Math 591 Lecture 17

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10/9/20

Recall that a submersion $F : M \to N$ satisfies $\forall p \in M$, $F_{*,p}$ is onto.

Fibrations are a special class of submersions. For any submersion, its fibers are $F^{-1}(q)$, $\forall q \in N$. $F^{-1}(q)$ is a regular submanifold by the regular value theorem. But not all fibers are diffeomorphic to each other. And $F^{-1}(q) = \emptyset$ is possible.

Also, the normal form theorem for submersions (also called "the submersion theorem") states that locally, a submersion is a map of the form $(x', x'') \mapsto x'$.

Immersions

Defn: A smooth map $F : M \to N$ is an <u>immersion at $p \in M$</u> $\Leftrightarrow$ $F_{*,p}$ is injective. Note that this requires $\dim M \leq \dim N$.

Defn: F is an <u>immersion</u> iff $\forall p \in M$, F is an immersion at p.

Ex: Our model case is $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ with $r \mapsto (r, 0)$, where $k \leq n$.

Ex:

1) If $S \subset N$ is a regular submanifold, the inclusion $\iota : S \hookrightarrow N$ is an immersion, since $\forall p \in S$, $\iota_{*,p} : T_pS \hookrightarrow T_pN$.
2) Immersions don't have to be injecive. Consider the lemniscate $F : \mathbb{R} \to \mathbb{R}^2$. $\forall t \in \mathbb{R}$, $\dot{F}(t) \neq 0$, so F is an immersion. But it's not injective!
3) Restrict the domain of the lemniscate to obtain a one-to-one map whose image is a regular submanifold.
4) Can also restrict the domain of the lemniscate to obtain a one-to-one map whose image is <u>not</u> a regular submanifold. (I.e. it goes right up to the point where the curve would self intersect, but doesn't map to the intersection point more than once.)
5) Let $N = \mathbb{R}^2/\mathbb{Z}^2$ (where $\mathbb{Z}^2$ is the integer lattice). Define $\pi : \mathbb{R}^2 \to N$, and declare this to be a local diffeomorphism, giving us a smooth structure on N. This is the 2-Torus. Let $F : \mathbb{R} \to \mathbb{R}^2 \xrightarrow{\pi} N$. Fix $v = (a, b) \in \mathbb{R}^2 \setminus \{0\}$. Sy $F(t) = \pi(tv)$, $\forall t \in \mathbb{R}$. If a and b are rational, you get a periodic curve around the torus. These are all immersions. If a/b is irrational, then the image of F in $\mathbb{R}^2/\mathbb{Z}^2$ is dense! With the subspace topology, $F(\mathbb{R})$ is very much so *not* locally Euclidean.

Thm: (Normal Form for Immersions) Let $F : M \to N$ be an immersion at $p \in M$. Then there are coordinates $p \in U, \phi = (x^1, \ldots, x^m)$ of M and $f(p) \in V, \psi = (y^1, \ldots, y^n)$ of N such that $U \subset F^{-1}(V)$ and $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n, \underbrace{0, \ldots, 0}_{n-m \text{ zeros}})$. We have

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\phi"] & V \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Proof: Next time...

Defn: An immersion $F : M \to N$ hich is a homeomorphism onto $F(M)$ with the subspace topology is called an <u>embedding</u>. This is a global property!

Observe: Embeddings are injective.