Chains and Antichains in Posets

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Defn: A chain in $P = (X, \preceq)$ is a sequence of elements $x_1 \prec x_2 \prec \cdots \prec x_k$.

Defn: An antichain in P is a subset of mutually incomparable elements.

Observe: In a pig boset, we have either a big chain, or a big antichain.

Defn: A chain decomposition of $P = (X, \preceq)$ is a way to write $X = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ where each C_i is a chain. The size of this decomposition is k.

Defn: An antichain decomposition of $P = (X, \preceq)$ is a way to write $X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_k$, where each A_i is an antichain. The size of this decomposition is k.

What about minimal chain/antichain decompositions?

Defn: The maximum antichain size in P is $\alpha(P)$.
The maximum chain size in P is $\beta(P)$.
The minimum antichain decomposition size in P is $\gamma(P)$.
The minimum chain decomposition size in P is $\delta(P)$.

Thm: (Mirsky) $P = (X, \preceq)$ then $\beta(P) = \gamma(P)$.
Proof: Induction on $|X|$. Base case $n = 1$ is trivial. Assume the theorem holds for all posets on $\leq n$ elements. Consider $|X| = n + 1$.
Observe that $\beta(P) \leq \gamma(P)$ always holds.
Let $m = \beta(P)$ (the longest chain has m elements). Let $X_{\max} = \{x \in X : x \text{ maximal}\}$; observe that $X_{\max}$ is an antichain. Indeed, if $a, b \in X_{\max}$, then $a \not\prec b$ and $b \not\prec a$, so a and b are incomparable.
Let $X' = X \setminus X_{\max}$, so $|X'| \leq n$. Let $P' = (X', \preceq)$. We have $\beta(P') = \beta(P) - 1 = m - 1$. The longest chain in P' is the longest chain in P minus one element. We can apply our induction hypothesis to P', so we have $X' = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_{m-1}$, an antichain decomposition of size $m - 1$ for P'. So we have $X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_{m-1} \sqcup X_{\max}$, an antichain decomposition of size $m = \beta(P)$. $\square$

Thm: (Dilworth) $P = (X, \preceq)$, then $\alpha(P) = \delta(P)$.
Proof $LHS \leq RHS$: $A = \{a_1, \ldots, a_k\}$ antichain, and $X = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_\ell$. Then each chain C_i contains at most one elt from A, so $k \leq \ell$.
The main idea: Let $A = \{a_1, \ldots, a_m\}$ be the max antichain in X. Since A is maximal, if $x \notin A$, then $x \prec a_i$ or $x \succ a_j$ (but not both). Let $X^+ = \{x \notin A : x \succ a_i \text{ for some } i\}$ and
$X^- = \{x \notin A : x \prec a_i \text{ for some } j\}$
$X = A \sqcup X^+ \sqcup X^-$. Let $X_1 = A \sqcup X^+$, $P_1 = (X_1, \preceq)$ and $X_2 = A \sqcup X^-$, $P_2 = (X_2, \preceq)$. Both P_1 and P_2 have smaller size than X. By induction: $X_1 = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m$, and $X_2 = C'_1 \sqcup C'_2 \sqcup \cdots \sqcup C'_m$.
So $X = (C_1 \cup C'_1) \sqcup (C_2 \cup C'_2) \sqcup \cdots \sqcup (C_m \cup C'_m)$. $\square$

Cor: Let $P = (X, \preceq)$ be a poset. Then $\alpha(P) \cdot \beta(P) \geq |X|$.
Proof: Let $m = \alpha(P)$. By Dilworth's theorem, $X = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m$.
So $|X| = |C_1| + \cdots + |C_m| = m \cdot \beta(P) = \alpha(P) \cdot \beta(P)$. $\Box$

Thm: Let $r, s \geq 1$. Consider any sequence $S = a_1, a_2, \ldots, a_{rs+1} \in \mathbb{R}$. Then S has an increasing subsequence of $r + 1$ elements or a decreasing subsequence of $s + 1$ elements.
Proof: $X = \{1, \ldots, rs + 1\}$. Let $i \preceq j \leftrightarrow i \leq j \wedge a_i \leq a_j$. Then $P = (X, \preceq)$ is a poset. Apply the corollary; then $\alpha(P) \cdot \beta(P) \geq |X| = rs + 1$. So either $\alpha(P) \geq s + 1$ or $\beta(P) \geq r + 1$.
Case 1: $\beta(P) \geq r + 1$). Then there is a chain of at least $r + 1$ elements $i_1 \prec i_2 \prec \cdots \prec i_{r+1}$, so $i_1 < i_2 < \cdots < i_{r+1}$ and $a_{i_1} \leq a_{i_2} \leq \cdots \leq a_{i_{r+1}}$. Thus, we have an increasing sequence of length $r + 1$.
Case 2: $\alpha(P) \geq s + 1$. Then there is an antichain $A = \{i_1, i_2, \ldots, i_{s+1}\}$. Rearrange so that $i_1 < i_2 < \cdots < i_{s+1}$ (as natural numbers). $\Box$