Math 591 Lecture 4

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Defn: A space X is locally Euclidean iff every point in X has a neighborhood homeomorphic to $\mathbb{R}^n$, for some fixed n.

Defn: A topological manifold is a space that is locally Euclidean, Hausdorff, and second countable.

Thm: If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic nonempty open sets, then $m = n$. In other words, "dimension is topological".

The idea of this proof is to show that any open set in $\mathbb{R}$ can be covered by families of open sets with overlaps of at most 2 sets, any open set in $\mathbb{R}^2$ can be covered by families of open sets with overlaps of at most 3 sets, and so on.

Observe that in the definition of locally Euclidean, it's equivalent to ask that $\forall p \in X$, p has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Defn: Let M be a topological manifold. If $U \subseteq M$ is open, and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$, then the pair (U, ϕ) is a chart of M.

Defn: Let (U, ϕ) and (V, ψ) be charts, with $U \cap V \neq \emptyset$. The transition function (from ϕ to ψ) is a map

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap V)} : \phi(U \cap V) \to \psi(U \cap V)$$

Note: $\phi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^n$, because ϕ and ψ are homeomorphisms.
Note: Transition functions are automatically homeomorphisms.

Defn: Two charts of a topological manifold are C^∞-compatible (or just compatible) iff their transition functions are C^∞. That is,

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap V)} \qquad \text{and} \qquad \phi \circ \psi^{-1}\big|_{\psi(U\cap V)}$$

are both C^∞ diffeomorphisms.

Defn: An atlas of a topological manifold M is a collection $\mathscr{A} = \{(U_i, \phi_i)\}_{i\in I}$ of charts s.t. $M = \bigcup_{i\in I} U_i$.

Preliminary "definition": An atlas of M s.t. $\forall i, j \in I$, the transition function $\phi_i \circ \phi_j^{-1}$ is C^∞ determines a differentiable structure on M. Note that the condition is vacuous if $U_i \cap U_j = \emptyset$.

Ex: Some topological manifolds and atlases satisfying the preliminary definition:

- A trivial example: $M \subseteq \mathbb{R}^n$ any open set, $\mathscr{A} = \{M \hookrightarrow \mathbb{R}^n \text{ (inclusion)}\}$.
- Let $A \subseteq \mathbb{R}^n$ be an open set, and $G : A \to \mathbb{R}^k$ a C^∞ map. Let M be the graph of G, i.e., $M = \{(x, G(x)) \in \mathbb{R}^{n+k} \mid x \in A\} \subseteq \mathbb{R}^{n+k}$ with the subspace topology. Then let $\mathscr{A} = \{\pi : M \to \mathbb{R}^n \mid \pi : (x, G(x)) \mapsto x\}$.
- Cases of $M \subseteq \mathbb{R}^N$ which are locally graphs. (Note: $\mathbb{R}^N$ is known as the "ambient space".)
  - S^1. Let

$$U_1 = \left\{(x, \sqrt{1-x^2}); x \in (-1,1)\right\}$$
$$U_2 = \left\{(y, \sqrt{1-y^2}); y \in (-1,1)\right\}$$
$$U_3 = \left\{(x, -\sqrt{1-x^2}); x \in (-1,1)\right\}$$
$$U_4 = \left\{(y, -\sqrt{1-y^2}); y \in (-1,1)\right\}$$
$$\mathscr{A} = \{(U_1, (x,y) \mapsto x), (U_2, (x,y) \mapsto y), (U_3, (x,y) \mapsto x), (U_4, (x,y) \mapsto y)\}$$

Let's explicitly compute a transition map. $\phi_1^{-1}(x) = (x, \sqrt{1-x^2})$, so $\phi_2 \circ \phi_1^{-1}(x) = \sqrt{1-x^2}$. Note: this is C^∞ on $(0,1)$.

– S^1 with a new atlas. Let $p = (u, v)$. Let $U_+ = \{S^1 \setminus \{(0,1)\}\}$ and $U_- = \{S^1 \setminus \{(1,0)\}\}$.
  Then let $\phi_+(p) = x = \frac{u}{1-v}$ and $\phi_-(p) = y = \frac{u}{1+v}$. Another atlas: $\mathscr{B} = \{(U_1, \phi_1), (U_2, \phi_2)\}$. We claim that ϕ_1 and ϕ_2 are C^∞-compatible.
  In fact, it turns out that $\mathscr{A} \cup \mathscr{B}$ consists of compatible charts. So $\mathscr{A}$ and $\mathscr{B}$ define the same differentiable structure on S^1.