Matchings

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Defn: Let $G = (V_1 \sqcup V_2, E)$ be a bipartite graph. A matching $M \subset E$ is any collection of vertex-disjoint egdes. M is a perfect matching iff $|M| = |V_1| = |V_2|$.

Thm: (Halls) Let $S_1, \ldots, S_m \subset [n]$. There is a way to pick $x_i \in S_i$ s.t. $x_i \neq x_j$, $\forall i, j$, $\Leftrightarrow$ $\forall$ subcollections $S_{i_1}, \ldots, S_{i_k}$, we have $|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k$ $(k \leq m)$.

Cor: $G = (V_1 \cup V_2, E)$ with $|V_1| = |V_2| = n$ has a perfect matching $\Leftrightarrow$ $|N(x)| \geq |X|$ $\forall X \subset V_1$.

Proof (Halls):
Necessary: Assume we can pick $x_i \in S_i$. Then $\forall S_{i_j}$, $x_{i_j} \in S_{i_j}$.

Sufficient: Induction on m.
$m = 1$: $|S_1| \geq 1$, so pick any $x_1 \in S_1$.
$m = k$ true, let's show $m = k + 1$.

Case 1: $|S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}| \geq k$. Then we can choose any $x_{m+1} \in S_{m+1}$ and the statement still holds.
Case 2: $\exists$ some subcollection $|S_{i_1} \cup \cdots \cup S_{i_k}| = k$ for $x_{i_1} \in S_{i_1}, \ldots, x_{i_k} \in S_{i_k}$. S_{m+1} must have aneighbor outside this subcollection, or otherwise, the inductive hypothesis fails.

Define $S'_j = S_j \backslash \{S_{i_1}, \ldots, S_{i_k}\}$. We know $\left|S_{i_1} \cup \cdots \cup S_{i_k} \cup S_{i_{k+1}} \cup \cdots \cup S_{i_{k+\ell}}\right| \geq k+\ell$. So $\left|S'_{i_{k+1}} \cup \cdots \cup S'_{i_{k+\ell}}\right| = \ell + k - k = \ell$.

So by induction, we can pick $x_{k+1} \in S'_{k+1}, \ldots, x_{k+\ell} \in S'_{k+\ell}$. $\square$

Defn: A maximal matching M is a matching with $|M| \geq |M'|$ for all matchings M'.

Defn: A subset $X \subset V$ is an edge cover if every edge is incident to at least one vertex $x \in X$.

Thm: (König) Assume $G = (V_1 \sqcup V_2, E)$ is a bipartite graph, perhaps without a perfect matching. Let $M \subset E$ be a maximal matching, and $C \subset V$ be a minimal edge cover. Then $|M| = |C|$.
Proof:
$|M| \leq |C|$: This is easily true; if our matching has size $|C| + 1$ or larger, then by the pigeonhole principle, two edges hit the same vertex.
$|M| \geq |C|$: $\forall X \subset X_1$, $|N(x) \setminus Y_1| \geq |X|$. If not, we can decrease the size of the edge cover. **Sort of a proof by picture.** So for each element in C, pick a match.
$\square$