

# Matchings

Dr. Danny Nguyen

*Transcribed by Thomas Cohn*

10/2/18

**Defn:** Let  $G = (V_1 \sqcup V_2, E)$  be a bipartite graph. A matching  $M \subset E$  is any collection of vertex-disjoint edges.  $M$  is a perfect matching iff  $|M| = |V_1| = |V_2|$ .

**Thm:** (Halls) Let  $S_1, \dots, S_m \subset [n]$ . There is a way to pick  $x_i \in S_i$  s.t.  $x_i \neq x_j, \forall i, j, \Leftrightarrow \forall$  subcollections  $S_{i_1}, \dots, S_{i_k}$ , we have  $|S_{i_1} \cup \dots \cup S_{i_k}| \geq k$  ( $k \leq m$ ).

**Cor:**  $G = (V_1 \cup V_2, E)$  with  $|V_1| = |V_2| = n$  has a perfect matching  $\Leftrightarrow |N(X)| \geq |X| \forall X \subset V_1$ .

Proof (Halls):

Necessary: Assume we can pick  $x_i \in S_i$ . Then  $\forall S_{i_j}, x_{i_j} \in S_{i_j}$ .

Sufficient: Induction on  $m$ .

$m = 1$ :  $|S_1| \geq 1$ , so pick any  $x_1 \in S_1$ .

$m = k$  true, let's show  $m = k + 1$ .

Case 1:  $|S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_k}| \geq k$ . Then we can choose any  $x_{m+1} \in S_{m+1}$  and the statement still holds.

Case 2:  $\exists$  some subcollection  $|S_{i_1} \cup \dots \cup S_{i_k}| = k$  for  $x_{i_1} \in S_{i_1}, \dots, x_{i_k} \in S_{i_k}$ .  $S_{m+1}$  must have a neighbor outside this subcollection, or otherwise, the inductive hypothesis fails.

Define  $S'_j = S_j \setminus \{S_{i_1}, \dots, S_{i_k}\}$ . We know  $|S_{i_1} \cup \dots \cup S_{i_k} \cup S_{i_{k+1}} \cup \dots \cup S_{i_{k+\ell}}| \geq k + \ell$ . So  $|S'_{i_{k+1}} \cup \dots \cup S'_{i_{k+\ell}}| = \ell + k - k = \ell$ .

So by induction, we can pick  $x_{k+1} \in S'_{k+1}, \dots, x_{k+\ell} \in S'_{k+\ell}$ .  $\square$

**Defn:** A maximal matching  $M$  is a matching with  $|M| \geq |M'|$  for all matchings  $M'$ .

**Defn:** A subset  $X \subset V$  is an edge cover if every edge is incident to at least one vertex  $x \in X$ .

**Thm:** (König) Assume  $G = (V_1 \sqcup V_2, E)$  is a bipartite graph, perhaps without a perfect matching. Let  $M \subset E$  be a maximal matching, and  $C \subset V$  be a minimal edge cover. Then  $|M| = |C|$ .

Proof:

$|M| \leq |C|$ : This is easily true; if our matching has size  $|C| + 1$  or larger, then by the pigeonhole principle, two edges hit the same vertex.

$|M| \geq |C|$ :  $\forall X \subset X_1, |N(X) \setminus Y_1| \geq |X|$ . If not, we can decrease the size of the edge cover. **Sort of a proof by picture.** So for each element in  $C$ , pick a match.

$\square$