Math 591 Lecture 25

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Thm: If M is compact, then every $X \in \mathfrak{X}(M)$ is complete.

Proof: We know (by the existence of local flows) that $\forall p \in M$, there exists a neighborhood V of p, with $\varepsilon_p > 0$, such that the flow of X is defined on $(-\varepsilon_p, \varepsilon_p) \times V_p \to M$. We can extract a finite subcover of $\{V_p \mid p \in M\}$, say $\{V_{p_1}, \ldots, V_{p_k}\}$. Let $\varepsilon = \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\}$. Then the flow of X is defined on $(-\varepsilon, \varepsilon) \times M$, $\forall p$. So by the uniform time lemma, X is complete. $\square$

Question: How do vector fields relate with smooth maps? The answer is "not well".

In general, vector fields cannot be pushed forward or pulled back. Let $F : M \to N$ smooth. We can certainly push forward single tangent vectors: $F_{*,p} : T_pM \to T_{F(p)}N$. But F may not be injective or surjective. If F is not injective, we have $F(p_1) = F(p_2)$ for some $p_1, p_2 \in M$. If F is not injective, there's a $q \in N$ such that $F(p) \neq q$, $\forall p \in M$. In either case, it's not clear what the vector field should be at that point.

Observe: If F is a diffeomorphism, thne given $X \in \mathfrak{X}(M)$, we *can* define $(F_*X)_q$ by $\forall q \in N$, $(F_*X)_q = F_{*,F^{-1}(q)}(X_{F^{-1}(q)})$.

Defn: Given $F : M \to N$, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, we say X and Y are <u>F-related</u> iff $\forall p \in M$, $F_{*,p}(X_p) = Y_{F(p)}$.

Ex: Let $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x$. Take $X \in \mathfrak{X}(\mathbb{R}^2)$ to be $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, $f, g \in C^\infty(\mathbb{R}^2)$. When is X F-related to some $Y \in \mathfrak{X}(\mathbb{R})$?

Well, $F_{*,(x,y)}(X_{(x,y)}) = f(x, y)\frac{\partial}{\partial x}$. So only when f doesn't depend on y.

Prop: Let $F : M \to N$ be smooth, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. X and Y are F-related iff (1) $\forall c$, an integral curve of X, $F \circ c$ is an integral curve of Y (if there are no domain issues) iff (2) the following diagram commutes:

$$\begin{tikzcd}
C^\infty(M) \arrow[d, "X"] & C^\infty(N) \arrow[l, "F^*"'] \arrow[d, "Y"] \\
C^\infty(M) & C^\infty(N) \arrow[l, "F^*"']
\end{tikzcd}$$

where $F^*(g) = g \circ F$, i.e., $\forall g \in C^\infty(N)$, $X(g \circ F) = Y(g) \circ F$.

Proof: (1) essentially follows directly from the definition, and uniqueness of integral curves.

For (2), recall how fields act as operators.

$$X(g \circ F)(p) = d(g \circ F)(X_p) = dg(F_{*,p}(X_p))$$

$$(Y(g) \circ F)(p) = Y(g)(F(p)) = dg(Y_{F(p)})$$

These are equal $\forall g$ iff $F_{*,p}(X_p) = Y_{F(p)}$, which is precisely the condition that X and Y are F-related. $\square$

Prop: Given $F : M \to N$ smooth, $X_1, X_2 \in \mathfrak{X}(M)$, $Y_1, Y_2 \in \mathfrak{X}(N)$, if X_1 is F-related to Y_1 and X_2 is F-related to Y_2, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

Proof: Left as an exercise. Use condition (2) from the previous proposition.

Cor: If $X_1, X_2 \in \mathfrak{X}(G)$, G a Lie group, and X_1 and X_2 are left-invariant, then $[X_1, X_2]$ is left-invariant.

Proof: $\forall g \in G$, we know X_1 and X_2 are L_g-related to themselves. Therefore, $[X_1, X_2]$ is L_g-related to itself, i.e., $[X_1, X_2]$ is left-invariant. $\square$

Question: Given $X, Y \in \mathfrak{X}(M)$, we defined $[X, Y]$ regarding X and Y as operators. What is the dynamical interpretation/meaning of the commutator?

Thm: Let $p \in M$, let ϕ be the flow of X. Form the curve in T_pM:

$$(-\varepsilon, \varepsilon) \ni t \mapsto (d(\phi_t)_p)^{-1}(Y_{\phi_t(p)}) \stackrel{\text{def}}{=} v_t$$

(with $d(\phi_t)_p : T_pM \to T_{\phi_t(p)}M$). Then $\frac{d}{dt}v_t = d(\phi_t)_p^{-1}([X, Y]_{\phi_t(p)})$ (the derivative of a curve in a vector space). At $t = 0$, $\frac{d}{dt}v_t = [X, Y]_p$.

Proof: Next time.

Cor: If $[X, Y] \equiv 0$ (everywhere), then $\forall t, s$, $\phi_t \circ \phi_s = \phi_s \circ \phi_t$, where ϕ is the flow of X and ψ is the flow of Y.

$$\begin{tikzcd} p \arrow[r, "\phi"] \arrow[d, "\psi"] & \phi_t(p) \arrow[d, "\psi"] \\ \psi_s(p) \arrow[r] & \phi_t(\psi_s(p)) = \psi_s(\phi_t(p)) \end{tikzcd}$$

Proof: The assumption $[X, Y] = 0$ implies the curves v_t from above are constant. So $\forall t$, $d(\phi_t)^{-1}(Y_{\phi_t(p)}) = Y_p$. Thus, Y is ϕ_t-related to itself, $\forall t$. So ϕ_t maps integral curves of Y to integral curves of Y. This is equivalent to the commutativity we're trying to show. $\square$