

Maxflow, Mincut

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Defn: Given a network \vec{N} , and a flow f , an augmenting path is a sequence of edges $s = v_0 - v_1 - \dots - v_k = t$ such that $f(e) > 0$ if e is backwards in the path and $f(e) < c(e)$ if e is forwards in the path.

Observe: If f is a flow on \vec{N} , and there is an augmenting path, then f is not maximal.

Ford-Fulkerson Algorithm

Thm: If there is no augmenting path, then f is max.

Defn: A cut is a partition $V = X \sqcup Y$ s.t. $s \in X, t \in Y$.

Lemma: (1) Given $X \sqcup Y$ a cut, and a flow f , then $|f| = f(X, Y) - f(Y, X)$ where $f(A, B) = \sum_{e \text{ edge from } A \rightarrow B} f(e)$.
Proof: Let $E_0 = \{\text{edges in } X\}$, $E_1 = \{\text{edges from } X \rightarrow Y\}$, and $E_2 = \{\text{edges from } Y \rightarrow X\}$.

$$\begin{aligned} \text{Then } RHS &= \sum_{e \in E_1} f(e) - \sum_{e \in E_2} f(e) \\ &= \sum_{e \in E_1} f(e) - \sum_{e \in E_2} f(e) + \sum_{e \in E_0} (f(e) - f(e)) \\ &= f^-(s) + \sum_{x \in X \setminus \{s\}} f^-(x) - f^+(x) \\ &= f^-(s) = |f|. \quad \square \end{aligned}$$

Defn: For a cut $X \sqcup Y$, its capacity is $c(X, Y) = \sum_{e \text{ edge from } X \rightarrow Y} c(e)$.

Proof of Theorem: Assume f has no augmenting paths $s \rightarrow t$. Define $X = \{x \in V : \text{there is an augmenting path } s \rightarrow x\}$. Let $Y = V \setminus X$, so that $X \sqcup Y$ is a cut. By lemma (1), we know $|f| = f(X, Y) - f(Y, X)$. For any edge $e = \vec{xy}$ for $x \in X, y \in Y$, $f(e) = c(e)$. For any edge $e' = \vec{y'x'}$, $f(e') = 0$.

Lemma: (2) Given \vec{N} , flow f on \vec{N} , and a cut $X \sqcup Y$, $|f| \leq c(X, Y)$.

Proof: By lemma (1), $|f| = f(X, Y) - f(Y, X) \leq f(X, Y) \leq c(X, Y)$. \square

$|f| = f(X, Y) - \cancel{f(Y, X)} = \sum_{e=\vec{xy}} c(e) = c(X, Y)$. If f' is any other flow, by lemma (2), $|f'| \leq c(X, Y)$. Therefore, f is a max flow. \square

Thm: (Maxflow-Mincut) Given \vec{N} , then $\max_{\text{flow } f} |f| = \min_{\text{cut } X \sqcup Y} c(X, Y)$.

Proof: If f is a flow, $X \sqcup Y$ is a cut, by lemma (2), $|f| \leq c(X, Y)$.

Conversely, if f is a max flow, then there is no augmenting path $s \rightarrow t$. By the previous argument, there is some cut $X \sqcup Y$ and $|f| = c(X, Y)$. \square

Thm: (Integer Property of Flows) Given \vec{N} s.t. $c(e) \in \mathbb{Z}_+$, then there is a max flow s.t. $f(e) \in \mathbb{Z}_+$.

Proof: Start with $f = 0$. Keep repeating Ford-Fulkerson. By theorem, if f is not maximal, then there is an augmenting path.

Ford-Fulkerson repeats at most $O(|E| M)$ times, where $M = \max_e c(e) \in \mathbb{Z}_+$. \square

Cor: This also holds for rational flows.

Defn: Let \vec{G} be a directed graph, $s, t \in \vec{G}$.

$a_e(s, t)$ is the maximum number of edge-disjoint directed paths from s to t .

$a_v(s, t)$ is the maximum number of vertex-disjoint directed paths from s to t .

$b_e(s, t)$ is the minimum number of edges that can be deleted to disconnect s and t .

$b_v(s, t)$ is the minimum number of vertexes that can be deleted to disconnect s and t .

Thm: (Menger, directed) For any \vec{G} , and $s, t \in \vec{G}$, $a_e(s, t) = b_e(s, t)$ and $a_v(s, t) = b_v(s, t)$.

Thm: (Menger, undirected) For any G , and $s, t \in G$, $a_e(s, t) = b_e(s, t)$ and $a_v(s, t) = b_v(s, t)$.

Proof of directed version: Given \vec{G} , WOLOG we can assume s has only outgoing edges and t has only incoming edges. So s is the source, t is the sink. Let $c(e) = 1$. First, we will prove that $a_e(s, t) = b_e(s, t)$.

$LHS \leq RHS$: If there are k edge-disjoint paths $s \rightarrow t$, then we need to delete at least k edges to disconnect each path.

$LHS \geq RHS$: We claim LHS is the max flow in \vec{G} .

So $LHS = \text{max flow} = \text{min cut} = c(X, Y) = \text{the number of edges from } X \text{ to } Y \geq RHS$.

Therefore, $LHS = RHS$.