Math 591 Lecture 2

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Quotient Spaces

Defn: Let X be a tangent space, $\sim$ an equivalence relation on X. Then $X/\sim$ is the set of equivalence classes of $\sim$, and

$$\begin{aligned} \pi : X &\to X/\sim \\ x &\mapsto [x] \end{aligned}$$

The quotient topology is defined by $W \subseteq X/\sim$ is open iff $\pi^{-1}(W) \subseteq X$ is open.

We just assume that this is a topology – is it?

Observe: π is continuous by definiion of the quotient topology. And the quotient topology is he finest topology on $X/\sim$ for which π is continuous.

Prop: Let Y be a topological space, $f : X/\sim \to Y$. Then f is continuous iff $f \circ \pi : X \to Y$ is continuous.

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "f\circ\pi"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof: $\Rightarrow$ is trivial, because π is continuous.
$\Leftarrow$ is the real content of the proof (left as an exercise).

Observe: it's not true that X being Hausdorff implies $X/\sim$ being Hausdorff.
And it's not true that X being second countable implies $X/\sim$ being second countable.

Defn: An equivalence relation $\sim$ on a tangent space X is open iff $\pi : X \to X/\sim$ is an open map (i.e. $\forall U \subseteq X$ open, $\pi(U)$ is open).

Let's investigate: $\pi(U)$ is open, by definition, if $\pi^{-1}(\pi(U)) \subseteq X$ is open. So let $\hat{U} = \pi^{-1}(\pi(U)) = \{x \in X \mid \exists y \in U \text{ s.t. } x \sim y\}$.
To recap, $\sim$ is open $\Leftrightarrow \forall U \subseteq X$ open, $\hat{U}$ is open.

Thm: Let $\sim$ be an equivalence relation. Assume it is open, then
1) If X is second countable, then so is $X/\sim$.
2) $X/\sim$ is Hausdorff iff the graph of the relation, Γ, is closed in $X \times X$. $\Gamma = \{(x, y) \in X \times X \mid x \sim y\} \subseteq X \times X$.

Proof:
1) If $\sim$ is open, then for any basis $\mathscr{B} = \{B_j\}_{j \in J}$ of X, $\{\pi(B_j)\}_{j \in J}$ is a basis for $X/\sim$. $\square$
2) Γ is closed iff $\forall (x, y) \in (X \times X) \setminus \Gamma$, $\exists U, V \subseteq X$ open with $x \in U$, $y \in V$ s.t. $(U \times V) \cap \Gamma = \emptyset$. And this is true iff $\forall (u, v) \in U \times V$, $\pi(u) \neq \pi(v)$, which is true iff $\pi(U) \cap \pi(V) = \emptyset$. And $\pi(U)$ is a neighborhood of $\pi(x)$, $\pi(V)$ is a neighborhood of $\pi(y)$, because π is an open map.

Some remarks:

1. In general, if $X/\sim$ is Hausdorff, then for any point $p \in X/\sim$, $\{p\}$ is closed. Thus, $\pi^{-1}(p)$ (an equivalence class of $\sim$) is closed, because set complements ork nicely. So we conclude that a necessary condition for $X/\sim$ to be T_2 is all equivalence classes $[x] \subseteq X$ are closed. But this isn't sufficient!

2. If $P : X \to S$ is a surjective map...

Defn: $\forall x, y \in X$, $x \sim_P y \overset{\text{def}}{\Leftrightarrow} P(x) = P(y)$ $(\star)$ is an equivalence relation. If P is surjective, there's a natural idenification of $S \cong X/\sim_P$. If X, S are tangent spaces, P is called a quotient map iff $\star$ is a homeomorphism, i.e., iff $\forall W \subseteq S$, W is open iff $P^{-1}(W)$ is open.

Ex: $X = S^n \subseteq \mathbb{R}^{n+1}$, the n-dimensional sphere. Consider $\tau : S^n \to S^n$, where $x \mapsto -x$.
Observe that τ is continuous, and $\tau^2 = \mathrm{Id}$ (so $\tau = \tau^{-1}$).
Define the following equivalence relation: $\forall x, y \in S^n$, $x \sim y \Leftrightarrow x = y$ or $y = \tau(x)$.

Claim: $\sim$ is open.
Proof: Let $U \subseteq S^n$ be open. $\hat{U} = \pi^{-1}(\pi(U)) = U \cup \tau(U)$. So $\hat{U}$ is open because $\tau(U)$ is open because τ is a homeomorphism because τ is continuous and $\tau = \tau^{-1}$. Thus, $S^n/\sim$ is second-countable (because we know S^n is second-countable). Write

$$\begin{aligned}\Gamma &\subseteq S^n \times S^n \\ &= \{(x,y) \in S^n \times S^n \mid x \sim y\} \\ &= \{(x,y) \in S^n \times S^n \mid x = y \text{ or } y = \tau(x)\} \\ &= \underbrace{\{(x,x) \mid x \in S^n\}}_{\text{diagonal}} \cup \underbrace{\{(x,\tau(x)) \mid x \in S^n\}}_{\text{graph of } \tau}\end{aligned}$$

The graph of τ is closed.
Consider $F : S^n \times \{0,1\} \to S^n$ (note: $\{0,1\} = \mathbb{Z}_2$), where $(x,0) \mapsto x$ and $(x,1) \mapsto \tau(x)$. This is a group action.
So the graph of τ is the iamge of the map $S^n \to S^n \times S^n$, where $x \mapsto (x, \tau(x))$. This is a continuous map, and S^n is compact, so its image is compact, so its image is closed.
Thus, Γ is the finite union of closed sets, so Γ is closed, so $S^n/\sim$ is Hausdorff.
Note: $S^n/\sim = \mathbb{RP}^n$, the n-dimensional real projective space. This is isomorphic to the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$.