Math 591 Lecture 13

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9/30/20

Let $F : M \to N$ C^∞, with $p \in M$. Last time, we defined $F_{*,p} = df_p : T_pM \to T_{F(p)}N$. Our first question today is: How do properties of $F_{*,p}$ reflect properties of F?

Thm: If $F_{*,p}$ is bijective (i.e. $\dim M = \dim N$), then F is a local diffeomorphism at p, i.e., there exist open neighborhoods U of p and V of $F(p)$ such that $F(U) = V$ and $F|_U^V : U \to V$ has a smooth inverse.

Proof: Start with coordinate charts (U, ϕ) near p and (V, ψ) near $F(p)$, so that $U \subseteq F^{-1}(V)$.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "m=\dim M \ \ (x^1,\dots,x^m)=\phi"'] & V \arrow[d, "\psi=(y^1,\dots,y^n) \ \ n=\dim N"] \\ \phi(U) \arrow[r, "\tilde{F}=\psi\circ F\circ\phi^{-1}"] & \psi(V) \end{tikzcd}$$

The matrix of $F_{*,p}$ is $\left(\frac{\partial F^i}{\partial x^j}(p)\right)$, where $F^i = y^i \circ F$ for $i \in \{1, \dots, n\}$. This matrix is the Jacobian of $\tilde{F}$. By assumption (that $m = n$), this matrix is invertible. So by the inverse function theorem in Euclidean space, by shrinking $\phi(U)$ and $\psi(V)$ if necessary, $\tilde{F}$ has a smooth inverse. (This is equivalent to shrinking U and V if necessary.)
So $(F|_U^V)^{-1} = \phi^{-1} \circ \tilde{F}^{-1} \circ \psi$. $\square$

Cor: $F : M \to N$ is a local diffeomorphism iff $\forall p \in M$, $F_{*,p}$ is bijective.

Proof: $\Rightarrow$ $\forall p \in M$, there are neighborhoods U of p and V of $F(p)$ such that $F|_U^V$ is a diffeomorphism. So $F_{*,p}$ has an inverse, $((F|_U^V)^{-1})_{*,p}$ by the chain rule.

$\Leftarrow$ We already showed this.

$\square$

Observe: We now have the notion of a *smooth* covering map.

Defn: $F : M \to N$ is a <u>smooth covering map</u> iff $\forall q \in N$, there is a neighborhood V of q s.t. $F^{-1}(V) = \bigsqcup_{i\in I} U_i$ s.t. $\forall i \in I$, $V = F(U_i)$ and $F|_{U_i}^V$ is a diffeomorphism. Such a V is said to be <u>evenly covered</u>.

Ex: $S^n \to \mathbb{RP}^n$.
The quotient map $S^n \to S^n/S^0 \cong \mathbb{RP}^n$ is a smooth covering map.

A smooth covering map is always a local diffeomorphism, but the converse is false.

Ex: Let

$$\begin{aligned} f : (0, 4\pi) &\to S^1 \subseteq \mathbb{C} \\ t &\mapsto e^{it} \end{aligned}$$

This is a local diffeomorphism, but not a covering map: $F^{-1}(1) = \{2\pi\}$, but every neighborhood of 1 is not evenly covered.

Defn: A smooth function $F : M \to N$ is called a <u>diffeomorphism</u> if it has a smooth inverse.

Defn: Let $F : M \to N$ be smooth.

a) A point $p \in M$ is a <u>regular point</u> of F $\Leftrightarrow$ $F_{*,p}$ is onto.
b) F is a <u>submersion</u> $\Leftrightarrow$ $\forall p \in M$, $F_{*,p}$ is onto.

Thm: (Normal Form for Submersions) Let $F : M \to N$ be a submersion. Then $\forall p \in M$, there are coordinate charts (U, ϕ) around p and (V, ψ) around $F(p)$ such that $U \subseteq F^{-1}(p)$ and $\tilde{F} = \psi \circ F \circ \phi^{-1}$ satisfies $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n)$.

Observe: $F_{*,p} : T_pM \to T_{F(p)}N$ surjective implies that $m \geq n$. Define $r' = (r^1, \ldots, r^n)$ and $r'' = (r^{n+1}, \ldots, r^m)$, so $(r^1, \ldots, r^m) = (r', r'')$. Then $\tilde{F}(r', r'') = r'$.

Cor: A submersion is an open map.

Preliminary Observation: (This is a corollary of the inverse function theorem.) Suppose $p \in U \overset{\text{open}}{\subseteq} M$, and $F : U \to \mathbb{R}^m$ $(m = \dim M)$ such that $F_{*,p}$ is bijective. Then we claim that (after shrinking U if necessary) (U, F) is a coordinate chart.

Proof: By the implicit function theorem, since we can shrink U, WOLOG $F : U \to F(U)$ is a diffeomorphism. So it's a continuous chart (homeomorphism), and by definition of C^∞, (U, F) is compatible with the smooth charts in an atlas. So (U, F) is in the C^∞ structure.