Math 591 Lecture 10

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9/23/20

Review: Partial Derivatives

Given $p \in U \subseteq M$ and $\phi : U \to \mathbb{R}^n$, a coordinate chart, we can write $\phi = (x^1, \dots, x^n)$ where each $x^i : U \to \mathbb{R}$. Then we defined

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial f_\phi}{\partial r^i}(\phi(p))$$

where $f_\phi = f \circ \phi^{-1}$, and r^i is simply the coordinate in $\mathbb{R}^n$.

Ex: Take $M \subseteq \mathbb{R}^3$ defined as the graph of $G : A \to \mathbb{R}$ where A is an open subset of $\mathbb{R}^2$ and G is C^∞. Then we can write $M = \{(r^1, r^2, G(r^1, r^2)) \mid (r^1, r^2) \in A\}$. There is one chart: $U = M$, ϕ is projection onto the (r^1, r^2) plane. $\phi^{-1}(r^1, r^2) = (r^1, r^2, G(r^1, r^2))$.

Let $f : M \to \mathbb{R}$ be the restriction of some $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ C^∞. Then

$$f_\phi(^1, r^2) = (f \circ \phi^{-1})(r^1, r^2) = \tilde{f}(\underbrace{r^1, r^2, G(r^1, r^2)}_{p \in M})$$

Compute:

$$\frac{\partial f}{\partial x^1}(p) = \frac{\partial \tilde{f}}{\partial r^1}(p) + \frac{\partial G}{\partial r^1}(r^1, r^2)\frac{\partial \tilde{f}}{\partial r^3}(p)$$

Observe: $\frac{\partial f}{\partial x^1}(p) = \nabla \tilde{f} \cdot \dot{\gamma}$, for $\gamma(t) = (r^1 + t, r^2, G(r^1 + t, r^2)) \in M$, so $\dot{\gamma}(t)|_{t=0} = (1, 0, \frac{\partial g}{\partial r^1}(r^1, r^2))$.

Now, back to the theorem from last time:

Thm: If $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ is a chart, and $\phi = (x^1, \dots, x^n)$, then $\forall D \in T_pM$, one has

$$D = \sum_{j=1}^n D([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p, \quad [x^i] \in C_p^\infty(M)$$

Proof: It's based on the following observations:

- The set of derivations, T_pM, is an $\mathbb{R}$-vector space.
- For any constant function k, $\forall D \in T_pM$, $D[k] = 0$.
  Proof: $D([1]) = D([1^2]) = 2D([1])$ by the product rule, so $D([1]) = 0$. Then linearity implies $D[k] = 0$.
- $\forall D \in T_pM$, if $[f](p) = [g](p) = 0$, then $D([f][g]) = 0$.

We'll start with what we had last time.

$$f_\phi(r) = f(p) + \sum_{i=1}^n (r^i - r_0^i)\frac{\partial f_\phi}{\partial r^i}(r_0) + \frac{1}{2}\sum_{i,j=1}^n (r^i - r_0^i)(r^j - r_0^j) \cdot g_{ij}(r)$$

Composing with ϕ yields

$$f(r) = f(p) + \sum_{i=1}^n (x^i - x_0^i)\frac{\partial f}{\partial x^i}(p) + \frac{1}{2}\sum_{i,j=1}^n (x^i - x_0^i)(x^j - x_0^j) \cdot g_{ij}(x)$$

Apply D. Well, $f(p)$ is constant, so it vanishes. And the last term is second order, so based on the above observation, it vanishes as well. We're left with

$$D([f]) = \sum_{i=1}^n D([x^i - x_0^i])\frac{\partial f}{\partial x}^i(p) = \sum_{i=1}^n D([x^i])\frac{\partial f}{\partial x^i}(p)$$

Thus,

$$D = \sum_{i=1}^{n} D([x^i]) \left. \frac{\partial}{\partial x^i} \right|_p$$

$\square$

Cor: It's easy to check that $\dfrac{\partial x^i}{\partial x^j} = \delta_{ij}$, so $\left\{ \left. \dfrac{\partial}{\partial x^i} \right|_p \right\}_{i=1}^{k}$ is a basis of T_pM over $\mathbb{R}$.

In summary,

- We are defining tangent vectors by T_pM, which is the set of derivations at p. We'll be changing our notation: $u, v, w, \ldots \in T_pM$.
- Representation of vectors in coordinates:

$$v = \sum_{i=1}^{n} v_i \left. \frac{\partial}{\partial x^i} \right|_p \qquad (v_1, \ldots, v_n) \in \mathbb{R}^n$$

- Also, $v = \dot{\gamma}$ for some $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ C^∞, with $\gamma(t_0) = p$.
- Note for later: $p \neq q \Rightarrow T_pM \cap T_qM = \emptyset$.

Lemma: $\dot{\gamma}(t_0) = \displaystyle\sum_{i=1}^{n} \frac{\partial x^i}{\partial t}(t_0) \left. \frac{\partial}{\partial x} \right|_p$ if $\gamma(t) = (x^1(t), \ldots, x^n(t)) \in \mathbb{R}^n$.

Differentials of Functions

Defn: Let $p \in U \overset{\text{open}}{\subseteq} M$, $f : U \to \mathbb{R}$ C^∞. Then we define

$$\begin{aligned} df_p : T_pM &\to \mathbb{R} \\ v &\mapsto v[f] \end{aligned}$$

Notation: we say $T_pU \overset{\text{def}}{=} T_pM$.

Note: $df_p \in (T_pM)^*$, the dual of the tangent space.

Defn: $T_p^*M = (T_pM)^*$ is called the cotangent space of M at p. $df_p \in T_p^*M$.

Note: $df_p(v) = v[f]$.

In coordinates, we saw that if $\phi = (x^1, \ldots, x^n)$ and $\phi(p) = (x_0^1, \ldots, x_0^n) \in \mathbb{R}^n$, then

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - x_0^i) + O(2)$$

(With $O(2)$ denoting something that vanishes in the second order at p.) Then

$$v[f] = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v([x^i])$$

By definition, $v[x^i] = dx^i(v)$. We conclude that

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) dx_p^i$$

This is just like it was in Calc III!

Note that, in some ways, it makes more sense to first define T_p^*M, and then obtain T_pM as its dual.

Defn: $I_p = \left\{ [f] \in C_p^\infty(M) \mid [f](p) = 0 \right\}$, an ideal in the ring of germs.

$$I_p^2 = \left\{ \sum_{i,j} [f_i][g_j] : [f_i], [g_j] \in I_p \right\}$$

is the set of "$O(2)$ germs". We claim that $T_p^*M \cong I_p / I_p^2$. Then df is the class of $[f - f(p)] \in I_p / I_p^2$.