Math 591 Lecture 36

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Continuing from last time, W.T.S. $\sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$ is independent of choice of coordinates and partition of unity. Say $\left\{\tilde{\phi}_\beta\right\}, \{\tilde{\chi}_\beta\}$ is another choice. Then $\forall \alpha$,

$$\chi_\alpha \mu = \sum_\beta \tilde{\chi}_\beta \chi_\alpha \mu \qquad \sum_\beta \tilde{\chi}_\beta = 1$$

Thus,

$$\int (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \sum_\beta \int (\phi_\alpha^{-1})^*(\underbrace{\tilde{\chi}_\beta \chi_\alpha \mu}_{\text{supp in } (\operatorname{dom} \tilde{\phi}_\beta) \cap (\operatorname{dom} \phi_\alpha)})$$

So $\forall \beta$,

$$\int (\phi_\alpha^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu) = \int (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu)$$

because of the invariance of integrals of top-degree forms in Euclidean space, under orientation-preserving diffeomorphisms. (In this case, the transition function is that orientation-preserving diffeomorphism.) Now, back to the full term in the integral:

$$\sum_\alpha (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \sum_{\alpha,\beta} (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu)$$

$\forall \beta$, sum over α first, and use the fact that $\sum_\alpha \chi_\alpha = 1$. This yields

$$\sum_\alpha (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \cdots = \sum_\beta (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \mu)$$

$\square$

Observe: In practice, don't use partitions of unity. Use parameterizations that partition M (or at least, $\operatorname{supp} \mu \subset M$), up to sets of measure 0.

Ex: On a torus, pullback μ to $\mathbb{R}^n$ and integrate over a fundamental domain.

Defn: Let $S \subset M$ be an oriented k-dimensional submanifold. Define

$$\int_S : \Omega_0^k(M) \to \mathbb{R}$$
$$\alpha \mapsto \int_S \iota^* \alpha$$

where $\iota : S \hookrightarrow M$ is the inclusion map. In general, we omit the ι^*, and just say

$$\int_S \alpha \overset{\text{def}}{=} \int_S \iota^* \alpha$$

Manifolds with Boundary

These are needed for Stokes' theorem. We begin with some preliminary definitions...

Defn: Let $S, T \subset \mathbb{R}^n$ (with no assumptions of their properties). S and T are <u>diffeomorphic</u> iff $\exists U, V \overset{\text{open}}{\subset} \mathbb{R}^n$ s.t. $S \subset U$, $T \subset V$, and $\exists F : U \to V$ a diffeomorphism s.t. $F(S) = T$. We will say $F|_S^T : S \to T$ is a <u>diffeomorphism</u> between the sets.

Lemma: If $U \subset \mathbb{R}^n$ is open and diffeomorphic to $T \subset \mathbb{R}^n$, then T is open.

Proof: By definition, $\exists V, \tilde{U}$ open with $T \subset V$ and $U \subset \tilde{U}$, and $\exists F : \tilde{U} \to V$ diffeomorphic such that $F(U) = T$. Then $F|_U : U \to \mathbb{R}^n$ is an open map, so T is open. $\square$

Defn: $\mathbb{H}^n = \left\{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x_n \geq 0\right\}$.

Ex: $B_r(0) \cap \mathbb{H}^n$. A diffeomorphism on this set must map interior points to interior points, so it maps $B_r(0) \cap \{x^n = 0\}$ to the boundary of its image.

In fact, the same works for any relatively open sets of $\mathbb{H}^n$! And such diffeomorphisms restrict to diffeomorphisms of $W \cap \partial\mathbb{H}^n$ onto $F(W) \cap \partial\mathbb{H}^n$ (for any W relatively open).

Defn: A <u>topological manifold with boundary</u> M is a second-countable, Hausdorff topological space that is local homeomorphic to relatively open sets in $\mathbb{H}^n$.

Note: This includes ordinary open sets in $\mathbb{R}^n$, and the sets with boundary.

Defn: A <u>C^∞ atlas</u> on a topological manifold with boundary is an atlas such that all transition functions are C^∞.

Defn: A <u>C^∞ manifold with boundary</u> is a topological manifold with boundary, together with a maximal C^∞ atlas.

Defn: Let M be a smooth manifold with boundary. Then the <u>boundary</u> ∂M is the set of points $p \in M$ which have a nearby chart ϕ such that $\phi(p)$ is in the boundary of $\mathbb{H}^n$.

Note that this is true for one coordinate chart iff it's true for all coordinate charts.

Lemma: ∂M inherits a C^∞ manifold structure (without boundary) by restricting charts of M defined near boundary points.

Lemma: $\partial(\partial M) = \emptyset$.

Tangent Spaces

They're defined as before!

Defn: Given $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^k$ is <u>smooth</u> iff $\exists U \subset \mathbb{R}^n$ open with $S \subset U$, and $\exists \tilde{f} : U \to \mathbb{R}^k$ smooth such that $\tilde{f}\Big|_S \equiv f$. We call $\tilde{f}$ a <u>smooth extension</u>.

So, for f smooth, $f \circ \phi^{-1} : W \to \mathbb{R}$, $W \subset \mathbb{H}^n$ must be C^∞ for all charts $(\phi^{-1}(W), \phi)$. This gives us germs: $\forall p \in M$, $C_p^\infty(M)$. We define T_pM to be the set of all derivations on $C_p^\infty(M)$.

But what happens at the boundary? If $p \in \partial M$, what is T_pM?

Claim: Well, if $p \in \partial\mathbb{H}^n$, $T_p\mathbb{H}^n$ is still spanned by $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$.

The question is how do we define $\frac{\partial}{\partial x^n}$, and how does it act? If $f \in C^\infty$ near p, then $\exists \tilde{f}$, an extension of f to an open set of $\mathbb{R}^n$. Then define $\frac{\partial f}{\partial x^n}(p) \overset{\text{def}}{=} \frac{\partial \tilde{f}}{\partial x^n}(p)$.

Claim: The RHS is independent of choice of extension. Well,

$$\frac{\partial \tilde{f}}{\partial x^n}(p) = \lim_{h \to \infty} \frac{\tilde{f}(0, \ldots, h) - \tilde{f}(0, \ldots, 0)}{h}$$

This limit will always be the same, since the limit always exists (by definition of the extension).