Math 493 Lecture 11

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10/9/19

Defn: A subgroup of M is **discrete** if $\exists \varepsilon > 0$ s.t.

1) If $t_a \in G$ (with $a \neq 0$), then $|a| > \varepsilon$.
2) If $\rho_\theta \in G$ with $\theta \in (0, 2\pi)$, then $|\theta| > \varepsilon$.

There's an analogous definition for discrete subgroups of $\mathbb{R}$, $\mathbb{R}^2$, $O(2)$, etc.

Ex:

- Any finite subgroup is discrete.
- $G = \left\{t_a \mid a \in \mathbb{Z}^2\right\}$ is discrete.
- $SO(2)$ is not discrete.
- If θ is not a rational multiple of 2π, then $G = \{\rho_{n\theta} \mid n \in \mathbb{Z}\}$ is not discrete.

Goal: classify the discrete subgroups of M.

Figures with infinite discrete symmetry groups:

- Frieze patterns, such as the graph of sin
- Wallpaper patterns, such as the triangle and square lattices

Prop: Let G be a discrete subgroup of $\mathbb{R}^2$. Then exactly one of the following is true:

- $G = \{0\}$
- $G = \{na \mid n \in \mathbb{Z}\}$ for some $a \in \mathbb{R}^2$
- $G = \{na + mb \mid n, m \in \mathbb{Z}\}$ for some $a, b \in \mathbb{R}^2$ with a and b linearly independent

Proof: Assume $G \neq \{0\}$. Pick $a \in G$ nonzero.

Case 1: $G \subset \mathbb{R} - a = \text{span}(a)$. Let $\lambda > 0$ be minimal s.t. $\lambda a \in G$. We claim that λa generates G, i.e., $G = \{n\lambda a \mid n \in \mathbb{Z}\}$. Let $v \in G$. Write $v = \mu - a$ for $\mu \in \mathbb{R}$. Write $\mu = n\lambda + \varepsilon$, for $n \in \mathbb{Z}$, $\varepsilon \in [0, \lambda)$. Then $v = n(\lambda a) + \varepsilon a$, so $\varepsilon a = v - n\lambda a \in G$. Thus, $\varepsilon = 0$, because λ is minimal. So $v = n\lambda a$.

Case 2: $G \not\subset \text{span}(a)$. Assume $a \in G \setminus \{0\}$ of minimal length. Let $b \in G$ not be a scalar multiple of a. Choose $b' \in G$ in the region bounded by $0, a, b, a + b$ not on the line $\mathbb{R}a$ but closest to it. Claim: $G = \langle a, b' \rangle$. Let $v \in G$. Write $v = xa + yb'$, with $x, y \in \mathbb{R}$. Write $x = n + x_0$ and $y = m + y_0$, for $n, m \in \mathbb{Z}$, $x_0, y_0 \in [0, 1)$. So $v - (na + mb) = x_0 a + y_0 b'$, which must be 0 because it's in G and the a, b' parallelogram. So $v = na + mb'$. $\square$

Recall that $T \subset M$ is the group of all translations, and it's isometric to $\mathbb{R}^2$. If G is a discrete subgroup of M, then $G \cap T$ is a discrete subgroup of T.
In fact, there are three possibilities:

1. $G \cap T = \{0\}$ means G is finite.

2. $G \cap T = \langle t_a \rangle$ for some $a \in \mathbb{R}^2$ nonzero gives us a frieze.
3. $G \cap T = \langle t_a, t_b \rangle$ for some $a, b \in \mathbb{R}^2$ nonzero gives us a wallpaper.

From now on, assume we're dealing with case 3.

Defn: Let $L = \left\{a \in \mathbb{R}^2 \mid t_a \in G\right\}$. This is called a **lattice**.

Defn: Let H be the image of G under the map $M \to O(2)$. This is called the **point group** of G, and it's finite.

By the classification of finite subgroups of $O(2)$, we know that $M = D_n$ or the group of rotations in D_n, for some n.

Prop: If $h \in$ and $a \in L$, then $h \cdot a \in L$.
Proof: Note that h is a 2×2 matrix and a is a vector in $\mathbb{R}^2$. Let $\tilde{h} \in G$ be a lift of h. Then $\tilde{h} = t_b h$ for some b. $\tilde{h}, t_a \in G \Rightarrow \tilde{h} t_a \tilde{h}^{-1} \in G$.
By our previous calculations, $h t_a h^{-1} = t_{h(a)}$, so $\tilde{h} t_a \tilde{h}^{-1} = t_b h t_a h^{-1} t_b^{-1} = t_b t_{ha} t_b^{-1} = t_{ha}$.
$t_{ha} \in G$, so $ha \in L$. $\square$

Thm: (Crystallographic Restriction) Any rotation in H has order 1, 2, 3, 4, or 6.
(This implies $H \subset D_n$, for $n \in \{1, 2, 3, 4, 6\}$.)
Proof: Let $a \in L$ be a nonzero vector of minimum length. Let $h \in H$ be a rotation by θ. We just showed $ha \in L$. Thus, because L is a group, $ha - a \in L$, so $(h-1)a \in L$. Thus, $\theta \geq \frac{2\pi}{6}$, or else $|ha - a| < |a|$. $\square$

Thm: (Classification of Wallpaper Groups) There are exactly 17 wallpaper groups.