))$ are not isometric.

Another invariant: Let V be a quadratic space. The **kernel** of V is $\{v \in V \mid \langle v, w\rangle = 0, \forall w \in V\}$. This is a subspace of V, and $\dim\ker(V)$ is an invariant.

Defn: Let V be a quadratic space, and let $a \in F$. We say V **represents** a if $\exists v \in V \setminus \{0\}$ s.t. $\langle v, v \rangle = a$. The set of all elements of F represented by V is an isometry invariant of V.

Ex: $F = \mathbb{R}$, $V = \mathbb{R}^2$, $\langle v, w \rangle = v \cdot w = v_1 w_1 + v_2 w_2$, $(v, w) = v_1 w_1 - v_2 w_2$. Then $(V, \langle\ ,\ \rangle)$ represents positive real numbers, but $(V, (\ ,\ ))$ represents all real numbers. So they're not isometric.

Defn: Let V and W be quadratic spaces. Their **orthogonal direct sum** is $V \perp W$.

- The vector space is $V \oplus W$.
- The form is $\langle v + w, v' + w' \rangle = \langle v, v' \rangle + \langle w, w' \rangle$ for $v, v' \in V, w, w' \in W$.

If V is a quadratic space, and $U, W \subseteq V$ are subspaces, then $V = U \perp W$ if $V = U \oplus W$ and they're orthogonal, i.e., $\forall u \in U, w \in W$, $\langle u, w \rangle = 0$.

Lemma: Let V be a quadratic space s.t. $\langle\ ,\ \rangle \neq 0$. Then $\exists v \in V$ s.t. $\langle v, v \rangle \neq 0$.

Proof: By the assumption, $\exists u, w \in V$ s.t. $\langle u, w \rangle \neq 0$. If $\langle u, u \rangle \neq 0$ or $\langle w, w \rangle \neq 0$, we're done. If $\langle u, u \rangle = \langle w, w \rangle = 0$, then compute

$$\langle u + w, u + w \rangle = \cancel{\langle u, u \rangle} + \langle u, w \rangle + \langle w, u \rangle + \cancel{\langle w, w \rangle} = 2 \langle u, w \rangle = 2 \langle u, w \rangle \neq 0$$

$\square$

Defn: Let V be a quadratic space, and $W \subseteq V$ a subspace. Define $W^\perp = \{v \in V \mid \langle v, w \rangle = 0, \forall w \in W\}$.

By our lemma, if $\langle\ ,\ \rangle \neq 0$, then $\exists v$ s.t. $v \notin (Fv)^\perp = (\mathrm{span}(v))^\perp$.
Assume $\langle\ ,\ \rangle \neq 0$. Pick v s.t. $\langle v, v \rangle \neq 0$. Then $\mathrm{span}(v)^\perp = \ker \begin{pmatrix} V \to F \\ w \mapsto \langle v, w \rangle \end{pmatrix}$.
So $\dim(\mathrm{span}(v)^\perp) = \dim V - 1$, so $\dim(\mathrm{span}(v)) = 1$, so $\mathrm{span}(v) \cap \mathrm{span}(v)^\perp = \{0\}$.
Thus, $V = \mathrm{span}(v) \perp \mathrm{span}(v)^\perp$, and we conclude that if $\langle\ ,\ \rangle \neq 0$, then we have a $V \cong L \perp V'$, where $\dim(L) = 1$ and $\dim(V') = \dim(V) - 1$. This conclusion is obvious if $\langle\ ,\ \rangle = 0$ also.

By induction on $\dim V = n$, we can find $L_1, \ldots, L_n$, each dimension 1 quadratic spaces, such that $V \cong L_1 \perp \cdots \perp L_n$.

Alternative statement 1: Given a quadratic space V, there is an orthogonal basis $e_1, \ldots, e_n$ where $\langle e_i, e_j \rangle = 0$ if $i \neq j$.

Alternative statement 2: Given an $n \times n$ symmetric matrix A, $\exists Q \in \mathrm{GL}_n(F)$ s.t. $Q^T A Q$ is diagonal.