

# Math 493 Lecture 11

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**Defn:** A subgroup of  $M$  is **discrete** if  $\exists \varepsilon > 0$  s.t.

- 1) If  $t_a \in G$  (with  $a \neq 0$ ), then  $|a| > \varepsilon$ .
- 2) If  $\rho_\theta \in G$  with  $\theta \in (0, 2\pi)$ , then  $|\theta| > \varepsilon$ .

There's an analogous definition for discrete subgroups of  $\mathbb{R}$ ,  $\mathbb{R}^2$ ,  $O(2)$ , etc.

**Ex:**

- Any finite subgroup is discrete.
- $G = \{t_a \mid a \in \mathbb{Z}^2\}$  is discrete.
- $SO(2)$  is not discrete.
- If  $\theta$  is not a rational multiple of  $2\pi$ , then  $G = \{\rho_{n\theta} \mid n \in \mathbb{Z}\}$  is not discrete.

Goal: classify the discrete subgroups of  $M$ .

Figures with infinite discrete symmetry groups:

- Frieze patterns, such as the graph of  $\sin$
- Wallpaper patterns, such as the triangle and square lattices

**Prop:** Let  $G$  be a discrete subgroup of  $\mathbb{R}^2$ . Then exactly one of the following is true:

- $G = \{0\}$
- $G = \{na \mid n \in \mathbb{Z}\}$  for some  $a \in \mathbb{R}^2$
- $G = \{na + mb \mid n, m \in \mathbb{Z}\}$  for some  $a, b \in \mathbb{R}^2$  with  $a$  and  $b$  linearly independent

Proof: Assume  $G \neq \{0\}$ . Pick  $a \in G$  nonzero.

Case 1:  $G \subset \mathbb{R} - a = \text{span}(a)$ . Let  $\lambda > 0$  be minimal s.t.  $\lambda a \in G$ . We claim that  $\lambda a$  generates  $G$ , i.e.,  $G = \{n\lambda a \mid n \in \mathbb{Z}\}$ . Let  $v \in G$ . Write  $v = \mu - a$  for  $\mu \in \mathbb{R}$ . Write  $\mu = n\lambda + \varepsilon$ , for  $n \in \mathbb{Z}$ ,  $\varepsilon \in [0, \lambda)$ . Then  $v = n(\lambda a) + \varepsilon a$ , so  $\varepsilon a = v - n\lambda a \in G$ . Thus,  $\varepsilon = 0$ , because  $\lambda$  is minimal. So  $v = n\lambda a$ .

Case 2:  $G \not\subset \text{span}(a)$ . Assume  $a \in G \setminus \{0\}$  of minimal length. Let  $b \in G$  not be a scalar multiple of  $a$ . Choose  $b' \in G$  in the region bounded by  $0, a, b, a + b$  not on the line  $\mathbb{R}a$  but closest to it. Claim:  $G = \langle a, b' \rangle$ . Let  $v \in G$ . Write  $v = xa + yb'$ , with  $x, y \in \mathbb{R}$ . Write  $x = n + x_0$  and  $y = m + y_0$ , for  $n, m \in \mathbb{Z}$ ,  $x_0, y_0 \in [0, 1)$ . So  $v - (na + mb) = x_0a + y_0b'$ , which must be 0 because it's in  $G$  and the  $a, b'$  parallelogram. So  $v = na + mb'$ .  $\square$

Recall that  $T \subset M$  is the group of all translations, and it's isometric to  $\mathbb{R}^2$ . If  $G$  is a discrete subgroup of  $M$ , then  $G \cap T$  is a discrete subgroup of  $T$ .

In fact, there are three possibilities:

1.  $G \cap T = \{0\}$  means  $G$  is finite.
2.  $G \cap T = \langle t_a \rangle$  for some  $a \in \mathbb{R}^2$  nonzero gives us a frieze.

3.  $G \cap T = \langle t_a, t_b \rangle$  for some  $a, b \in \mathbb{R}^2$  nonzero gives us a wallpaper.

From now on, assume we're dealing with case 3.

**Defn:** Let  $L = \{a \in \mathbb{R}^2 \mid t_a \in G\}$ . This is called a **lattice**.

**Defn:** Let  $H$  be the image of  $G$  under the map  $M \rightarrow O(2)$ . This is called the **point group** of  $G$ , and it's finite.

By the classification of finite subgroups of  $O(2)$ , we know that  $M = D_n$  or the group of rotations in  $D_n$ , for some  $n$ .

**Prop:** If  $h \in$  and  $a \in L$ , then  $h \cdot a \in L$ .

Proof: Note that  $h$  is a  $2 \times 2$  matrix and  $a$  is a vector in  $\mathbb{R}^2$ . Let  $\tilde{h} \in G$  be a lift of  $h$ . Then  $\tilde{h} = t_b h$  for some  $b$ .  $\tilde{h}, t_a \in G \Rightarrow \tilde{h} t_a \tilde{h}^{-1} \in G$ .

By our previous calculations,  $h t_a h^{-1} = t_{h(a)}$ , so  $\tilde{h} t_a \tilde{h}^{-1} = t_b h t_a h^{-1} t_b^{-1} = t_b t_{ha} t_b^{-1} = t_{ha}$ .  $t_{ha} \in G$ , so  $ha \in L$ .  $\square$

**Thm:** (Crystallographic Restriction) Any rotation in  $H$  has order 1, 2, 3, 4, or 6. (This implies  $H \subset D_n$ , for  $n \in \{1, 2, 3, 4, 6\}$ .)

Proof: Let  $a \in L$  be a nonzero vector of minimum length. Let  $h \in H$  be a rotation by  $\theta$ . We just showed  $ha \in L$ . Thus, because  $L$  is a group,  $ha - a \in L$ , so  $(h - 1)a \in L$ . Thus,  $\theta \geq \frac{2\pi}{6}$ , or else  $|ha - a| < |a|$ .  $\square$

**Thm:** (Classification of Wallpaper Groups) There are exactly 17 wallpaper groups.