Math 493 Lecture 14

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Midterm Review

Topics:

1. Group Theory
2. Linear Algebra
3. Rigid Motions of the Plane
4. Group Actions

I Group Theory

The main focus of this is groups themselves, group homomorphisms, and group isomorphisms.

Ex: The most important groups:

- The trivial group
- $\mathbb{Z}/n\mathbb{Z}$ or $\mathbb{Z}$ (cyclic groups)
- S_n the symmetric group and A_n the alternating group
- $\mathrm{GL}_n(F)$ the general linear group over field F
- D_n the dihedral group of order $2n$

Constructing Groups

Ex: G any group, the group of automorphisms $\mathrm{Aut}(G)$
$\exists \gamma : G \to \mathrm{Aut}(G)$ a group homomorphism where $g \mapsto \gamma_g$ where $\gamma_g(h) = ghg^{-1}$ (conjugation by g). The γ_g are inner automorphisms.
$\ker \gamma = Z(G)$, the center of G
$\mathrm{im}\gamma = \mathrm{Inn}(G)$, the group of inner automorphisms of G, is a normal subgroup of $\mathrm{Aut}(G)$.

Given groups G and H, we can build a new group $G \times H$ called the direct product of G and H, with group law $(g, h)(g', h') \mapsto (gg', hh')$.

If $N \subseteq G$ is a normal subgroup, we can form the quotient gruop G/N. Elements are left cosets (or right cosets, because N is normal) gN. We have the projection map $\pi : G \to G/N$, a surjective group homomorphism where $g \mapsto gN$. $\ker(\pi) = N$. From π, we have the mapping property: given a group H, we have a bijection

$$\left\{\text{homomorphisms } G/N \xrightarrow{\bar{f}} H\right\} \xrightarrow{\sim} \left\{\text{homomorphisms } G \xrightarrow{f} H \text{ s.t. } N \subseteq \ker(f)\right\}$$
$$\bar{f} \mapsto f$$

$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ G/N \arrow[r, "\bar{f}"] & H \end{tikzcd}$$

If G is any group, $S \subset G$ a set of elements in G, then the subgroup of G generated by S is $\langle S \rangle$. We have two perspectives of this:
Top-Down: $\langle S \rangle$ is the intersection of all subgroups of G that contain S.
Bottom-Up: $\langle S \rangle$ is the set of finite products of elements of S and S^{-1}.

Results about Groups

Thm: (First Isomorphism Theorem) If $f : G \to H$ is a surjective group homomorphisms, then via the mapping property, f induces an isomorphism $\bar{f} : G/\ker f \xrightarrow{\sim} \mathrm{im} f = H$.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\ker f \arrow[ur, "\sim", "\bar{f}"'] & \end{tikzcd}$$

Thm: (Correspondence Theorem) Let $N \subseteq G$ a normal subgroup, $\pi : G \to G/N$ the quotient map. We have

$$\{\text{subgroups of } G/N\} \xrightarrow{\sim} \{\text{subgroups of } G \text{ containing } N\}$$
$$H \mapsto \pi^{-1}(H)$$

Let p be prime. Groups of small order:

- Every group of order p is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
- Every group of order p^2 is abelian and isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p^2\mathbb{Z}$.
- There are 2 groups of order 6: $\mathbb{Z}/6\mathbb{Z}$ and $S_3 = D_3$.

II Linear Algebra

Fields

Ex:

- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{F}_p$
- $\mathbb{Q}$

Some important things:

- Vector spaces and linear transformations
- Span, linear independence. Both together implies a basis.
- Every vector space has a basis. (Thank you axiom of choice!)
- A vector space is finite dimensional if it can be spanned by finitely many elements.
- If $\dim(V) = n$, choosing a basis is equivalent to choosing an isomoprhism $V \xrightarrow{\sim} F^n$.

We have several ways of constructing vector spaces:

- Direct sum
- Quotient vector spaces
- Span

Thm: (Rank-Nullity) Given $T : V \to W$ with V finite-dimensional, then

$$\dim v = \dim(\mathrm{im}(T)) + \dim(\ker(T)) = \mathrm{rank}(T) + \mathrm{nullity}(T)$$

Eigen-stuff

Let $T : V \to V$ be a linear operator, with V finite dimensional.

- $v \in V \setminus \{0\}$ is an eigenvector for T if $\exists \lambda$ s.t. $Tv = \lambda v$.
- The characteristic polynomial of T is $\det(T - tI)$.
- We say T is diagonalizable if there is a basis for V s.t. the matrix for T is diagonal.
- T is diagonalizable iff V has a basis of eigenvectors.

III Rigid Motions of the Plane

M is the group of rigid motions of the plane P.

We can look at the structure of M:

- Elements ρ_θ rotation, t_a translation, r reflection
- T is the group of translations, and is a normal subgroup.
- $\mathrm{O}(2)$ is the subgroup of M fixing the origin, generated by ρ_θ and r.

$$\begin{tikzcd} \mathrm{O}(2) \arrow[r, "\mathrm{Id}"] \arrow[rr, bend right, "\sim"'] & M \arrow[r, "\pi"] & M/T \end{tikzcd}$$

Every finite subgroup of M is conjugate to $\mathbb{Z}/n\mathbb{Z}$ or D_n inside of $\mathrm{O}(2)$.

Given a subset S of P (i.e. a plane figure), its symmetry group is the subgroup of M preserving S.

IV Group Actions

- G-sets, homomorphisms/isomorphisms
- Orbits and stabilizers. Let X be a G-set, $x \in X$. Then
  - Orbit $O_x = \{gx \mid g \in G\}$
  - Stabilizer $G_x = \{g \in G \mid gx = x\}$

Let X be a G-set. The orbits partition X into disjoint, transitive G-sets.

Every transitive G set is isomorphic to G/H for some subgroup H of G.

Counting formula: let X be a G-set with G finite, and $x \in X$. Then $\#O_x \cdot \#G_x = \#G$.