Graph Colorings

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Defn: Given $G = (V, E)$ and some $k \in \mathbb{N}$, a (proper) $\underline{k\text{-coloring}}$ of G is a mpa $f : V \to [k]$ s.t. for every edge $v_i v_j \in E$, $f(v_i) \neq f(v_j)$.

Defn: If there exists a k-coloring on G, we say G is $\underline{k\text{-colorable}}$.

Ex: 1-colorable $\Rightarrow$ only the empty graphs ($E = \emptyset$).
2-colorable $\Rightarrow$ bipartite graphs. All trees are 2-colorable using BFS.
3-colorable graphs have no nice characterization, and are *hard* to check.

Lemma: G is bipartite $\leftrightarrow$ G has no odd-length cycles.

Defn: $D = \max\limits_{v \in V} \{\deg(v)\}$ is the $\underline{\text{max degree}}$.

Ex: $D = 1 \leftrightarrow G$ is disjoint union of edges and vertices.
$D = 2 \leftrightarrow G$ is disjoint union of paths, cycles, and vertices.

Prop: If $D = \max\limits_{v \in V} \{\deg(v)\}$, then G is $(D+1)$-colorable.
Proof: Start with v_1, color it c_1. If $v_1, \ldots, v_k$ are colored, look at v_{k+1}. We know $|N(V_{k+1})| \leq D$, so if we exclude at most D colors for v_{k+1}, there exists some remaining color for v_{k+1}. Color v_{k+1} with that color. $\square$

Can we improve the coloring "index" from $D+1$ to D?
$G = K_{n+1}$, $D = n$, but G is not D-colorable.

Thm: If G is connected and not complete, and $D = \max \deg(G) \geq 3$, then G is D-colorable.
Proof: By contradiction, assume $\exists G$ such that
(1) G is not complete and is connected, (2) $D = \max \deg(G) \geq 3$,
(3) G is not D-colorable, and
(4) $|V| = n$ is smallest possible,

pick any $x \in G$ and consider its neighbors. Let $G' = G - x$.
Then $\max \deg(G') \leq D \to G'$ is D-colorable due to (4).
If G' is not connected, look at the components of G'. Each has less than n vertices, and so must be D-colorable. So D' is colorable.
Consider any D-coloring on G'. If $|N(x)| < D$, we can color x with some remaining color, so G is D-colorable. But this and (3) imply that $|N(x)| = D$ $\forall x \in G$.

We know x has D neighbors: $x_1, \ldots, x_D$. Let $c_i = \text{color}(x_i)$. If $c_i = c_j$ $(i \neq j)$, then we can color x, in contradiction with (3). Oops!
So $c_i \neq c_j$, $\forall i \neq j$. Take any x_i and x_j. Let H_{ij} be the subgraph consisting of c_i and c_j.
Observation 1: x_i and x_j are in the same connected component of H_{ij}.

Proof: If not, then x_i is in some component C of H_{ij}, $x_j \notin C$. Flip colors in C. Since this doesn't affect coloring property, $c_i = c_j$. But $i \neq j$. Oops!

Observation 2: The component C connecting $x_{i,\ j}$ in H_{ij} is a path $x_i \rightsquigarrow x_j$.

Proof: **Not exactly sure what this part is saying, since it was something of a picture proof.**

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Then $N(u)$ must have the same colors. Recolor u to some (c_k).

This went unfinished.