Math 591 Lecture 33

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Global formulae for d

First, we want to consider the pairing of forms and vector fields. If $\alpha \in \Omega^k$ and $X_1, \dots, X_k \in \mathfrak{X}(M)$, then $\alpha(X_1, \dots, X_k) : M \to \mathbb{R}$ is a function on M.

Prop: (1) If $\alpha \in \Omega^1(M)$, $Y_0, Y_1 \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, Y_1) = Y_0(\alpha(Y_1)) - Y_1(\alpha(Y_0)) - \alpha([Y_0, Y_1])$$

Proof: WOLOG assume $\alpha = f\,dg$. Then $d\alpha = df \wedge dg$. Thus,

$$(d\alpha)(Y_0, Y_1) = df(Y_0)dg(Y_1) - df(Y_1)dg(Y_0) = Y_0(f)Y_1(g) - Y_1(f)Y_0(g)$$

Well, the RHS of this is

$$Y_0(fY_1(g)) - Y_1(fY_0(g)) - f\,[Y_0, Y_1]\,g = (Y_0f)(Y_1g) + \cancel{fY_0Y_1g} - (Y_1fY_0g + \cancel{fY_1Y_0g}) - f(\cancel{Y_0Y_1g} - \cancel{Y_1Y_0g})$$

$\square$

Prop: (2) If $\alpha \in \Omega^2(M)$, $Y_0, Y_1, Y_2 \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, Y_1, Y_2) = Y_0\alpha(Y_1, Y_2) - Y_1\alpha(Y_0, Y_2) + Y_2\alpha(Y_0, Y_1) - \alpha([Y_0, Y_1], Y_2) + \alpha([Y_0, Y_2], Y_1) - \alpha([Y_1, Y_2], Y_0)$$

In general, if $\alpha \in \Omega^k(M)$, $Y_0, \dots, Y_k \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, \dots, Y_k) = \sum_{j=0}^{k} (-1)^j Y_j \alpha(Y_0, \dots, \widehat{Y_j}, \dots, Y_k) + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([Y_i, Y_j], Y_0, \dots, \widehat{Y_i}, \dots, \widehat{Y_j}, \dots, Y_k)$$

where $\widehat{\cdot}$ indicates that we omit that term.

Refer to Proposition 20.14 in Tu/Proposition 14.32 in Lee.

Another way to think of this is, gven $\alpha \in \Omega^k(M)$, $X_1, \dots, X_k \in \mathfrak{X}(M)$, and $p \in M$, we have $(X_i)_p \in T_pM$, so

$$\alpha_p : \underbrace{T_pM \times \cdots \times T_pM}_{k} = (T_pM)^k \to \mathbb{R}$$
$$p \mapsto \alpha_p((X_1)_p, \dots, (X_k)_p) \in \mathbb{R}$$

Lie Derivatives of Forms

Defn: Let $X \in \mathfrak{X}(M)$, $\alpha \in \Omega^k(M)$, and $p \in M$. Let φ be the local flow of X. Form the curve $(-\varepsilon, \varepsilon) \ni t \mapsto (\varphi_t^*\alpha)_p \in \bigwedge^k(T_p^*M)$. (Note that $\bigwedge^k(T_p^*M)$ is independent of t.) Then we defined

$$(\mathcal{L}_X\alpha)_p = \frac{d}{dt}(\varphi_t^*\alpha)_p\bigg|_{t=0}$$

Observe: If $k = 0$, then α is a function on M, so

$$(\mathcal{L}_X\alpha)_p = \frac{d}{dt}(\alpha \circ \varphi_t)_p\bigg|_{t=0} = \frac{d}{dt}(\alpha(\varphi_t(p)))\bigg|_{t=0} = \dot{\varphi}_t\bigg|_{t=0}(\alpha)$$

Thus, $(\mathcal{L}_X\alpha)_p = X(\alpha)(p)$ if α is a function.

First Properties

Prop:

a) The expression $\mathcal{L}_X(\alpha)$ is $\mathbb{R}$-linear w.r.t X and α.
b) $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$ – $\mathcal{L}_X$ is a derivation.
c) $\mathcal{L}_X(d\alpha) = d\mathcal{L}_X(\alpha)$.
d) If $F : M \to M$ is a diffeomorphism, then $\mathcal{L}_X(F^*\alpha) = F^*\left(\mathcal{L}_{F_*(X)}\alpha\right)$.

Proof: (b) holds because $\varphi_t^*(\alpha \wedge \beta) = (\varphi_t^*\alpha) \wedge (\varphi_t^*\beta)$. We just need to show the right hand side satisfies he usual product rule with respect to $\frac{d}{dt}$.
(c) holds because $\varphi_t^*(d\alpha) = d(\varphi_t^*\alpha)$, $\forall t$. This uses the fact that d and $\frac{d}{dt}$ commute, i.e., $\frac{\partial^2}{\partial t \partial x^j} = \frac{\partial^2}{\partial x^j \partial t}$.
(d) holds because for $p \in M$, $v_1, \ldots, v_k \in T_pM$,

$$\mathcal{L}_X(F^*\alpha)_p(v_1, \ldots, v_k) = \frac{d}{dt}\underbrace{\varphi_t^*(F^*\alpha)_p}_{(\varphi_t^*\circ F^*)(\alpha)=(F\circ\varphi_t)^*(\alpha)}(v_1, \ldots, v_k)\bigg|_{t=0} = \frac{d}{dt}\alpha((F\circ\varphi_t)_{*,p})(v_1), \ldots, (F\circ\varphi_t)_{*,p}(v_k)\bigg|_{t=0}$$

Observe: $t \mapsto F \circ \varphi_t$ is the integral curve of $F_*(X)$ at $F(p)$. So

$$\mathcal{L}_X(F^*\alpha)_p(v_1, \ldots, v_k) = \cdots = (\mathcal{L}_{F_*X}\alpha)(F_{*,p}(v_1), \ldots, F_{*,p}(v_k)) = F^*(\mathcal{L}_{F_*(X)}\alpha)(v_1, \ldots, v_k)$$

□

Cor: If $\mathcal{L}_X\alpha = 0$ (everywhere), then $\varphi_t^*\alpha = \alpha$, $\forall t$.

Proof: We will show $\forall p \in M$, the curve $t \mapsto (\varphi_t^*\alpha)_p \in \bigwedge^k T_p^*M$ is constant.

Proof: $\forall s$,

$$\frac{d}{dt}\varphi_t^*\alpha\bigg|_{t=s} = \frac{d}{dt}(\varphi_{t+s}^*\alpha)\bigg|_{t=0} = \frac{d}{dt}\varphi_t^*(\varphi_s^*(\alpha))\bigg|_{t=0} = \mathcal{L}_X(\varphi_s^*\alpha) = \varphi_s^*\mathcal{L}_{\underbrace{(\varphi_s)_*(X)}_{=X}}\alpha = \varphi_s^*\underbrace{\mathcal{L}_X\alpha}_{=0} = 0$$

□

Towards Cartan's Formula

We need interior multiplication/contraction.

Defn: Given $\alpha \in \Omega^k$, $X \in \mathfrak{X}(M)$, we define $\iota_X\alpha \in \Omega^{k-1}$ by, $\forall p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$,

$$(\iota_X\alpha)_p(v_1, \ldots, v_{k-1}) = \alpha_p(X_p, v_1, \ldots, v_k)$$

Thm: (Cartan's Magic Formula) $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d : \Omega^k \to \Omega^k$.

Picture:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1} \arrow[r, "d"] & \Omega^k \arrow[r, "d"] \arrow[dl, "\iota_X"'] \arrow[d, Rightarrow, "\mathcal{L}_X"] & \Omega^{k+1} \arrow[r, "d"] \arrow[dl, "\iota_X"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1} \arrow[r, "d"] & \Omega^k \arrow[r, "d"] & \Omega^{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

where $\mathcal{L}_X$ is the "sum of two paths".

On Friday, we'll show that when $\alpha = dx \wedge dy$ and $M = \mathbb{R}^2$, then $\mathcal{L}_X(\alpha) = (\operatorname{div} X)\alpha$, in the calc III sense.