Lecture 25

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Characters

Let V be a finite-dimensional complex representation of a group G.

Defn: The **character** of V is the function $\begin{aligned}\mathcal{X}_V : G &\to \mathbb{C} \\ g &\mapsto \operatorname{tr}(g|_V)\end{aligned}$.

Lemma:

- $\mathcal{X}_{V\oplus W} = \mathcal{X}_V + \mathcal{X}_W$
- $\mathcal{X}_{\mathrm{Hom}(V,W)} = \overline{\mathcal{X}_V}\mathcal{X}_W$
- $\mathcal{X}_{V^*} = \overline{\mathcal{X}_V}$

Lemma: $\mathcal{X}_V$ satisfies $\mathcal{X}_V(ghg^{-1}) = \mathcal{X}_V(h)$, $\forall g, h \in G$.

Defn: A **class function** on G is a function $f : G \to \mathbb{C}$ s.t. $f(ghg^{-1}) = f(h)$, $\forall g, h \in G$.

Ex: $\mathcal{X}_V$ is a class function.

Defn: $\mathscr{C}(G)$ is the space of all class functions. It is a $\mathbb{C}$-vector space, and its dimension is the number of conjugacy classes in G.

For $\varphi, \psi \in \mathscr{C}(G)$, define

$$\langle \varphi, \psi \rangle = \frac{1}{\#G} \sum_{g\in G} \overline{\varphi(g)}\psi(g)$$

This is a positive definite Hermitian inner product on $\mathscr{C}(G)$.

Defn: $\mathcal{X}_{\mathrm{triv}} = \mathbb{1} \in \mathscr{C}(G)$ is the function $g \mapsto 1$, $\forall g \in G$.

Thm: If V is a representation of G, then $\dim V^G = \langle \mathbb{1}, \mathcal{X}_V \rangle$.
Idea of a proof: Define $\pi : V \to V$ where $v \mapsto \frac{1}{\#G}\sum_{g\in G} gv$.
Clearly, $\operatorname{tr}(\pi) = \langle \mathbb{1}, \mathcal{X}_V \rangle$. On the other hand, $\pi^2 = \pi$, and $\operatorname{im}(\pi) = V^G$, so $\operatorname{tr}(\pi) = \dim V^G$.

Cor: For V, W representations of G, dim $\mathrm{Hom}_G(V,W) = \langle \mathcal{X}_V, \mathcal{X}_W \rangle$.
Proof: Recall $\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G$. By the theorem,

$$\dim \mathrm{Hom}_G(V,W) = \left\langle \mathbb{1}, \mathcal{X}_{\mathrm{Hom}(V,W)} \right\rangle = \left\langle \mathbb{1}, \overline{\mathcal{X}_V}\mathcal{X}_W \right\rangle = \langle \mathcal{X}_V, \mathcal{X}_W \rangle$$

$\square$

Thm: (Schur's Lemma) For V, W irreducible representations of G,

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 0 & V \ncong W \\ 1 & V \cong W \end{cases}$$

Moreover, if $V = W$, $\operatorname{Hom}_G(V, V) = \operatorname{span}(\operatorname{Id}_V)$.

Cor: (Schur Orthogonality) If V, W are irreducible representations, then

$$\langle \mathcal{X}_V, \mathcal{X}_W \rangle = \begin{cases} 0 & V \ncong W \Leftrightarrow \mathcal{X}_V \neq \mathcal{X}_W \\ 1 & V \cong W \Leftrightarrow \mathcal{X}_V = \mathcal{X}_W \end{cases}$$

This implies the number of irreducible representations (up to isomorphism) is at most $\dim \mathscr{C}(G)$, which in turn is the number of conjugacy classes of G.

Let $L_1, \ldots, L_r$ be "the" irreducible representations of G (that is, choose one from each isomorphism class). Any representation V can be decomposed as

$$V \cong L_1^{\oplus m_1} \oplus L_2^{\oplus m_2} \oplus \cdots \oplus L_r^{\oplus m_r}$$

Defn: m_i is called the **multiplicity** of L_i in V.

$\mathcal{X}_V = m_1\mathcal{X}_1 + \cdots + m_r\mathcal{X}_r$. By Schur Orthogonality, $m_i = \langle \mathcal{X}_i, \mathcal{X}_r \rangle$.

Cor: If $\mathcal{X}_V = \mathcal{X}_W$, then $V \cong W$.