Math 591 Lecture 14

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Defn: Let $F : M \to N$ be smooth.

- F is a <u>submersion at $p \in M$</u> $\Leftrightarrow F_{*,p}$ is onto. We say p is a <u>regular point</u> of F.
- F is a <u>submersion</u> $\Leftrightarrow \forall p \in M$, F is a submersion at p.
- A <u>critical point</u> is a point which is not a regular point.
- $q \in N$ is a <u>regular value</u> $\Leftrightarrow \forall p \in F^{-1}(q)$, F is a submersion at p.

Observe: If $N = \mathbb{R}$, then either $F_{*,p} : T_pM \to T_{F(p)}\mathbb{R} \cong \mathbb{R}$ is onto, or it is zero. So our definition is consistent with that used in single-variable calculus.

Observe: If $q \in N$ satisfies $F^{-1}(q) = \emptyset$, then q is a regular value.

Thm: (Normal Form Theorem) Assume $F : M \to N$ is a submersion at $p \in M$. Let $m = \dim M$ and $n = \dim N$. Then there are coordinate charts (U, ϕ) around p and (V, ψ) around $F(p)$, with $U \subseteq F^{-1}(V)$, such that for $\tilde{F} = \psi \circ F \circ \phi^{-1}$, $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n)$.

$$\begin{array}{ccc} M & \xrightarrow{\quad F \quad} & N \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \psi} \\ \mathbb{R}^m \supseteq \phi(U) & \xrightarrow{\tilde{F}} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Proof: Start with any coordinates $\phi = (x^1, \ldots, x^m)$, $\psi = (y^1, \ldots, y^n)$. Wrie $F = (F^1, \ldots, F^n)$, with each $F^i = y^i \circ F$. Then the Jacobian of F at p is

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^m}(p) \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \cdots & \frac{\partial F^n}{\partial x^m}(p) \end{bmatrix}$$

We know J has full rank. So by permuting the x^j's, WOLOG $J : (M \; *)$ where M is $n \times n$ and full rank.
Define $\tilde{\phi} = (F^1, \ldots, F^n, x^{n+1}, \ldots, x^m)$. We claim that $\tilde{\phi} : U \to \mathbb{R}^m$ is a local diffeomorphism at p, and these are the desired coordinates.
Well, the Jacobian of $\tilde{\phi}$ in the x coordinates is

$$\begin{bmatrix} M & * \\ 0 & I_{m-n} \end{bmatrix}$$

because $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$. And this matrix is invertible because M is. $\square$

Defn: Let M be a manifold. Let $S \subseteq M$. Then S is a (<u>regular</u>) <u>submanifold</u> of M iff there is some $k \in \mathbb{N}$ s.t. $\forall p \in S$, there is a chart (U, ϕ) of M with $p \in U$, such that $S \cap U = \left\{q \in U \mid x^j(q) = 0 \text{ for } j \in \{k+1, \ldots, m\}\right\}$. Such a coordinate system (U, ϕ) is <u>adapted</u> to the submanifold and defines an <u>induced chart</u> on S by $\phi_S : S \cap U \to \mathbb{R}^k$, where $\phi_S = (x^1, \ldots, x^k)\big|_{S \cap U}$.

Claim: $\phi_S : S \cap U \to \phi(S \cap U) \overset{\text{open}}{\subseteq} \mathbb{R}^k$ is a homeomorphism with respect to the subspace topology.

Claim: With the subspace topology and an atlas of adapted charts, S inherits a C^∞ structure. That is, any two charts on S arising from adapted coordinate systems are C^∞-compatible.

The idea of the proof is given two charts (U, ϕ) and (V, ψ), $\psi_S \circ (\phi_S)^{-1}$ is the restriction of $\psi \circ \phi^{-1}$, so the former is C^∞.

We also need to check the point-set topology conditions: Hausdorff and second-countable. But they follow directly from the subspace topology.

Can a chart on the submanifold be extended to a chart on the original manifold? Yes!
Observe: If we start with $(x^1, \ldots, x^k) : W \to \mathbb{R}^k$, a chart of S, then there is a chart ϕ on $U \overset{\text{open}}{\subseteq} M$ s.t. $W = U \cap S$ and $(x^1, \ldots, x^k) = \phi_S$.
Proof: Skip for now...

Prop: If $F : \mathbb{R}^N \to \mathbb{R}^{N-k}$ and 0 is a regular value of F, then $F^{-1}(0)$ is a regular submanifold of $\mathbb{R}^N$.

> Proof: WOLOG S is the graph of $G : A \to \mathbb{R}^{N-k}$, with $A \overset{\text{open}}{\subseteq} \mathbb{R}^k$. $S = \{(a, G(a)) \mid a \in A\}$.
> Define $x^i = x^i$ for $1 \leq i \leq k$, and $x^j = x^j - G^j(x^1, \ldots, x^k)$ for $k+1 \leq j \leq N$, with $G = (G^1, \ldots, G^{N-k})$. These are adapted coordinates! $\square$

The same proof, using the normal form theorem, gives:
Thm: If $q \in N$ is a regular value of $F : M \to N$, then $S = F^{-1}(q) \subseteq M$ is a regular submanifold.