Math 591 Lecture 35

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Final remarks on orientation:

Recall: If M is oriented, $\exists\{(U_\alpha, \varphi_\alpha)\}$, a positive atlas of M. This means all transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ have the determinant of their Jacobian positive at every point, and the coordinate frames are positive.

Conversely, if M is an atlas satisfying he above property, then one can define an orientation of M by requiring the coordinate frames are positive.

In general, to show a manifold is orientable, exhibit such an atlas.

Ex: Check: The atlas of $\mathbb{RP}^n$ used in homework has this condition, so it is orientable.

Observe: If $S \subset M$ is a codim-1 submanifold, and M is oriented, and there exists a continuous field of normal vectors on S,

$$S \ni p \mapsto \vec{n}_p \in T_pM \quad \text{s.t.} \quad T_pM = T_pS \oplus \mathbb{R}\vec{n}_p$$

then S is orientable, and the convention for its orientation is: a basis $\{b_1, \dots, bn-1\}$ of T_pS is positive iff $\{\vec{n}_p, b_1, \dots, b_{n-1}\}$ is positive w.r.t. M.
TL;DR, put the normal vector first.

Ex: We can embed the Klein bottle in a dim-3 manifold M s.t. there exists a continuous $\vec{n}$, but M is non-orientable.

Partitions of Unity

This is a very technical, but very useful tool. We begin with point-set topology.

Defn: An indexed covering (not necessarily open) $\{S_\alpha\}_{\alpha\in A}$ of (a manifold) M (doesn't have to be a manifold) (with $S_\alpha \subset M$) is said to be locally finite iff every $p \in M$ has a neighborhood U s.t. $\{\alpha \in A \mid S_\alpha \cap U \neq \emptyset\}$ is finite. That is, every p is in only finitely-many S_α.

Thm: (Thm 1.15 in Lee) Any topological manifold is paracompact: every open cover $\{U_\alpha\}_{\alpha\in A}$ has a countable, locally-finite refinement $\{V_i\}_{i\in\mathbb{N}}$. That is, $\forall i \in \mathbb{N}$, V_i is open, and $\exists \alpha \in A$ s.t. $V_i \subset U_\alpha$ and M is covered by $\{V_i\}_{i\in\mathbb{N}}$.

Proof: This proof is long and complex, but it only uses point-set topology. This is the first time we're using the fact that M is second-countable!

Observe: If $\mathscr{B}$ is any basis of M, the V_i can be chosen to be in $\mathscr{B}$.

Defn: Let M be a smooth manifold. A partition of unity on M is an indexed family $\{\chi_\alpha\}_{\alpha\in A}$ of C^∞ functions on M s.t.

(1) $\{\operatorname{supp}(\chi_\alpha)\}_{\alpha\in A}$ is a locally finite cover of M.
(2) $\forall p \in M$, $\sum_{\alpha\in A} \chi_\alpha(p) = 1 \in \mathbb{R}$. (Note that this is a finite sum by (1).)

Thm: Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then $\exists\{\chi_\alpha\}_{\alpha\in A}$, a partition of unity, that is subordinate to $\{U_\alpha\}_{\alpha\in A}$, i.e., $\forall \alpha \in A$, $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$.

Proof: We'll use paracompactness. (It may be easier to start by just thinking of a compact manifold.)
Let $\mathscr{B}$ be the set of normal coordinate balls in M; we define $B \subset M$ to be a normal coordinate ball in M iff there's a chart (U, ϕ) s.t. $\overline{B} \subset U$ and $\phi(B) = B_r(0) \subset \mathbb{R}^n$, the ball of radius r centered at 0 in $\mathbb{R}^n$, and also $\exists r' > r$ s.t. $\overline{B_r(0)} \subset B_{r'}(0) \subset \phi(U)$.

We claim that $\mathscr{B}$ is a basis of the topology of M. Let $\{U_\alpha\}_{\alpha\in A}$ be any open cover. Use the theorem on paracompactness: $\exists \{B_i\}_{i\in\mathbb{N}}$, a locally-finite refinement, and $\forall i \in \mathbb{N}$, B_i is a normal coordinate ball. $\forall i \in \mathbb{N}$, let

$$\phi_i(B_i) = B_{r_i}(0) \subset \overline{B_{r_i}(0)} \subset B_{r_i'}(0)$$

and H_i be a function on $\mathrm{Im}(\phi_i)$ such that $H_i : \mathrm{Im}(\phi_i) \to \mathbb{R}$ is smooth, with

- $H_i > 0$ on $B_{r_i}(0)$
- $H_i = 0$ on $B_{r_i}(0)^\complement$

Thus, $\operatorname{supp} H_i = \overline{B_{r_i}(0)}$.

Define $\psi_i \in C^\infty(M)$ by $\psi_i = H_i \circ \phi_i$ on $\operatorname{dom}\phi_i$, and 0 everywhere else. Then $\operatorname{supp}\psi_i = \overline{B_i} \subset M$. We claim that $\left\{\overline{B_i}\right\}_{i\in\mathbb{N}}$. Observe that $\forall p$, $\sum_{i\in\mathbb{N}} \psi_i(p) > 0$, because $\{B_i\}_{i\in\mathbb{N}}$ forms a cover of M, and $\psi_i|_{B_i} > 0$.

Now, define

$$f_i = \frac{1}{\sum\limits_{j\in\mathbb{N}} \psi_j} \psi_i$$

so that $\{\operatorname{supp} f_i\}_{i\in\mathbb{N}} = \left\{\overline{B_i}\right\}$ is locally finite, and $\sum_{i\in\mathbb{N}} f_i = 1$, $\forall p \in M$. Then, we just have to fix it so that he indexing sets are the same as $\{U_\alpha\}_{\alpha\in A}$ by $\forall i \in \mathbb{N}$, pick $\alpha(i) \in A$ such that $B_i \subset U_{\alpha(i)}$ and $\forall \alpha \in A$, let

$$\chi_\alpha = \sum_{\substack{i \text{ s.t.} \\ \alpha(i)=\alpha}} f_i$$

(Note that $\chi_\alpha = 0$ if the sum is empty.)

We claim that $\left\{\operatorname{supp}\chi_{\alpha(i)}\right\}_{i\in\mathbb{N}}$ is still locally finite. This follows from $\{\operatorname{supp} f_i\}_{i\in\mathbb{N}}$ being locally finite. $\square$

There are many applications of partitions of unity!
Ex: Existence of C^∞ sections of any vector bundle.

Say $E \to M$ is a vector bundle of rank r. Then there exist $\{(U_\alpha, f_\alpha)\}$ local trivializations:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "f_\alpha"] \arrow[dr] & & U_\alpha \times \mathbb{R}^n \arrow[dl] \\
& U_\alpha \arrow[ul, bend left, "s_\alpha"] \arrow[ur, bend right, "\text{pick any section } s"'] &
\end{tikzcd}$$

Let $\{\chi_\alpha\}$ be a partition of unity on M subordinate to $\{U_\alpha\}$. Then let $s = \sum_{\alpha\in A} \chi_\alpha \cdot s_\alpha$ (we interpret $\chi_\alpha \cdot s_\alpha$ as a C^∞ section on M).

The main application of partitions of unity is integrating forms.

Defn: Let M be an oriented n-dimensional manifold. Let $\mu \in \Omega_0^n(M)$ be a top degree form with compact support. Let $\{\phi_\alpha\}$ be a positive atlas, and $\{\chi_\alpha\}$ a subordinate partition of unity (i.e. $\operatorname{supp}\chi_\alpha \subseteq \operatorname{supp}\phi_\alpha$, $\forall\alpha$). Then we define

$$\int\limits_M \mu = \sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$$

We have to check that the right hand side is independent of choice of coordinates. We'll do this next time...