BEST Theorem

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Defn: Let $\vec{G} = (V, \vec{E})$ be a directed graph. Forward/backward edges are allowed; self loops are not. An <u>Eulerian circuit</u> in $\vec{G}$ is a sequence going through all edges, each only once, starting with some specified edge $v_1\vec{v}_2$ and ending at $\vec{v_1}$.

We're left with the obvious question: When does $\vec{G}$ have an Eulerian circuit?
Answer: If and only if $\forall v \in V$, $\deg^+(v) = \deg^-(v)$ and G is connected.

Sufficient: Starting at $v_1\vec{v}_2$, pick arbitrary edges. Either
(a) Cover all edges and return to v_1 (an Eulerian circuit).
(b) Come back to v_1 and get stuck. But $\vec{G}$ is connected, so $\exists v \in C$ with unused edges. Trace another circuit C' at v, and then attach C' to C at v. Repeat this process until the whole graph is connected; then C is an Eulerian circuit.

Thm: For any connected digraph $\vec{G}$ with $\deg^+(v) = \deg^-(v)$ for all $v \in V$, we have the number of Eulerian circuits starting at $v_1\vec{v}_2$ is

$$t_1(\vec{G}) \prod_{v \in V} (\deg^+(v) - 1)!$$

where $t_1(\vec{G})$ is the number of directed spanning trees rooted at v_1.

Proof: We want to construct an N-to-1 map $\{\text{E.c.}\} \to \{\text{dir. trees}\}$, where N is the multiplier.
<u>E.c.→dir. tree</u>: Given *E.c.* C, for each $i \neq 1$, let $\vec{e}(v_1)$ be the last outgoing edge of v_i that C has. We claim that $\{\vec{e}(v_i) | i \neq 1\}$ forms a directed tree with root v_i. Proof by contradiction: assume it doesn't. Then we must have a cycle. If $\vec{e}(v_i) = v_i\vec{v}_j$, then we won't see v_i again, so we can't have a cycle, i.e., no edge $v_u\vec{v}_i$ later in C. Oops.

<u>dir. tree→E.c.</u>: Let T be a directed tree. For each v_i, let $\text{Out}(v_i) = \left\{\text{all outgoing edges in } \vec{G} \text{ from } v_i\right\}$.
Pick permutation Π_i on $\text{Out}(v_i)$ s.t.

- If $i = 1$, then $v_1\vec{v}_2$ is the first in Π_1
- If $i \neq 1$, and $v_i\vec{v}_j \in T$, then $v_i\vec{v}_j$ is last in Π_i.

Since $|\text{Out}(v_i)| = \deg^+(v_i)$ and we've fixed one element in each permutation, the number of possible permutations Π_i is $(\deg^+(v_i) - 1)!$

$(T, \Pi_1, \Pi_2, \ldots, \Pi_n) \to$ E.c. C

- Start with v_1, go to v_2 via $v_1\vec{v}_2$.
- When at v_i, pick the next unused edge $v_i\vec{v}_j$ in Π_i, add $v_i\vec{v}_j$ to C, go to v_j.
- Repeat.

Outcome 1: End at v_1, all edges are used. So we have an E.c.
Outcome 2: Stuck at v_1 with unused edges on some v_j. But this is impossible! Proof:
If this is the case, each edge $v_i\vec{v}_1$ in T was used, where v_i child of v_1 in T. But $v_i\vec{v}_1$ was the last unused edge in Π_i, so we've used all edges of v_i. Recurse. $\square$

End of proof. $\square$