Matroids

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This command may be useful:
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\newcommand{\precdot}{\mathrel{\ooalign{$\prec$\cr\hidewidth\hbox{$\cdot\mkern0.5mu$}\cr}}}
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From last time:
Prop: A lattice $\mathcal{L}$ can be ranked if and only if every maximal chain $\hat{0} \precdot x_1 \precdot x_2 \precdot \cdots \precdot x_\ell = \hat{1}$ has the same length.
Proof "$\Rightarrow$": $\text{rank}(x_{i+1}) = \text{rank}(x_i) + 1$. So $\ell = \text{rank}(x_\ell) = \text{rank}(\hat{1})$.
Proof "$\Leftarrow$": Inductive, starting with $\text{rank}(\hat{0}) = 0$.
$\square$

Defn: A matriod $M = (E, \mathcal{I})$ is a finite set E and a family $\mathcal{I}$ of subsets of E with 2 properties:

M0) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

M1) If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, then $\exists e \in Y \setminus X$ s.t. $X \cup \{e\} \in \mathcal{I}$.

We call an $X \in \mathcal{I}$ an independent set in E. A maximal independent set is called a base.

Ex: $M = (E, \mathcal{I})$ where $E = \{1, 2, \ldots, n\}$ and $\mathcal{I} = \{X \subseteq E : |X| \leq k\}$. (Uniform matroid).
$|I| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$.

Ex: $M = (E, \mathcal{I})$ where $E \subseteq \mathbb{R}^n$ (finite set of n-dimensional vectors). $E = \{v_1, \ldots, v_m\}$.
$\mathcal{I} = \{X \subseteq E : X \text{ linearly independent}\}$. Then M is called a linear matroid.

M0) X linearly independent, $Y \subseteq X$ means Y is also linearly independent, so $Y \in \mathcal{I}$.

M1) X, Y linearly independent, $|X| < |Y|$. Then $\dim(\text{span}(X)) < \dim(\text{span}(Y))$. So by basic linear algebra, we're done.

Ex: $G = (V, E)$ graph, $M = (E, \mathcal{I})$. $\mathcal{I} = \{X \subset E : X \text{ is acyclic}\}$ (or equivalently, X forms a forest in G). This is a graphic matroid.

M0) X acyclic, $Y \subseteq X \to Y$ acyclic.

M1) X, Y forests, $|X| < |Y|$. Recall that a forest with n vertices and ℓ components has $n - \ell$ edges. So assume $|X| = a < b = |Y|$. Then X has $n - a$ components and Y has $n - b$ components. So there exists an edge e in Y connecting two components in X, so $X \cup \{e\}$ is still a forest.

Prop: All bases in a matroid have the same size.
Proof: Assume X and Y are bases and $|X| < |Y|$. By M0, $\exists e \in Y \setminus X$ s.t. $X \cup E \in \mathcal{I}$. Thus, X is not maximal, and is therefore not a base. Oops! $\square$

Defn: $M = (E, \mathcal{I})$, let $S \subseteq E$. Then $r(S) = \max\{|X| : X \subseteq S$ is an independent set$\}$ is called the rank function. In particular, $r(M) = r(E)$ is the size of any base.

If $S \subseteq T$, then $r(S) \leq r(T)$.

Lemma: Matroid $M = (E, \mathcal{I})$, $A \subseteq S \subseteq E$, A is an independent set.
Then there is a set B with $A \subseteq B \subseteq S$ such that B is also an independent set and $|B| = r(S)$.
Proof: Let $C \subseteq S$ s.t. C is independent and $|C| = r(S)$. If $|A| = |C|$, we're done. If $|A| < |C|$, by M1, there exists $e \in C \setminus A$ s.t. $A \cup \{e\}$ independent. We can repeat this until eventually we get $A \subset B$ independent with $|B| = |C|$. $\square$

Thm: (Rank semimodularity) Let $M = (E, \mathcal{I})$ be a matroid, $S, T \subseteq E$.
Then $r(S) + r(T) \geq r(S \cap T) + r(S \cup T)$.
Proof: Consider $S \cap T$, let $A \subseteq S \cap T$ such that A is independent and $|A| = r(S \cap T)$. $A \subseteq T \cup T$.
By our lemma, we can find $A \subseteq B \subseteq S \cup T$ such that B independent, and $|B| = r(S \cup T)$.
Thus, $|A| = r(S \cap T)$ and $|B| = r(S \cup T)$. Let $B_1 \subseteq S$ and $B_2 \subseteq T$ both be independent sets. Then $r(S) \geq |B_1|$ and $r(T) \geq |B_2|$. So $r(S) + r(T) \geq |B_1| + |B_2|$.
Therefore, $|A| + |B| \geq r(S \cap T) + r(S \cup T)$. $\square$

Defn: $M = (E, \mathcal{I})$, $S \subseteq E$. We define the closure of S $\mathrm{cl}(S) = \{x \in E : r(S) = r(S \cup x)\}$.
We say S is closed or flat if $\mathrm{cl}(S) = S$. We say S is k-flat if $\mathrm{cl}(S) = S$ and $r(S) = k$.

Some properties of the closure:

- $S \subseteq \mathrm{cl}(S)$
- $S \subseteq t \to \mathrm{cl}(S) \subseteq \mathrm{cl}(T)$
- $\mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}(S)$