Math 591 Lecture 24

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Recall: If $X \in \mathfrak{X}(M)$, $p \in M$, then there exists a neighborhood V of p, $\varepsilon > 0$, and a function $\phi : (-\varepsilon, \varepsilon) \times V \to M$ such that $\forall q \in V$, $t \mapsto \phi(t, q)$ is an integral curve of X with $\phi(0, p) = p$.

Notation: $\phi(t, q) = \phi_t(q)$. So $\forall t \in (-\varepsilon, \varepsilon)$, we can think of $\phi_t : V \to M$ (a "time t map").

Notation: Fix $X \in \mathfrak{X}(M)$. Then $\forall p \in M$, $(\alpha(p), \beta(p))$ is the domain of the (unique) maximal integral curve of X through p.

$\forall t \in \mathbb{R}$, let $M_t = \{p \in M \mid t \in (\alpha(p), \beta(p))\}$. M_t is the set of points whose integral curves are defined at time t. This is a little fussy for our purposes, since most of the vector fields we care about are complete.

From last time, $\mathcal{W} = \{(p, t) \in M \times \mathbb{R} \mid p \in M_t\}$. Recall that $\mathcal{W}$ is open in $M \times \mathbb{R}$. The map $\phi : \mathcal{W} \to M$ is the global flow of X.

Observe: For our purposes, we don't need the global theory as much. We'll concentrate on:

a) Local flows $\phi : (-\varepsilon, \varepsilon) \times V \to M$ (uniform time)
b) Complete fields, i.e., those $X \in \mathfrak{X}(M)$ for which $\forall p \in M$, $(\alpha(p), \beta(p)) = \mathbb{R}$.

Recall the example from last friday.

Ex: $X = yx^2 \frac{\partial}{\partial x}$. Then $x(t) = \frac{x(0)}{1 - x(0)y(0)t}$ and $y(t) = y(0)$. This is not a complete vector field. $\phi_t(x, y) = (\frac{x}{1 - xyt}, y)$.

Ex: (From physics) Let $M = \mathbb{R}^2$ with coordinates (x, p). Let $\dot{x} = p$ and $\dot{p} = 0$. Then the integral curves of $x = p\frac{\partial}{\partial x} + 0\frac{\partial}{\partial p}$ are of the form $x(t) = tp + x(0)$, $p(t) = p(0)$. Thus,

$$\phi_t(x, p) = (x + tp, p) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

This is a linear shear. Note that the integral curves are just horizontal lines.

Prop: If $t, s, t + s \in (\alpha(p), \beta(p))$, then $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

We call this our group law. This means, where defined, $\phi_{t+s} = \phi_t \circ \phi_s$, $\forall t, s \in \mathbb{R}$.

Let's amplify this idea. Suppose the vector field is complete. From the group law, $\forall t \in \mathbb{R}$, $\phi_t \circ \phi_{-t} = I$, i.e., $\phi_t : M \to M$ is a diffeomorphism with inverse $(\phi_t)^{-1} = \phi_{-t}$.

Also, the mapping $\mathbb{R} \to \mathrm{Diff}(M)$ is a group morphism from $(\mathbb{R}, +)$ to $(\mathrm{Diff}(M), \circ)$. In this case, ϕ is called a one-parameter group of diffeomorphisms.

More generally, one can consider smooth maps $\phi : \mathbb{R} \times M \to M$, and define $\phi_t(p) = \phi(t, p)$. Then $\{\phi_t\}_{t \in \mathbb{R}}$ is a smooth one-parameter family of maps $M \to M$. The ones which satisfy the group law $\phi_{t+s} = \phi_t \circ \phi_s$ correspond precisely to vector fields. Specifically, let X_p be the velocity at $t = 0$ of the integral curve $t \mapsto \phi_t(p)$.

Defn: X is the <u>infinitesimal generator</u> of the 1-parameter subgroup ϕ_t.

Lemma: (Translation Lemma) Let ϕ be the 1-parameter group (flow) generated by $X \in \mathfrak{X}(M)$. Then $\forall s \in \mathbb{R}, p \in M$, $t \mapsto \phi_{t+s}(p)$ is the integral curve of X through $\phi_s(p)$.

Proof: Use the calc 1 chain rule and the group law. $\phi_{t+s}(p) = \phi_t(\phi_s(p))$. Now differentiate both sides with respect to t. $\square$

Thm: If M is compact, any $X \in \mathfrak{X}(M)$ is complete, i.e., all maximal integral curves of X have domain $\mathbb{R}$.

We're not quite ready to prove this yet, but we will soon.

Lemma: (Uniform Time Lemma) For any M, for any $X \in \mathfrak{X}(M)$, if $\exists \varepsilon > 0$ such that all maximal integral curves' domains contain $(-\varepsilon, \varepsilon)$, then X is complete.

Proof: Assume X is not complete. THen $\exists p \in M$ s.t. $\beta(p) < \infty$ (the argument would follow identically if instead $\alpha(p) > -\infty$). Let $t_0 \in \mathbb{R}$ s.t. $\beta(p) - \varepsilon < t_0 < \beta(p)$, and consider the curve

$$c(t) = \begin{cases} \phi_t(p) & \alpha(p) < t < t_0 \\ \phi_{t-t_0}(\phi_{t_0}(p)) & -\varepsilon < t - t_0 < \varepsilon \end{cases}$$

Then this is an integral curve of X, with $c(0) = p$, and it is defined for all t such that $t_0 - \varepsilon < t < \varepsilon + t_0$. Note that $\varepsilon + t_0 > \beta(p)$. Since it is defined for all $t \in (\alpha(p), \varepsilon + t_0)$, this is a contradiction, as $(\alpha(p), \beta(p))$ is the maximal domain of an integral curve through p. $\square$