Matrix-Tree Theorem

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Defn: Given $G = (V, E)$, the adjacency matrix $A = (a_{ij})$ with $a_{ij} = \begin{cases} 1 & (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$

Defn: The laplacian $L = (\ell_{ij})$ is defined by $\ell_{ij} = \begin{cases} \deg(v_i) & i = j \\ -a_{ij} & i \neq j \end{cases}$

Observe: $\det(L) = 0$. This is because $L \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, so $\ker L \neq 0$.

Defn: For each $1 \leq i \leq n$, the minor $L_{i,j}$ is the determinant of L where the i-th row and j-th column are deleted.

Ex:

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \qquad L_{11} = \det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} = 8 \qquad L_4 = \det \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = 8$$

Notice that every minor of the Laplacian is the same.

Thm: (Matrix Tree) Given any graph G, the number of spanning trees in $G = L_{i,i}$ for any $1 \leq \ell \leq n$. (L is the Laplacian.)

Proof: Let $e_1, \ldots, e_m$ be all the edges in G. Orient each edge e_i arbitrarily (denote the oriented edge $\vec{e_i}$).

Define the incidence matrix $N \in \mathbb{R}^{n \times m}$ by $n_{ij} = \begin{cases} 0 & x_i \notin \vec{e_j} \\ 1 & x_i = \text{head}(\vec{e_j}) \\ -1 & x_i = \text{tail}(\vec{e_j}) \end{cases}$

Observe that $NN^T = L$.

Proof: $\ell_{i,i} = r_i(N) \cdot r_i(N) = \deg(v_i)$. For $i \neq j$, $\ell_{i,j} = r_i(N) \cdot r_j(N) = \begin{cases} -1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$ $\square$

So $\det(L_{1,1}) = \det(N_1 N_1^T)$, where N_1 is N with row 1 deleted.

Lemma: (Cauchy-Binet) Let $A \in \mathbb{R}^{\ell \times m}$, $B \in \mathbb{R}^{m \times \ell}$, with $\ell \leq m$.

$$\det(AB) = \sum_{I = \{i_1, \ldots, i_\ell\} \subset [m]} \det(A^I) \det(B^I)$$

where A^I is A with cols $i_1, \ldots, i_\ell \in I$ and B^I is B with rows $i_1, \ldots, i_\ell \in I$.
This is proved in the textbook.

So we now have $\det(L_{1,1}) = \det(N_1 N_1^T) = \sum_{I=\{i_1,\ldots,i_{n-1}\}\subset[m]} \det(N_1^I)\det((N_1^I)^T) = \sum_{I=\{i_1,\ldots,i_{n-1}\}\subset[m]} \det(N_1^I)^2.$

Lemma: For each $I = \{i_1, \ldots, i_{n-1}\} \subset [m]$, $\det(N_1^I) = \begin{cases} 0 & \{e_{i_1}, \ldots, e_{i_{n-1}}\} \text{ is not a tree} \\ \pm 1 & \{e_{i_1}, \ldots, e_{i_{n-1}}\} \text{ is a tree} \end{cases}$

Proof: If $\{e_{i_1}, \ldots, e_{i_{n-1}}\}$ is disconnected, then there is a cycle in one of the connected components. Adding up the columns for that cycle and multiplying by ± 1 as needed gives us a column which is $\vec{0}$. So the determinant of the matrix is 0.

If $\{e_{i_1}, \ldots, e_{i_{n-1}}\}$ is a tree, then $\exists$ some leaf $v \neq 1$ because the first row is gone. In the row of v, there exists a single ± 1 entry; perform Laplace Expansion along that row to calculate the determinant, and recurse. □

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