Planar Graphs

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Defn: $G = (V, E)$ is planar if there is a drawing of the graph in the 2D plane s.t. no 2 edges cross each other.

Ex: $V = [n]$ and $E = \emptyset$ is planar.

Ex: C_n and P_n are planar.

Ex: K_1, K_2, and K_3 are planar.

Ex: K_4 is planar. The normal way we would draw it does not work, but if we draw it like a tetrahedron, it does.

Ex: K_5 is *not* planar.

Ex: The Petersen graph is not planar. We can see this by contracting the edge connecting the outer corners to each of the corners of the inner shape, leaving us with K_5.

Defn: Consider a drawing of a planar graph G in $\mathbb{R}^2$. Remove the edges and vertices in the drawing. Then the connected components of $\mathbb{R}^2 \setminus (V \cup E)$ are called the faces of the drwaing. We denote the number of faces as f.

Thm: (Euler) If G is a connected planar graph, then in any planar drawing, we have $v - e + f = 2$.
Proof: If x is a leaf in G, delete x and the attached edge. Then $v' = v - 1$, $e' = e - 1$, and $f' = f$. So $v' - e' + f' = (v - 1) - (e - 1) + f = 2$.
Assume there are no leaves. Then $\forall x \in V$, $\deg(()\,x) \geq 2$. So it is not a tree, so $\exists$ a cycle C in our graph. Assume WOLOG it is the smallest cycle – that is, there is no smaller cycle contained inside it. Delete an arbitrary edge d in the cycle. Then $v' = v$, $e' = e - 1$, and $f' = f - 1$. So $v' - e' + f' = v - (e - 1) + (f - 1) = 2$.
Therefore, by induction, $\square$

Prop: If G is planar, then $2e = \sum_{\varphi \in \{faces\}} |\{\text{edges bounding } \varphi\}|$. This works because every edge is incident to exactly two faces (note that these two faces may actually be the same face).

Ex: Prove that K_5 is not planar.
Proof by contradiction: assume K_5 is planar. We have $v = 5$ and $e = 10$, so $f = 7$. But by our proposition, we have $2e = 20 = \sum_{\varphi \in \{faces\}} |\{\text{edges bounding } \varphi\}| \geq \sum_{\varphi} e = ef = 21$. So $20 \geq 21$. Oops! $\square$

Ex: $K_{3,3}$ is not planar.
Proof by contradiction: Assume $K_{3,3}$ is planar. We have $v = 6$, $e = 9$, so $f = 5$. Since $K_{3,3}$ is a bipartite graph, every cycle must have an even number of edges. So every face must have at least 4 edges. But by our proposition, we have $2e = 18 = \sum_{\varphi \in \{faces\}} |\{\text{edges bounding } \varphi\}| \geq \sum_{\varphi} 4 = 4f = 20$. So

$18 \geq 21$. Oops! $\square$

Question: How many colors do we need to properly color a planar graph? We can observe that 4 is enough.

Thm: 4 colors is enough to properly color any planar graph.
Proof: *Way* too hard for class.

Instead, we will prove two claims.
Claim 1: Every planar graph is 6-colorable.
Claim 2: Every planar graph is 5-colorable.

Thm: Claim 1: Every planar graph is 6-colorable.
Proof: Observe that in any simple planar graph, $\exists x \in V$ s.t. $\deg(x) \leq 5$. To prove this observation to be true, we by contradiction assume $\forall x_i \in V$ $\deg(x_i) \geq 6$.
Then $2e \geq \sum_{x_i \in V} 6 = 6v \to e \geq 3v$.
And $2e \geq \sum_{\varphi} 3 = 3f \to e \geq \frac{3}{2}f$.
So $\frac{1}{3}e \geq v$ and $\frac{2}{3}e \geq f$. So $-e + v + f \leq 0$. Oops!
Therefore, $\exists x \in V$ s.t. $\deg(x) \geq 5$.

Delete x. Then we now have $G' = G - x$. If G' is 6 colorable, then so is G. G' is still planar, so recurse. $\square$

Remark: The planar graph of the icosahedron has $\deg(v) = 5$ for all $v \in V$.

Thm: Claim 2: Every planar graph is 5-colorable.
Proof: WOLOG, add edges to G until all faces are triangles (except the big outer face). We still have G planar. Adding edges cannot decrease χ_G, so it is enough to show that $\chi_{G'} \leq 5$. We will use induction.

Let $V = V_b \sqcup V_i$, with V_b boundary vertexes and V_i interior vertexes.
For any $x \in V_i$, the set of possible colors $C(x) = \{c_1, \ldots, c_5\}$.
For any $x \in V_b$, the set of possible colors $C(x)$ follows $|C(x)| = 3$.
For some adjacent $x_1, x_2 \in V_b$, we fix $C(x_1) = \{c_1\}$ and $C(x_2) = \{c_2\}$.
We claim $\exists f : V \to \{c_1, \ldots, c_5\}$ s.t. $f(x_i) \in C(x_i)$ and x_i, x_j adjacent $\to f(x_i) \neq f(x_j)$.

If $v \leq 3$ (the base case), all is good!
Case 1: $\exists x_i, x_j \in V^b$ s.t. x_j and x_i are non-adjacent on the boundary, but adjacent in the graph. Then $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \{x_i, x_j\}$. So $\forall (x_1, x_2) \neq (x_i, x_j)$ we have $x_1, x_2 \in G_1$ or $x_1, x_2 \in G_2$. By induction, G_1 has a coloring. In that coloring, assume $f(x_i) = c$ and $f(x_j) = c'$. Then let x_i, x_j be two special boundary vertices for G_2, with $C(x_i) = \{c\}$ and $C(x_j) = \{c'\}$. So by induction, G_2 has a coloring. So we can combine G_1 and G_2 to color G.

Case 2: No such x_i, x_j exist. Then look at x_2 (with preceeding and succeeding vertexes x_1 and x_3 on the boundary), and $u_1, \ldots, u_k$ interior adjacent points.
We ran out of time in class here. Danny will be sending out the rest of the proof.
$\square$