Math 493 Lecture 10

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Recall: $P = \mathbb{R}^2$ is the plane. M is the group of rigid motions of P.
From last time, we have M generated by t_a translations, ρ_θ rotations about 0, and r reflections across the x-axis.

Defn: Let $S \subset P$ be a "planar figure". The **symmetry group** of S is $\Gamma_S = \{m \in M \mid mS = S\}$. This is a subgroup of M.

Ex:

1. Let S be the unit circle. $\Gamma_S = O(2)$ rotations and reflections through the origin.
2. $S = \mathbb{R}^2 \Rightarrow \Gamma_S + M$.
3. S is the unit circle with an arrow at $\frac{2\pi k}{n}$ for each k. Then $\Gamma_S = \left\{\text{rotations by } \frac{2\pi k}{n}\right\} \cong \mathbb{Z}/n\mathbb{Z}$.
4. S a regular n-gon. We have rotations by $\frac{2\pi k}{n}$, for $0 \leq k \leq n-1$, reflect through certain lines (lines that go through the origin and a vertex or midpoint of an edge).

Defn: For S a regular n-gon, Γ_S is denoted D_n and called the n**th dihedral group**. It's generated by $a = \rho_{\frac{2\pi}{n}}$ and $b = r$.

Relations: $a^n = 1$, $b^2 = 1$, $bab^{-1} = a^{-1}$. Every element of D_n can be written uniquely as a^k or $a^k b$ for $0 \leq k < n$. Note that $|D_n|\, 2n$, and $D_3 \cong S_3$.

Question: can we classify the symmetry groups of plane figures? A good first step is classifying the finite subgroups of M.

Thm: Suppose G is a finite subgroup of M. Then G has a fixed point, i.e., $\exists x \in P$ s.t. $gx = x, \forall g \in G$.
Proof: Let S be a finite subset of P. Define the center of gravity of S by $COG(S) = \frac{1}{n}\sum_{i=1}^n x_i$.

Lemma: Given $m \in S$, $COg(mS) = m(COG(s))$ (note: $mS = \{mx_1, \ldots, mx_n\}$).
Proof: Let $m \in O(2)$. Then $mCOG(s) = m(\frac{1}{n}\sum_{i=1}^n x_i) = \frac{1}{n}\sum_{i=1}^n mx_i = COG(mS)$. Let $m = t_a$. Then $COG(mS) = \frac{1}{n}\sum_{i=1}^n (x_i + a) = \frac{1}{n}\sum_{i=1}^n x_i + a = COG(S) + a = mCOG(s)$. $\square$

Continued: pick some $x_0 \in P$ at random. Let $S = \{gx_0 \mid g \in G\}$. Observe that for $h \in G$, $hS = S$. $hS = \{hgx_0 \mid g \in G\} = \{gx_0 \mid g \in G\}$. So $hCOG(S) = COS(hS) = COG(S)$. So $x = COS(S)$ is a fixed point of G. $\square$

Suppose $G \subset M$ is a subgroup that fixes $x \in P$. Then $t_x^{-1} G t_x$ fixes $0 \in P$, becuase, for $g \in G$,

$$(tx^{-1} g t_x)0 = t_x^{-1}(g(t_x(0))) = t_x^{-1}(g(x)) = t_x^{-1}(x) = 0$$

Cor: Let $G \subset M$ be a finite subgroup. Then $\exists x \in P$ s.t. $t_x G t_x^{-1} \subset O(2)$.

Problem: classify the finite subgroups of $O(2)$.

Defn: $SO(2) \subseteq O(2)$ is the group of rotations.

Note: every element of $O(2)$ can be written as ρ_θ or $\rho_\theta r$. This means 1 and r are the coset representatives for $O(2)$ with respect to $SO(2)$, so $[O(2) : SO(2)] = 2$.
First step: classify finite subgroups of $SO(2)$.

Prop: Let G be a finite subgroup of $SO(2)$. Then G is cyclic, and generated by ρ_θ, with $\theta = \frac{2\pi}{n}$ for some $n \in \mathbb{N}$.
Proof: let $0 < \theta$ be minimal such that $\rho_\theta \in G$. Then because G is finite, ρ_θ has finite order, so θ is a rational multiple of 2π.
We can write $\theta = \frac{2\pi p}{q}$, with p, q real, coprime, and positive. ρ_θ then generates $\rho_{\frac{2\pi}{q}}$, because we can write $ab + pq = 1$ for some $a, b \in \mathbb{Z}$, so $\rho_{\frac{2\pi}{q}} = \rho_{\frac{2\pi}{q}}^{ap+bq} = (\rho_{\frac{2\pi p}{q}})^a$.
By the minimality of ρ, $\theta = \frac{2\pi}{q}$. So G is generated by ρ_θ; suppose $\rho_\psi \in G$, with $\psi \geq 0$. Pick n s.t. $\rho_\psi \rho_\theta^{-n}$ is a rotation by an angle in $[0, \frac{2\pi}{q})$. By minimality of θ, $\rho_\psi \rho_\theta^{-n} \Rightarrow \rho_\psi = \rho_\theta^n$. $\square$

Prop: Let G be a finite subgroup of $O(2)$. Then either G is cyclic and generated by $\rho_{\frac{2\pi}{n}}$, or G is dihedral.
Proof: Case 1: G does not contain a reflection. Then $G \subset SO(2)$. So we're done by our classification of subgroups of $SO(2)$.
Case 2: G contains a reflection b. Let $H = G \cap SO(2)$. We know H is cyclic, and generated by $\rho_{\frac{2\pi}{n}}$ for some n. If $g \in G$, then either $g \in H$ or $gb \in H$. Thus, $[G : H] = 2$, so $G = D_n$.
We conclude every finite subgroup of M is either cyclic or dihedral. $\square$