

Math 591 Lecture 36

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Continuing from last time, W.T.S. $\sum_{\alpha} \int (\phi_{\alpha}^{-1})^*(\chi_{\alpha}\mu)$ is independent of choice of coordinates and partition of unity. Say $\{\tilde{\phi}_{\beta}\}, \{\tilde{\chi}_{\beta}\}$ is another choice. Then $\forall \alpha$,

$$\chi_{\alpha}\mu = \sum_{\beta} \tilde{\chi}_{\beta}\chi_{\alpha}\mu \quad \sum_{\beta} \tilde{\chi}_{\beta} = 1$$

Thus,

$$\int (\phi_{\alpha}^{-1})^*(\chi_{\alpha}\mu) = \sum_{\beta} \int (\phi_{\alpha}^{-1})^* \underbrace{(\tilde{\chi}_{\beta}\chi_{\alpha}\mu)}_{\text{supp in } (\text{dom } \tilde{\phi}_{\beta}) \cap (\text{dom } \phi_{\alpha})}$$

So $\forall \beta$,

$$\int (\phi_{\alpha}^{-1})^*(\tilde{\chi}_{\beta}\chi_{\alpha}\mu) = \int (\tilde{\phi}_{\beta}^{-1})^*(\tilde{\chi}_{\beta}\chi_{\alpha}\mu)$$

because of the invariance of integrals of top-degree forms in Euclidean space, under orientation-preserving diffeomorphisms. (In this case, the transition function is that orientation-preserving diffeomorphism.) Now, back to the full term in the integral:

$$\sum_{\alpha} (\phi_{\alpha}^{-1})^*(\chi_{\alpha}\mu) = \sum_{\alpha, \beta} (\tilde{\phi}_{\beta}^{-1})^*(\tilde{\chi}_{\beta}\chi_{\alpha}\mu)$$

$\forall \beta$, sum over α first, and use the fact that $\sum_{\alpha} \chi_{\alpha} = 1$. This yields

$$\sum_{\alpha} (\phi_{\alpha}^{-1})^*(\chi_{\alpha}\mu) = \cdots = \sum_{\beta} (\tilde{\phi}_{\beta}^{-1})^*(\tilde{\chi}_{\beta}\mu)$$

□

Observe: In practice, don't use partitions of unity. Use parameterizations that partition M (or at least, $\text{supp } \mu \subset M$), up to sets of measure 0.

Ex: On a torus, pullback μ to \mathbb{R}^n and integrate over a fundamental domain.

Defn: Let $S \subset M$ be an oriented k -dimensional submanifold. Define

$$\int_S : \Omega_0^k(M) \rightarrow \mathbb{R}$$

$$\alpha \mapsto \int_S \iota^* \alpha$$

where $\iota : S \hookrightarrow M$ is the inclusion map. In general, we omit the ι^* , and just say

$$\int_S \alpha \stackrel{\text{def}}{=} \int_S \iota^* \alpha$$

Manifolds with Boundary

These are needed for Stokes' theorem. We begin with some preliminary definitions...

Defn: Let $S, T \subset \mathbb{R}^n$ (with no assumptions of their properties). S and T are diffeomorphic iff $\exists U, V \subset \mathbb{R}^n$ s.t. $S \subset U$, $T \subset V$, and $\exists F : U \rightarrow V$ a diffeomorphism s.t. $F(S) = T$. We will say $F|_S^T : S \rightarrow T$ is a diffeomorphism between the sets.

Lemma: If $U \subset \mathbb{R}^n$ is open and diffeomorphic to $T \subset \mathbb{R}^n$, then T is open.

Proof: By definition, $\exists V, \tilde{U}$ open with $T \subset V$ and $U \subset \tilde{U}$, and $\exists F : \tilde{U} \rightarrow V$ diffeomorphic such that $F(U) = T$. Then $F|_U : U \rightarrow \mathbb{R}^n$ is an open map, so T is open. \square

Defn: $\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Ex: $B_r(0) \cap \mathbb{H}^n$. A diffeomorphism on this set must map interior points to interior points, so it maps $B_r(0) \cap \{x^n = 0\}$ to the boundary of its image.

In fact, the same works for any relatively open sets of \mathbb{H}^n ! And such diffeomorphisms restrict to diffeomorphisms of $W \cap \partial\mathbb{H}^n$ onto $F(W) \cap \partial\mathbb{H}^n$ (for any W relatively open).

Defn: A topological manifold with boundary M is a second-countable, Hausdorff topological space that is local homeomorphic to relatively open sets in \mathbb{H}^n .

Note: This includes ordinary open sets in \mathbb{R}^n , and the sets with boundary.

Defn: A C^∞ atlas on a topological manifold with boundary is an atlas such that all transition functions are C^∞ .

Defn: A C^∞ manifold with boundary is a topological manifold with boundary, together with a maximal C^∞ atlas.

Defn: Let M be a smooth manifold with boundary. Then the boundary ∂M is the set of points $p \in M$ which have a nearby chart ϕ such that $\phi(p)$ is in the boundary of \mathbb{H}^n .

Note that this is true for one coordinate chart iff it's true for all coordinate charts.

Lemma: ∂M inherits a C^∞ manifold structure (without boundary) by restricting charts of M defined near boundary points.

Lemma: $\partial(\partial M) = \emptyset$.

Tangent Spaces

They're defined as before!

Defn: Given $S \subset \mathbb{R}^n$, $f : S \rightarrow \mathbb{R}^k$ is smooth iff $\exists U \subset \mathbb{R}^n$ open with $S \subset U$, and $\exists \tilde{f} : U \rightarrow \mathbb{R}^k$ smooth such that $\tilde{f}|_S \equiv f$.

We call \tilde{f} a smooth extension.

So, for f smooth, $f \circ \phi^{-1} : W \rightarrow \mathbb{R}^k$, $W \subset \mathbb{H}^n$ must be C^∞ for all charts $(\phi^{-1}(W), \phi)$. This gives us germs: $\forall p \in M$, $C_p^\infty(M)$. We define $T_p M$ to be the set of all derivations on $C_p^\infty(M)$.

But what happens at the boundary? If $p \in \partial M$, what is $T_p M$?

Claim: Well, if $p \in \partial\mathbb{H}^n$, $T_p \mathbb{H}^n$ is still spanned by $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$.

The question is how do we define $\frac{\partial}{\partial x^n}$, and how does it act? If $f \in C^\infty$ near p , then $\exists \tilde{f}$, an extension of f to an open set of \mathbb{R}^n . Then define $\frac{\partial f}{\partial x^n}(p) \stackrel{\text{def}}{=} \frac{\partial \tilde{f}}{\partial x^n}(p)$.

Claim: The RHS is independent of choice of extension. Well,

$$\frac{\partial \tilde{f}}{\partial x^n}(p) = \lim_{h \rightarrow \infty} \frac{\tilde{f}(0, \dots, h) - \tilde{f}(0, \dots, 0)}{h}$$

This limit will always be the same, since the limit always exists (by definition of the extension).