Math 493 Lecture 17

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Group Presentations

D_n (dihedral group) generated by a (rotation by $\frac{2\pi}{n}$) and b (reflection across x-axis).
Relations: $a^n = 1$, $b^2 = 1$, $bab^{-1} = a^{-1}$.

M (rigid motions of the plane) generated by t_a (translation by a), ρ_θ (rotation by θ), r (reflection across x-axis).
Relations: lots (see notes from lecture 10/2).

Free Groups

Defn: A **free group** has some set of generators, S, with no relations.

Ex: $\mathbb{Z}$, $S = \{1\}$. $\mathbb{Z}$ is the free group with one generator.

Fix set S (elements are symbols). Define $S' = \left\{x, x^{-1} : x \in S\right\}$ (note: x^{-1} is a new formal symbol). Let W' be the set of words in S' (a word is a finite string $x_1 x_2 \cdots x_n$, $x_j \in S'$). Allow the empty word ($n = 0$).

Defn: A word $w \in W'$ is called **reduced** if there are no 2 adjacent letters of the form xx^{-1} (for $x \in S'$).

Note: for $x \in S$, x^{-1} is an element of S'. $(x^{-1})^{-1}$ is taken to mean x.

Defn: Starting with any word, we can cancel xx^{-1} in it to get a reduced word. This is called the **reduced form** of a word.

Ex: $S = \{a, b, c\}$, $S' = \left\{a, a^{-1}, b, b^{-1}, c, c^{-1}\right\}$. Consider $W = bacc^{-1}a^{-1}a$ (W is not reduced). Go to reduced form:

$$ba(cc^{-1})a^{-1}a \to ba(a^{-1}a) \to ba$$
$$ba(cc^{-1})a^{-1}a \to b(aa^{-1})a \to ba$$

Note that different letters survive, but the reduced form always appears the same.

Prop: Let w be a word. Then any two reduced forms of w are equal.

Proof: Induction on the length of w. If w is reduced, there's nothing to prove. So say we have xx^{-1} in w somewhere. It's enough to show that any reduced form of w can start by canceling this xx^{-1}.

Consider some reduced form w_0 of w. We have two cases:

1. At some step, we cancel the xx^{-1} in question. $w = (\text{stuff})xx^{-1}(\text{more stuff})$. First cancel in (stuff) and (more stuff). Then cancel xx^{-1}. Then cancel more. As the three parts of w are disjoint, we can switch the first two steps.
2. Not 1., i.e., we never cancel the xx^{-1} in question. As w_0 is reduced, one of x and x^{-1} must be canceled. WOLOG, x is canceled. Then $w = (\text{stuff})xx^{-1}(\text{stuff})$. After cancellations, it becomes $(\text{stuff})x^{-1}xx^{-1}(\text{stuff})$. Cancel $x^{-1}x$, and continue. We get the same result.

$\square$

Defn: Define an equivalence relation on W' by $w_i \sim w_j$ if their reduced forms are equal.

W' has a binary operation given by concatenating words: given $w, w' \in W'$, $w = x_1 \cdots x_n$, $w' = y_1 \cdots y_m$, $x_*, y_* \in S'$. Let $ww' = x_1 \cdots x_n y_1 \cdots y_m$. This binary operation is associative, and has the identity element (the empty string). But it does not have inverses – xx^{-1} is not the empty word. But obviously, $xx^{-1} \sim$ empty word.

Prop: Given equivalent words $a \sim a'$, $b \sim b'$, then $ab \sim a'b'$.

Proof: Let a_0 be the reduced form of a and a', b_0 the reduced form of b and b'. Then $ab \to a_0b_0$ and $a'b' \to a_0b_0$. Let c be the reduced form of a_0b_0. Then $ab \to c$, $a'b' \to c$, so $ab \sim a'b'$. $\square$

By this proposition, concatenation induces a binary map on $F = W'/\sim$. This is associative and has an identity element, just as the binary operation was. It also has inverses now:

$$w = x_1 \cdots x_n, \quad w' = x_n^{-1} \cdots x_1^{-1}, \quad \Rightarrow \quad ww' \sim w'w \sim \text{empty word}$$

Thus, F is a group under this composition law.

Defn: $F = W'/\sim$ is the free group with generating set S.

Notation: F_S, $\langle S \rangle$, and F_n if $\#S = n$.

Prop: (Mapping Property for Free Groups) Let S be a set, $F = F_S$, and G a group. Then

$$\{\text{group homomorphisms } F \to G\} \overset{\text{restriction}}{\underset{\Phi}{\to}} \{\text{Functions } S \to G\}$$

is a bijection. Concretely, if $F = \langle x_1, \ldots, x_n \rangle$ a free group on n generators, then to give a homomorphism $F \to G$, pick $g_1, \ldots, g_n \in G$ and $\exists! \varphi : F \to G$ group homomorphism.
$$x_i \mapsto g_i$$

Proof: We know Φ is injective because S generates F. Let $f_0 : S \to G$ be given. We need to show $\exists f : F \to G$ a group homomorphism extending f_0 (so then $f_0 = \Phi(f)$, so Φ is surjective).

Define $\tilde{f} : W' \to G$ by $\tilde{f}(x_1 \cdots x_n) = f_0(x_1) \cdots f_0(x_n)$ for $x_i \in S'$, (for $x \in S$, let $f(x^{-1}) = f(x)^{-1}$). It's clear that if w_0 is the reduced form of w, then $\tilde{f}(w_0) = \tilde{f}(w)$. Thus, $\tilde{f}$ is well-defined on the equivalence classes, i.e., $w \sim w' \Rightarrow \tilde{f}(w) = \tilde{f}(w')$. So $\tilde{f}$ induces $f : F \to G$, and since $\tilde{f}(ww') = \tilde{f}(w)\tilde{f}(w')$, it follows that f is a group homomorphism. $\square$

Remark: Let G be a group, $S \subset G$, $F = \langle S \rangle$. We have a natural map $F \to G$. The image of this map is the subgroup of G generated by S. Particularly, $F \to G$ is surjective if and only if S generates G.

Ex: $F_2 = \langle x_1, x_2 \rangle$. We have a surjection

$$\begin{aligned} f : F_2 &\to D_n \\ x_1 &\mapsto a \quad \text{rotation} \\ x_2 &\mapsto b \quad \text{reflection} \end{aligned}$$

The relations between a and b give elements in the kernel of f.

$$x_1^n, x_2^2, x_2x_1x_2x_1 \in \ker(f)$$

Because f is surjective, the first isomorphism theorem says $F_2/\ker(f) \cong D_n$.

Fact: $\ker(f)$ is the smallest normal subgroup generated by our three elements.

Defn: Let S be a set, $F = \langle S \rangle$. Let $R \subset F$ (subset). Then the smallest normal subgroup containing R, N, is the subgroup of F generated by all conjugates of elements of R. Let $\langle S \mid R \rangle = F/N$ (the group generated by S with relations R). If G is a group, a **presentation** of G is an isomorphism $G \cong \langle S \mid R \rangle$ for some S and R.

Ex: $D_n \cong \left\langle x_1, x_2 \mid x_1^n, x_2^2, x_1x_2x_1x_2 \right\rangle$. This is a presentation. We will prove this soon.

Prop: (Mapping Property) Given S set, $F = F_S$, $R \subset F$, G a group, we have a bijection.

$$\{\text{group homomorphisms } \langle S \mid R \rangle \to G\} \underset{\Phi}{\xrightarrow{\sim}} \{\text{functions } S \to G \text{ with } R \mapsto \{\text{id}\}\}$$

Proof: Given a homomorphism $f : \langle S \mid R \rangle \to G$, we get a composition $F \to \langle S \mid R \rangle \xrightarrow{f} G$ (recall: $\langle S \mid R \rangle = F/N$, where N is the smallest normal subgroup containing R). This sends everything in R to $1 \in G$, so Φ is well-defined.
Also, Φ is injective, as S generates $\langle S \mid R \rangle$.
Suppose $f_0 : S \to G$ s.t. $f : F \to G$ is the corresponding group homomorphism. Then $f : R \to 1$, because $R \subseteq \ker(f) \Rightarrow N \subseteq \ker(f)$. So we have

$$\begin{tikzcd}
F \arrow[d, "f_0"'] \arrow[dr, "f"] & \\
\langle S \mid R \rangle \arrow[r, "\exists!"'] \arrow[d, equal] & H \\
F/N &
\end{tikzcd}$$

How to find a presentation for G:

1. Find a set S of generators for G, and a set R of relations.
2. By the mapping property, we get a natural homomorphism $f : \langle S \mid R \rangle \to G$, which is surjective as S generates G.
3. Show it's a bijection:
   (a) $\ker(f) = 1$
   (b) $\# \langle S \mid R \rangle \leq \#G$ if $\#G < \infty$.

Let's carry out this process for D_n:

1. $S = \{a, b\}$, F is the free group on symbols $A \leftrightarrow a$, $B \leftrightarrow b$. $R = \left\{A^n, B^2, ABAB\right\}$.
2. We have a surjection $f : \langle S \mid R \rangle \to D_N$ where $A \mapsto a, B \mapsto b$.
3. We will show $\# \langle S \mid R \rangle \leq 2n$, by showing every element has the form A^k or BA^k for $0 \leq k < n$.

As $A^n = 1$, $B^2 = 1$, in $\langle S \mid R \rangle$, every element can be written as $A^{k_1}BA^{k_2}B \cdots A^{k_\ell}$. $A^{k_{n-1}}BA^{k_n} = BA^{k_n - k_{n-1}}$. If negative, $A^{-k} = A^{n-k}$. So we can simplify to get the result we want.