

Math 591 Lecture 32

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The Exterior Differential

Thm: Let M be a manifold. Then there exists a unique operator d

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k$$

s.t.

- (1) d is \mathbb{R} -linear.
- (2) $d : \Omega^0 \rightarrow \Omega^1$ is the usual differential of a function.
- (3) $d^2 = 0$.
- (4) The anti-derivation property: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. (This is also known as the super-symmetric version of Leibniz' rule.)
- (5) If $F : M \rightarrow N$ is smooth, then $\forall \alpha \in \Omega^k N$, $d(F^* \alpha) = F^*(d\alpha)$. (This property is called “Naturality”.)

Note: In theory, the d is a different operator for each Ω^i . But it would be inconvenient to call them different names.

Proof: We'll begin by proving this for $U \subseteq \mathbb{R}^n$, and then use coordinates. This works because the anti-derivation property implies locality.

For \mathbb{R}^n : If $\alpha = \sum_I a_I dx^I \in \Omega^k(\mathbb{R}^n)$, then the properties imply that we must have $d\alpha = \sum_I \sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$.

Use this as our definition of d on \mathbb{R}^n . Now, we must check the 5 properties. 1 and 4 follow directly from its definition. By linearity, we can check this solely on “monomials”.

Anti-derivation: Let $\alpha = a dx^I$, $\beta = b dx^J$, $a, b \in C^\infty$. Then

$$\begin{aligned} d(\alpha \wedge \beta) &= d(a dx^I \wedge b dx^J) \\ &= d(ab dx^I \wedge dx^J) \\ &= d(ab dx^{IJ}) \\ &= \sum_{j=1}^n \frac{\partial(ab)}{\partial x^j} dx^j \wedge dx^{IJ} \\ &= \sum_{j=1}^n \left[b \frac{\partial a}{\partial x^j} + a \frac{\partial b}{\partial x^j} \right] dx^j \wedge dx^{IJ} \\ &= \sum_{j=1}^n \frac{\partial a}{\partial x^j} dx^j \wedge dx^I \wedge (b dx^J) + a \sum_{j=1}^n \left(\frac{\partial b}{\partial x^j} dx^j \wedge dx^I \right) \wedge dx^J \\ &= \sum_{j=1}^n \frac{\partial a}{\partial x^j} dx^j \wedge dx^I \wedge (b dx^J) + (-1)^k a dx^I \sum_{j=1}^n \frac{\partial b}{\partial x^j} dx^j \wedge dx^I \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \end{aligned}$$

$d^2 = 0$:

$$\begin{aligned}
d [d(a dx^I)] &= d \left[\sum_{j=1}^n \frac{\partial a}{\partial x^j} dx^j \wedge dx^I \right] \\
&\stackrel{(1)}{=} \sum_{j=1}^n d \left(\frac{\partial a}{\partial x^j} \right) \wedge dx^j \wedge dx^I \\
&= \underbrace{\left(\sum_{j=1}^n \sum_{\ell=1}^n \frac{\partial^2 a}{\partial x^\ell \partial x^j} dx^\ell \wedge dx^j \right)}_{\text{Observe: Terms with } j = \ell \text{ vanish.}} \wedge dx^I \\
&= \left(\sum_{1 \leq j < \ell \leq n} \frac{\partial^2 a}{\partial x^\ell \partial x^j} dx^\ell \wedge dx^j + \frac{\partial^2 a}{\partial x^j \partial x^\ell} dx^j \wedge dx^\ell \right) \wedge dx^I \\
&= \left(\sum_{1 \leq j < \ell \leq n} \left(\frac{\partial^2 a}{\partial x^\ell \partial x^j} - \frac{\partial^2 a}{\partial x^j \partial x^\ell} \right) dx^\ell \wedge dx^j \right) \wedge dx^I \\
&\stackrel{(2)}{=} 0
\end{aligned}$$

with (1) because $dx^j \wedge dx^I$ is constant, so $d(dx^j \wedge dx^I) = 0$, and (2) by Clairaut's Theorem, which tells us that $\frac{\partial^2 a}{\partial x^\ell \partial x^j} = \frac{\partial^2 a}{\partial x^j \partial x^\ell}$, so their difference is 0.

Naturality: Say $F : V \rightarrow U$ is C^∞ , $V \subseteq \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$. Let $\alpha \in \Omega^k(U)$ of the form $\alpha = a dx^I$. We need to prove that $d(F^*\alpha) = F^*(d\alpha)$. Say $F = (F^1, \dots, F^n)$ with $F^j : V \rightarrow \mathbb{R}$. We proved that

$$F^*(dx^{i_1}) \wedge \dots \wedge F^*(dx^{i_k}) = F^*(dx^I) = dF^{i_1} \wedge \dots \wedge dF^{i_k} \stackrel{\text{def}}{=} dF^I$$

So $F^*\alpha = (\alpha \circ F)F^*(dx^I) = (\alpha \circ F)dF^I$, and thus,

$$d(F^*\alpha) = d(\alpha \circ F) \wedge dF^I + (\alpha \circ F) \underbrace{d(dF^I)}_{=0} = d(\alpha \circ F) \wedge dF^I$$

$$F^*(d\alpha) = F^* \left[\sum_{j=1}^n \frac{\partial a}{\partial x^j} dx^j \wedge dx^I \right] = \sum_{j=1}^n F^* \left(\frac{\partial a}{\partial x^j} \right) dx^j \wedge dx^I = \sum_{j=1}^n \left(\frac{\partial a}{\partial x^j} \circ F \right) \underbrace{(F^*(dx^j))}_{dF^j} \wedge dF^I = d(\alpha \circ F) \wedge dF^I$$

So we conclude that $d(F^*\alpha) = F^*(d\alpha)$.

Now, we use coordinates to do it on a manifold.

Defn: Given $\alpha \in \Omega^k(M)$, $\phi : U \rightarrow \mathbb{R}^n$ a coordinate system, define $d\alpha \in \Omega^{k+1}(U)$ by $\phi^*[d(\phi^{-1})^*\alpha]$, where d is the Euclidean version, and $(\phi^{-1})^*\alpha \in \Omega^k(\phi(U))$.

Now, we need to prove that $d\alpha$ on U is independent of choice of coordinates. Suppose $\psi : V \rightarrow \mathbb{R}^n$ is another coordinate chart (WOLOG $U \cap V \neq \emptyset$). Call U the intersection of their domains. Write $\psi = F \circ \phi$

$$\begin{array}{ccc}
& U & \\
\phi \swarrow & & \searrow \psi \\
\phi(U) & \xrightarrow{F} & \psi(U)
\end{array}$$

Then

$$\begin{aligned}
\psi^*[d(\psi^{-1})^*\alpha] &= \phi^* \circ F^*[d(\phi^{-1} \circ F^{-1})^*\alpha] \\
&= \phi^* \circ F^*[d((F^{-1})^* \circ (\phi^{-1})^*)\alpha] \\
&= \phi^* \circ F^* \circ (F^{-1})^*[d(\phi^{-1})^*\alpha] \\
&= \phi^*[d(\phi^{-1})^*\alpha]
\end{aligned}$$

As for showing uniqueness on manifolds, use that anti-derivation implies locality, as with vector fields. \square

Ex: \mathbb{R}^3

$$\begin{aligned} d(f dx + g dy + h dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= (\partial_x f dx + \partial_y f dy + \partial_z f dz) + \cdots \\ &= \partial_y f dy \wedge dx + \partial_z f dz \wedge dx + \cdots \\ &= (\partial_x g - \partial_y f) dx \wedge dy + (\partial_x h - \partial_z f) dx \wedge dz + \cdots \end{aligned}$$

We get the components of the curl of the vector field $\langle f, g, h \rangle$.

Defn: $\alpha \in \Omega^k(M)$ is closed if $d\alpha = 0$, and exact if $\exists \beta \in \Omega^{k-1}(M)$ such that $\alpha = d\beta$.

Note: $d^2 = 0$ means exact implies closed.

Defn: The k th DeRahm Cohomology $H^k(M) = \ker(d : \Omega^k \rightarrow \Omega^{k+1}) / \text{im}(d : \Omega^{k-1} \rightarrow \Omega^k)$ (with the vector space quotient).

Ex: If $M = S^1$, $d\theta$ dual to $\frac{\partial}{\partial \theta}$ is closed. (Any top-degree form is closed.)
 Claim: $d\theta$ is not exact, and in fact, $H^1(S^1) = \mathbb{R}$.