Math 591 Lecture 9

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Ex: (of germs)

Let $p \in \mathbb{R}^n$. Then $C_p^\infty(\mathbb{R}^n) = \left\{(U, f) \mid p \in U \overset{\text{open}}{\subseteq} \mathbb{R}^n, f : U \to \mathbb{R}\ C^\infty\right\} / \sim$.

Observe: There is a well-defined map

$$\begin{aligned} C_p^\infty(\mathbb{R}^n) &\to \mathbb{R}[[r]], r = (r^1, \ldots, r^n) \\ [f] &\mapsto f(p) + \textstyle\sum_{j=1}^n (r^j - r_0^j) \frac{\partial f}{\partial r^j}(p) + \cdots \end{aligned}$$

where $\mathbb{R}[[r]]$ is the set of formal power series in the r^i variables, and $[f]$ maps to the Taylor series of f at $p = (r_0^1, \ldots, r_0^n)$. Why is this well defined? Well, if $[f] = [g]$, then f and g agree on a neighborhood of p.

Prop:

(1) This map is a surjection, i.e., any formal power series is the Taylor series of some smooth functions.

(2) This map is *not* injective, i.e., there exist C^∞ functions f defined near p s.t. $\forall \alpha$ multi-indices, $\frac{\partial^\alpha f}{\partial r^\alpha}(p) = 0$, but f is not zero near p.

This is just an FYI – we're not going to use this for a while.

Now, back to manifolds...
Let M be a C^∞ manifold, and $p \in M$. We defined $T_pM = \{\text{all derivations } D : C_p^\infty(M) \to \mathbb{R}\}$.

Ex: (of derivations)

Start with a curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ smooth s.t. $\gamma(t_0) = p$. Define

$$\begin{aligned} \dot\gamma(t_0) : C_p^\infty(M) &\to \mathbb{R} \\ [f] &\mapsto \tfrac{d}{dt}(f \circ \gamma)(t)\big|_{t=t_0} \end{aligned}$$

It's easy to check that $\dot\gamma(t_0)$ is a derivation (by calc III stuff). Note that this defines $\dot\gamma(t_0) \in T_{\gamma(t_0)}M$.
We will see today that *all* derivations are of this form.

Observe: In the case where $M \subseteq \mathbb{R}^N$ is a local graph, then $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M \hookrightarrow \mathbb{R}^N$ can be interepreted as a smooth curve in $\mathbb{R}^N$. $\dot\gamma(t_0)$ was defined in calc III as an element in $\mathbb{R}^N$. These definitions are consistent! But our definition doesn't need an ambient space.

Introducting Local Coordinates and Partial Derivatives

Let $p \in U \subseteq M$, with $\phi : U \to \mathbb{R}^N$ a chart. Then we use the notation $r^i : \mathbb{R}^N \to \mathbb{R}$ is the ith component/coordinate.
We say $x^i = r^i \circ \phi : U \to \mathbb{R}$, so we can write $\phi = (x^1, x^2, \ldots, x^N)$. (The x^is are defined on $U \subseteq M$.)

Defn: Given $f : U \to \mathbb{R}$ smooth, $p \in U$,

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial r^i}(f \circ \phi^{-1})[\phi(p)] \in \mathbb{R}$$

Some notation: we write $f_\phi \overset{\text{def}}{=} f \circ \phi^{-1}$. Observe that $\frac{\partial f}{\partial x^i} = \frac{\partial f_\phi}{\partial x^i} \circ \phi$.

Lemma: $\forall i \in \{1, \ldots, n\}$, the map

$$\left.\frac{\partial}{\partial x^i}\right|_p : C_p^\infty(M) \ni [f] \mapsto \frac{\partial f}{\partial x^i}(p)$$

is a derivation at p, and moreover,

$$\Phi \overset{\text{def}}{=} \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis (over $\mathbb{R}$) of T_pM.

Observe: $\left.\frac{\partial}{\partial x^i}\right|_p$ are velocities of curves. Let $\phi(p) = (r_0^1, \ldots, r_0^n)$. Then if $\gamma_i : t \mapsto \phi^{-1}(r_0^1, \ldots, r_0^i + t, \ldots, r_0^n)$ for $t \in (-\varepsilon, \varepsilon)$, we claim that $\dot{\gamma}(p) = \left.\frac{\partial}{\partial x^i}\right|_p$.

To actually prove that Φ is a basis, we need:

Thm: Let g be a C^∞ function defined in a neighborhood of a point $r_0 \in \mathbb{R}^n$. Then $\exists g_{ij}$, with $i, j \in \{1, \ldots, n\}$, that is smooth and defined near r_0, such that $\forall r \in \mathrm{dom}(g)$,

$$g(r) = \underbrace{g(0) + \sum_{j=1}^n (r^j - r_0^j) \frac{\partial g}{\partial r^j}}_{\text{First degree Taylor polynomial}} + \underbrace{\frac{1}{2} \sum_{i,j=1}^n (r^i - r_0^i)(r^j - r_0^j) \cdot g_{ij}(r)}_{\text{``An interesting way of writing the remainder''}}$$

(Moreover, $g_{ij}(r_0) = \frac{\partial^2 g}{\partial r^i \partial r^j}(r_0)$.)

Let $D \in T_pM$, $[f] \in C_p^\infty(M)$. Apply this to $g = f_\phi$. We claim that

$$D[f] = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) \cdot D([x^i])$$

This implies

$$D = \sum_{i=1}^n D([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p$$