

Math 591 Lecture 6

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9/14/20

Many examples of C^∞ manifolds are produced by the implicit function theorem. Reminder:

Thm: (Implicit Function Theorem) (Theorem B2 in the textbook)

Let $U \subseteq \mathbb{R}^N$ open, $F : U \rightarrow \mathbb{R}^k$ C^∞ , and $x_0 \in U$ s.t. $F(x_0) = 0$.

Split: for $x \in U$, write $x = (x', x'')$, where $x' \in \mathbb{R}^{N-k}$ and $x'' \in \mathbb{R}^k$. Accordingly, the Jacobian of F at x_0 splits:

$$F'(x_0) = \left(\underbrace{\frac{\partial F}{\partial x'}}_{N-k} \quad \underbrace{\frac{\partial F}{\partial x''}}_k \right) \begin{pmatrix} x_0' \\ x_0'' \end{pmatrix}$$

Assume $\left[\frac{\partial F}{\partial x''}(x_0) \right] (k \times k)$ is invertible. Then there exist open sets A, B with $x_0' \in A \subseteq \mathbb{R}^{N-k}$, $x_0'' \in B \subseteq \mathbb{R}^k$ and $g : A \rightarrow B$ C^∞ s.t. $\{F^{-1}(0)\} \cap (A \times B) = \{(x', g(x')) \mid x' \in A\}$.

Application: Recall that given $F : U \rightarrow \mathbb{R}^k$, $U \subseteq \mathbb{R}^N$ open, zero is a regular value of F iff $\forall x \in F^{-1}(0)$, $F'(x)$ has rank k .

Cor: If 0 is a regular value for F , then $F^{-1}(0) = M$ is locally a graph. Moreover, this structure of local graph gives M a C^∞ atlas, and therefore a smooth manifold structure.

(Note: we will define “submanifold”, and then $F^{-1}(0)$ will be examples of submanifolds of \mathbb{R}^N .)

Proof/Explanation: Assume 0 is a regular value of F . Then $\forall x_0 \in F^{-1}(0) = M$,

$$F'(x_0) = \begin{pmatrix} - & \nabla f^1(x_0) & - \\ & \vdots & \\ - & \nabla f^k(x_0) & - \end{pmatrix}$$

(for $F = (f^1, \dots, f^k)$). After permuting the indices among the x_i , without loss of generality $\left[\frac{\partial F}{\partial x''}(x_0) \right]_{k \times k}$ is non-degenerate. (Think of this as swapping the columns around so the block is invertible.)

The independent variables will depend on x_0 . The number of independent variables is $N-k = \dim M$. k is the codimension of $M \subseteq \mathbb{R}^N$.

Statement about “Atlas”

Recall: A graph $\{(x', g(x')) \mid x' \in A\} = \Gamma$ has a global chart: just the projection onto the domain. $\Gamma \ni (x', x'') \mapsto x'$. Its inverse is $x' \mapsto (x', g(x'))$.

For two local representations of M as a local graph, transition functions are of the form $x' \mapsto (x', g(x')) \xrightarrow{\star} \mathbb{R}^{N-k}$. \star is a projection onto an $N - k$ -dimensional coordinate plane of \mathbb{R}^N . This is smooth, and a transition map.

Ex: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in \mathbb{R}^3 is the zero set of $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$.
(Check that 0 is a regular value.)

Ex: $O(n) = \{g \in \text{Mat}(n, \mathbb{R}) \mid g^{-1} = g^T\}$. $\text{Mat}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, so $O(n) \subseteq \mathbb{R}^{n^2}$.

In fact, $O(n) \subseteq \text{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (and $\text{GL}(n, \mathbb{R})$ is open in \mathbb{R}^{n^2}).

Define

$$F : \text{GL}(n, \mathbb{R}) \rightarrow \text{Symm}(n, \mathbb{R}) \\ g \mapsto gg^T - I$$

where $\text{Symm}(n, \mathbb{R}) = \{g \in \text{Mat}(n, \mathbb{R}) \mid g = g^T\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$, and I is the identity matrix. Note: We have to choose the codomain carefully so that 0 is a regular value.

So, $O(n) = F^{-1}(0)$.

Check: is 0 a regular value? To see if $F'(g)$, for $g \in O(n)$, has maximal rank, let $M \in \text{Mat}(n, \mathbb{R})$. Then compute $\frac{d}{dt}F(g + tM)\big|_{t=0}$. (Compute this as a matrix to avoid \mathbb{R}^{n^2} coords.) Then $(-\varepsilon, \varepsilon) \ni t \mapsto \text{GL}(n, \mathbb{R}) \xrightarrow{F} \text{Symm}(n, \mathbb{R})$.

$$\frac{d}{dt}F(g + tM)\bigg|_{t=0} = \frac{d}{dt}(g + tM)(g + tM)^T\bigg|_{t=0} = \frac{d}{dt}(gg^T + t(Mg^T + gM^T) + t^2MM^T)\bigg|_{t=0} = Mg^T + gM^T$$

Question: Is $\text{Mat}(n, \mathbb{R}) \ni M \mapsto Mg^T + gM^T$ onto? Yes! (This is true iff $F'(g)$ has rank equal to the dimension of $\text{Symm}(n, \mathbb{R})$. Let $S \in \text{Symm}(n, \mathbb{R})$. What can M be?)

By the implicit function theorem, $O(n) \subseteq \mathbb{R}^{n^2}$ is locally a graph. So it has a natural C^∞ structure.