Math 493 Lecture 6

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9/23/19

Continuing from last time...

(1) $\Rightarrow$ (3): S is a basis. We need to show no superset of S is linearly independent. Say $x \in V \setminus S$. S spans, so $X = \sum_{i=1}^{n} a_i v_i$ for some $a_i \in K$, $v_i \in S$. So $x - a_1 v_1 - a_2 v_2 - \cdots - a_n v_n = 0$. Thus, $S \cup \{x\}$ is not linearly independent.

(3) $\Rightarrow$ (1): S is a maximal linearly independent set. Suppose S did not span. Let $x \notin \text{span}(S)$. Then we claim $S \cup \{x\}$ is linearly independent. Suppose we have $bx + a_1 v_1 + \cdots + a_n v_n = 0$, with $a_i, b \in K$, $v_i \in S$. If $b = 0$, then we have a linear relation on the elements of S, so $a_i = 0$, because S is linearly independent. If $b \neq 0$, then $x = \frac{-1}{b}(a_1 v_1 + \cdots + a_n v_n)$, so $x \in \text{span}(S)$. Oops!

$\square$

Prop: V is a vector space, $B \subseteq V$. Then any maximally independent subset of B spans B.
Proof: Let $S \subset B$ be a maximal linearly independent set. Suppose $x \in B$ s.t. $x \notin \text{span}(S)$. Then $S \cup \{x\}$ is linearly independent. This contradicts S being a maximal linearly independent subset of B. $\square$

Lemma: (Zorn's Lemma) Suppose S is a partially ordered set with binary relation $\succeq$ s.t.

1. $\succeq$ is reflexive: $x \succeq x, \forall x \in S$.
2. $\succeq$ is antisymmetric: $x \succeq y, y \succeq x \Rightarrow x = y, \forall x, y \in S$.
3. $\succeq$ is transitive: $x \succeq y, y \succeq z \Rightarrow x \succeq z, \forall x, y, z \in S$.

Assume $S \neq \emptyset$ and every chain is bounded, i.e., given $x_1 \succeq x_2 \succeq \cdots \in S$, $\exists y \in S$ s.t. $x_i \succeq y, \forall i$, then there exists a maximal element $z \in X$, i.e., $z \succeq x \Rightarrow z = x$.

Cor: Let $A \subseteq B \subseteq V$, with A linearly independent and B spanning. Then there is a basis S contained in B which contains A.
Proof: define X to be the set of all independent subsets of B containing A.

- X is partially ordered by inclusion.
- X is nonempty because $A \in X$.

We will apply Zorn's Lemma, by showing all chains are bounded. Let $C_1 \subseteq C_2 \subseteq \cdots \in X$. We need to prove $\exists C \in C$ s.t. $C_i \subseteq C$, $\forall i$.
Take $C = \bigcup_{i \geq 1} C_i \subseteq B$. We need to show C is linearly independent. Suppose $a_n x_1 + \cdots + a_n x_n$ is a linear relation with $x_i \in C$. Then $\exists m$ s.t. $x_1, \ldots, x_n \in C_m$. So the relation is trivial, because $C_m \in X$, so C bounds the independent sets.
Thus, by Zorn's Lemma, there is a maximal element of X. $\square$

Cor: If $A \subseteq V$ is any independent set, there is a basis containing A.
Proof: take $B = V$. $\square$

Cor: If $B \subseteq V$ is any spanning set, there is a basis contained in B.
Proof: Take $A = \emptyset$. $\square$

Prop: (Basis Exchange Lemma) V vector space, B basis of V, C spanning set of V. Given $x \in B$, $\exists y \in C$ s.t. $(B \setminus \{x\}) \cup \{y\}$ is a basis.
Proof: $B \setminus \{x\}$ is not maximally independent, but is independent. So it must not span.
So $C \not\subseteq \operatorname{span}(B \setminus \{x\})$ (if we did have containment, then we'd have $\operatorname{span}(C) \subseteq \operatorname{span}(B \setminus \{x\})$).
So $\exists y \in C$ s.t. $y \notin \operatorname{span}(B \setminus \{x\})$. We need to show $B' = (B \setminus \{x\}) \cup \{y\}$ is a basis, because $y \in \operatorname{span}(B)$, $y = a_1 z_1 + \cdots + a_n z_n$ with $z_i \in B$.

Must have $z_i = x$, $a_i \neq 0$, for some i. Say $i = 1$ WOLOG (otherwise, $y \in \operatorname{span}(B \setminus \{x\})$).
$y = a_1 x + a_2 z_2 + \cdots + a_n z_n$. $x \in \operatorname{span}(B')$, because $x = \frac{1}{a}(y - a_1 z_1 - a_2 z_2 - \cdots - a_n z_n) \in \operatorname{span}(B')$.

$B \setminus \{x\} \subseteq \operatorname{span}(B') \Rightarrow B \subseteq \operatorname{span}(B)'$. So $V = \operatorname{span}(B) \subseteq \operatorname{span}(B')$, and thus, B' spans V.

Claim: B' is independent. Consider a linear relation $cy + d_1 v_1 + \cdots + d_m v_m = 0$, with $c, d_i \in K$, $v_1, \ldots, v_m \in B \setminus \{x\}$.

Case1: $c = 0$. Then this is a relation between elements of $B \setminus \{x\}$, which we know to be independent. So each $d_i = 0$.
Case2: $c \neq 0$. Then $c(a_1 x + a_2 z_2 + \cdots + a_n z_n) + d_1 v_1 + \cdots = d_m v_m = 0$. This is a linear relation between elements of B. So the coefficients must all be 0.

Therefore, B' is a basis. $\square$

Defn: A vector space V is **finite dimensional** if it has a finite spanning set.

Thm: (Dimension Theorem) Suppose V is a finite dimensional vector space. Then all bases are finite and have the same size.
Proof: Note that, by definition, V has a finite spanning set. This contains a basis, so there exists a finite basis.
Say $B = \{x_1, \ldots, x_n\}$ is a basis. C is some other basis. Define $B_0 = B$. Given B_k, define B_{k+1} by replacing $x_k \in B_k$ with something from C, while maintaining it as a basis.
Thus, $B_k \subseteq C$, so $C = B_k$. We have $\#C = \#B_k = \#B_0 = \#B$. $\square$

Defn: Suppose V is finite dimensional. Then we say the **dimension** of V, denoted $\dim V$, to be the size of a basis.

Ex: $V = K^n$. For $1 \leq i \leq n$, let e_i be the vector of zeros, with 1 in the ith position. $\{e_1, \ldots, e_n\}$ forms a basis of V, so $\dim V = n$.

Ex: V is the set of all polynomials in X with coefficients in K. This is not finite dimensional.

Ex: Let V be the set of all polynomials of degree $\leq d$. Then $\left\{1, x, x^2, \ldots, x^n\right\}$ is a basis, so $\dim V = n+1$.

Let V be a vector space, and let W be a subspace. We can then form V/W, the quotient space.

- As an abelian group, it's the usual quotient group.
- Given $a \in K$, $v \in V$, define $a(v \cdot W) = av + W$. Check that this is well defined:
  Suppose $v + W = v' + W$. Then $v - v' \in W$, so $a(v - v') = av - av' \in W$. Thus, $av + W = av' + W$.

We have the quotient map $\begin{aligned} \pi : V &\to V/W \\ v &\mapsto \bar{v} = v + W \end{aligned}$.
This is a linear map, because it is a group homomorphism and compatible with scalar multiplication.

Prop: Let V be a finite dimensional vector space, and $W \subseteq V$ a subspace. Then W and V/W are finite dimensional, and $\dim V = \dim W + \dim V/W$.

Proof: Let S be a basis for W. We can extend S to be a basis T of V. Thus, T (and S) are finite. $S = \{x_1, \ldots, x_n\}$. $T = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. So $\dim W = n$, $\dim V = n + m$.

We claim $\bar{y}_1, \ldots, \bar{y}_m$ form a basis of V/W. Let $\bar{v} \in V/W$. We can write $v = a_1x_1 + \cdots + a_nx_n + b_1y_1 + \cdots + b_my_m$. Apply π, so $\bar{v} = b_1\bar{y}_1 + \cdots + b_m\bar{y}_m$. Thus, $\bar{y}_1, \ldots, \bar{y}_m$ spans.

Say $b_1\bar{y}_1 + \cdots + b_m\bar{y}_m = 0$. Consider $b_1y_1 + \cdots + b_my_m \in \ker(\pi) = W$. This is equal to $a_1x_1 + \cdots + a_nx_n$ for some $a_1, \ldots, a_n$. So $-a_1x_1 - \cdots - a_nx_n + b_1y_1 + \cdots + b_my_m = 0$. So all coefficients are 0, so $\bar{y}_1, \ldots, \bar{y}_m$ are linearly independent.

Therefore, they're a basis. $\square$