Math 591 Lecture 20

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Smooth Sections of Vector Bundles

Start with a rank ρ vector bundle, with section s, i.e., $\pi \circ s = I_M$.

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

Let χ be a local trivialization

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr] & & U \times \mathbb{R}^\rho \arrow[dl] \\ & U & \end{tikzcd}$$

with χ a diffeomorphism, and linear on each fiber. Then $s|_U : U \to \pi^{-1}(U)$ satisfies

$$\begin{aligned} \chi \circ (s|_U) : U &\to U \times \mathbb{R}^\rho \\ p &\mapsto (p, F(p)) \end{aligned}$$

where $F : U \to \mathbb{R}^\rho$. We write $F = (F^1, \ldots, F^\rho)$ with each $F^i : U \to \mathbb{R}$.

Lemma: (From last time) $s|_U$ is smooth iff $\forall i$, F^i is smooth.

Proof: $\Rightarrow$ is trivial.

$\Leftarrow$: It's a fact from analysis that F is C^∞ iff $\forall i$, F^i is C^∞. So $s|_U(p) = \chi^{-1}(p, F(p))$, which is smooth.

$\square$

Observe: The trivializationabove corresponds to a "moving frame" on U.

Defn: A moving frame on U i a collection of ρ smooth sections on U, $\{e_1, \ldots, e_\rho\}$, s.t. $\forall p \in U$, $\{e_1(p), \ldots, e_\rho(p)\}$ is a basis of the fiber $\pi^{-1}(p)$.

Ex: If $(U, \phi = (x^1, \ldots, x^n))$ is a coordinate chart, let $e_i(p) = \frac{\partial}{\partial x^i}\Big|_p \in T_pM$. This defines a moving frame of TM on U.

Given a trivialization χ over U as above, how do we get a moving frame? Well, $\forall i \in \{1, \ldots, \rho\}, p \in U$, let $e_i(p) \overset{\text{def}}{=} \chi^{-1}(p, (0, \ldots, 1, \ldots 0))$ (with the 1 in the ith entry).

Observe: If $s|_U$ corresponds to $F = (F^1, \ldots, F^\rho) : U \to \mathbb{R}^\rho$, then $s|_U = \sum_{i=1}^\rho F^i e_i$, where the F^i are scalar-valued functions and the e_i are sections. So $\forall p \in U$, $s(p) \in \pi^{-1}(p)$, and $s(p) = \sum_{i=1}^\rho F^i(p) e_i(p)$ (using the vector space structure of $\pi^{-1}(p)$).

Conversely, we can also define a trivialization from a moving frame. (This is left as an exercise.)

Observe: If $C^\infty(M, \mathcal{E})$ is the space of C^∞ sections of $\mathcal{E} \to M$ vector bundles, then $C^\infty(M, \mathcal{E})$ is a module over $C^\infty(M)$. We can multiply a section s by a function $f \in C^\infty(M)$ fiber-wise, with $(fs)(p) = f(p)s(p)$.

Vector Fields

Let $\mathcal{E} = TM$.

Defn: $\mathfrak{X}$ is the set of all smooth vector fields on M.

$\forall X \in \mathfrak{X}$, with a coordinate system on U, $\exists a_i \in C^\infty(U)$ s.t. $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$. X is C^∞ iff $\forall i, a_i \in C^\infty$.

Prop: Any $X \in \mathfrak{X}(M)$ defines an operator

$$\begin{aligned} C^\infty(M) &\to C^\infty(M) \\ f &\mapsto X(f) \end{aligned}$$

which

a) is $\mathbb{R}$-linear.
b) satisfies Leibniz' rule: $\forall f, g \in C^\infty(M)$, $X(fg) = fX(g) + gX(f)$.

(An aside: As a section, the value of X at $p \in M$ is denoted $X_p \in T_pM$.)

Proof: $X(f)(p) = X_p([f])$, where $[f]$ is the germ of f at p. Thus, X is a derivation on $C^\infty(M)$, because X_p is a derivation on $C_p^\infty(M)$ germs. $\square$

Defn: Such an operator is called a derivation of $C^\infty(M)$.

Prop: (1) The operator defined by $X \in \mathfrak{X}(M)$ is local, i.e., $\forall f \in C^\infty(M), U \subseteq M$ open such that $f|_U \equiv 0$, then $X(f)|_U \equiv 0$.

Observe: This "locality" characterizes differential operators.

Observe: In local coordinates, if $X = \sum_i a_i \frac{\partial}{\partial x^i}$, then $X(f)(p) = \sum_i a_i \frac{\partial f}{\partial x^i}(p)$.

Thm: (2) Any operator $D : C^\infty(M) \to C^\infty(M)$ that is a derivation is given by a vector field.

Thm: (3) The commutator of two derivations is a derivation.

Together, we have: If $X, Y \in \mathfrak{X}(M)$, then there is a vector field denoted $[X, Y] \in \mathfrak{X}(M)$ (said "X bracket Y" or "X commutator Y") such that $\forall f \in C^\infty(M)$, $[X, Y](f) = X(Y(f)) - Y(X(f))$.

Proof of (3): Define $[X, Y]$ as the operator commutator above. Clearly this is linear. Verify Leibniz' rule:

$$[X,Y](fg) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) = \cdots = f[X,Y](g) + g[X,Y](f)$$

$\square$

In local coordinates, say $X = \sum_i a_i \frac{\partial}{\partial x^i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x^j}$. Then

$$[X,Y] = \sum_{ij} \left[a_i \frac{\partial}{\partial x^i}, b_j \frac{\partial}{\partial x^j} \right]$$

And

$$\begin{aligned} \left[a_i \frac{\partial}{\partial x^i}, b_j \frac{\partial}{\partial x^j} \right] &= a_i \frac{\partial}{\partial x^i}(b_j \frac{\partial f}{\partial x^j}) - b_j \frac{\partial}{\partial x^j}(a_i \frac{\partial f}{\partial x^i}) \\ &= \cancel{a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}} + a_i \frac{\partial b_j}{\partial x^i} \frac{\partial f}{\partial x^j} - \left(\cancel{b_j a_i \frac{\partial^2 f}{\partial x^j \partial x^i}} + b_j \frac{\partial a_i}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\ &= a_i \frac{\partial b_j}{\partial x^i} \frac{\partial f}{\partial x^j} - b_j \frac{\partial a_i}{\partial x^j} \frac{\partial f}{\partial x^i} \end{aligned}$$

This gives the commutator.