Partially Ordered Sets

Thomas Cohn

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Defn: A partially ordered set (also known as a poset) is a pair $P = (X, \preceq)$ with X a set and $\preceq$ a relation on X with the following properties:

1. Reflexivity: $\forall x \in X$, $x \preceq x$
2. Anti-Symmetry: $x \preceq y \wedge y \preceq x \rightarrow x = y$
3. Transitivity: $x \preceq y \wedge y \preceq z \rightarrow x \preceq z$

If $x \preceq y$, but $x \neq y$, we write $x \prec y$.

Ex: $(\mathbb{R}, \leq)$
$(\mathbb{N}, \leq)$
$([a, b], \leq)$ where $[a, b] = \{a, a + 1, \ldots, b\}$

In these examples, every pair of numbers is comparable. But this is not required!

Ex: $(\{a, b, c, d\}, \preceq = \{(a, b), (a, c), (b, d), (c, d)\})$

Defn: For $P = (X, \preceq)$ and some $x, y \in X$, if neither $x \preceq y$ nor $y \preceq x$, then we say x and y are incomparable (or independent).

Defn: A poset $P = (X, \preceq)$ is a total ordering if every pair $x, y \in X$ is comparable.

Ex: Our first three examples of posets above are total orderings. The last one is not.

Can we define a total order on $\mathbb{R}^2$? Yes!
Dictionary (lexicographic) ordering:
$(x_1, y_1) \preceq (x_2, y_2)$ if

- $x_1 < x_2$
- $x_1 = x_2$ and $y_1 \leq y_2$

We could also use polar coordinates:
$(r_1, \theta_1) \preceq (r_2, \theta_2)$ if

- $r_1 < r_2$
- $r_1 = r_2$ and $\theta_1 \leq \theta_2$

Defn: Let $P = (X, \preceq)$. An element $x \in X$ is minimal if there is no $y \in X$ s.t. $y \prec x$.

Thm: (a) If $P = (X, \preceq)$ is a finite poset, then there exists a total ordering $\leq$ on X which extends $\preceq$.

(b) If $x, y \in X$ are incomparable, then there are 2 total orderings $\leq_1$ and $\leq_2$ s.t. $x \leq_1 y$ and $y \leq_2 x$.

Proof (a): By induction on $|X| = n$.
Base case: $|X| = 1$ is trivial.

Induction: Assume that for all $|X| \leq n$. Then consider $|X| = n+1$. We claim that if $P = (X, \preceq)$ is a finite poset, there is some minimal $x \in X$.

Proof of claim: Pick a random $y \in X$. If y is minimal, we're done. Otherwise, pice $y_1 \prec y$. Repeat. Since X is finite, we cannot have an infinite chain, so *some* y_k is the minimal element in P.

Consider $x \in X$ minimal. Let $P' = (X - x, \preceq)$. Then P' has n elements, so there is a total order $\leq'$ on P' which extends $\preceq$. Because x is minimal, it is less than all elements in $X - x$, so we can add it into P' and still have a total order. $\square$

Proof (b): By induction on $|X| = n$.
Base case: $|X| = 2$ is trivial.

Induction: Assume that the assumption is true if $|X| \leq n$. Let X have size $n+1$. Then $\exists z \in X$ minimal.
Case 1: $z \neq x$ and $z \neq y$. By induction, $P' = (X - z, \preceq)$ has 2 total orderings $\leq_1'$ and $\leq_2'$ with $x \leq_1' y$ and $y \leq_2' x$. Complete $\leq_1', \leq_2'$ to $\leq_1, \leq_2$ by putting z last.
Case 2: x, y are both minimal elements. Let $P'' = (X - \{x, y\}, \preceq)$. Then P'' has the total ordering $\leq''$. Complete $\leq''$ to $\leq_1$ or $\leq_2$ by putting $x < y$ last or $y < x$ last.
Case 3: x is the only minimum element in X. This is not possible, because $x \prec y$. $\square$

Boolean posets: $P = (2^Y, \subseteq)$, where 2^Y is the power set of Y and $\subseteq$ is the subset relation.

Ex: $Y = \{1\}$. Then $2^Y = \{\emptyset, \{1\}\}$
$Y = \{1, 2\}$. Then $2^Y = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Thm: If $P = (X, \preceq)$ is any finite poset, then it can be embedded into some boolean poset.
Proof: For $x \in X$, let $S_x = \{y \in X : y \preceq x\}$. Consider $(2^X, \subseteq)$. The map $x \mapsto S_x$ is an embedding of P into $(2^X, \subseteq)$. $\square$