Math 591 Lecture 5

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Defn: Let M be a topological manifold. A $\underline{C^\infty \text{ atlas}}$ on M, $\mathscr{A} = \{(U_i, \phi_i)\}_{i \in I}$, is a collection of charts on M such that $M = \bigcup_{i \in I} U_i$ and $\forall i, j \in I$, (U_i, ϕ_i) and (U_j, ϕ_j) are C^∞ compatible.

Last time, we considered $M = S^1 \subseteq \mathbb{R}^2$ with the subspace topology. We constructed two atlases, $\mathscr{A}$ and $\mathscr{B}$, and claimed every chart in $\mathscr{B}$ is compatible with every chart in $\mathscr{A}$, i.e., $\mathscr{A} \cup \mathscr{B}$ is a C^∞ atlas.

We want a definition of smooth structure on M that includes $\mathscr{A}$ and $\mathscr{B}$.

Defn: A smooth structure on a topological manifold M is a maximal smooth atlas, $\mathscr{M}$, i.e., a smooth atlas such that: If (U, ϕ) is any continuous chart on M s.t. $\forall (V, \psi) \in \mathscr{M}$, (U, ϕ) and (V, ψ) are compatible, then $(U, \phi) \in \mathscr{M}$.

Prop: If $\mathscr{A}$ is a smooth atlas on M, $\exists! \mathscr{M}$, a maximal smooth atlas, s.t. $\mathscr{A} \subseteq \mathscr{M}$.
Proof: Let $\mathscr{M} = \{(U, \phi)$ continuous chart s.t. $\forall (V, \psi) \in \mathscr{A}, (U, \phi)$ and (V, ψ) are compatible$\}$. We need to show $\mathscr{M}$ is a smooth atlas, and that it is maximal.
First, check that it's a smooth atlas. Clearly $\mathscr{A} \subseteq \mathscr{M}$. So charts in $\mathscr{M}$ cover M. We just need to show compatibility. Let (U_1, ϕ_1) and (U_2, ϕ_2) be charts in $\mathscr{M}$. We need to show they're compatible with each other. Let $p \in U_1 \cap U_2$, and show $\phi_2 \circ \phi_1^{-1}$ is C^∞ at $\phi_1(p)$. Well, $\exists (V, \psi) \in \mathscr{A}$ s.t. $p \in V$. And every (U_i, ϕ_i) is compatible with charts in $\mathscr{A}$, so $\phi_2 \circ \phi_1^{-1} = (\phi_2 \circ \psi^{-1}) \circ (\psi \circ \phi_1^{-1})$ on some neighborhood of $\phi_1(p)$. This is a smooth map, so we conclude $\mathscr{M}$ is a smooth atlas.
Proving it's maximal just uses topology, and is left as an exercise.
□

Defn: A differentiable manifold is a pair $(M, \mathscr{M})$, where M is a topological manifold, and $\mathscr{M}$ is a maximal C^∞ atlas.

Ex: $M = \mathbb{R}$, $\mathscr{M}$ is the maximal atlas containing $(\mathbb{R}, x \mapsto \sqrt[3]{x})$.
This is a different smooth structure on $\mathbb{R}$. However, the standard $\mathbb{R}$ and this $\mathbb{R}$ are isomorphic in the category of smooth manifolds: they're diffeomorphic! The isomorphism is $x \mapsto \sqrt[3]{x}$.

Question (very hard): Given a topological manifold M, are there non-isomorphic smooth structures on M? Yes, it can happen!

Consider $U \subseteq \mathbb{R}^N$ open (recall: $\mathbb{R}^N$ is called the ambient space), and $F : U \to \mathbb{R}^k$ smooth.
$F = (f^1, \ldots, f^k)$, with $f^i : U \to \mathbb{R}$ smooth.
Recall: If $x_0 \in U$, the Jacobian of F at x_0 is the matrix

$$J(F)(x_0) = F'(x_0) = \begin{pmatrix} — & \nabla f^1(x_0) & — \\ & \vdots & \\ — & \nabla f^k(x_0) & — \end{pmatrix}$$

Defn: $0 \in \mathbb{R}^k$ is a regular value of F iff $\forall x \in F^{-1}(0)$, $F'(x)$ has rank k. (I.e. it defines a surjective linear map onto $\mathbb{R}^k$.)

$$F' = \underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f^k}{\partial x_1} & \cdots & \frac{\partial f^k}{\partial x_N} \end{pmatrix}}_{N} \left.\vphantom{\begin{pmatrix} \frac{\partial f^1}{\partial x_1} \\ \vdots \\ \frac{\partial f^k}{\partial x_1} \end{pmatrix}}\right\} k$$

Thm: Assume $0 \in \mathbb{R}^k$ is a regular value of F. Then $\forall x_0 \in F^{-1}(0)$, there's some neighborhood W of x_0 in the ambient space s.t. $F^{-1}(0) \cap W$ is the graph of a function of $N-k$ variables into $\mathbb{R}^k$, up to a permutation of the coordinates $x_1, \dots, x_N$. *This is just the implicit function theorem!*

Ex: An example of "permutation of coordinates".
Let $k = 2$ and $N = 5$ (so $N - k = 3$). Let $A \subseteq \mathbb{R}^3$ open, $G = (g^1, g^2) : A \to \mathbb{R}^2$. Then

$$F^{-1}(0) \cap W = \big\{(x_1, x_2, x_3, x_4, x_5) \mid x_2 = g^1(x_1, x_3, x_4), x_5 = g^2(x_1, x_3, x_4)\big\}$$