Chromatic Polynomials

Thomas Cohn

9/20/18

Defn: $G = (V, E)$, $k \in \mathbb{N}$.
$\chi_G(k)$ is the number of proper colorings of G with $\leq k$ colors.

Ex: If G has n vertexes, k colors, and 0 edges, then $\chi_G(k) = k^n$.

Ex: If $G = K_n$, k colors, then $\chi_G(k) = \underbrace{k(k-1)(k-2)\cdots(k-(n-1))}_{n}$.

Note: If $k < n$, there is no proper coloring on $G = K_n$ (the pigeonhole principle can be used to prove this). Notice that $k < n \to \exists j \in \mathbb{N}$ s.t. $k + j = n$, so $\chi_G(k) = 0$ as expected.

Ex: G a tree on n vertices. Then $\chi_G(k) = \underbrace{k(k-1)(k-1)\cdots(k-1)}_{n-1} = k(k-1)^{n-1}$

Notice that χ_G is a polynomial of degree n.

Thm: Give any graph G on n vertexes, $\chi_G(k)$ is a polynomial of degree n in k.
$\chi_G(k) = c_n k^n + c_{n-1}k^{n-1} + \cdots + c_1 k + c_0$, with $c_i \in \mathbb{Z}$ and $c_n > 0, c_{n-1} \leq 0, c_{n-2} \geq 0, \ldots$
Proof: Strong induction on $(n, m) = (|V|, |E|)$. If $m = 0$, $\chi_G(k) = k^n$. If $n = 1$, $\chi_G(k) = k$.
Assume the thm holds for all graphs with $<\ n$ vertices and all graphs on n vertices with $<\ m$ edges.
Consider any edge $e = (x, y) \in E$.
Let $G_1 = G - e$ (removing e); $n \to n$, $m \to m - 1$.
Let $G_1 = G/e$ (contracting e); $n \to n - 1$, $m \to m' < m$. $m' = m - 1 - |N(x) \cap N(y)|$.

$$\left.\begin{array}{l} G_1 \text{ has } < m \text{ edges} \\ G_2 \text{ has } < n \text{ vertices} \end{array}\right\} \to \text{induction hypothesis holds for } G_1 \text{ and } G_2.$$

Claim: $\chi_G(k) = \chi_{G_1}(k) - \chi_{G_2}(k)$. This makes sense because the number of ways to color G_1 is equal to the number of ways to color G_1 with $c(x) \neq c(y)$ plus the number of ways to color G_1 with $c(x) = c(y)$. And the number of ways to color G_2 is equal to the number of ways to color G with $c(x) = c(y)$. So $\chi_{G_1}(k) - \chi_{G_2}(k)$ should be the number of ways to color G with $c(x) \neq c(y)$, as intended.

$\chi_{G_1}(k) = d_n k^n + d_{n-1}k^{n-1} + \cdots + d_0$
$\chi_{G_2}(k) = e_{n-1}k^{n-1} + e_{n-2}k^{n-2} + \cdots + e_0$
So $\chi_G(k) = \chi_{G_1}(k) - \chi_{G_2}(k) = d_n k^n + (d_{n-1} - e_{n-1})k^{n-1} + (d_{n-2} - e_{n-2})k^{n-2} + \cdots$

$d_n > 0$, so $d_n > 0$.
$d_{n-1} \leq 0$, and $e_{n-1} > 0$, so $(d_{n-1} - e_{n-1}) \leq 0$.
$d_{n-2} \geq 0$, and $e_{n-2} \leq 0$, so $(d_{n-2} - e_{n-2}) \geq 0$.
$\vdots$
$\square$

Ex: $G = C_n$, the cycle of size n. $\chi_{C_n} = ?$.

$\chi_{C_n}(k) = \chi_{C_n - c}(k) - \chi_{C_n/e}(k) = \chi_{P_n}(k) - \chi_{C_{n-1}}(k) = k(k-1)^{n-1} - \chi_{C_{n-1}}(k)$
$= k(k-1)^{n-1} - k(k-1)^{n-2} + k(k-1)^{n-3} - \cdots \pm k(k-1)$
$= k((k-1)^{n-1} - (k-1)^{n-2} + (k-1)^{n-3} - \cdots \pm (k-1))$
$= k\left((k-1)\frac{1-(k-1)^n}{n-(k-1)}\right) = \cdots = (k-1)^n + (-1)^n(k-1)$

Defn: Given $G = (V, E)$, an <u>acyclic orientation</u> on G is a way to orient the edges in E so that we have no directed cycle.

Defn: Given $G = (V, E)$, let $a(G)$ be the number of acyclic orientations on G.

Ex: $a(K_n) = n!$