

BEST Theorem

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Defn: Let $\vec{G} = (V, \vec{E})$ be a directed graph. Forward/backward edges are allowed; self loops are not. An Eulerian circuit in \vec{G} is a sequence going through all edges, each only once, starting with some specified edge $v_1\vec{v}_2$ and ending at v_1 .

We're left with the obvious question: When does \vec{G} have an Eulerian circuit?

Answer: If and only if $\forall v \in V, \deg^+(v) = \deg^-(v)$ and G is connected.

Sufficient: Starting at $v_1\vec{v}_2$, pick arbitrary edges. Either

- (a) Cover all edges and return to v_1 (an Eulerian circuit).
- (b) Come back to v_1 and get stuck. But \vec{G} is connected, so $\exists v \in C$ with unused edges. Trace another circuit C' at v , and then attach C' to C at v . Repeat this process until the whole graph is connected; then C is an Eulerian circuit.

Thm: For any connected digraph \vec{G} with $\deg^+(v) = \deg^-(v)$ for all $v \in V$, we have the number of Eulerian circuits starting at $v_1\vec{v}_2$ is

$$t_1(\vec{G}) \prod_{v \in V} (\deg^+(v) - 1)!$$

where $t_1(\vec{G})$ is the number of directed spanning trees rooted at v_1 .

Proof: We want to construct an N -to-1 map $\{\text{E.c.}\} \rightarrow \{\text{dir. trees}\}$, where N is the multiplier.

E.c. \rightarrow dir. tree: Given E.c. C , for each $i \neq 1$, let $\vec{e}(v_i)$ be the last outgoing edge of v_i that C has. We claim that $\{\vec{e}(v_i) | i \neq 1\}$ forms a directed tree with root v_1 . Proof by contradiction: assume it doesn't. Then we must have a cycle. If $\vec{e}(v_i) = v_i\vec{v}_j$, then we won't see v_i again, so we can't have a cycle, i.e., no edge $v_u\vec{v}_i$ later in C . Oops.

dir. tree \rightarrow E.c.: Let T be a directed tree. For each v_i , let $\text{Out}(v_i) = \{\text{all outgoing edges in } \vec{G} \text{ from } v_i\}$.

Pick permutation Π_i on $\text{Out}(v_i)$ s.t.

- If $i = 1$, then $v_1\vec{v}_2$ is the first in Π_1
- If $i \neq 1$, and $v_i\vec{v}_j \in T$, then $v_i\vec{v}_j$ is last in Π_i .

Since $|\text{Out}(v_i)| = \deg^+(v_i)$ and we've fixed one element in each permutation, the number of possible permutations Π_i is $(\deg^+(v_i) - 1)!$

$(T, \Pi_1, \Pi_2, \dots, \Pi_n) \rightarrow \text{E.c. } C$

- Start with v_1 , go to v_2 via $v_1\vec{v}_2$.
- When at v_i , pick the next unused edge $v_i\vec{v}_j$ in Π_i , add $v_i\vec{v}_j$ to C , go to v_j .
- Repeat.

Outcome 1: End at v_1 , all edges are used. So we have an E.c.

Outcome 2: Stuck at v_1 with unused edges on some v_j . But this is impossible! Proof:

If this is the case, each edge $v_i\vec{v}_1$ in T was used, where v_i child of v_1 in T . But $v_i\vec{v}_1$ was the last unused edge in Π_i , so we've used all edges of v_i . Recurse. \square

End of proof. \square