Math 591 Lecture

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Recall: For $F : M \to N$ C^∞, and $p \in M$, we have

$$\begin{aligned} F_p^* : C^\infty_{F(p)}(N) &\to C^\infty_p(M) \\ [f] &\mapsto [f \circ F] \end{aligned}$$

By duality, we get $F_{*,p} : T_pM \to T_{F(p)}N$ defined so that, for $v \in T_pM$, $F_{*,p}(v)[f] = v[f \circ F]$. This is also called the differential.

F_p^* is a ring morphism, which maps the ideal

$$I_{F(p)} = \left\{[f] \in C^\infty_p \mid f(F(p)) = 0\right\}$$

into

$$I_p = \left\{[f] \in C^\infty_p \mid f(p) = 0\right\}$$

This induces a map

$$\begin{array}{ccc} I_{F(p)}/I_{F(p)}^2 & \xrightarrow{F_p^*} & I_p/I_p^2 \\ \wr\Vert & & \wr\Vert \\ T^*_{F(p)} & \xrightarrow{F_p^*} & T_p^*M \end{array}$$

Check that F_p^* is dual to $F_{p,*}$.

Thm: (Chain Rule) Let $M \xrightarrow{F} N \xrightarrow{G} O$ be smooth, and $p \in M$. Then $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$.

Proof: Let $[f] \in C^\infty_{G(F(p))}(O)$. Then $f \circ (G \circ F) = (f \circ G) \circ F$. Now pick $v \in T_pM$. Then

$$(G \circ F)_{*,p}(v)[f] = v(f \circ (G \circ F)) = v((f \circ G) \circ F) = F_{*,p}(v)(f \circ G) = G_{*,F(p)}(F_{*,p}(v))[f]$$

So $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. $\square$

Ex: Let $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ a coordinate chart. As usual, write $\phi = (x^1, \dots, x^n)$, with $x^i : U \to \mathbb{R}$. Say $\mathbb{R}^n = N$, a manifold with a single chart, the identity map $r = (r^1, \dots, r^n)$, $r^i : \mathbb{R}^n \to \mathbb{R}$.
Claim: $\phi_{*,p}(\frac{\partial}{\partial x^i}\big|_p) = \frac{\partial}{\partial r^i}\big|_{\phi(p)}$. In other words, partial derivatives in $\mathbb{R}^n$ correspond with standard basis vectors of T_pM, via the pushforward.

Proof: We can form $f_\phi = f \circ \phi^{-1}$. So by definition, $(\phi^{-1})_{*,\phi(p)}(\frac{\partial}{\partial r^i}) = \frac{\partial}{\partial x^i}$. By the chain ule, $(\phi^{-1})_{*,\phi(p)} = (\phi_{*,p})^{-1}$. $\square$

Differentials of Functions

Let $f : M \to N = \mathbb{R}$ (with a single chart, the identity map). Note that $\forall a \in \mathbb{R}$, we can identify $T_a\mathbb{R} \cong \mathbb{R}$ using $\frac{\partial}{\partial r}\big|_a$ as a basis of $T_a\mathbb{R}$.

Claim: Then $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is the same as $df_p : T_pM \to \mathbb{R}$ (defined as the class of $[f - f(p)] \in I_p$ in the quotient I_p/I_p^2).

Chain rule: $M \xrightarrow{F} N \xrightarrow{f} \mathbb{R}$, $v \in T_pM$ simply reads

$$(f \circ F)_{*,p}(v) = (f_{*,F(p)} \circ F_{*,p})(v) = df_p(F_{*,p}(v)) = F_p^*(df_p)(v)$$

by the duality between F_*^p and $F_{*,p}$.

Conclusion: The pullback map on differentials is the pushforward of the composition, i.e., $F_p^*(df_p) = (f \circ F)_{*,p} = d(f \circ F)_p$. So we frequently write $F_{*,p} = dF_p$.

Computation of $F_{*,p}$ in Coordinates

Let $F : M \to N$, $p \in M$. Let (V, ψ) be a coordinate chart near $F(p)$.

$$\begin{array}{ccc} p \in F^{-1}(V) \subseteq U \subseteq M & \xrightarrow{F\,:\,M \to N} & F(p) \in V \subseteq N \\ \Big\downarrow \scriptstyle{\phi = (x^1, \ldots, x^m)} & & \Big\downarrow \scriptstyle{\psi = (y^1, \ldots, y^n)} \\ \mathbb{R}^m & \xrightarrow{\tilde{F} = \psi \circ F \circ \phi^{-1} \,:\, \phi(U) \to \mathbb{R}^n} & \mathbb{R}^n \end{array}$$

Lemma: The matrix of $F_{*,p}$ in the ordered bases

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^m}\bigg|_p \right) \subseteq T_pM \qquad \text{and} \qquad \left(\frac{\partial}{\partial y^1}\bigg|_{F(p)}, \ldots, \frac{\partial}{\partial y^n}\bigg|_{F(p)} \right) \subseteq T_{F(p)}N$$

is simply $\left(\frac{\partial F^j}{\partial x^i}(p)\right)$, the Jacobian, where $F^j = y^j \circ F : U \to \mathbb{R}$, with $\psi \circ F = (F^1, \ldots, F^n)$.

Observe: This is the Jacobian of $\tilde{F}$ at $\phi(p)$ (in the Calc III sense).

Proof: We want to compute the component $F_{*,p}(\frac{\partial}{\partial x^i})$ with respect to $\frac{\partial}{\partial y^j}$. This is

$$F_{*,p}(\frac{\partial}{\partial x^i})([y^j]) = \frac{\partial}{\partial x^i}(y^j \circ F)\bigg|_p = \frac{\partial}{\partial x^i}(F^j)\bigg|_p$$

$\square$

Defn: A C^∞ map $F : M \to N$ is a <u>local diffeomorphism</u> iff $\forall p \in M$, there are open neighborhoods U of p and V of $F(p)$, such that $F(U) = V$ and $F|_U^V : U \to V$ has a smooth inverse $(F|_U^V)^{-1} : V \to U$.

Ex: $F : S^n \to \mathbb{RP}^n$ is a local diffeomorphism, but not a global diffeomorphism.