Math 591 Lecture 31

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11/13/20

To do today:
- Review differential forms
- Pullbacks
- Exterior derivatives

Let M be a manifold. Last time, we defined a smooth k-form α on M as an assignment $M \ni p \mapsto \alpha_p \in \bigwedge^k T_p^*M$.

In local coordinates $(x^1, \ldots, x^n)$, $\alpha = \sum_I{}' a_I dx^I$, $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$.

Smoothness: $\forall I$, for any coordinate chart, the a_I are C^∞.

Defn: $\Omega^k(M) \stackrel{\text{def}}{=} \{\text{all } C^\infty\ k\text{-forms}\}$.

Ex: On $\mathbb{R}^n$: volume form $dx^1 \wedge \cdots \wedge dx^n$.
On $\mathbb{R}^3$: $\Omega^1 = \{\alpha = f dx + g dy + h dz\}$.
$\Omega^2 = \{\alpha = f dx \wedge dy + g dy \wedge dz + h dx \wedge dz\}$.

Defn: Take $M \subseteq \mathbb{R}^3$ a surface such that ther is a smooth unit normal vector field $\vec{n}$ on M. Define a 2-form σ on M by $\forall p \in M$, $v, w \in T_pM \subset \mathbb{R}^3$, $\sigma_p(v, w) = \det(v, w, \vec{n}_p)$. So $\sigma_p(v, w)$ is the area of the parallelogram spanned by v, w, and $\vec{n}_p$. σ is called the area form of M, for the given $\vec{n}$ (orientation).

Pull-backs of Differential Forms

Defn: Let $F : N \to M$ be smooth, and $\alpha \in \Omega^k(M)$, We define the pullback of α by F, $(F^*\alpha) \in \Omega^k(N)$, by $\forall p \in N$, $v_1, \ldots, v_k \in T_pN$,
$$(F^*\alpha)_p(v_1, \ldots, v_k) = \alpha_{F(p)}(F_{*,p}(v_1), \ldots, F_{*,p}(v_k))$$

Lemma:
1. $F^*\alpha$ is C^∞.
2. $(F \circ G)^*\alpha = G^*(F^*\alpha)$ (the chain rule).
3. $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$.

Observe: $\Omega^0(M) = C^\infty(M)$. If f is a 0-form on M, then $F^*f = f \circ F$.

Pullbacks in Coordinates

Given
$$\mathbb{R}^n \xleftarrow{(y^1,\ldots,y^n)} V \subset N \xrightarrow{F} M \supset U \xrightarrow{(x^1,\ldots,x^m)} \mathbb{R}^m$$
and $\alpha = \sum_I{}' a_I dx^I$ with the $a_I \in C^\infty(U)$, then
$$F^*(\alpha) = \sum_I{}' (a_I \circ F) F^*(dx^I) = \sum_I{}' (a_I \circ F) F^*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_I{}' (a_I \circ F)(F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}))$$

Lemma: Let $F^i = x^i \circ F : V \to \mathbb{R}$ for each i. Then $F^*(dx^i) = dF^i$, the differential of F^i.

Proof: First, introduce some shorthand notation: $\partial_{y^j} = \frac{\partial}{\partial y^j}$. Now,

$$F^*(dx^i)(\partial_{y^j}) = dx^i(F_*\partial_{y^j}) \overset{(1)}{=} dx^i \sum_{\ell=1}^{m} \frac{\partial F^\ell}{\partial y^j} \partial_{x^\ell} = F^*(dx^i)(\partial_{y^j}) = \frac{\partial F^i}{\partial y^j}$$

with (1) because $F' = \left(\frac{\partial F^\ell}{\partial y^j}\right)_{(j,\ell)}$ is the matrix of F_* in $(\partial_{y^j}), (\partial_{x^i})$. Thus,

$$F^*(dx^i) = \sum_{j=1}^{n} \frac{\partial F^i}{\partial y^j} dy^j = dF^i$$

□

Now, back to the main computation:

$$F^*(\alpha) = \cdots = {\sum_I}' (a_I \circ F)(dF^{i_1} \wedge \cdots \wedge dF^{i_k})$$

Observe: The right hand side is a smooth form. There's a special case for $k = m = n$:

Prop: $F^*(dx^1 \wedge \cdots \wedge dx^n) = \det \underbrace{\left(\frac{\partial F^i}{\partial y^j}\right)_{(j,i)}}_{=F'=J(F)} (dy^1 \wedge \cdots \wedge dy^n)$

Proof: Well, the left hand side is

$$dF^1 \wedge \cdots \wedge dF^n = \underbrace{\left(\sum_{j_1=1}^{n} \frac{\partial F^1}{\partial y^{j_1}} dy^{j_1}\right)}_{dF^1} \wedge \cdots \wedge \underbrace{\left(\sum_{j_n=1}^{n} \frac{\partial F^n}{\partial y^{j_n}} dy^{j_n}\right)}_{dF^n} = \sum_{j_1,\ldots,j_n=1}^{n} \left(\prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{j_i}}\right) (dy^{j_1} \wedge \cdots \wedge dy^{j_n})$$

Observe that the terms of the sum with $j_a = j_b$ with $a \neq b$ vanish, so the sum is really over all orderings of $\{1, \ldots, n\}$.

$$dF^1 \wedge \cdots \wedge dF^n = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{\sigma(i)}}\right) \underbrace{\left(dy^{\sigma(1)}, \ldots, dy^{\sigma(n)}\right)}_{=\underbrace{(-1)^\sigma}_{\mathrm{sgn}(\sigma)} dy^1 \wedge \cdots \wedge dy^n} = \underbrace{\left(\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{\sigma(i)}}\right)}_{\det F'} (dy^1 \wedge \cdots \wedge dy^n)$$

□

Cor: If $\alpha = f dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, $f \in C_0^\infty(U)$ (i.e. f has compact support), and we define

$$\int \alpha = \underbrace{\int f dx^1 \wedge \cdots \wedge dx^n}_{\text{Riemann Integral}}$$

Then $\int F^*\alpha = \int \alpha$ by the change of variables formula, provided that $\det(F') > 0$.

Proof: $F^*\alpha = (f \circ F) \det(F') dy^1 \wedge \cdots \wedge dy^n$. □

The Exterior Differential

Thm: Let M be a manifold. Then there exists a unique operator d

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k$$

s.t.

(1) d is $\mathbb{R}$-linear.
(2) $d : \Omega^0 \to \Omega^1$ is the usual differential of a function.
(3) $d^2 = 0$.
(4) The anti-derivation property: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. (This is also known as the super-symmetric version of Leibniz' rule.)
(5) If $F : M \to N$ is smooth, then $\forall \alpha \in \Omega^k N$, $d(F^* \alpha) = F^*(d\alpha)$. (This property is called "Naturality".)

Proof: Next time...