Maxflow, Mincut

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Defn: Given a network $\vec{N}$, and a flow f, an augmenting path is a sequence of edges $s = v_0 — v_1 — \cdots — v_k = t$ such that $f(e) > 0$ if e is backwards in the path and $f(e) < c(e)$ if e is forwards in the path.

Observe: If f is a flow on $\vec{N}$, and there is an augmenting path, then f is not maximal.

Ford-Fulkerson Algorithm

Thm: If there is no augmenting path, then f is max.

Defn: A cut is a partition $V = X \sqcup Y$ s.t. $s \in X$, $t \in Y$.

Lemma: (1) Given $X \sqcup Y$ a cut, and a flow f, then $|f| = f(X,Y) - f(Y,X)$ where $f(A,B) = \sum_{e \text{ edge from } A \to B} f(e)$.
Proof: Let $E_0 = \{\text{edges in } X\}$, $E_1 = \{\text{edges from } X \to Y\}$, and $E_2 = \{\text{edges from } Y \to X\}$.
Then $RHS = \sum_{e \in E_1} f(e) - \sum_{e \in E_2} f(e)$

$$= \sum_{e \in E_1} f(e) - \sum_{e \in E_2} f(e) + \sum_{e \in E_0} (f(e) - f(e))$$
$$= f^-(s) + \cancel{\sum_{x \in X \setminus \{s\}} f^-(x) - f^+(x)}$$
$$= f^-(s) = |f|. \square$$

Defn: For a cut $X \sqcup Y$, its capacity is $c(X,Y) = \sum_{e \text{ edge from } X \to Y} c(e)$.

Proof of Theorem: Assume f has no augmenting paths $s \to t$. Define $X = \{x \in V : \text{ there is an augmenting path } s \to x\}$. Let $Y = V \setminus X$, so that $X \sqcup Y$ is a cut. By lemma (1), we know $|f| = f(X,Y) - f(Y,X)$. For any edge $e = \vec{xy}$ for $x \in X, y \in Y$, $f(e) = c(e)$. For any edge $e' = \vec{y'x'}$, $f(e) = 0$.

Lemma: (2) Given $\vec{N}$, flow f on $\vec{N}$, and a cut $X \sqcup Y$, $|f| \leq c(X,Y)$.
Proof: By lemma (1), $|f| = f(X,Y) - f(Y,X) \leq f(X,Y) \leq c(X,Y)$. $\square$

$|f| = f(X,Y) - \cancel{f(Y,X)} = \sum_{e = \vec{xy}} c(e) = c(X,Y)$. If f' is any other flow, by lemma (2), $|f'| \leq c(X,Y)$. Therefore, f is a max flow. $\square$

Thm: (Maxflow-Mincut) Given $\vec{N}$, then $\max\limits_{\text{flow } f} |f| = \min\limits_{\text{cut } X \sqcup Y} (c(X,Y)$.
Proof: If f is a flow, $X \sqcup Y$ is a cut, by lemma (2), $|f| \leq c(X,Y)$.
Conversely, if f is a max flow, then there is no augmenting path $s \to t$. By the previous argument, there is some cut $X \sqcup Y$ and $|f| = c(X,Y)$. $\square$

Thm: (Integer Property of Flows) Given $\vec{N}$ s.t. $c(e) \in \mathbb{Z}_+$, then there is a max flow s.t. $f(e) \in \mathbb{Z}_+$.
Proof: Start with $f = 0$. Keep repeating Ford-Fulkerson. By theorem, if f is not maximal, then there is an augmenting path.
Ford-Fulkerson repeats at most $O(|E|\, M)$ times, where $M = \max_e c(e) \in \mathbb{Z}_+$. $\square$

Cor: This also holds for rational flows.

Defn: Let $\vec{G}$ be a directed graph, $s, t \in \vec{G}$.
$a_e(s,t)$ is the maximum number of edge-disjoint directed paths from s to t.
$a_v(s,t)$ is the maximum number of vertex-disjoint directed paths from s to t.
$b_e(s,t)$ is the minimum number of edges that can be deleted to disconnect s and t.
$b_v(s,t)$ is the minimum number of vertexes that can be deleted to disconnect s and t.

Thm: (Menger, directed) For any $\vec{G}$, and $s, t \in \vec{G}$, $a_e(s,t) = b_e(s,t)$ and $a_v(s,t) = b_v(s,t)$.

Thm: (Menger, undirected) For any G, and $s, t \in G$, $a_e(s,t) = b_e(s,t)$ and $a_v(s,t) = b_v(s,t)$.

Proof of directed version: Given $\vec{G}$, WOLOG we can assume s has only outgoing edges and t has only incoming edges. So s is the source, t is the sink. Let $c(e) = 1$. First, we will prove that $a_e(s,t) = b_e(s,t)$.

$LHS \leq RHS$: If there are k edge-disjoint paths $s \to t$, ten we need to delete at least k edges to disconnect each path.
$LHS \geq RHS$: We claim LHS is the max flow in $\vec{G}$.
So $LHS =$max flow=min cut$= c(X,Y) =$the number of edges from X to $Y \geq RHS$.
Therefore, $LHS = RHS$.