

# Math 591 Lecture 30

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## Tensors

**Defn:** Let  $V$  be a finite-dimensional vector space. A tensor on  $V$  is an element of

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \otimes \underbrace{V \otimes \cdots \otimes V}_m$$

(where  $\otimes$  is the tensor product).

Observe: This space is isomorphic to the space of multilinear maps  $\underbrace{V \times \cdots \times V}_{\ell} \times \underbrace{V^* \times \cdots \times V^*}_m \rightarrow \mathbb{R}$ , because  $(V^*)^* \cong V$ .

Last time, we defined the set of *alternating* multilinear functions  $\bigwedge^k V^* \hookrightarrow \underbrace{V^* \otimes \cdots \otimes V^*}_k$ .

Reminder: A basis for  $\bigwedge^k V^*$ : choose  $\{\mathcal{E}^i\}_{1 \leq i \leq n}$  an ordered basis of  $V^*$ . For each  $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}$ , let  $\mathcal{E}^I(v_1, \dots, v_k) = \det(\mathcal{E}^{i_j}(v_\ell))_{(j,\ell)}$ .

**Prop:**  $\{\mathcal{E}^I : I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}, \#I = k\}$  is a basis of  $\bigwedge^k V^*$ .

In fact,  $\forall \alpha \in \bigwedge^k V^*$ ,  $\alpha = \sum_I' \alpha(e_{i_1}, \dots, e_{i_k}) \mathcal{E}^I$ , where  $\{e_j\}$  is the basis of  $V$  dual to  $\{\mathcal{E}^j\}$  (i.e.,  $\mathcal{E}^i(e_j) = \delta_{ij}$ ).

Note: the notation  $\sum_I'$  means sum over increasing  $I = \{i_1 < \cdots < i_k\}$ .

Also,  $\dim \bigwedge^k V^* = \binom{n}{k}$ .

**Defn:** By convention,  $\bigwedge^0 V^* = \mathbb{R}$ .

**Defn:** If  $\alpha \in \bigwedge^k V^*$ ,  $\beta \in \bigwedge^\ell V^*$ , we define the wedge product  $\alpha \wedge \beta \in \bigwedge^{k+\ell} V^*$  by

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

Note: We can define  $(\alpha \otimes \beta)(v_1, \dots, v_{k+\ell}) = \alpha(v_1, \dots, v_k) \beta(v_{k+1}, \dots, v_{k+\ell})$ , but this may not be alternating (in fact, it almost certainly isn't). But this can be skew symmetrized by forming the above sum (with appropriate normalization).

Note: An equivalent formula for the wedge product is

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum_{\sigma \in \text{Sh}(k, \ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

This is a much smaller sum, as it removes redundancies.

Recall:  $\sigma \in \text{Sh}(k, \ell)$  iff  $\sigma \in S_{k+\ell}$ ,  $\sigma(1) < \cdots < \sigma(k)$ ,  $\sigma(k+1) < \cdots < \sigma(k+\ell)$ .

Note:  $\# \text{Sh}(k, \ell) = \binom{k+\ell}{k} = \binom{k+\ell}{\ell}$ .

**Ex:** Say  $\alpha \in \bigwedge^2 V^*$ ,  $\beta \in \bigwedge^2 V^*$ . Then the elements of  $\text{Sh}(2, 2)$  are:

1	2	3	4	sgn
1	2	3	4	+
1	3	2	4	-
1	4	2	3	-
2	3	1	4	+
2	4	1	3	+
3	4	1	2	+

So

$$\begin{aligned}
 (\alpha \wedge \beta)(v_1, v_2, v_3, v_4) &= \alpha(v_1, v_2)\beta(v_3, v_4) \\
 &\quad - \alpha(v_1, v_3)\beta(v_2, v_4) \\
 &\quad - \alpha(v_1, v_4)\beta(v_2, v_3) \\
 &\quad + \alpha(v_2, v_3)\beta(v_1, v_4) \\
 &\quad + \alpha(v_2, v_4)\beta(v_1, v_3) \\
 &\quad + \alpha(v_3, v_4)\beta(v_1, v_2)
 \end{aligned}$$

### Properties of the Wedge Product

- The wedge product is bilinear:  $(\alpha_1 + \lambda\alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \lambda(\alpha_2 \wedge \beta)$ .
- The wedge product is associative:  $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma$ .
- The wedge product is anticommutative: for  $\alpha \in \bigwedge^k, \beta \in \bigwedge^\ell, \alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$ . This implies that even forms commute with any other form.
- If  $\alpha^1, \dots, \alpha^k \in V^*$ , then  $(\alpha^1 \wedge \dots \wedge \alpha^k)(v_1, \dots, v_k) = \det(\alpha^i(v_j))_{(i,j)}$ .  
In particular,  $\mathcal{E}^I = \mathcal{E}^{i_1} \wedge \dots \wedge \mathcal{E}^{i_k}$ .

**Ex:** Say  $\alpha^1, \alpha^2 \in V^*$ . Then  $(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1)$ .

**Defn:**  $\left( \bigoplus_{k=0}^n \bigwedge^k V^*, \wedge \right)$  is the exterior algebra or Grassmann algebra of  $V$ .

Now, back to manifolds...

**Defn:** Let  $M$  be a smooth manifold. A  $k$ -differential form (or  $k$ -form) on  $M$  is an assignment  $\forall p \in M, p \mapsto \alpha_p \in \bigwedge^k(T_p^*M)$ .

Note: When  $k = 1$ ,  $\alpha$  is a 1-form as before. When  $k = 0$ ,  $\alpha$  is just an  $\mathbb{R}$ -valued function.

### Local Expression

Let  $(U, (x^1, \dots, x^n))$  be a coordinate chart. Then  $\forall p \in U$ , we get  $\left\{ dx^i|_p \right\}_{i=1, \dots, n}$  a basis of  $T_p^*M$ , with corresponding dual basis  $\left\{ \frac{\partial}{\partial x^i}|_p \right\}$  of  $T_pM$ . If  $\alpha$  is a  $k$ -form, then  $\forall p \in U$ ,

$$\alpha_p = \sum_I' a_I(dx^I)_p \quad a_I(p) = \alpha_p \left( \frac{\partial}{\partial x^{i_1}}|_p, \dots, \frac{\partial}{\partial x^{i_k}}|_p \right)$$

This defines functions  $a_I : U \rightarrow \mathbb{R}$  for each  $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$ .

**Defn:** A  $k$ -form  $\alpha$  is smooth iff there exists a  $C^\infty$  atlas of  $M$  such that in each chart, each  $a_I$  is smooth.

**Lemma:**  $\alpha$  is smooth iff for every chart in the smooth structure, each function  $a_I$  is smooth.

Proof:  $\Leftarrow$  is trivial. For  $\Rightarrow$ , let  $(y^1, \dots, y^n) : V \rightarrow \mathbb{R}$  be an arbitrary coordinate chart. Let  $p \in V$ . By our assumption, there's a chart  $(x^1, \dots, x^n) : U \rightarrow \mathbb{R}$  near  $p$  such that  $\alpha = \sum_I' a_I dx^I$ , with  $a_I \in C^\infty(U)$ . Also,  $\alpha = \sum_J' b_J dy^J$ . We need to show  $b_J \in C^\infty(U \cap V)$ , but how? Well,

$$\begin{aligned} b_J = \alpha \left( \frac{\partial}{\partial y^{j_1}}, \dots, \frac{\partial}{\partial y^{j_k}} \right) &= \sum_I' a_I dx^I \underbrace{\left( \frac{\partial}{\partial y^{j_1}}, \dots, \frac{\partial}{\partial y^{j_k}} \right)}_{= \det \left( dx^{i_r} \left( \frac{\partial}{\partial y^{j_s}} \right) \right)_{(r,s)}} \\ &= \det \left( \frac{\partial x^{i_r}}{\partial y^{j_s}} \right)_{(r,s)} \end{aligned}$$

The  $\frac{\partial x^{i_r}}{\partial y^{j_s}}$  are smooth, so the whole determinant is smooth, so  $b_J$  is the sum of smooth functions. Thus, it's smooth.  $\square$

From the bundle point-of-view, we can define  $\bigwedge^k T^*M = \bigsqcup_{p \in M} \bigwedge^k (T_p^*M)$ .

**Prop:** One can make  $\bigwedge^k T^*M$  into a vector bundle, with trivializations given by the moving frames  $\{dx^I\}$  associated to coordinates. Then  $C^\infty$   $k$ -forms are  $C^\infty$  sections of this bundle.