Math 591 Lecture 21

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Last time, we showed that a smooth vector field $X \in \mathfrak{X}(M)$ defines a derivation

$$\begin{aligned} X : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto (p \mapsto X_p([f])) \end{aligned}$$

Here, we're thinking of X as an operator, i.e., $f \mapsto X(f)$. Note that $\forall p \in M$, $X_p \in T_pM$.

Prop: The commutator of any two derivations $C^\infty(M) \to C^\infty(M)$ is a derivation.
Proof: This is just an algebraic calculation.

Today, we'll prove the converse – that for any derivation D, there is a unique vector field $X \in \mathfrak{X}(M)$ such that $D = X$ (as an operator). So overall, we will have showed a one-to-one correspondence between derivations and vector fields. To do this, we need "bump functions".

Prop: Let $U \subseteq M$ open, $p \in U$. Then $\exists \chi \in C^\infty(M)$ s.t.

(1) $\operatorname{supp}(\chi) = \overline{\{q \in M : \chi(q) \neq 0\}} \subseteq U$ (and it is compact)
(2) $\exists V$ open with $p \in V$ such that $\chi|_V \equiv 1$.
Note: (1) implies that $\overline{V} \subseteq U$.

Defn: Such a χ is called a <u>bump function</u> at p.

Proof: It's enough to consider the case where $p = 0 \in \mathbb{R}^n$, as we can use a chart near p to define χ in some neighborhood of p, and then extend χ to be 0 outside that neighborhood.
Start with the case where $n = 1$ (i.e. $\mathbb{R}$). (See also §13 in the book.) Start with

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

We claim that f is C^∞ on $\mathbb{R}$. (This is because $\forall k \in \mathbb{N}$, $f^{(k)}(0)$ is defined.)

Note: f is a famous example of a non-analytic function.

Next, let $g(x) = \frac{f(x)}{f(x)+f(1-x)}$. Note: $\forall x \in \mathbb{R}$, $f(x) + f(1-x) \neq 0$, so g is well-defined, and C^∞. If $x \geq 1$, then $f(1-x) = 0$, so $g(x) = 1$. If $x \leq 0$, $f(x) = 0$, so $g(x) = 0$.

Next, choose some $a, b \in \mathbb{R}_{>0}$ with $0 < a^2 < b^2$, and define $h(x) = g(\frac{x-a^2}{b^2-a^2})$. Then finally, take $\rho(x) = 1 - h(x^2)$. Then we have $\rho|_{[-a,a]} \equiv 1$, and $\rho|_{(-\infty,-b]\cup[b,\infty)} \equiv 0$, and ρ is C^∞.

For $\mathbb{R}^n$, let $\chi(x) = \rho(||x||^2)$. Then $\operatorname{supp}\chi$ is a subset of a ball around the origin, and χ restricted to a smaller ball is always 1. $\square$

Defn: $D : C^\infty(M) \to C^\infty(M)$ is a <u>local operator</u> if $\forall f, g \in C^\infty(M)$, $\forall U \overset{\text{open}}{\subseteq} M$, if $f|_U = g|_U$, then $D(f)|_U = D(g)|_U$.

Prop: A derivation $D : C^\infty(M) \to C^\infty(M)$ is a local operator.

Proof: By linearity of D, WOLOG $g \equiv 0$. Assume that $f|_U \equiv 0$, and let $p \in U$. Let $\chi \in C^\infty(M)$ be a bump function at p with $\operatorname{supp}(\chi) \subset U$. Note: $\chi \cdot f \equiv 0$ on M, so $D(\chi f) = 0$. Well, by the chain rule, $D(\chi f) = \chi D(f) + f D(\chi)$. If we evaluate at p, we have $f(p) = 0$ and $\chi(p) = 1$, so $0 = 0 + D(f)(p)$, so $D(f)(p) = 0$. Thus, $D(f)|_U \equiv 0$. $\square$

Note: One can show that every local (linear) operator is a differential operator.

Thm: Let $D : C^\infty(M) \to C^\infty(M)$ be a derivation. Then $\exists X \in \mathfrak{X}(M)$ such that $D = X$ (as an operator).

Proof: Let $p \in M$. To define $X_p \in T_pM$, pick some $[f] \in C^\infty_p(M)$. Let $f : U \to \mathbb{R}$ represent this germ. Let χ be a bump function at p with $\text{supp}(\chi) \subseteq U$. Define $\tilde{f} : M \to \mathbb{R}$ where $\tilde{f} = \chi f$, i.e.,

$$\tilde{f}(p) = \begin{cases} \chi(p)f(p) & p \in U \\ 0 & p \in M \setminus U \end{cases}$$

Observe that $\tilde{f} \in C^\infty(M)$, and since $\tilde{f}$ agrees with f in some open neighborhood V of p, it's an extension of $f|_V$. Define $X_p([f]) = D(\tilde{f})(p)$. We need to justify that this is well-defined – what if we changed our representation of $[f]$, or chose a different χ? Is the number $D(\tilde{f})(p)$ invariant with respect to these changes? Yes! Under the above changes, there's no effect on the germ $[\chi f] \in C^\infty_p(M)$, and we just proved that D is local.

Next, we need to show that X, as it's defined above, is smooth. Let $\phi = (x^1, \dots, x^n)$ be any coordinate system on $U \subset M$. Then

$$X|_U = \sum_{j=1}^n X(x^j) \frac{\partial}{\partial x^j}$$

where $X(x^j)$ is a function on U. We need to check that each $X(x^j)$ is smooth. Again, we will use a bump function at $p \in U$. By definition, $X(x^j)(p) = D(\tilde{x}^j)(p)$, where $\tilde{x}^j = \chi \cdot x^j$ (extended by 0 outside of U). And by our assumption, $D(\tilde{x}^j) \in C^\infty(M)$.

We conclude that $X(x^j) \in C^\infty(M)$, so X is smooth. $\square$

Cor: If $X, Y \in \mathfrak{X}(M)$, then $[X, Y]$ (treating X and Y as operators) is itself a vector field.