Math 591 Lecture 15

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Thm: (Regular Value Theorem for Manifolds) Let M and N be manifolds, $F : M \to N$ C^∞, and $q \in N$ a regular value of F. Then $F^{-1}(q)$ is a regular submanifold of M.

Proof: Let $p \in F^{-1}(q)$. We want to show there are coordinates of M near p which are adapted to the preimage of $F^{-1}(q)$. Because q is a regular value, $F_{*,p} : T_pM \to T_qN$ is onto for any p. By the normal form for submersions, there are coordinates $(U, \phi = (x^1, \dots, x^m))$ near p and $(V, \psi = (y^1, \dots, y^n))$ near q, with $U \subseteq F^{-1}(V)$, such that $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n)$.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\phi"] & V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \end{tikzcd}$$

WOLOG assume $\psi(q) = 0$. Split ϕ, with $x' = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ and $x'' = (x^{n+1}, \dots, x^m) : U \to \mathbb{R}^{m-n}$. Then $F^{-1}(q) \cap U$ corresponds to $\tilde{F}^{-1}(0)$ by ϕ, i.e., $F^{-1}(q) \cap U = \{a \in U \mid x'(a) = 0\}$. Thus, (x'', x') are adapted coordinates to $F^{-1}(q) \cap U$. $\square$

Observe: (Keeping the notation of the proof) $x'' : F^{-1}(q) \cap U \to \mathbb{R}^{m-n}$ are coordinates on $F^{-1}(q) \cap U$. So $\dim F^{-1}(q) = m - n$. (Recall: $m \geq n$.)

Defn: The <u>codimension</u> of a submanifold is the dimension of the ambient space minus the dimension of the submanifold.

$\operatorname{codim} F^{-1}(q) = \dim M - \dim F^{-1}(q) = m - (m - n) = n$. This is the dimension of the target space.

Observe: $\forall p \in F^{-1}(q)$ (if q is a regular value), $T_p(F^{-1}(q)) \subseteq T_pM$ as a subspace. In fact, $T_p(F^{-1}(q))$ is the kernel of $F_{*,p}$.

A General Observation on Tangent Spaces of Submanifolds

Let $S \subseteq M$ be a submanifold, and $p \in S$. Then $T_pS \subseteq T_pM$ by: $\forall \gamma : (-\varepsilon, \varepsilon) \to S$ with $\gamma(0) = p$, we have

$$\begin{array}{ccc} \dot{\gamma}_S(0) & \xmapsto{\iota_{*,p}} & \dot{\gamma}_M(0) \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ T_pS & & T_PM \end{array}$$

by using the differential of the inclusion $\iota : S \hookrightarrow M$. The inclusion in adapted coordinates is $x' \mapsto (x', 0)$. If $[f] \in C_p^\infty(M)$, $\dot{\gamma}_M(0)[f] = \dot{\gamma}_S(0)[f \circ \iota]$. Observe: $f \circ \iota$ is the restriction of f to S.

Conclusion: Tangent spaces of submanifolds are subspaces of the tangent spaces of the original manifold.

Defn: A map $F : M \to N$ is a <u>submersion</u> iff $\forall p \in M$, $F_{*,p}$ is onto.

Cor: If F is a submersion, then $\forall q \in N$, q is a regular value, so $F^{-1}(q)$ ("the fiber of f over q") if either empty or a codimension n submanifold of M.

Ex: Let $M = \mathbb{R}^2 \setminus S^1$, $N = \mathbb{R}$, $F : M \to N$ with $F(x, y) = x$. What are the fibers?

- For $q \in (-\infty, 1) \cup (1, \infty)$, $F^{-1}(q) = \mathbb{R}$.
- For $q \in (-1, 1)$, $F^{-1}(q) = (-\infty, -\sqrt{1-q^2}) \cup (-\sqrt{1-q^2}, \sqrt{1-q^2}) \cup (\sqrt{1-q^2}, \infty)$.
- For $q \in \{-1, 1\}$, $F^{-1}(q) = \mathbb{R} \setminus \{0\}$.

Note that in this example, some of the fibers are different topologically!

Ex: Let $M = S^3$, $F : S^3 \to \mathbb{RP}^1 \cong S^2$ (the Riemann Sphere). Then the fibers are all circles, and the map from S^3 to S^2 is called the Hopf fibration.

Defn: A C^∞ map $F : M \to N$ is a <u>fibration</u> with fiber Φ, where Φ is a manifold, iff there is an open covering $\{U_\alpha\}$ of N (called the <u>base</u>) and diffeomorphic maps $\chi_\alpha : F^{-1}(U_\alpha) \to U_\alpha \times \Phi$ (called <u>trivializations</u>) such that the diagram

$$\begin{tikzcd} F^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "F|_{F^{-1}(U_\alpha)}"'] & & U_\alpha \times \Phi \arrow[dl, "\pi \text{ projection}"] \\ & U_\alpha & \end{tikzcd}$$

commutes (i.e. all paths are the same). We say that F is a <u>fiber bundle</u> with <u>fiber</u> Φ.

Let's unpack what this means. Commutativity of the diagram means $\forall p \in F^{-1}(U_\alpha)$, $\chi_\alpha(p) = (F(p), \star)$ where $\star \in \Phi$. So $\forall q \in U_\alpha$, χ_α restricts to the fiber $F^{-1}(q)$, where $\chi_\alpha(p) \mapsto \star$.

Ex: The tangent bundle TM is a fiber bundle, with fiber $\mathbb{R}^m$ (with $m = \dim M$).

$$\begin{tikzcd} TM \arrow[d] \\ M \end{tikzcd}$$

Note: This has additional structure: $\Phi \cong \mathbb{R}^m$ is a vector space, the fibers are all vector spaces, and there exist trivializations that are linear on the fibers.