# Graph Colorings

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**Defn:** Given $G = (V, E)$ and some $k \in \mathbb{N}$, a (proper) $\underline{k\text{-coloring}}$ of $G$ is a mpa $f : V \to [k]$ s.t. for every edge $v_i v_j \in E$, $f(v_i) \neq f(v_j)$.

**Defn:** If there exists a $k$-coloring on $G$, we say $G$ is $\underline{k\text{-colorable}}$.

**Ex:** 1-colorable $\Rightarrow$ only the empty graphs ($E = \emptyset$).
2-colorable $\Rightarrow$ bipartite graphs. All trees are 2-colorable using BFS.
3-colorable graphs have no nice characterization, and are *hard* to check.

**Lemma:** $G$ is bipartite $\leftrightarrow$ $G$ has no odd-length cycles.

**Defn:** $D = \max_{v \in V} \{\deg(v)\}$ is the $\underline{\text{max degree}}$.

**Ex:** $D = 1 \leftrightarrow G$ is disjoint union of edges and vertices.
$D = 2 \leftrightarrow G$ is disjoint union of paths, cycles, and vertices.

**Prop:** If $D = \max_{v \in V} \{\deg(v)\}$, then $G$ is $(D+1)$-colorable.
Proof: Start with $v_1$, color it $c_1$. If $v_1, \ldots, v_k$ are colored, look at $v_{k+1}$. We know $|N(V_{k+1})| \leq D$, so if we exclude at most $D$ colors for $v_{k+1}$, there exists some remaining color for $v_{k+1}$. Color $v_{k+1}$ with that color. $\square$

Can we improve the coloring "index" from $D+1$ to $D$?
$G = K_{n+1}$, $D = n$, but $G$ is not $D$-colorable.

**Thm:** If $G$ is connected and not complete, and $D = \max\deg(G) \geq 3$, then $G$ is $D$-colorable.
Proof: By contradiction, assume $\exists G$ such that
(1) $G$ is not complete and is connected, (2) $D = \max\deg(G) \geq 3$,
(3) $G$ is not $D$-colorable, and
(4) $|V| = n$ is smallest possible,

pick any $x \in G$ and consider its neighbors. Let $G' = G - x$.
Then $\max\deg(G') \leq D \to G'$ is $D$-colorable due to (4).
If $G'$ is not connected, look at the components of $G'$. Each has less than $n$ vertices, and so must be $D$-colorable. So $D'$ is colorable.
Consider any $D$-coloring on $G'$. If $|N(x)| < D$, we can color $x$ with some remaining color, so $G$ is $D$-colorable. But this and (3) imply that $|N(x)| = D\ \forall x \in G$.

We know $x$ has $D$ neighbors: $x_1, \ldots, x_D$. Let $c_i = \text{color}(x_i)$. If $c_i = c_j$ ($i \neq j$), then we can color $x$, in contradiction with (3). Oops!
So $c_i \neq c_j$, $\forall i \neq j$. Take any $x_i$ and $x_j$. Let $H_{ij}$ be the subgraph consisting of $c_i$ and $c_j$.
Observation 1: $x_i$ and $x_j$ are in the same connected component of $H_{ij}$.
Proof: If not, then $x_i$ is in some component $C$ of $H_{ij}$, $x_j \notin C$. Flip colors in $C$. Since this doesn't

affect coloring property, $c_i = c_j$. But $i \neq j$. Oops!
Observation 2: The component $C$ connecting $x_{i,\ j}$ in $H_{ij}$ is a path $x_i \rightsquigarrow x_j$.
Proof: **Not exactly sure what this part is saying, since it was something of a picture proof.**
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Then $N(u)$ must have the same colors. Recolor $u$ to some $(c_k)$.
**This went unfinished.**