Inclusion-Exclusion and the Möbius Function

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$|A \cup B| = |A| + |B| - |A \cap B|$
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
What is the general case?

Thm: (Inclusion-Exclusion) Let $A_1, \ldots, A_n \subseteq S$. For $0 \le K \le n$, we define $\Sigma_K = \displaystyle\sum_{1 \le i_1 < \cdots < i_K \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_K}|$.

Also, $\Sigma_0 \overset{\text{def}}{=} |S|$.
Then $|A_1 \cup \cdots \cup A_n| = \Sigma_1 - \Sigma_2 + \Sigma_3 - \Sigma_4 + \cdots + (-1)^{n-1}\Sigma_n$
and $|S - (A_1 \cup \cdots \cup A_n)| = \Sigma_0 - \Sigma_1 + \Sigma_2 - \Sigma_3 + \cdots + (-1)^n \Sigma_n$

Proof: Consider $x \in S$. Case 1: $x \notin A_1, \ldots, A_n$. Then x is only counted once in Σ_0, and not counted in any other Σ_K.
Case 2: $x \in A_{i_1}, \ldots, A_{i_m}$. Then it is counted $1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^m \binom{m}{m} = (1-1)^m = 0$. $\square$

Defn: d_n is defined to be the number of permutations on $[n]$ with no fixed points.

Ex: $d_1 = 0$, $d_2 = 1$. Let's find d_3:

- $\cancel{123}$
- $\cancel{132}$
- $\cancel{213}$
- 231 ✓
- 312 ✓
- $\cancel{321}$

So $d_3 = 2$.

Let S be the set of all permutations on $[n]$. $|S| = n!$.
Let A_i be the set of all permutations that fix i, for $1 \le i \le n$.
Then $d_n = |S - (A_1 \cup \cdots \cup A_n)| = \Sigma_0 - \Sigma_1 + \Sigma_2 - \cdots$.
$\Sigma_K = \displaystyle\sum_{1 \le i_1 < \cdots < i_K \le n} |A_{i_1} \cup \cdots \cup A_{i_K}| = \binom{n}{k} \cdot (n-k)!$
$d_n = \binom{n}{0} n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots = \frac{n!}{0!} - \frac{n!}{1!} + \frac{n!}{2!} - \cdots = n! \cdot \underbrace{\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right)}_{\approx e^{-1}}$

So $d_n \approx \frac{n!}{e}$

Ex: $n \in \mathbb{N}$, $\varphi(n) = |\{1 \leq i \leq n : \gcd(i, n) = 1\}|$

- $\varphi(1) = 1$
- $\varphi(2) = 1$
- $\varphi(3) = 2$
- $\varphi(4) = 2$
- $\varphi(5) = 4$
- $\varphi(6) = 2$

If n is prime, then $\varphi(n) = n - 1$.
If p, q prime, then $\varphi(p \cdot q) = (p \cdot q) - p - q + 1 = (p-1)(q-1)$
If $p_1, \ldots, p_n$ prime, then $\varphi(p_1 \cdots p_n) = n - \sum_i \frac{n}{p_i} + \sum_{i \neq j} \frac{n}{p_i p_j} - \cdots = (p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$
If $p_1, \ldots, p_n$ prime, then $\varphi(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = (p_1 - 1)p_1^{\alpha_1} \cdot (p_2 - 1)p_2^{\alpha_2} \cdots = n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_n}\right)$

Proof: Let $S = \{1, \ldots, n\}$ and $A_i = \{1 \leq x \leq n : p_i \mid x\}$ (for $1 \leq i \leq n$).
Then $\varphi(n) = |S - (A_1 \cup \cdots \cup A_n)| = \Sigma_0 - \Sigma_1 + \cdots$
$|A_{i_1} \cap \cdots \cap A_{i_k}| = ?$
$x \in A_{i_1} \cap \cdots \cap A_{i_k} \Leftrightarrow p_{i_1} p_{i_2} \cdots p_{i_k} \mid x \Leftrightarrow x = (p_{i_1} \cdots p_{i_k}) \cdot y$ for $1 \leq x \leq n$; $1 \leq y \leq \frac{n}{p_{i_1} \cdots p_{i_k}}$.
So $|A_{i_1} \cup \cdots \cup A_{i_k}| = \dfrac{n}{p_{i_1} \cdots p_{i_k}} \to \Sigma_k = \displaystyle\sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{n}{p_{i_1} \cdots p_{i_k}}$
$\to \varphi(n) = \Sigma_0 - \Sigma_1 + \Sigma_2 - \cdots = n - \sum_i \frac{n}{p_i} + \sum_{i_1 \neq i_2} \frac{n}{p_{i_1} p_{i_2}} - \cdots = n \left(1 - \sum_i \frac{i}{p_i} + \sum_{i_1 \neq i_2} \frac{1}{p_{i_1} p_{i_2}} - \cdots \right) =$
$= n \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_n}\right)$. $\square$

Thm: For $n \in \mathbb{N}$, $\displaystyle\sum_{\{d : d \mid n\}} \varphi(d) = n$.

Proof: The idea is to assign each $1 \leq x \leq n$ into some class $S_d = (d \mid n)$ so that $|S_d| = \varphi(d)$.
Let $y = \gcd(x, n)$. ($y \mid n$, $n = y \cdot d$).
Let $S_d = \left\{1 \leq x \leq n : \gcd(x, n) = \frac{n}{d}\right\}$. So $S_1 = \{n\}$, and $S_n = \{x : x, n \text{ coprime}\}$.
In general, $S_d = \{1 \leq x \leq n : x = y \cdot u \wedge \gcd(u, d) = 1\}$
So we have $\{1, \ldots, n\} = \bigsqcup_{\{d : d \mid n\}} S_d$, and so $n = \sum_{\{d : d \mid n\}} S_d = \sum_{\{d : d \mid n\}} \varphi(d)$. $\square$

Ex: $n = 10$, then $d = 1, 2, 5, 10$.
$\varphi(1) + \varphi(2) + \varphi(5) + \varphi(10) = 1 + 1 + 4 + 4 = 10$.

Ex: For p, q prime,
$\varphi(1) + \varphi(p) + \varphi(q) + \varphi(p \cdot q) = 1 + (p-1) + (q-1) + (p-1)(q-1) = p \cdot q$

$n = \sum_{\{d : d \mid n\}} \varphi(d)$. Then $\varphi(n) = \sum_{\{d : d \mid n\}} d \cdot \mu\left(\frac{n}{d}\right)$. μ is called the Möbius function.