Math 591 Lecture 16

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The Hopf Fibration

$$\begin{tikzcd} \mathbb{C}^2 \supset S^3 \arrow[d, "\pi"] \\ \mathbb{RP}^1 \cong S^1 \backslash S^3 \end{tikzcd}$$

$\pi(z_0, z_1) = \{(e^{i\theta} z_0, e^{i\theta} z_1) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$.
$\pi(z_0, z_1) = [z_0 : z_1]$.

Claim: This is a fiber bundle with fiber $\Phi = S^1$.

For the covering of $\mathbb{RP}^1$, we choose the same covering as used in the homework: $U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}$ and $U_1 = \{[z_0 : z_1] : z_1 \neq 0\}$.

Define $\mathscr{S}(\mathfrak{z}) = \frac{1}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then we have

$$\begin{tikzcd} & & S^3 \arrow[d, "\pi"] \\ \mathbb{R}^2 \cong \mathbb{C} & U_0 \arrow[l, "\cong"'] \arrow[ur, "\mathscr{S}"] \arrow[r, hook] & \mathbb{RP}^1 \\ \mathfrak{z} = \frac{z_1}{z_0} & {[z_0 : z_1]} \arrow[l, mapsto] & \end{tikzcd}$$

Note: $\pi \circ \mathscr{S} = I_{U_0}$, since $\pi(\mathscr{S}(\mathfrak{z})) = \mathfrak{z}$.

Defn: If $\pi : M \to N$ is a fiber bundle, a <u>section</u> of π is a C^∞ map $\mathscr{S} : N \to M$ s.t. $\pi \circ \mathscr{S} = I_N$.

Observe: The Hopf fibration does not have a global section.

Observe: Local sections always exist, because they always exist for the trivial bundle $N \to N \times \Phi$: fix $\nu \in \Phi$, and map $p \mapsto (p, \nu)$.

More on the Hopf fibration... Define a trivialization

$$\begin{tikzcd} \pi^{-1}(U_0) \arrow[rr, "\chi"] \arrow[dr] & & U_0 \times \Phi \arrow[dl] \\ & U_0 & \end{tikzcd}$$

with $\chi(\mathfrak{z}, e^{i\theta}) = e^{i\theta}\mathscr{S}(\mathfrak{z}) = \frac{e^{i\theta}}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then $\pi \circ \mathscr{S} = I_{U_0}$, and $\forall \mathfrak{z} \in U_0$, $\mathscr{S}(\mathfrak{z}) = \pi^{-1}(\mathfrak{z})$.

$\chi(z_0, z_1) = (\frac{z_1}{z_0}, \frac{z_0}{|z_0|})$ (note that $z_0 \neq 0$). Then $S^3 \cong \mathbb{R}^3 \cup \{\infty\} = \bigsqcup S^1$, an uncountable disjoint union.

Vector Bundles

Defn: $\pi : E \to B$ is a <u>vector bundle</u> with <u>base</u> B and <u>rank</u> $k \in \mathbb{N}$ iff

a) It's a fibration.
b) $\forall p \in B$, the fiber $\pi^{-1}(p)$ is a vector space.
c) There is an open covering $\{U_\alpha\}$ of B and all trivializations $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ restrict to linear maps on each fiber, i.e.,

$$\begin{tikzcd} \chi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \arrow[r, "\cong"] \arrow[rr, bend right, "\text{Linear Isomorphism}"'] & \{p\} \times \mathbb{R}^k \cong & \mathbb{R}^k \end{tikzcd}$$

Note: $\forall p \in B$, the zero section applied to $p \in B$ gives $0 \in \pi^{-1}(p)$.

Ex:

1) The tangent bundle $TB \to B$.
2) The cotangent bundle $T^*B = \bigsqcup_{p \in B} \{p\} \times T_p^*B \to B$.

Defn: If $S \subseteq M$ is a submanifold, then the <u>co-normal bundle of S</u> is $\mathcal{N} = \Big\{(p, \alpha) \in T^*M \mid \ \alpha|_{T_pS} = 0\Big\}$.

The co-normal bundle is a vector bundle.

Note: If we give T_pM a Euclidean inner product, we can identify

$$\begin{aligned} T_p^*M &\cong T_pM \\ \langle \cdot, v \rangle &\longleftarrow\!\shortmid v \end{aligned}$$

Claim: $\mathcal{N} \subset T^*M$ is a submanifold, and $\pi : \mathcal{N} \to S$ is a vector bundle of rank $\operatorname{codim} S = \dim M - \dim S$. The fiber of $\mathcal{N}$ over p is the annihilator of T_pS.