Math 591 Lecture 29

Thomas Cohn

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Motivation: Integration

How do we integrate on a manifold? Start with calc 3. Let $U \subseteq \mathbb{R}^n$ be an open set. Then we can change variables with

$$\int_U f(x)\,dx = \int_V f(x(y)) \underbrace{\det\left(\tfrac{\partial x}{\partial y}\right)}_{\text{if positive}} dy$$

for $x = x(y)$.

Now, for the general interpretation: let $\begin{aligned} F : V &\to U \\ y &\mapsto x = x(y) \end{aligned}$ be a diffeomorphism. Then $\left(\frac{\partial x}{\partial y}\right)$ is the Jacobian of F. We'll interpret this as the pullback of the RHS integral by $F - F^*(dx^1, \ldots, dx^n) = ?$. But what does det mean in general? It's an alternating, multilinear function, so let's work with that.

Defn: Let V be a n-dimensional vector space, and $k \in \mathbb{N}$. A <u>k-covector</u> or <u>k-form</u> on V is a multilinear map $\alpha : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$ that is alternating (i.e. if you swap two elements, the sign flips).

Observe: Let $\sigma \in S_k$ (the symmetric group). Given $v_1, \ldots, v_k$, α alternating, we have

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \underbrace{(-1)^\sigma}_{=\mathrm{sgn}(\sigma)} \alpha(v_1, \ldots, v_k)$$

Observe: If $\{v_1, \ldots, v_k\}$ is linearly dependent, and α is alternating, then $\alpha(v_1, \ldots, v_k) = 0$.
Proof: Say $v_1 = \lambda_2 v_2 + \cdots + \lambda_k v_k$. Then

$$\alpha(v_1, \ldots, v_k) = \alpha\left(\sum_{i=2}^k \lambda_i v_i, v_2, \ldots, v_k\right) = \sum_{i=2}^k \lambda_i \alpha(v_i, v_2, \ldots, v_i, \ldots, v_k) = 0$$

Defn: $\bigwedge^k V^*$ is the set of alternating $\mathbb{R}$-multilinear functions on V^k. We say that k is the <u>degree</u>.

Observe: k-forms can be pulled back by linear maps.

Defn: If $F : V \to W$ is linear, and $\alpha \in \bigwedge^k W^*$, $v_1, \ldots, v_k \in V$, then

$$(F^*\alpha)(v_1, \ldots, v_k) = \alpha(F(v_1), \ldots, F(v_k))$$

and $F^*\alpha \in \bigwedge^k V^*$.

Defn: Let $(\mathcal{E}^1, \ldots, \mathcal{E}^n)$ be an ordered basis of V^*. Let $A = \{a_1 < \cdots < a_k\} \subset \{1, \ldots, n\}$. Define $\mathcal{E}^A : V \times \cdots \times V \to \mathbb{R}$ by

$$\mathcal{E}^A(v_1, \ldots, v_k) = \det\left(\mathcal{E}^{a_i}(v_j)\right)_{(i,j)}$$

Ex: Say $V = \mathbb{R}^3$ with the standard basis. Then $\mathcal{E}^{13}((x_1, x_2, x_3), (y_1, y_2, y_3)) = \det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix}$.

Prop: For a given V and k, and an ordered basis $(\mathcal{E}^1, \ldots, \mathcal{E}^n)$ of V^*, the set $\left\{\mathcal{E}^I : I \subset \{1, \ldots, n\}, \#I = k\right\}$ is a basis of $\bigwedge^k V^*$. In particular, $\dim \bigwedge^k V^* = \binom{n}{k}$.

Observe: If $k > n$, $\bigwedge^k V^* = \{0\}$. If $k = n$, $\dim \bigwedge^k V^* = 1$. If $k = 1$, $\bigwedge^1 V^* = V^*$.

As a warmup, let $k = 2$¿ Let $\{e_1, \dots, e_n\}$ be a basis of V, and $\left\{\mathcal{E}^1, \dots, \mathcal{E}^n\right\}$ the corresonding dual basis of V^*. Note that $\mathcal{E}^i(e_j) = \delta_{ij}$.
Say $\alpha \in \bigwedge^2 V^*$, $v_1 = \sum_{a=1}^n v_1^a e_a$, $v_2 = \sum_{b=1}^n v_2^b e_b$. Then

$$\begin{aligned}
\alpha(v_1, v_2) &= \alpha\left(\sum_{a=1}^n v_1^a e_a, \sum_{b=1}^n v_2^b e_b\right) \\
&= \sum_{a=1}^n v_1^a \alpha\left(e_a, \sum_{b=1}^n v_2^b e_b\right) \\
&= \sum_{a=1}^n \sum_{b=1}^n v_1^a v_2^b \alpha(e_a, e_b) \\
&= \sum_{1 \le a < b \le n} v_1^a v_2^b \alpha(e_a, e_b) + v_1^b v_2^a \alpha(e_b, e_a) \\
&= \sum_{1 \le a < b \le n} \underbrace{(v_1^a v_2^b - v_1^b v_2^a)}_{= \begin{vmatrix} v_1^a & v_2^a \\ v_1^b & v_2^b) \end{vmatrix} = \mathcal{E}^{ab}(v_1, v_2)} \alpha(e_a, e_b) \\
&= \sum_{I = \{a < b\}} \alpha(e_a, e_b) \mathcal{E}^I(v_1, v_2)
\end{aligned}$$

In general, $\alpha = \sum_{1 \le a < b \le n} \alpha(e_a, e_b) \mathcal{E}^{ab}$.

Now, for general $k \in \mathbb{N}$, fix $\alpha \in \bigwedge^k V^*$. For $i = 1, \dots, k$, let $v_i = \sum_{a=1}^n v_i^a e_a$. Then

$$\begin{aligned}
\alpha(v_1, \dots, v_k) &= \alpha\left(\sum_{a_1=1}^n v_1^{a_1} e_{a_1}, \dots, \sum_{a_k} v_k^{a_k} e_{a_k}\right) \\
&= \sum_{a_1, \dots, a_k = 1}^n \left(\prod_{j=1}^k v_j^{a_j}\right) \alpha(e_{a_1}, \dots, e_{a_k}) \\
&= \sum_{1 \le a_1 < \dots < a_k \le n} \underbrace{\left(\sum_{\sigma \in S_k} \left(\prod_{j=1}^k v_j^{\sigma(a_i)}\right) (-1)^\sigma\right)}_{= \det\left(v_j^{a_i}\right)_{(i,j)} = \mathcal{E}^A(v_1, \dots, v_k),\, A = \{a_1, \dots, a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \\
&= \sum_{A = \{a_1 < \dots < a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \mathcal{E}^A(v_1, \dots, v_k)
\end{aligned}$$

So in general, $\alpha = \sum_{A = \{a_1, \dots < a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \mathcal{E}^A$.

The Wedge Product

We want to take a k-form and an ℓ-form and make a $k + \ell$-form.

Defn: For $\alpha \in \bigwedge^k V^*$, $\beta \in \bigwedge^\ell V^*$, define the wedge product of α and β, $\alpha \wedge \beta \in \bigwedge^{k+\ell} V^*$, by

$$\begin{aligned}
(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) &\overset{\text{def}}{=} \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\
&= \sum_{\sigma \in \mathrm{Sh}(k,\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})
\end{aligned}$$

Defn: $\mathrm{Sh}(k, \ell)$ is the set of $k - \ell$ shuffles, which are permutations $\sigma \in S(k + \ell)$ such that $\sigma(1) < \dots < \sigma(k)$ and $\sigma_{k+1} < \dots < \sigma_{k+\ell}$.

The Skew-Symmetrizer

Given $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^\ell V^*$, we can define

$$(\alpha \otimes \beta)(v_1, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+\ell})$$

As a map, $\alpha \otimes \beta : V^{k+\ell} \to \mathbb{R}$ is $k+\ell$-multilinear, but not alternating/skew-symmetric.

Defn: The skew-symmetrizer of a multilinear map $f : V^m \to \mathbb{R}$ is defined by

$$A(f)(v_1, \ldots, v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma f(v_{\sigma(1)}, \ldots, v_{\sigma(m)})$$

Lemma: $A(f)$ is skew-symmetric/alternating, and if f is already skew-symmetric, $A(f) = f$.

Proof: Let $\tau \in S_m$. Then

$$\begin{aligned} A(f)(v_{\tau(1)}, \ldots, v_{\tau(m)}) &= \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma f(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(m))}) \\ &\overset{(1)}{=} \frac{1}{m!} \sum_{\mu \in S_m} \underbrace{(-1)^{\mu\tau}}_{=(-1)^\tau(-1)^\mu} f(v_{\mu(1)}, \ldots, v_{\mu(m)}) \\ &= (-1)^\tau A(f) \end{aligned}$$

with (1) true because if $\mu = \sigma\tau$, then $\sigma = \mu\tau^{-1}$. $\square$

Thus, we have $\alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!} A(\alpha \otimes \beta)$.

Ex: $k = \ell = 1$. Then $\mathrm{Sh}(1,1) = S_2$, so

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

Ex: $k = 1$, $\ell = 2$. Then the elements of S_3 are

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\in \mathrm{Sh}(1,2)$?	sgn
1	2	3	Yes	$+$
1	3	2	No	
2	1	3	Yes	$-$
2	3	1	No	
3	1	2	Yes	$+$
3	2	1	No	

so $\mathrm{Sh}(1,2) = \{(1\ 2\ 3), (2\ 1\ 3), (3\ 1\ 2)\}$, so

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1)\beta(v_2, v_3) - \alpha(v_2)\beta(v_1, v_3) + \alpha(v_3)\beta(v_1, v_2)$$

If $\beta = \gamma \wedge \delta$, then we get

$$\begin{aligned} (\alpha \wedge (\gamma \wedge \delta))(v_1, v_2, v_3) =& \alpha(v_1)(\gamma(v_2)\delta(v_3) - \gamma(v_3)\delta(v_2)) \\ &- \alpha(v_2)(\gamma(v_1)\delta(v_3) - \gamma(v_3)\delta(v_1)) \\ &+ \alpha(v_3)(\gamma(v_1)\delta(v_2) - \gamma(v_2)\delta(v_1)) \\ =& \begin{vmatrix} \alpha(v_1) & \alpha(v_2) & \alpha(v_3) \\ \gamma(v_1) & \gamma(v_2) & \gamma(v_3) \\ \delta(v_1) & \delta(v_2) & \delta(v_3) \end{vmatrix} \end{aligned}$$

Lemma: $\mathcal{E}^I \wedge \mathcal{E}^J = \mathcal{E}^{IJ}$.

Lemma: $\mathcal{E}^I = \mathcal{E}^{I_1} \wedge \cdots \wedge \mathcal{E}^{I_k}$ ($\wedge$ is associative).

Prop: The wedge product is

- Bilinear
- Associative
- Anti-commutative: $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$, for $\alpha \in \bigwedge^k$, $\beta \in \bigwedge^\ell$
- If $F : V \to W$, $\alpha \in \bigwedge^k W^*$, $\beta \in \bigwedge^\ell W^*$, then $F^*(\alpha \wedge \beta) = (F^\alpha) \wedge (F^*\beta)$