

Math 591 Lecture 19

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

10/14/20

Thm: The image of an embedding $F : M \rightarrow N$ is a regular submanifold of N .

Proof: Let $q \in F(M)$. We need to show that there are coordinates of N near q adapted to $F(M)$. Let $p \in M$ s.t. $F(p) = q$. By the immersion theorem, there are coordinates $(U, \phi = (x^1, \dots, x^m))$ of M with $p \in U$, and $(V, \psi = (y^1, \dots, y^n))$ of N with $q \in V$, with $U \subseteq F^{-1}(V)$, such that $F(U) = (I, 0)$ (with $m - n$ zeros).

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \downarrow \phi & & \downarrow \psi \\ \mathbb{R}^m \supseteq \phi(U) & \xrightarrow{\tilde{F}} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Well, $F(U) = \{w \in V \mid y^j(w) = 0, \forall j \in \{1, \dots, n\}\}$. The point is that because F is an embedding (i.e. $F|_{F(M)} : M \rightarrow F(M)$ is a homeomorphism), then $F(U)$ is a relatively open set of $F(M)$. So there is an open set $W \subseteq N$ such that $F(U) = F(M) \cap W$. Therefore, $F(M) \cap W = \{w^{-1}V \mid y^j(w) = 0, \forall j \in \{1, \dots, n\}\}$. So ψ on $V \cap W$ is adapted to $F(M)$ at q . \square

We want a stronger statement for when an injective immersion is an embedding.

Prop: Let $F : M \rightarrow N$ be a continuous proper map between manifolds. Then F is closed.

Proof: Let $C \subseteq M$ be a closed set, and $q \in \overline{F(C)}$. We need to show $q \in F(C)$. Well, let $V \subseteq N$ be an open neighborhood of q such that \overline{V} is compact. (We can do this because Euclidean spaces are locally compact.) Observe then that $q \in \overline{F(C) \cap \overline{V}}$. $F^{-1}(\overline{V})$ is compact, since F is proper, so $C \cap F^{-1}(\overline{V})$ is compact. Therefore, $F(C \cap F^{-1}(\overline{V}))$ is compact, so it's closed. But $F(C \cap F^{-1}(\overline{V})) = F(C) \cap \overline{V}$ (this is a set-theoretic fact). So $F(C) \cap \overline{V}$ is closed. Since $q \in \overline{F(C) \cap \overline{V}}$, $q \in F(C) \cap \overline{V}$, so $q \in F(C)$. Thus, F is closed. \square

Vector Fields

Defn: Let M be a smooth manifold of dimension n . A vector field on M is a section of $TM \xrightarrow{\pi} M$. That is, it's a map $X : M \rightarrow TM$ s.t. $\forall p \in M, X_p \in T_p M$.

This is a slight abuse of notation. Update: Write $X(p) = (p, X_p)$. The p is the same because X is a section $\pi \circ X = I_M$.

Basically, a vector field on a manifold assigns a tangent vector to every point in a manifold.

Defn: A vector field X on M is smooth (C^∞) iff it is smooth as a map between manifolds $X : M \rightarrow TM$.

What does this mean (concretely)?

Let's think about smooth sections of vector bundles in general (e.g. TM and T^*M). Let

$$\begin{array}{c} \mathcal{E} \\ \downarrow \pi \\ M \end{array}$$

be a C^∞ vector bundle of rank $\rho \in \mathbb{N}$. This means \mathcal{E} is a manifold, π is C^∞ and onto, $\forall p \in M, \pi^{-1}(p)$ has the structure of a vector space over \mathbb{R} of dimension ρ , and there exists a family $\{(U_\alpha, \chi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of M and $\forall \alpha$,

χ_α is a diffeomorphism, and we have the vector bundle trivialization

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\chi_\alpha} & U_\alpha \times \mathbb{R}^\rho \\ & \searrow \pi & \swarrow \\ & U_\alpha & \end{array}$$

s.t. χ_α is linear on fibers, i.e., $\forall p \in U_\alpha$, we have

$$\chi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \xrightarrow{\cong} \{p\} \times \mathbb{R}^\rho \cong \mathbb{R}^\rho$$

The mapping from $\pi^{-1}(p)$ to \mathbb{R}^ρ is a linear isomorphism.

Ex: Let $\mathcal{E} = TM$. The trivializations are induced by coordinate charts. Let $U \subseteq M$, with $\phi = (x^1, \dots, x^n) : U \rightarrow \mathbb{R}^n$ be a coordinate chart. Then $\forall p \in U, i \in \{1, \dots, n\}$, $\frac{\partial}{\partial x^i}|_p \in T_p U = T_p M$. We have

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\chi} & U \times \mathbb{R}^n \\ & \searrow \pi & \swarrow \\ & U & \end{array}$$

defined by

$$\chi^{-1}(p, v) = \left(p, \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \Big|_p \right)$$

with $p \in U$ and $v = (v^1, \dots, v^n) \in \mathbb{R}^n$.

We proved that $\forall p \in M$, $\left\{ \frac{\partial}{\partial x^i} \Big|_p : i = 1, \dots, n \right\}$ is a basis of $T_p M$. So χ^{-1} is invertible and a linear isomorphism of the fibers.

(The definition of the smooth structure on TM is such that χ as above is a diffeomorphism.)

Ex: $T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$

Again, a coordinate chart on M induces a trivialization.

Use: $\forall \alpha \in T_p^*M$, $\alpha = \sum_{i=1}^n \alpha_i dx^i|_p$, where $dx^i|_p$ is dual to $\frac{\partial}{\partial x^i}|_p$, and $\{dx^i|_p \mid i = 1, \dots, n\}$ is a basis of T_p^*M .

Back to

$$\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \quad \left. \begin{array}{c} \nearrow s \\ \searrow \end{array} \right)$$

Let s be a section (i.e. $\pi \circ s = I_M$). When is s smooth? We will answer this question using trivializations. Given a trivialization χ ,

$$\begin{array}{ccc} \pi^{-1}(U) & \xleftarrow{\chi} & U \times \mathbb{R}^p \\ & \searrow \pi & \swarrow \pi \\ & U & \end{array} \quad \begin{array}{c} s|_U \\ \chi \circ (s|_U) \stackrel{\text{def}}{=} s_\chi \end{array}$$

We have $\forall p \in U$, $s_\chi(p) = (p, \star)$, where \star is given by a function $F : U \rightarrow \mathbb{R}^p$. F is just the projection onto \mathbb{R}^p composed with s_χ . So $\forall p \in U$, $s_\chi(p) = (p, F(p))$.

Claim: $s|_U$ is smooth iff $F : U \rightarrow \mathbb{R}^p$ is smooth.

Proof: Next time...