Math 493 Lecture 23

Thomas Cohn

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Representation Theory

Recall from last time,

Defn: For G a group, a **representation** of G is a pair (V, ρ), where V is a vector space, and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. In our case, we assume V is finite dimensional, and specifically, a K-vector space (for some field K).

Our problem is now understanding what representations of G look like.

Defn: If V, W are two representations of G. A **map of representations** (also known as a **G-map**) is a linear map $f : V \to W$ s.t. $f(gv) = gf(v)$, $\forall g \in G, v \in V$.

Defn: An **isomorphism of representations** is a bijective map of representations.

The usual constructions of linear algebra apply to representations. Let V, W be representations of G.

- $V \oplus W$ is naturally a representation of G – $g(v, w) = (gv, gw)$.
- $\mathrm{Hom}_G(V, W) = \{f : V \to W \mid f \text{ is a } G\text{-map}\}$ is a vector space.
- $\mathrm{Hom}(V, W) = \{f : V \to W \mid f \text{ is a linear map}\}$ is a vector space.

In fact, $\mathrm{Hom}(V, W)$ is naturally a representation. Given $g \in G$, $f \in \mathrm{Hom}(V, W)$, define $gf \in \mathrm{Hom}(V, W)$ by $(gf)(v) = gf(g^{-1}v)$.

Defn: $V^G = \{v \in V \mid gv = v, \forall g \in G\}$ is the **space of invariants**.

Fact: $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$.
Reason: Say $f \in \mathrm{Hom}(V, W)$. Then f is G-invariant (i.e. $f \in \mathrm{Hom}(V, W)^G$) iff $gf = f$, $\forall g \in G$

$$\begin{aligned} &\Leftrightarrow gf(g^{-1}v) = f(v), \forall g \in G, v \in V \\ &\Leftrightarrow gf(v) = f(gv), \forall g, v \\ &\Leftrightarrow f \text{ is a } G\text{-map} \end{aligned}$$

Defn: $V^* = \mathrm{Hom}(V, K)$ (where K is the one dimensional trivial representation over our field) is the **dual representation**.

Pick bases for V and W, with $\dim V = n$, $\dim W = m$. The action of G on V corresponds to $\rho : G \to \mathrm{GL}_n(K)$, and the action of G on W corresponds to $\sigma : G \to \mathrm{GL}_m(K)$. Then $V \oplus W$ has dimension $n + m$, and the matrix for $(\rho \oplus \sigma)(g)$ is

$$\begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \left[\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \sigma(g) \end{array}\right] \end{array}$$

$\mathrm{Hom}(V, W) \cong M_{m,n}(K)$, the set of $m \times n$ matrices. For $\mathcal{A} \in M_{m,n}(K)$, $g \cdot A = \sigma(g) A \rho(g)^{-1}$.

If $V, W \subseteq U$ are subrepresentations, then

- $V \cap W$ is a subrepresentation.
- $V + W = \{v + w \mid v \in V, w \in W\}$ is a subrepresentation.
- U/V is naturally a representation – $g(u+V) = gu + V$.
- V and W are **complementary** if $U = V \oplus W$, which is true iff $V \cap W = \{0\}$ and $V + W = U$.

Defn: A representation V is called **irreducible** if $V \neq 0$ and its only subrepresentations are 0 and V.

Our approach to understanding representations is as follows. First, we will try to understand irreducible representations, and then we will understand how a general representation is build out of irreducible representations.

Ex: $G = D_n = \left\{a, b \mid a^2, b^n, (ab)^2\right\}$. We have a 2D representation $\rho : G \to \mathrm{GL}_2(\mathbb{R})$, where $a \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $b \mapsto \left(\begin{smallmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{smallmatrix}\right)$.
Fact: ρ is an irreducible representation. Why? Because $\rho(b)$ doesn't have any real eigenvalues (all lines are rotated), so there isn't any line in $\mathbb{R}^2$ that's fixed by G.

Ex: $G = S_n$, $V = \mathbb{C}^n$, $\sigma e_i = e_{\sigma(i)}$.
This is *not* irreducible. Let $L = \mathrm{span}(v)$, where $v = e_1 + \cdots + e_n = \left[\begin{smallmatrix} 1 & \cdots & 1 \end{smallmatrix}\right]^T$. Then $\sigma v = v$, $\forall \sigma \in S_n$. Thus, L is a subrepresentation.

Does L have a complementary subrepresentation?
Let $T : \mathbb{C} \to \mathbb{C}^n$, where $f \mapsto e_1 + \cdots + e_n = v$. ($f$ is the basis vector for $\mathbb{C}$.) This is a map of representations – $T(\sigma f) = T(f) = v$, and $\sigma T(f) = \sigma v = v$.
Now consider $S : \mathbb{C}^n \to \mathbb{C}$, where $e_i \mapsto 1$. $\ker(S) = \{\sum_{i=1}^n a_i e_i \mid \sum_{i=1}^n a_i = 0\}$ is a subrepresentation of $\mathbb{C}^n$, and has basis $(e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n)$.
These are complementary subrepresentations, so $\mathbb{C}^n = \ker(S) \oplus L$.

Fact: Not every subrepresentation has a complementary subrepresentation.

Ex: $\rho : \mathbb{Z} = G \to \mathrm{GL}_2$, where $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Note that $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & n+m \\ 0 & 1 \end{smallmatrix}\right)$. $L = \mathrm{span}(e_1) = \left[\begin{smallmatrix} * \\ 0 \end{smallmatrix}\right]$ is a subrepresentation, but no other 1D space is mapped to itself, so L has no complementary subrepresentation.

Ex: $G = \mathbb{Z}/p\mathbb{Z}$ (for p prime), $K = \mathbb{F}_p$.
Let $\rho : G \to \mathrm{GL}_2(\mathbb{F}n)$ where $n \mapsto \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. This is also an example, as $\mathrm{span}(e_1)$ has no complement.

Thm: If G is finite and $\mathrm{char}(K) = 0$, then every subrepresentation has a complement.
Proof: Let V be some representation, and W a subrepresentation of V. Let $\pi : V \to V/W$ be the projection map. Let $S : V/W \to V$ s.t. $\pi(S(x)) = x$. Note that S is not necessarily a G-map.
We will show that, for any choice of S, $\mathrm{im}(S)$ is a complementary subspace to W: Given some $v \in V$, we have

$$\pi(v - S(\pi(v))) = \pi(v) - \pi(S(\pi(v))) = \pi(v) - \pi(v) = 0$$

So $v - S(\pi(v)) \in W = \ker(\pi)$. So $v = \underbrace{(v - S(\pi(v)))}_{\in W} + \underbrace{S(\pi(v))}_{\in \mathrm{im}(S)}$.
If S is a G-map, then $\mathrm{im}(S)$ is a subrepresentation, and it's a complementary subrepresentation to W. To ensure S is a G-map, we can average it. Specifically, define

$$S' : V/W \to V \qquad S'(x) = \frac{1}{\#G} \sum_{g \in G} g S(g^{-1}x)$$

We claim that (1) S' is a G-map, and (2), that $\pi \circ S' = \mathrm{Id}$.

(1) For $h \in G$,

$$S'(hx) = \frac{1}{\#G} \sum_{\substack{g \in G \\ g = hg'}} gS(g^{-1}hx) = \frac{1}{\#G} \sum_{g' \in G} (hg')S((g')^{-1}h^{-1}hx) = hg'(x)$$

(2) Because π is a G-map,

$$\pi(S'(x)) = \pi\left(\frac{1}{\#G} \sum_{g \in G} gS(g^{-1}x)\right) = \frac{1}{\#G} \sum_{g \in G} g\pi(S(g^{-1}x)) = \frac{1}{\#G} \sum_{g \in G} g(g^{-1}(x)) = x$$

Thus, $\mathrm{im}(S')$ is a complementary subrepresentation to W. $\square$

Note: The theorem and proof are also valid if $p = \mathrm{char}(K) > 0$ and $p \nmid \#G$.

Cor: Let G be a finite group, $\mathrm{char}(K) = 0$ (or it doesn't divide G). Then every finite dimensional representation of G is the direct product of irreducible representations.
Proof: We'll proceed by induction on the dimension of V. Let V be given. If V is irreducible, then there is nothing to prove. If $V = \{0\}$, then there is nothing to prove.
Assume $V \neq \{0\}$, and V is not irreducible. Then $\exists W \subset V$ a subrepresentation, with $W \neq \{0\}, W \neq V$. By the above theorem, $\exists U$ a complementary subspace, so $V = U \oplus W$. By induction, U and W are the direct products of irreducible representations, so V is too. $\square$

The previous remark is still true!

Given the corollary, the main problem (for G finite, $\mathrm{char}(K) = 0$) is to understand the irreducible representations. There is a beautiful solution when $K = \mathbb{C}$ using *character theory*. For now, assume $K = \mathbb{C}$, $\#G < \infty$.

Defn: Given a representation V of G, we define its **character** to be the map $\begin{aligned} \chi_V : G &\to \mathbb{C} \\ g &\mapsto \mathrm{tr}(g|_V) \end{aligned}$

($g|_V$ is the linear operator g acting on V.)

We have the really cool fact that $V \cong W$ iff $\chi_V = \chi_W$.