Math 493 Lecture 9

Thomas Cohn

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Group Actions

Defn: Let $P = \mathbb{R}^2$, a plane. A **rigid motion** or **isometry** of P is a distance-preserving bijective map $m : P \to P$ where $d(m(x), m(y)) = d(x, y)$.

The set of rigid motions forms a group M under composition.

Ex: Some elements of M:

- Identity
- Rotation about a point by some amount
- Translation
- Reflection about any line
- Glide, i.e., translation along a line, then reflect over it

Translation: $a \in \mathbb{R}^2$, $t_a \in M$ be translation by a. $t_a(x) = x + a$.

Rotation: $\theta \in \mathbb{R}$, $\rho_\theta \in M$ be rotation by θ around 0.

$$\rho_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reflection: $r \in M$ is a reflection about the x-axis.

$$r\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Prop: Every element of M can be written uniquely in the form $t_a \circ \rho_\theta \circ r^i$, where $a \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$, and $i \in \{0, 1\}$.

Proof: Let $m \in M$ Suppose $m(0) = 0$, but m does not preserve orientation. Then mr fixes 0 and preserves orientation. So $mr = \rho_\theta$, for some θ, so we can write $m = \rho_\theta \circ r$.
Let $m \in M$ be arbitrary. $a = m(\theta) \Rightarrow t_{-a} \circ m$ fixes 0. So $m = t_a \circ \rho_\theta \circ r^i$.

Now, we must show uniqueness. Suppose $t_a\rho_\theta r^i = t_b\rho_\psi r^j$. Evaluate at 0. Then $a = b$, so $\rho_\theta r^i = \rho_\psi r^j$. Both maps are orientation preserving or orientation reversing, so $i = j$. Thus, we have $\rho_\theta = \rho_\psi$, so $\theta = \psi$. $\square$

Identities

- $t_a t_b = t_{a+b}$
- $\rho_\theta \rho_\psi = \rho\theta + \psi$
- $\rho_\theta = \rho_{\theta \bmod 2\pi}$
- $(\rho_\theta t_a \rho_\theta^{-1})(x) = (\rho_\theta t_a)(\rho_\theta^{-1}(x)) = \rho_\theta(\rho_\theta^{-1}(x) + a) = x + \rho_\theta(a) = t_{\rho_\theta(a)}(x)$
- Similarly, $r t_a r^{-1} = t_{r(a)}$

- $r\rho_\theta r^{-1} = \rho_{-\theta}$

This is a complete list of identities

$$
\begin{aligned}
(t_a\rho_\theta r^i)(t_b\rho_\psi r^j) &= t_a\rho_\theta(r^i t_b r^{-i})r^i\rho_\psi r^j \\
&= t_a\rho_\theta t_{r^i(b)} r^i\rho_\psi r^j \\
&= t_a(\rho_\theta t_{r^i(b)}\rho_{-\theta})\rho_\theta r^i\rho_\psi r^j \\
&= t_a t_{\rho_\theta(r^i b)}\rho_\theta r^i\rho_\psi r^j \\
&= t_{a+\rho_\theta(r^i b)}\rho_\theta r^i\rho_\psi r^j \\
&= t_{a+\rho_\theta(r^i b)}\rho_\theta(r^i\rho_\psi r^{-i})r^{i+j} \\
&= t_{a+\rho_\theta(r^i b)}\rho_\theta\rho_{(-1)^i\psi} r^{i+j} \\
&= t_{a+\rho_\theta(r^i b)}\rho_{\theta+(-1)^i\psi} r^{i+j}
\end{aligned}
$$

Some consequences of what we've learned:

- The map $M \xrightarrow{f} \{\pm 1\}$ detecting orientation is a group homomorphism. $f(t_a\rho_\theta r^i) = (-1)^i$. By the above computation, $f(xy) = f(x)f(y)$.
- $T \subset M$ is the subgroup consisting of translations. Then we have the group isomorphism $\mathbb{R}^2 \to T$ where $A \mapsto t_a$. By the identities we have above, T is a normal subgroup of M.
- $O(2) \subset M$ is the subgroup consisting of $m \in M$ s.t. $m(0) = 0$. We have a surjective group homomorphism $f : M \to O(2)$ where $t_a \mapsto 1$, $\rho_\theta \mapsto \rho_\theta$, $r \mapsto r$. Thus, $\ker(f) = T$, so $M/T \cong O(2)$.