Math 591 Lecture 19

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Thm: The image of an embedding $F : M \to N$ is a regular submanifold of N.

Proof: Let $q \in F(M)$. We need to show that there are coordinates of N near q adapted to $F(M)$. Let $p \in M$ s.t. $F(p) = q$. By the immersion theorem, there are corodinates $(U, \phi = (x^1, \ldots, x^m))$ of M with $p \in U$, and $(V, \psi = (y^1, \ldots, y^n))$ of N with $q \in V$, with $U \subseteq F^{-1}(V)$, such that $\tilde{F}(I) = (I, 0)$ (with $m - n$ zeros).

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ \mathbb{R}^m \supseteq \phi(U) & \xrightarrow{\tilde{F}} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Well, $F(U) = \left\{w \in V \mid y^j(w) = 0, \forall j \in \{1, \ldots, n\}\right\}$. The point is that because F is an embedding (i.e. $F|^{F(M)} : M \to F(M)$ is a homeomorphism), then $F(U)$ is a relatively open set of $F(M)$. So there is an open set $W \subseteq N$ such that $F(U) = F(M) \cap W$. Therefore, $F(M) \cap W \left\{w^{-1}V \mid y^j(w) = 0, \forall j \in \{1, \ldots, n\}\right\}$. So ψ on $V \cap W$ is adapted to $F(M)$ at q. $\square$

We want a stronger statement for when an injective immersion is an embedding.

Prop: Let $F : M \to N$ be a continuous proper map between manifolds. Then F is closed.

Proof: Let $C \subseteq M$ be a closed set, and $q \in \overline{F(C)}$. We need to show $q \in F(C)$. Well, let $V \subseteq N$ be an open neighborhood of q such that $\overline{V}$ is compact. (We can do this because Euclidean spaces are locally compact.) Observe then that $q \in \overline{F(C) \cap \overline{W}}$. $F^{-1}(\overline{V})$ is compact, since F is proper, so $C \cap F^{-1}(\overline{V})$ is compact. Therefore, $F(C \cap F^{-1}(\overline{V}))$ is compact, so it's closed. But $F(C \cap F^{-1}(\overline{V})) = F(C) \cap \overline{V}$ (this is a set-theoretic fact). SO $F(C) \cap \overline{V}$ is closed. Since $q \in \overline{F(C) \cap \overline{V}}$, $q \in F(C) \cap \overline{V}$, so $q \in F(C)$. Thus, F is closed. $\square$

Vector Fields

Defn: Let M be a smooth manifold of dimension n. A <u>vector field on M</u> is a secion of $TM \xrightarrow{\pi} M$. That is, it's a map $X : M \to TM$ s.t. $\forall p \in M$, $X_p \in T_pM$.

This is a slight abuse of notation. Update: Write $X(p) = (p, X_p)$. The p is the same because X is a section $\pi \circ X = I_M$.

Basically, a vector field on a manifold assigns a tangent vector to every point in a manifold.

Defn: A vector field X on M is <u>smooth</u> (C^∞) iff it is smooth as a map between manifolds $X : M \to TM$.

What does this mean (concretely)?

Let's think about smooth sections of vector bundles in general (e.g. TM and T^*M). Let

$$\begin{array}{c} \mathcal{E} \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$$

be a C^∞ vector bundle of rank $\rho \in \mathbb{N}$. This means $\mathcal{E}$ is a manifold, π is C^∞ and onto, $\forall p \in M$, $\pi^{-1}(p)$ has the structure of a vector space over $\mathbb{R}$ of dimension ρ, and there exists a family $\{(U_\alpha, \chi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of M and $\forall \alpha$,

χ_α is a diffeomorphism, and we have the vector bundle trivialization

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^\rho \arrow[dl] \\
& U_\alpha &
\end{tikzcd}$$

s.t. χ_α is linear on fibers, i.e., $\forall p \in U_\alpha$, we have

$$\chi_\alpha\big|_{\pi^{-1}(p)} : \pi^{-1}(p) \overset{\cong}{\to} \{p\} \times \mathbb{R}^\rho \cong \mathbb{R}^\rho$$

The mapping from $\pi^{-1}(p)$ to $\mathbb{R}^\rho$ is a linear isomorphism.

Ex: Let $\mathcal{E} = TM$. The trivializations are induced by coordinate charts. Let $U \subseteq M$, with $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be a coordinate chart. Then $\forall p \in U, i \in \{1, \ldots, n\}$, $\frac{\partial}{\partial x^i}\big|_p \in T_pU = T_pM$. We have

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl] \\
& U &
\end{tikzcd}$$

defined by

$$\chi^{-1}(p, v) = \left(p, \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p\right)$$

with $p \in U$ and $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$.
We proved that $\forall p \in M$, $\left\{ \frac{\partial}{\partial x^i}\big|_p : i = 1, \ldots, m \right\}$ is a basis of T_pM. So χ^{-1} is invertible and a linear isomorphism of the fibers.
(The definition of the smooth structure on TM is such that χ as above is a diffeomorphism.)

Ex: $T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$
Again, a coordinate chart on M induces a trivialization.
Use: $\forall \alpha \in T_p^*M$, $\alpha = \sum_{i=1}^n \alpha_i \, dx^i\big|_p$, where $dx^i\big|_p$ is dual to $\frac{\partial}{\partial x^i}\big|_p$, and $\left\{ dx^i\big|_p \mid i = 1, \ldots, n \right\}$ is a basis of T_p^*M.

Back to

$$\begin{tikzcd}
E \arrow[d, "\pi"] \\
M \arrow[u, bend right, "s"']
\end{tikzcd}$$

Let s be a section (i.e. $\pi \circ s = I_M$). When is s smooth? We will answer this question using trivializations. Given a trivialization χ,

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^p \arrow[dl, "\pi"] \\
& U \arrow[ul, bend right, "s|_U"'] \arrow[ur, bend right, "\chi \circ (s|_U) \overset{\mathrm{def}}{=} s_\chi"'] &
\end{tikzcd}$$

We have $\forall p \in U$, $s_\chi(p) = (p, \star)$, where $\star$ is given by a function $F : U \to \mathbb{R}^\rho$. F is just the projection onto $\mathbb{R}^\rho$ composed with s_χ. So $\forall p \in U$, $s_\chi(p) = (p, F(p))$.

Claim: $s|_U$ is smooth iff $F : U \to \mathbb{R}^\rho$ is smooth.

Proof: Next time...