Ramsey's Theorem

Thomas Cohn

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Extremal Problems: If G on n vertices avoids the same structure, how big is $|E|$?

Thm: (Turán) If G on n vertices avoids K_k, then $|E| \leq \frac{k-2}{2(k-1)}n^2 + C$, where C is a constant term.

If both G and $\overline{G}$ on n vertices avoid K_k, then how big is n?

Ex: $k = 2$, then $n < 2$.

Prop: If $n = 6$, and the edges of K_n are colored R/B, then we can find a monochromatic triangle.
Proof: Look at vertex v. $N(v) = X \sqcup Y$, where $X = \{x : xv \text{ is red}\}$ and $Y = \{y : yv \text{ is blue}\}$.
$|X| + |Y| = 5$, so one of them (WOLOG X) has size at least 3.

Say $x_1, x_2, x_3 \in X$. Then vx_1x_2, vx_1x_3, and vx_2x_3 are all triangles. If any x_1x_2, x_1x_3, and x_2x_3 are red, then we have a red triangle vx_ix_j. If all are blue, then $x_1x_2x_3$ is a blue triangle. $\square$

Defn: For $r, s \in \mathbb{N}$, the Ramsey Number $R(r, s)$ is the minimum $n \in \mathbb{N}$ s.t. any R/B coloring of the edges in K_n contains a red K_r or a blue K_s.

Ex: $R(3, 3) = 6$

Thm: (Ramsey) $R(r, s) < \infty$, $\forall r, s \in \mathbb{N}$.
Proof: Induction on r, s.
Assume we know $R(r-1, s)$ and $R(r, s-1)$ are finite.
Claim: $R(r, s) \leq R(r-1, s) + R(r, s-1)$. Let $n = R(r-1, s) + R(r, s-1) \in \mathbb{N}$. Consider a R/B coloring of K_n. Let $X = \{x : xv \text{ is red}\}$ and $Y = \{y : yv \text{ is blue}\}$. Then $|X|+|Y| = n-1 = a+b-1$. Thus $|X| \geq a$ or $|Y| \geq b$.

Case 1: Let $|X| \geq a$. Let G_x be the complete subgraph on X. Then G_x has a red K_{r-1} or a blue K_s. If G_x contains a red K_{r-1}, then G has a K_r. If G_x contains a blue K_s, then G has a K_s.

Case 2: Just do the same thing.

We conclude that if $n = a + b$, then any R/B coloring of edges in K_n contains a red K_r or blue K_s, and $n \geq R(r, s)$. $\square$

Ex: $R(4, 4) = 18$
$R(3, 5) = 14$
$R(3, 6) = 18$
$R(3, 7) = 23$
$R(3, 8) = 28$
$R(3, 9) = 36$

Cor: $R(r,s) \leq \binom{r+s-2}{r-1}$. If $r = s$, then $R(r,s) \leq \binom{2r-2}{r-1} \leq 2^{2r-2}$.

Proof: Assume $R(r-1,s) \leq \binom{r+s-3}{r-2}$ and $R(r,s-1) \leq \binom{r+s-3}{r-1}$.

Then $R(r,s) \leq R(r-1,s) + R(r,s-1) = \binom{r+s-3}{r-2} + \binom{r+s-3}{r-1} = \binom{r+s-2}{r-1}$. $\square$

Thm: $R(r,r) \geq 2^{(r-2)/2}$.

Proof: $n = 2^{(r-2)/2}$. There is a R/B coloring of edges in $K)_n$ with <u>no</u> red K_r or blue K_r. Consider $X \subset [n]$ with $|X| = r$. Then $\mathbb{P}(G_x \text{ is mono}) = \frac{1}{2^{\binom{r}{2}}} + \frac{1}{2^{\binom{r}{2}}} = 2^{1-\binom{r}{2}}$

Let $Y = |\{X \subset [n] : |X| = r \wedge G_x \text{ is mono}\}|$.

Then $\mathbb{E}[y] = \sum_{X \subset [n], |X|=r} \mathbb{P}(G_x \text{ is mono}) = \binom{n}{r} \cdot 2^{1-\binom{r}{2}}$, and $\binom{n}{r} < n^r$

So $\mathbb{E}[y] < n^r \cdot 2^{1-\binom{n}{r}} = \left(2^{(r-2)/2}\right)^r = 2^{1-\frac{r(r-1)}{2}} = 2^{r(r-2)/2 - r(r-1)/2+1} = 2^{-r/2+1} < 1$

So there exists a set of coin flips for which $y = 0$, and therefore, there is a R/B coloring on K_n for which no G_x is mono. $\square$