

Math 591 Lecture 37

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Briefly, think back to integration of forms. Observe:

$$\int_M \mu = \sum_{\alpha} \int (\phi_{\alpha}^{-1})^* (\chi_{\alpha} \mu)$$

We claimed (without proof) that this sum has finitely-many nonzero summands.

Proof: Let $K = \text{supp } \mu$ (note that K must be compact), and $U_{\alpha} = \text{supp } \chi_{\alpha}$. Note that $\{U_{\alpha}\}$ is locally finite: $\forall p \in K$, $\exists V_p$, a neighborhood of p , such that $\{\alpha \mid V_p \cap U_{\alpha} \neq \emptyset\}$ is finite. Also, $\{V_p\}_{p \in K}$ is naturally a cover of K , so by compactness, $\exists p_1, \dots, p_n \in K$ s.t. $\{V_{p_1}, \dots, V_{p_n}\}$ is a cover of K . Finally,

$$\{\alpha \mid K \cap U_{\alpha} \neq \emptyset\} \subset \bigcup_{j=1}^n \underbrace{\{\alpha \mid V_{p_j} \cap U_{\alpha} \neq \emptyset\}}_{\text{finite}}$$

□

Now, we return to working with manifolds with boundary...

The principle is that all definitions on manifold with boundary are exact analogues to the case where $\partial M = \emptyset$. The key difference is the model spaces are open subsets of $\mathbb{H}^n = \{x^n \geq 0\} \subseteq \mathbb{R}^n$, and transition maps are diffeomorphisms between open subsets of \mathbb{H}^n (which carry boundary points to boundary points, and restrict to diffeomorphisms between open sets of $\partial \mathbb{H}^n \cong \mathbb{R}^{n-1}$).

Ex: Notions of smoothness. $f : M \rightarrow \mathbb{R}$ is smooth iff for any chart ϕ_{α} , $f_{\alpha} = f \circ \phi_{\alpha}^{-1} : U \rightarrow \mathbb{R}$ is smooth (with $U \subseteq^{\text{open}} \mathbb{H}^n$). In case $U \cap \partial \mathbb{H}^n \neq \emptyset$, smoothness of $f_{\alpha} : U \rightarrow \mathbb{R}$ means $\exists \tilde{f}_{\alpha} : \tilde{U}_{\alpha} \rightarrow \mathbb{R}$, a smooth extension of f_{α} to $\tilde{U}_{\alpha} \subseteq^{\text{open}} \mathbb{R}^n$. Then $U = \tilde{U}_{\alpha} \cap \mathbb{H}^n$.

Ex: Existence of partitions of unity, just as before.

Tangent Spaces

$\forall p \in \partial M$, $T_p M$ is still n -dimensional, and is still spanned by $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$. $\frac{\partial}{\partial x^n}$ is well-defined on smooth functions, because such functions extend across the boundary, and last time, we showed the choice of extension doesn't matter.

Note: ∂M inherits a C^{∞} manifold structure, by restricting charts of M .

$\forall p \in \partial M$, $T_p(\partial M) \xrightarrow{\iota_{*,p}} T_p M$, where $\iota : \partial M \rightarrow M$ is the inclusion. Identify $T_p(\partial M) \subset T_p M$ as a subspace – the image of $\iota_{*,p}$. It's a hyperplane, i.e., $T_p(\partial M)$ has codimension 1.

Note: $(T_p M) \setminus (T_p \partial M)$ has two components: “inward-pointing” vectors and “outward-pointing” vectors.

How do we characterize outward-pointing vectors? Well, say $p \in \partial M$. Let $\gamma : (-\varepsilon, 0] \rightarrow M$ smooth with $\gamma(0) = p$. Then define $\dot{\gamma}(0)$ to be outward pointing.

Orientation of Manifolds with Boundary

Once again, this is an exact analogue of manifolds without boundary. $\forall p \in M$, $T_p M$ has dimension n , two orientations, etc.

Lemma: If M is orientable, then ∂M is orientable.

By convention, $\forall p \in \partial M$, a basis (b_1, \dots, b_{n-1}) of $T_p(\partial M)$ is positive iff $(\nu, b_1, \dots, b_{n-1})$ is a positive basis of $T_p M$, for any outward-pointing $\nu \in T_p M$. This defines the boundary's orientation.

Ex: Say $M = \mathbb{H}^n$, oriented such that $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ is positive. What is the boundary orientation?

For $n = 2$, $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$ is positive. So if ν is outward-pointing, $\{\nu, \frac{\partial}{\partial x^2}\}$ is positive, so $\{\frac{\partial}{\partial x^1}\}$ is positive on $\partial \mathbb{H}^2 \cong \mathbb{R}$. For $n = 3$, is (\vec{i}, \vec{j}) positive? Look at $v = -\vec{k} - (-\vec{k}, \vec{i}, \vec{j})$ is negative. So we claim that $\partial \mathbb{H}^n$ is $(-1)^n$ times the standard orientation of \mathbb{R}^n .

Stokes' Theorem

Thm: (Stokes' Theorem) Let M be a manifold (possibly with boundary). Let $\mu \in \Omega_0^{n-1}(M)$. Assume M is oriented; give ∂M the boundary orientation. Then

$$\int_{\partial M} \iota^* \mu = \int_M d\mu$$

(where $\iota : \partial M \hookrightarrow M$ is the inclusion map). One often omits the ι^* , so we say

$$\int_{\partial M} \mu = \int_M d\mu$$

(If $\partial M = \emptyset$, then $\int_M d\mu = 0$.)

Observe: $\iota^* \mu$ is a top degree form on the boundary.

Ex: $M = [a, b]$, $n = 1$, $\mu \in \Omega_0^0(M) \cong C^\infty(M)$. Then $\int_{\partial M} \mu = \mu(b) - \mu(a) = \int_M df$. The “ $-$ ” sign comes because of the orientation of ∂M is outward-pointing.

Ex: $n = 2$. Then we get Green's Theorem:

$$\mu = P dx + Q dy, \quad d\mu = (Q_x - P_y) dx \wedge dy.$$

The orientation on ∂M comes from the right-hand rule.

$$\int_{\partial M} P dx + Q dy = \iint_M (Q_x - P_y) dx \wedge dy$$

Exer: For $n = 3$, we get the usual Stokes' theorem.

Proof of Stokes' Theorem

First, assume it holds for \mathbb{H}^n . Then it follows for $\mu \in \Omega_0^{n-1}(M)$ if μ is supported in the domain of a chart ϕ :

$$\int_{\partial M} \mu = \int_{\partial \mathbb{H}^n} (\phi^{-1})^* \iota^* \mu \stackrel{(1)}{=} \int_{\mathbb{H}^n} d(\phi^{-1})^* \mu = \int_{\mathbb{H}^n} (\phi^{-1})^* d\mu = \int_M d\mu$$

with (1) by the definition of integrals, plus our assumption about μ . In general (still under the assumption that Stokes' theorem holds for \mathbb{H}^n), we use a partition of unity $\{\chi_j\}$ subordinate to an atlas.

$$\int_{\partial M} \iota^* \mu = \sum_j \int_{\partial M} \iota^* (\chi_j \mu) = \sum_j \int_M d(\chi_j \mu) = \sum_j \int_M d(\chi_j) \wedge \mu + \int_M \chi_j d\mu = \int_M d\mu + \int_M \left(\underbrace{\sum_j d\chi_j}_{\substack{=d \sum_j \chi_j = 0 \\ \text{b/c } \sum_j \chi_j \equiv 1}} \right) \wedge \mu = \int_M d\mu$$

So finally, it's enough to prove Stokes' Theorem for $M = \mathbb{H}^n$. Write

$$\mu = \sum_{i=1}^n a_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad a_i \in C^\infty(\mathbb{H}^n)$$

Then

$$d\mu = \left(\sum_{i=1}^n \frac{\partial a_i}{\partial x^j} (-1)^{j-1} \right) \underbrace{dx^1 \wedge \cdots \wedge dx^n}_{\text{standard volume form}}$$

Because we assume μ has compact support, $\text{supp } \mu$ is bounded, so $\exists R > 0$ s.t.

$$\text{supp } \mu \subseteq \underbrace{[-R, R] \times \cdots \times [-R, R]}_{n-1 \text{ times}} \times [0, R]$$

Thus,

$$\int_{\mathbb{H}^n} d\mu = \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial a_i}{\partial x^j} dx^1 \cdots dx^n$$

So $\forall i$, first do $\int_{-R}^R \frac{\partial a_i}{\partial x^i} dx^i$. For $i = 1, \dots, n-1$,

$$\int_{-R}^R \frac{\partial a_i}{\partial x^i} dx^i = a(x^1, \dots, R, \dots, x^n) - a(x^1, \dots, -R, \dots, x^n) = 0 - 0 = 0 \quad \int_0^R \frac{\partial a_n}{\partial x^n} dx^n = -a_n(x^1, \dots, x^{n-1}, R)$$

Thus,

$$\int_{\mathbb{H}^n} d\mu = \underbrace{-(-1)^{n-1}}_{=(-1)^n} \int_{\mathbb{R}^{n-1}} a_n(x^1, \dots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}$$

And on the other hand,

$$\int_{\partial \mathbb{H}^n} \mu = \underbrace{(-1)^n}_{\substack{\uparrow \\ \text{boundary orientation}}} \int_{\mathbb{R}^{n-1}} a_n(x^1, \dots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}$$

(with the 0 appearing in the a_n term because $\iota^*(dx^n) = 0$.)

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