Lattices

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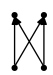
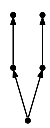
Defn: A lattice $\mathcal{L}$ is a poset with two operations $x \vee y$ (join) and $x \wedge y$ (meet) with the following properties:

- $x, y \preceq z \Leftrightarrow x \vee y \preceq z$
- $w \preceq x, y \Leftrightarrow w \preceq x \wedge y$

$x \vee y$ is the least upper bound for x and y.
$x \wedge y$ is the greatest lower bound for x and y.

For example:





These are not lattices.

Ex: $P = (2^{[n]}, \subseteq)$. Then we can define $S \vee T = S \cup T$ and $S \wedge T = S \cap T$. This is a lattice.

Ex: $P = (\mathbb{N}, |)$. Then we can define $x \vee y = \text{lcm}(x, y)$ and $x \wedge y = \gcd(x, y)$. This is a lattice.

Prop: (a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
(b) $a \wedge (a \vee b) = a = a \vee (a \wedge b)$
Proof (a): $z_1 = (a \wedge b) \wedge c$, $z_2 = a \wedge (b \wedge c)$. We need to show $z_1 = z_2$.
$z_1 \preceq z_2$: $z_1 \preceq a \wedge b \preceq a$, $z_1 \preceq a \wedge b \preceq b$, $z_1 \preceq c$, so $z_1 \preceq b \wedge c$, so $z_1 \preceq a \wedge (b \wedge c) = z_2$.
$z_1 \succeq z_2$: follows similarly.
Therefore $z_1 = z_2$. $\square$
Proof (b): $a \preceq a \vee b \to a \wedge (a \vee b) = a$. $a \wedge b \preceq a \to a \vee (a \wedge b) = a$. $\square$

Remember, we can write $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ no matter the ordering of the x_i's. The same goes for $\vee$.

Prop: If $\mathcal{L}$ is a *finite* lattice, then it has a unique smallest element $\hat{0}$ and a unique largest element $\hat{1}$.
Proof: Let $\hat{0} = \bigwedge_{x \in \mathcal{L}} x$, $\hat{1} = \bigvee_{x \in \mathcal{L}} x$. These are well defined by the previous proposition. $\hat{0} \preceq x$ and $\hat{1} \succeq x$, for all $x \in \mathcal{L}$. $\square$

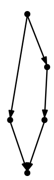
Defn: We say x covers y, denoted $y \lessdot x$, if $y \prec x$ and there is no z such that $y \prec z \prec x$.

Prop: (a) If $z \lessdot x, y$, then $z = x \wedge y$.
(b) If $x, y, \lessdot z$, then $z = x \vee y$.
Proof (a): $z \lessdot x, y \rightarrow z \prec x, y \rightarrow z \preceq x \wedge y$. If $z \neq x \wedge y$, then $z \prec x \wedge y \preceq x, y$, so $z \not\lessdot x, y$. Oops!
Therefore, $z = x \wedge y$. $\square$
(b) follows similarly

Defn: A finite lattice $\mathcal{L}$ is ranked if there is a function rank $: \mathcal{L} \rightarrow \mathbb{N}$ such that

- rank $(\hat{0}) = 0$
- rank $(x) =$ rank $(y) + 1$ if $y \lessdot x$.

Not all lattices can be ranked. For example, consider



Prop: $\mathcal{L}$ can be ranked if and only if all maximal chains $\hat{0} \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot \hat{1}$ have the same length.
Proof: Assume $\mathcal{L}$ is ranked. Then a given chain must have a length of rank $(\hat{1})$. So all chains must have a length of rank $(\hat{1})$.