Polytopes

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Thm: (Helly's Theorem) Let $X_1, \dots, X_M \subseteq \mathbb{R}^n$ be convex sets.
If any $n+1$ of them intersect, then they all intersect.
Proof: If $m \leq n+1$, we're done. So assume $m = n+2$. We have $X_1, \dots, X_{n+2}$.
Let $x_i \in X_1 \cap \cdots \cap X_{i-1} \cap X_{i+1} \cap \cdots \cap X_{n+2}$ (skipping X_i), and consider $\{x_1, \dots, x_{n+2}\}$.

Recall Radon's thm: $Y \subseteq \mathbb{R}^n$, $|Y| = n+2$, then we can partition $Y = S \sqcup T$ s.t. $\mathrm{ch}(S) \cap \mathrm{ch}(T) \neq \emptyset$.
Applying this gives us $\{x_1, \dots, x_{n+2}\} = S \sqcup T$ with $\mathrm{ch}(S) \cap \mathrm{ch}(T)$ nonempty.

Let $y \in \mathrm{ch}(S) \cap \mathrm{ch}(T)$, and let $1 \leq i \leq n+2$. Then X_i contains all x for $j \neq i$, so either $S \subseteq X_i$ or $T \subseteq X_i$. So $y \in \mathrm{ch}(S) \subseteq X_i$ or $y \in \mathrm{ch}(T) \subseteq X_i$ for any $1 \leq i \leq n+2$. So $m = n+2$ is done.

Let $m \geq n+2$ be arbitrarily large, let $X_1' = X_1 \cap X_2 \neq \emptyset$. Replace X_1 and X_2 by X_1'; we claim that any $n+1$ of the new sets also intersect. If we take X_1' and n of $X_3, \dots, X_m$, then by the case $m = n+2$, we know that $(X_1' \cap \cdots) = (X_1 \cap X_2 \cap \cdots) \neq \emptyset$. So we can perform induction on m. $\square$

Defn: A <u>polytope</u> is a convex hull of finitely many points.

Ex: 1-dimensional polytopes: closed intervals

Ex: 3-dimensional polytopes: simplex, cube, octahedron, etc. (the platonic solids)

Defn: Let $X \subseteq \mathbb{R}^n$. The <u>affine dimension</u> of X is $\mathrm{affdim}(X) = \begin{cases} -1 & x = \emptyset \\ \dim(\mathrm{span}\,\{y - x : y \in X\}) & x \neq \emptyset \end{cases}$

Defn: An <u>n-dimensional simplex</u> $S \subseteq \mathbb{R}^n$ is a convex hull of $n+1$ points $\{x_0, \dots, x_{n+1}\}$ with $\mathrm{affdim}(S) = n$.

If $P = \mathrm{ch}(X)$ is a polytope, then P can always be triangulated $P = \bigcup_{i=1}^m \triangle_i$, where each $\triangle_i$ is a simplex with vertices in X, and $\mathrm{Int}\,\triangle_i \cap \mathrm{Int}\,\triangle_j = \emptyset$ for any $i \neq j$.
Defn: Faces:
(-1)-dim face: $\emptyset$
0-dim faces: vertices
1-dim faces: edges
2-dim faces: (traditional) faces
$\vdots$
n-dim faces: the polytope itself

The number of i-dimensional faces is $\binom{n}{i} 2^{n-i}$.

Thm: (Euler-Poincare) If P is an n-dimensional polytope, then $f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = 1$.

Ex: P is an n-dimensional simplex, $f_i = \binom{n+1}{i+1}$. Then $\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \cdots + (-1)^n \binom{n+1}{n+1} = 1$.
$P = C_n$ (cube) means $f_i = \binom{n}{i} 2^{n-i}$, so $\binom{n}{0} 2^n - \binom{n}{1} 2^{n-1} + \cdots + (-1)^n \binom{n}{n} 2^0 = (2-1)^n = 1$.