Math 591 Lecture 38

Thomas Cohn

12/7/20

Sard's Theorem (with an Application)

As a preliminary, we have to talk about sets of measure 0.

Defn: Informally speaking, $S \subset \mathbb{R}^n$ has measure zero iff $\forall \varepsilon > 0$, S can be covered by countably many n-cubes of total volume less than ε.

Prop: If S has measure 0, and $F : S \to \mathbb{R}^m$ is smooth, then $F(S)$ has measure 0.

Proof: Based on the fact that C^∞ functions are Lipschitz on compact sets. I.e., $||F(p) - F(q)|| < C\,||p - q||$ for some constant $C \in \mathbb{R}_{>0}$.

Defn: A subset $S \subset M$ has measure zero iff $\forall (U, \phi)$, a coordinate chart, the set $\phi(U \cap S) \subseteq \mathbb{R}^n$ has measure zero.

Prop: Equivalently, S can be covered by countably many charts $\{(U_i, \phi_i)\}$ s.t. $\forall i$, $\phi_i(U_i \cap S)$ has measure zero.

Thm: (Sard's Theorem) If $F : M \to N$ is smooth, the set of crtical values of F has measure 0.

Reminder: $q \in N$ is a regular value iff $\forall q \in F^{-1}(p)$, $F_{*,p}$ is surjective.
$q \in N$ is a critical value iff q is not a regular value.

Note: If $q \notin \mathrm{Im}(F)$, then q is a regular value.

Cor: The set of regular values of F is dense in N. (It's the complement of a set of measure zero.) In particular, if $F : M \to N$ is smooth, and $\dim M < \dim N$, then the only regular values are $N \setminus \mathrm{Im}(F)$, so we conclude that $N \setminus \mathrm{Im}(F)$ is dense, and $\mathrm{Im}(F)$ has measure zero. In particular, submanifolds of nonzero codimension have measure zero.

(Recall: A set S is dense if $\forall U$ open, $U \cap S \neq \emptyset$.)

The Embedding Theorem

Thm: (Whitney Embedding Theorem) Let M be an n-dimensional manifold. Then M can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$. (This is the weak version.)

Thm: M can be embedded in $\mathbb{R}^{2n}$. (This is the strong version.)

Proof of the weak version: We start by embedding M into some $\mathbb{R}^N$. Then we successively project $M \subset \mathbb{R}^N$ onto "good hyperplanes". The first step is to cover M with an atlas $\{(U_i, \phi_i)\}_{i=1,\dots,k}$. Let $\{\chi_i\}_{i=1,\dots,k}$ be a subordinate partition of unity: $\forall i$, $\operatorname{supp} \chi_i \subset U_i$.

Now, define

$$\forall i,\ \psi_i : M \to \mathbb{R}^n \qquad p \mapsto \begin{cases} \chi_i(p)\phi_i(p) & p \in U_i \\ 0 & p \notin U_i \end{cases}$$

$$F : M \to \mathbb{R}^{kn+k \overset{\text{def}}{=} N} \qquad p \mapsto (\psi_1(p), \dots, \psi_k(p), \chi_1(p), \dots, \chi_k(p))$$

We claim that F is injective, and an immersion. Assume $p, q \in M$ s.t. $F(p) = F(q)$.
Then $\exists i$ s.t. $\chi_i(p) = \chi_i(q) \neq 0$. Thus, $p, q \in U_i$. So $F(p) = F(q) \Rightarrow \psi_i(p) = \psi_i(q) \Rightarrow \phi_i(p) = \phi_i(q) \Rightarrow p = q$.

Now, to show that F is an immersion. Assume $v \in T_pM$ s.t. $F_{*,p}(v) = 0$. Again, choose i s.t. $\chi_i(p) \neq 0$, so $p \in U_i$. Then

$$0 = (\psi_i)_{*,p}(v) = d\chi_i(v)\phi_i(p) + \chi_i(p)d\phi_p(v)$$

since $\psi_i = \chi_i \cdot \phi_i$. But also $d\chi_i(v) = 0$, so $d(\phi_i)_p(v) = 0$, so we must have $v = 0$ (as ϕ_i is a diffeomorphism). $\square$

The next step is lowering the dimension. Let $\mathbb{P} = \left\{\ell \subseteq \mathbb{R}^N \text{ subspaces of dimension } 1\right\}$. $\forall \ell \in \mathbb{P}$, let $\pi_\ell : \mathbb{R}^N \to \ell^\perp$, where $\ell^\perp$ is the orthogonal complement (and a hyperplane). We claim that $\exists \ell \in \mathbb{P}$ s.t. $\pi_\ell|_M : M \to \ell^\perp$ is an embedding, provided that $N > 2n + 1$.

Well, we need to find an ℓ s.t. $\pi_\ell|_M$ is injective and an immersion. Consider

$$\begin{aligned} G : M \times M \setminus \{(p,p) \mid p \in M\} &\to \mathbb{P} \\ \underbrace{(p,q)}_{p,q \in \mathbb{R}^N} &\mapsto \mathbb{R}\underbrace{(p-q)}_{\neq 0} \end{aligned}$$

G is smooth. If $\dim(M \times M) < \dim(P)$, then $\mathbb{P} \setminus \mathrm{Im}(G)$ is dense, so if we pick $\ell \in \mathbb{P} \setminus \mathrm{Im}(G)$, then $\pi_\ell|_M$ is injective. Because $dim(M \times M) = 2n$, and we need $2n < N - 1$, we require $N > 2n + 1$.

Now, to ensure $\pi_\ell|_M$ is an immersion, let

$$\begin{aligned} H : SM \overset{\text{def}}{=} \{(p,v) \in TM \mid ||v|| = 1\} &\to \mathbb{P} \\ (p,v) &\mapsto \mathbb{R}v \end{aligned}$$

We claim that if $\ell \in \mathbb{P} \setminus \mathrm{Im}(H)$, then $\pi_\ell|_M$ is an immersion. We perform a similar dimension count as before, and we get to $N > 2n$. $\square$