Math 591 Lecture 27

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Lie Groups and Their Algebras

Reminder/Review: Given G a Lie group, $\forall g \in G$, the map $L_g : G \to G$ where $L_g(k) = gk$. $X \in \mathfrak{X}(M)$ is left-invariant iff $\forall g \in G$, X is L_g-related to itself.

Prop: (HW 8 Problem 4) There is a bijective linear correspondence between $\mathfrak{g} = T_eG$, the Lie algebra, and the set of left-invariant fields on G, where $T_eG \ni A \mapsto A^\sharp \in \mathfrak{X}(G)$. $A^\sharp$ is defined by $\forall g \in G$, $A^\sharp_g = (L_g)_{*,e}(A)$. $A^\sharp$ is smooth.

Observe: $\forall X, Y \in \mathfrak{X}(G)$ left-invariant, $[X,Y]$ is also left-invariant, because being related by L_g preserves commutators.

Defn: Under this correspondence, we can define the <u>bracket of fields</u>

$$\begin{aligned} [\,\cdot\,,\,\cdot\,] &: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ (A,B) &\mapsto [A,B] \stackrel{\text{def}}{=} \left[A^\sharp, B^\sharp\right]_e \end{aligned}$$

Defn: $(g, [\,\cdot\,,\,\cdot\,]$ is the <u>Lie algebra</u> of G.

$[\,\cdot\,,\,\cdot\,]$ is $\mathbb{R}$-bilinear and satisfies the Jacobi identity.

The Exponential Map

Notation: $\forall A \in \mathfrak{g}$, let F^A be the flow of $A^\sharp$.

Defn: $\forall A \in \mathfrak{g}$, the <u>exponential map</u> is defined to be $\exp t\,A \stackrel{\text{def}}{=} F^A_t(e)$.

Prop: Given $A \in \mathfrak{g}$:

(1) $\exp t\,A$ is defined $\forall t \in \mathbb{R}$.
(2) $\exp(t+s)\,A = (\exp t\,A) \cdot (\exp s\,A)$, $\forall s,t \in \mathbb{R}$ (with $\cdot$ being group multiplication).

Proof (2): Assume $t+s$ is small. Then

$$\begin{aligned} \exp{(t+s)}\,A &= F^A_{t+s}(e) = F^A_t(F^A_s(e)) \\ (\exp t\,A) \cdot (\exp s\,A) &= L_{\exp t\,A}(\exp s\,A) \end{aligned}$$

So $L_{\exp t\,A}$ maps integral curves of $A^\sharp$ to integral curves of $A^\sharp$, because $A^\sharp$ is $L_{\exp t\,A}$-related to itself. Thus, the map $s \mapsto L_{\exp t\,A}(\exp s\,A)$ is the integral cuve of $A^\sharp$ through $\exp t\,A$, so it must agree with $F^A_s(\exp t\,A)$. This proves (2) for small s,t. $\square$

Proof (1): Well, we know $\exists \varepsilon > 0$ s.t. $\exp t\,A$ is defined for $t \in (-\varepsilon, \varepsilon)$. So we'll make use of the fact that $\exp(t+s)\,A = (\exp t\,A) \cdot (\exp s\,A)$. Note: the right-hand side is defined for $t+s \in (-\varepsilon,\varepsilon)$, so extend the left-hand side to $t+s \in (-2\varepsilon, 2\varepsilon)$. This is somewhat sketchy, but it works. Then, we just have to check that this extension is an integral curve of $A^\sharp$, and it must agree with $\exp(t+s)\,A$. Now, we have $\exp t\,A$ defined for $t \in (-2\varepsilon, 2\varepsilon)$. Repeat ad nauseum... $\square$

Cor: (2) $\Rightarrow \exp t\,A, \exp s\,A \in G$ commute.

Ex: $G = \mathrm{GL}(n, \mathbb{R}) \overset{\text{open}}{\subseteq} \mathbb{R}^{n^2}$. $\mathfrak{g} = \mathrm{gl}(n, \mathbb{R}) = \mathbb{R}^{n^2}$, the set of $n \times n$ real matrices. Then

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$$

We need to check that this series converges absolutely (i.e. for some matrix norm). Well, $||AB|| \leq ||A||\,||B||$, and $\frac{d}{dt}(\exp t\,A) = A \exp t\,A = (\exp t\,A)A = A^\sharp A$.
(Claim: $\forall g \in \mathrm{GL}(n, \mathbb{R})$, $L_g(A) = A^\sharp g$. Proof: $L_g : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is linear, so its differential is itself, i.e., $(L_g)_{*,e} = L_g$.)

Defn: $\exp : \mathfrak{g} \to G$ is defined by $\exp(A) \overset{\text{def}}{=} \exp(t)\,A|_{t=1}$.

Prop: $(\exp)_{*,0} : T_0\mathfrak{g} \to \mathfrak{g}$ is the identity map $\mathfrak{g} \to \mathfrak{g}$, so exp is a local diffeomorphism at $0 \in \mathfrak{g}$.

Proof:

$$(\exp)_{*,0}(A) \overset{(1)}{=} \frac{d}{dt} \exp t\,A \bigg|_{t=0} = A^\sharp_e = A$$

where (1) holds by using the curve $t \mapsto tA$, in $\mathfrak{g}$ adapted to $(0, A)$. $\square$

Prop: $\forall A \in \mathfrak{g}$, $A^\sharp$ is complete.

Proof: $\forall g \in G$, $L_g(\exp t\,A) = g \cdot \exp t\,A$ is the integral curve of $A^\sharp$ starting at g. $\square$

Subgroups (Part 1)

Defn: A regular (or closed, or embedded) subgroup H of G is a regular submanifold that is also a subgroup. It follows directly that H is a lie group in its own right, and $\mathfrak{h} = T_eH \hookrightarrow \mathfrak{g} = T_eG$.

Prop: $\mathfrak{h}$ is closed under $[\,\cdot\,,\,\cdot\,]$ of $\mathfrak{g}$. This means, $\forall A, B \in \mathfrak{h}$, $\left[A^\sharp, B^\sharp\right]$ is tangent to H, and $\left[A^\sharp, B^\sharp\right]_e \in \mathfrak{h}$.