Probabilistic Combinatorics

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Defn: A <u>discrete random variable</u> X takes a finite number of possible values $a_1, \ldots, a_n$, with
$$\mathbb{P}(X = a_i) = p_i$$
where $0 \leq p_i \leq 1$ and $\sum p_i = 1$.

Defn: The <u>average</u> of X is
$$\mathbb{E}[X] = \sum_{i=1}^{n} a_i \cdot \mathbb{P}(X = a_i)$$

Defn: 2 random variables X, Y are <u>independent</u> if
$$\mathbb{P}(X = a, Y = b) = \mathbb{P}(X = a) \cdot \mathbb{P}(Y = b)$$

Ex: $\{O, O, O, A, A, A\}$
$\mathbb{P}(X = A, Y = A) = \frac{1}{2} \cdot \frac{2}{5} \neq \mathbb{P}(X = A) \cdot \mathbb{P}(Y = A) = \frac{1}{2} \cdot \frac{1}{2}$
$\Rightarrow X$ and Y are dependent.

Some properties:
X, Y random variables.

$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$
$$\mathbb{E}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$
If X and Y are independent, then
$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.

If $M = \mathbb{E}[X]$, then X must attain some value $a \leq M$ and $b \geq M$.

Ex: $\mathbb{P}(X_i = 1) = \frac{1}{2}$, $\mathbb{P}(X_i = 0) = \frac{1}{2}$.
$X = X1 + \cdots + X_n$.
$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \left(1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}\right) = \frac{n}{2}.$$
$$\mathbb{E}[X] = \sum_{i=0}^{n} i \cdot \mathbb{P}(X = i).$$
$$\mathbb{P}(X = i) = \frac{\binom{n}{i}}{2^n}, \text{ so } \mathbb{E}[X] = \sum_{i=0}^{n} i \cdot \frac{\binom{n}{i}}{2^n} = \frac{n}{2^n} \sum_{i=0}^{n} \binom{n-1}{i-1} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}.$$

Ex: Flip n fair coins, compute the average number of runs (T T H H H T T would give us 3 runs).

$n = 1$: $\begin{matrix} T \\ H \end{matrix} \rightarrow \mathbb{E} = 1.$

$n = 2$: $\begin{matrix} HH \rightarrow 1 \\ HT \rightarrow 2 \\ TH \rightarrow 2 \\ TT \rightarrow 1 \end{matrix} \rightarrow \mathbb{E} = 1.5$

$n = 3$: $\ldots \rightarrow \mathbb{E} = 2.$

So we guess $\frac{n+1}{2} = 1 + \frac{n-1}{2}$.

The number of runs $X = 1 + Y = 1 + Y_1 + Y_2 + \cdots + Y_{n-1}$, where Y is the number of changes in between Heads and Tails.

$\mathbb{E}[X] = \mathbb{E}[1+Y] = 1 + \mathbb{E}[Y] = 1 + \sum_{i=1}^{n-1} \mathbb{E}[Y_i] = 1 + (n-1)(\frac{1}{2}) = 1 + \frac{n-1}{2}.$

Ex: n hunters $x_1, \ldots, x_n$.
n rabbits $y_1, \ldots, y_n$.
Each hunter has one shot, and is aiming at a specific rabbit.

$\mathbb{E}[\#\text{ surviving rabbits}] = ?\ \mathbb{E}[\text{when } n = 100] \sim 37$

$X_i = 1$ when y_i survives, $X_i = 0$ otherwise.

$\Rightarrow \mathbb{E}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \prod_{j=1}^{n} \frac{n-1}{n} = \sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^n = n \cdot \left(1 - \frac{1}{n}\right)^n \rightarrow \frac{n}{e}.$

Prop: If G is a graph on n vertices, with e edges, then G contains a bipartite subraph with at least $e/2$ edges.
Proof: For each v_i, let $\mathbb{P}(v_i \in A) = \frac{1}{2}$, $\mathbb{P}(v_i \in B) = \frac{1}{2}$. $e = v_i v_j$.
$\mathbb{P}(e \text{ goes between 2 parts}) = \mathbb{P}(v_i, v_j \text{ are in different parts}) = \frac{1}{2}$
$\mathbb{E}[\#\text{ of edges between 2 parts}] = \sum_e \frac{1}{2} = \frac{E}{2}$. $\square$

Defn: Given a graph G, a set of vertices S is <u>independent</u> if there are no edges between them.

Defn: $\alpha(G) = \max_{S \text{ independent set}} |S|$. $\alpha(G)$ is called the <u>independent number</u>, <u>coclique number</u>.

Ex: $E = 0 \rightarrow \alpha(G) = n$
$E = \frac{n}{2} \rightarrow \alpha(G) = \frac{n}{2}$
$E = \binom{k}{2} \cdot \frac{n}{k} = \frac{n(k-1)}{2} \rightarrow \alpha(G) = k$

Thm: (Turán) If G has n vertices, E edges, then $\alpha(G) \geq \frac{n^2}{2E+n}$.
Pf: Consider a random permutation of $v_1, \ldots, v_n$. Let $\pi : [n] \leftrightarrow [n]$ be a random permutation.
Let $v \in S$ if $\pi(v) > \pi(w)$, $\forall w \in N(v)$. Then for each vector $v \in S$, $S \cap N(v) = \emptyset$.

$\mathbb{E}[|S|] = \mathbb{E}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \mathbb{E}[X_i]$. $\mathbb{E}[X_i] = \mathbb{P}(v_i \in S) = \frac{1}{\deg(v_i)+1}$.

$$\mathbb{E}[|S|] = \left(\sum_{i=1}^{n} \frac{i}{\deg(v_i)+1}\right)\left(\sum \deg(v_i) + 1\right)\left(\sum \deg(v_i) + 1\right)^{-1}$$
$$\geq (1 + \cdots + 1)^2 \cdot \left(\sum \deg(v_i) + 1\right)^{-1} = \frac{n^2}{\sum \deg(v_i) + 1} = \frac{n^2}{2|E| + n}. \square$$