

Math 591 Lecture 23

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Uniqueness of Integral Curves

Last time, given $X \in \mathfrak{X}(M)$, we defined integral curves of X , and proved *local* existence and uniqueness by reducing to the Euclidean case and using theory from ordinary differential equations. Today, we'll start with global uniqueness.

Lemma: Assume that $c_1, c_2 : (\alpha, \beta) \rightarrow M$ (with $\alpha < 0 < \beta$) are integral curves of X , and $c_1(0) = c_2(0) = p$. Then $\forall t \in (\alpha, \beta), c_1(t) = c_2(t)$.

Proof: Assume not. Then $S \stackrel{\text{def}}{=} \{t \in (\alpha, \beta) \mid t > 0, c_1(t) \neq c_2(t)\} \neq \emptyset$. Well, this set is bounded below and nonempty, so let $\tau = \inf S$.

We claim that $c_1(\tau) = c_2(\tau)$ – assume not. Then because M is Hausdorff, there exist neighborhoods U_1 around $c_1(\tau)$ and U_2 around $c_2(\tau)$ such that $U_1 \cap U_2 = \emptyset$. By the continuity of c_1 and c_2 , $\exists t < \tau$ such that $c_1(t) \neq c_2(t)$. Thus, τ is not the infimum of S . Oops! This is a contradiction, so we must have $c_1(\tau) = c_2(\tau)$.

Now, we use local uniqueness of integral curves of X . Let c be an integral curve with initial condition $c(\tau) = c_1(\tau) = c_2(\tau)$. By local uniqueness, c must agree with c_1 and c_2 on a neighborhood of τ , so $\tau < \inf S$. Oops!

Therefore, we must have $c_1(t) = c_2(t), \forall t \in (\alpha, \beta)$. \square

Cor: Given $X \in \mathfrak{X}(M)$ and $p \in M$, there is an interval $(\alpha(p), \beta(p))$ containing 0 (possibly with $\alpha(p) = -\infty$ and/or $\beta(p) = +\infty$), and an integral curve $c : (\alpha(p), \beta(p)) \rightarrow M$ of X with $c(0) = p$, such that for any other integral curve $\tilde{c} : I \rightarrow M$ (with I an open interval) with $\tilde{c}(0) = p$, one has $I \subset (\alpha(p), \beta(p))$ and on I , $c|_I = \tilde{c}$.

Proof: Let \mathcal{I} be the set of intervals which are domains for some integral curve c of X with $c(0) = p$. Then we have $(\alpha(p), \beta(p)) = \bigcup_{I \in \mathcal{I}} I$. Any two $c_1 : I_1 \rightarrow M, c_2 : I_2 \rightarrow M$ agree on their overlap, $I_1 \cap I_2$, so they define an integral curve on their union, $I_1 \cup I_2$. Doing this for all $I \in \mathcal{I}$ gives us the desired integral curve. \square

Defn: Such an integral curve is the unique maximal integral curve of X through p at $t = 0$.

We can refer to a chapter on vector fields in the book by Boothby, but it's a little too detailed.

Defn: Given $X \in \mathfrak{X}(M)$, define $\mathcal{W} = \{(t, p) \in \mathbb{R} \times M \mid t \in (\alpha(p), \beta(p))\} \subseteq \mathbb{R} \times M$. This is the domain of a map $\phi : \mathcal{W} \rightarrow M$, which takes a pair (t, p) to the unique maximal integral curve of X with initial condition p at time t . In other words, $\forall (t, p) \in \mathcal{W}, \phi(0, p) = p$ and $\frac{\partial \phi}{\partial t}(t, p) = X_{\phi(t, p)}$. ϕ is called the flow of X .

Thm: $\mathcal{W} \subseteq \mathbb{R} \times M$ is open, and ϕ is a smooth map (of t, p).

This general theorem is rather challenging to prove, but the local version (which is included in our textbook) is sufficient for our purposes.

Thm: Let $X \in \mathfrak{X}(M), p \in M$. Then there exists a neighborhood V of $p, \varepsilon > 0$, and a smooth map $\phi : (-\varepsilon, \varepsilon) \times V \rightarrow M$ such that

- $\phi(0, q) = q, \forall q \in V$
- $\frac{\partial \phi}{\partial t}(t, q) = X_{\phi(t, q)}$. (Note: This is the velocity of the curve $t \mapsto \phi(t, q)$ at time t . That is, $t \mapsto \phi(t, q)$ is an integral curve of X with initial condition q .)

Proof: Just quote Calc IV/diffeq. In case $M = \mathbb{R}^n$, this is a theorem. Then just use local coordinates to reduce any manifold to the Euclidean case. \square

Note: This isn't as fancy as the previous (global) version, but it's enough for our purposes. $(-\varepsilon, \varepsilon) \times V$ is sometimes referred to as a “flow box”. \mathcal{W} may be quite complicated, but the flow boxes are always easy to work with.

Main points:

1. ϕ is C^∞ in (t, p) . We refer to this as “smooth dependence on initial conditions”.
2. $\varepsilon > 0$ can be uniform on V .

Notation: It is standard to write $\phi(t, p) = \phi_t(p)$. This emphasizes, in the local flow theorem, that $\forall t \in (-\varepsilon, \varepsilon)$, we can think of the map $\phi_t : V \rightarrow M$. This is called the “time t map”. Think about it as moving every point in V by time t along their respective integral curves. In other words, it takes the blob V to a new blob $\phi_t(V)$.

Thm: Given $X \in \mathfrak{X}(M)$, $p \in M$. If $t, s, t + s \in (\alpha(p), \beta(p))$, then

$$\phi_t(\phi_s(p)) = \phi_{t+s}(p) = \phi_{s+t}(p) = \phi_s(\phi_t(p)).$$

This is known as the “1-parameter group”.

Proof: Fix s , and consider the curves $t \mapsto \phi_{t+s}(p)$ and $t \mapsto \phi_t(\phi_s(p))$. Both are integral curves of X , with the same initial conditions (check that this is true), so by uniqueness, they're the same curve. The rest follows from commutativity of addition. \square

So altogether, if there are no domain issues,

$$\phi_t \circ \phi_s = \phi_{t+s} = \phi_{s+t} = \phi_s \circ \phi_t.$$

For complete fields, this shows that $t \mapsto \phi_t$ is a map $\mathbb{R} \rightarrow \{\text{All diffeomorphisms } M \rightarrow M\} \stackrel{\text{def}}{=} \text{Diff}(M)$, and this map is a group morphism from $(\mathbb{R}, +) \rightarrow \text{Diff}(M)$!