Network Flow

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Defn: An edge cover F is minimal if $|F|$ is minimal.

Defn: An independent set $I \subset V$ is a collection of vertices with no edges among them.

Defn: An independent set is a maximum independent set if $|I|$ is max.

Thm: (König) If G is bipartite with no isolated vertices, then the size of the maximum independent set equals the size of the minimal edge cover.

Lemma: In any graph G (not necessarily bipartite) with n vertexes, we have $|I| = n - |X|$, where I is a minimum independent set and X is a minimum vertex cover.
Proof: If $S \subset V$ is an independent set, then $V \setminus S$ is a vertex cover, and vice-versa. $\square$

Lemma: (Gallai) If G has no isolated vertices, then $|E'| = n - |M|$, where E' is a minimum edge cover set and M is a maximal matching.
Proof:
$|E'| + |M| \leq n$: Take a maximal matching. No vertexes are isolated, so we can add one edge per vertex. And if the max matching has size k, and there are ℓ remaining vertexes (that is, $2k + \ell = n$), we can pick an edge cover of size $k + \ell$. $(k + \ell) + k = 2k + \ell = n$. So $|E'| \leq n - |M|$

$|E'| + |M| \geq n$: Observe that if $F \subset E$ is a minimum edge cover, and $xy \in F$, then either x or y is incident to no other edges in F. If F is a minimum edge cover, then F is a disjoint union of "stars". Assume that F is a minimum edge cover with ℓ stars. Then pick 1 edge from each star. This gives us a matching of size ℓ. So $\ell + (k_1 + \cdots + k_\ell) = (k_1 + 1) + \cdots + (k_\ell + 1) = n$. And therefore, $|E'| + |M| \geq n$.

Therefore, $|E'| = n - |M|$. $\square$

Network Flow

Defn: A network $\vec{N} = (V, E)$ is a directed graph with 2 special vertices:

- "source" s, which has only outgoing edges
- "sink" t, which has only incoming edges

and a capacity function $c : E \to \mathbb{R}_+$.

Defn: A flow f through $\vec{N}$ is a function $f : E \to \mathbb{R}_+$ s.t.

- $f(e) \leq c(e)$, $\forall e \in E$
- $\forall v \in V$ with $v \neq s, v \neq t$, we have $f^+(v) = \displaystyle\sum_{e \in \mathrm{In}(v)} f(e) = f^-(v) = \sum_{e \in \mathrm{Out}(v)} f(e)$

Observe: If f is a flow on $\vec{N}$, then $f^-(s) = f^+(t)$.

Proof: $0 = \sum_{e \in E}(f^+(e) - f^-(e)) = \sum_{v \in V} \left(\underbrace{\sum_{e \in \mathrm{In}(v)} f(e)}_{f^+(v)} - \underbrace{\sum_{e \in \mathrm{Out}(v)} f(e)}_{f^-(v)} \right) = -f^-(s) + f^+(t)$. $\square$

Defn: If f is a flow through $\vec{N}$, then the strength of f is $|f| = f^+(t) = f^-(s)$.

Question: Given $\vec{N}$, can we find a flow f with maximum $|f|$? Yeah probably. Consider an arbitrary network $\vec{N}$ with flow f.

Observe: If there is a directed path $s = x_0 \to x_1 \to \cdots \to x_k = t$ with $f(e) < c(e)$, $\forall e$ in the path, then we can increase the flow.

Observe: If there is a path $s = x_0 \overset{?}{—} x_1 \overset{?}{—} x_2 \overset{?}{—} \cdots \overset{?}{—} x_k = t$ and $f(e) < c(e)$, $\forall e$ in the path which are "forward", and $f(e) > 0$, $\forall e$ in the path which are "backward", then we can increase f.

Proof: For e in the path, define $\mathcal{K}(e) = \begin{cases} c(e) - f(e) & e \text{ is "forward"} \\ f(e) & e \text{ is "backward"} \end{cases}$ Note that $\mathcal{K}(e) > 0$ for every e in the path.

Define $\varepsilon = \min\limits_{e \in \text{path}} \mathcal{K}(e)$, and define a new flow $f'(e) = \begin{cases} f(e) + \varepsilon & e \text{ is "forward"} \\ f(e) - \varepsilon & e \text{ is "backward"} \\ f(e) & e \text{ not in the path} \end{cases}$

Then $|f'| = |f| + \varepsilon$, and f' still satisfies conservation of flow. $\square$