

Math 591 Lecture 16

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The Hopf Fibration

$$\begin{array}{ccc} \mathbb{C}^2 & \supset & S^3 \\ & \downarrow \pi & \\ \mathbb{RP}^1 & \cong & S^1 \backslash S^3 \end{array}$$

$$\begin{aligned} \pi(z_0, z_1) &= \{(e^{i\theta} z_0, e^{i\theta} z_1) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}. \\ \pi(z_0, z_1) &= [z_0 : z_1]. \end{aligned}$$

Claim: This is a fiber bundle with fiber $\Phi = S^1$.

For the covering of \mathbb{RP}^1 , we choose the same covering as used in the homework: $U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}$ and $U_1 = \{[z_0 : z_1] \mid z_1 \neq 0\}$.

Define $\mathcal{S}(\mathfrak{z}) = \frac{1}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then we have

$$\begin{array}{ccccc} & & & & S^3 \\ & & & \nearrow \mathcal{S} & \downarrow \pi \\ \mathbb{R}^2 \cong \mathbb{C} & \xleftarrow{\cong} & U_0 & \xrightarrow{\quad} & \mathbb{RP}^1 \\ \mathfrak{z} = \frac{z_1}{z_0} & \longleftrightarrow & [z_0 : z_1] & & \end{array}$$

Note: $\pi \circ \mathcal{S} = I_{U_0}$, since $\pi(\mathcal{S}(\mathfrak{z})) = \mathfrak{z}$.

Defn: If $\pi : M \rightarrow N$ is a fiber bundle, a section of π is a C^∞ map $\mathcal{S} : N \rightarrow M$ s.t. $\pi \circ \mathcal{S} = I_N$.

Observe: The Hopf fibration does not have a global section.

Observe: Local sections always exist, because they always exist for the trivial bundle $N \rightarrow N \times \Phi$: fix $\nu \in \Phi$, and map $p \mapsto (p, \nu)$.

More on the Hopf fibration... Define a trivialization

$$\begin{array}{ccc} \pi^{-1}(U_0) & \xrightarrow{\chi} & U_0 \times \Phi \\ & \searrow \quad \swarrow & \\ & U_0 & \end{array}$$

with $\chi(\mathfrak{z}, e^{i\theta}) = e^{i\theta} \mathcal{S}(\mathfrak{z}) = \frac{e^{i\theta}}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then $\pi \circ \chi = I_{U_0}$, and $\forall \mathfrak{z} \in U_0$, $\mathcal{S}(\mathfrak{z}) = \pi^{-1}(\mathfrak{z})$.

$\chi(z_0, z_1) = (\frac{z_1}{z_0}, \frac{z_0}{|z_0|})$ (note that $z_0 \neq 0$). Then $S^3 \cong \mathbb{R}^3 \cup \{\infty\} = \bigsqcup S^1$, an uncountable disjoint union.

Vector Bundles

Defn: $\pi : E \rightarrow B$ is a vector bundle with base B and rank $k \in \mathbb{N}$ iff

- a) It's a fibration.
- b) $\forall p \in B$, the fiber $\pi^{-1}(p)$ is a vector space.
- c) There is an open covering $\{U_\alpha\}$ of B and all trivializations $\chi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ restrict to linear maps on each fiber, i.e.,

$$\begin{array}{ccc} \chi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) & \xrightarrow{\cong} & \{p\} \times \mathbb{R}^k \cong \mathbb{R}^k \\ & \searrow \text{Linear Isomorphism} & \nearrow \end{array}$$

Note: $\forall p \in B$, the zero section applied to $p \in B$ gives $0 \in \pi^{-1}(p)$.

Ex:

- 1) The tangent bundle $TB \rightarrow B$.
- 2) The cotangent bundle $T^*B = \bigsqcup_{p \in B} \{p\} \times T_p^*B \rightarrow B$.

Defn: If $S \subseteq M$ is a submanifold, then the co-normal bundle of S is $\mathcal{N} = \left\{ (p, \alpha) \in T^*M \mid \alpha|_{T_p S} = 0 \right\}$.

The co-normal bundle is a vector bundle.

Note: If we give $T_p M$ a Euclidean inner product, we can identify

$$\begin{aligned} T_p^* M &\cong T_p M \\ \langle \cdot, v \rangle &\longleftrightarrow v \end{aligned}$$

Claim: $\mathcal{N} \subset T^*M$ is a submanifold, and $\pi : \mathcal{N} \rightarrow S$ is a vector bundle of rank $\text{codim } S = \dim M - \dim S$. The fiber of \mathcal{N} over p is the annihilator of $T_p S$.