Math 591 Lecture 23

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Uniqueness of Integral Curves

Last time, given $X \in \mathfrak{X}(M)$, we defined integral curves of X, and proved *local* existence and uniqueness by reducing to the Euclidean case and using theory from ordinary differential equations. Today, we'll start with global uniqueness.

Lemma: Assume that $c_1, c_2 : (\alpha, \beta) \to M$ (with $\alpha < 0 < \beta$) are integral curves of X, and $c_1(0) = c_2(0) = p$. Then $\forall t \in (\alpha, \beta)$, $c_1(t) = c_2(t)$.

Proof: Assume not. Then $S \stackrel{\text{def}}{=} \{t \in (\alpha, \beta) \mid t > 0, c_1(t) \neq c_2(t)\} \neq \emptyset$. Well, this set is bounded below and nonempty, so let $\tau = \inf S$.

We claim that $c_1(\tau) = c_2(\tau)$ – assume not. Then because M is Hasudorff, there exist neighborhoods U_1 around $c_1(\tau)$ and U_2 around $c_2(\tau)$ such that $U_1 \cap U_2 = \emptyset$. By the continuity of c_1 and c_2, $\exists t < \tau$ such that $c_1(t) \neq c_2(t)$. Thus, τ is not the infimum of S. Oops! This is a contradiction, so we must have $c_1(\tau) = c_2(\tau)$.

Now, we use local uniqueness of integral curves of X. Let c be an integral curve with initial condition $c(\tau) = c_1(\tau) = c_2(\tau)$. By local uniqueness, c must agree with c_1 and c_2 on a neighborhood of τ, so $\tau < \inf S$. Oops!

Therefore, we must have $c_1(t) = c_2(t)$, $\forall t \in (\alpha, \beta)$. $\square$

Cor: Given $X \in \mathfrak{X}(M)$ and $p \in M$, there is an interval $(\alpha(p), \beta(p))$ containing 0 (possibly with $\alpha(p) = -\infty$ and/or $\beta(p) = +\infty$), and an integral curve $c : (\alpha(p), \beta(p)) \to M$ of X with $c(0) = p$, such that for any other integral curve $\tilde{c} : I \to M$ (with I and open interval) with $\tilde{c}(0) = p$, one has $I \subset (\alpha(p), \beta(p))$ and on I, $c|_I = \tilde{c}$.

Proof: Let $\mathcal{I}$ be the set of intervals which are domains for some integral curve c of X with $c(0) = p$. Then we have $(\alpha(p), \beta(p)) = \bigcup_{I \in \mathcal{I}} I$. Any wo $c_1 : I_1 \to M$, $c_2 : I_2 \to M$ agree on their overlap, $I_1 \cap I_2$, so they define an integral curve on their union, $I_1 \cup I_2$. Doing this for all $I \in \mathcal{I}$ gives us the desired integral curve. $\square$

Defn: Such an integral curve is the unique <u>maximal integral curve</u> of X through p at $t = 0$.

We can refer to a chapter on vector fields in the book by Boothby, but it's a little too detailed.

Defn: Given $X \in \mathfrak{X}(M)$, define $\mathcal{W} = \{(t, p) \in \mathbb{R} \times M \mid t \in (\alpha(p), \beta(p))\} \subseteq \mathbb{R} \times M$. This is the domain of a map $\phi : \mathcal{W} \to M$, which takes a pair (t, p) to he unique maximal integral curve of X with initial condition p at time t. In other words, $\forall (t, p) \in \mathcal{W}$, $\phi(0, p) = p$ and $\frac{\partial \phi}{\partial t}(t, p) = X_{\phi(t,p)}$. ϕ is called the <u>flow</u> of X.

Thm: $\mathcal{W} \subseteq \mathbb{R} \times M$ is open, and ϕ is a smooth map (of t, p).

This general theorem is rather challenging to prove, but the local version (which is included in our textbook) is sufficient for our purposes.

Thm: Let $X \in \mathfrak{X}(M)$, $p \in M$. Then there exists a neighborhood V of p, $\varepsilon > 0$, and a smooth map $\phi : (-\varepsilon, \varepsilon) \times V \to M$ such that

- $\phi(0, q) = q$, $\forall q \in V$
- $\frac{\partial \phi}{\partial t}(t, q) = X_{\phi(t,q)}$. (Note: This is the velocity of the curve $t \mapsto \phi(t, q)$ at time t. That is, $t \mapsto \phi(t, q)$ is an integral curve of X with initial condition q.)

Proof: Just quote Calc IV/diffeq. In case $M = \mathbb{R}^n$, this is a theorem. Then just use local coordinates to reduce any manifold to the Euclidean case. $\square$

Note: This isn't as fancy as the previous (global) version, but it's enough for our purposes. $(-\varepsilon, \varepsilon) \times V$ is sometimes referred to as a "flow box". $\mathcal{W}$ may be quite complicated, but the flow boxes are always easy to work with.

Main points:

1. ϕ is C^∞ in (t, p). We refer to this as "smooth dependence on initial conditions".
2. $\varepsilon > 0$ can be uniform on V.

Notation: It is standard to write $\phi(t, p) = \phi_t(p)$. This emphasizes, in the local flow theorem, that $\forall t \in (-\varepsilon, \varepsilon)$, we can think of the map $\phi_t : V \to M$. This is called the "time t map". Think about it as moving every point in V by time t along their respective integral curves. In other words, it takes the blob V to a new blob $\phi_t(V)$.

Thm: Given $X \in \mathfrak{X}(M)$, $p \in M$. If $t, s, t+s \in (\alpha(p), \beta(p))$, then

$$\phi_t(\phi_s(p)) = \phi_{t+s}(p) = \phi_{s+t}(p) = \phi_s(\phi_t(p)).$$

This is known as the "1-parameter group".

Proof: Fix s, and consider the curves $t \mapsto \phi_{t+s}(p)$ and $t \mapsto \phi_t(\phi_s(p))$. Both are integral curves of X, with the same initial conditions (check that this is true), so by uniqueness, they're the same curve. The rest follows from commutativity of addition. $\square$

So altogether, if there are no domain issues,

$$\phi_t \circ \phi_s = \phi_{t+s} = \phi_{s+t} = \phi_s \circ \phi_t.$$

For complete fields, this shows that $t \mapsto \phi_t$ is a map $\mathbb{R} \to \{\text{All diffeomorphisms } M \to M\} \stackrel{\text{def}}{=} \text{Diff}(M)$, and this map is a group morphism from $(\mathbb{R}, +) \to \text{Diff}(M)$!