Math 591 Lecture 1

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What is a differentiable manifold? It's a bit of a long answer. Here's something that's true, but not complete:

A differentiable manifold is a topological space that is
(a) Hausdorff (T_2)
(b) Second-Countable
(c) Locally Euclidean (Informally, this means every point has a neighborhood that is homeomorphic to a Euclidean space.)
(d) Has a C^∞ atlas (This is additional structure, not a topological property.)
Note that (a), (b), and (c) together define a topological manifold.

Hausdorff Spaces

Defn: A topological space X is Hausdorff (or T_2) if $\forall x, y \in X$ with $x \neq y$, there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Ex: Any metric space is Hausdorff.
Proof: Fix distinct x, y. Let $r = \frac{1}{3}d(x,y) > 0$ (because $x \neq y$). Let $U = B(x,r)$, $V = B(y,r)$. Then by the triangle inequality, $U \cap V = \emptyset$. $\square$

Ex: The following topological space is *not* Hausdorff.
Let $X = (-\infty, 0) \cup \{A, B\} \cup (0, \infty)$ (with $A \neq B$), with the standard topology on $(-\infty, 0)$ and $(0, \infty)$, and neighborhoods of A and B contain $(-\varepsilon, 0)$ and $(0, \varepsilon)$.
This space is not Hausdorff, because you can't separate A and B.

Second-Countable Spaces

Defn: Let X be a topological space. A basis of X is a collection $\mathscr{B}$ of open sets such that every open set in X can be written as the (possibly infinite) union of elements in $\mathscr{B}$.

Defn: X is second-countable if there is a countable basis of X.

Ex: $X = \mathbb{R}^n$ (with the standard topology) is second countable.
Proof: Let $\mathscr{B} = \{B(q,r) \mid q \in \mathbb{Q}^n, r \in \mathbb{Q}\}$. We know $\mathscr{B}$ is countable, so we need to show $\mathscr{B}$ is a basis.
Let $U \subseteq \mathbb{R}^n$ be open. Let $\mathscr{B}_U = \{B \in \mathscr{B} \mid B \subseteq U\}$. We claim that $U = \bigcup_{B \in \mathscr{B}_U} B$.
Well, $\supseteq$ is trivial. To show $\subseteq$, let $x \in U$. Since U is open, $\exists r \in \mathbb{R}_{>0}$ s.t. $B(x,r) \subseteq U$. Let $q \in \mathbb{Q}^n$ s.t. $d(x,q) < \frac{r}{43}$. Let $\rho \in \mathbb{Q}$ such that $\frac{r}{43} \overset{(1)}{<} \rho \overset{(2)}{<} \frac{r}{42}$.

(1) implies that $B(q,\rho)$ contains x.
(2) implies (by the triangle inequality and the fact that $x \in B(q,\rho)$) that $B(q,\rho) \subseteq U$, so $B(q,\rho) \in \mathscr{B}_U$.

$\square$

Ex: The following topological space is *not* second-countable.
Let $X = \mathbb{R}$ with the discrete topology (i.e. singletons are open).
A basis must have every singleton, so it's obviously not second-countable.

Relationships of These Properties with "New Spaces from Old"

Product Spaces

Defn: Let X, Y be topological spaces. Then $X \times Y$ has a natural topology called the product topology, defined by the basis $\{U \times V \mid U \subseteq X, V \subseteq Y \text{ open}\}$.

Thm: (A.21, A.22) If X and Y are second-countable topological spaces, then so is $X \times Y$. If X and Y are T_2, then so is $X \times Y$.

Quotient Spaces

Next time...