Math 591 Lecture 7

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Observe:

(1) $\dim O(n) = \dim(\text{ambient}) - \dim(\text{Symm}(n,\mathbb{R})) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.
(2) $\ker F'(I) = \left\{M \in \text{Mat}(n,\mathbb{R}) \mid M + M^T = 0\right\} = \{\text{skew-symmetric matrices}\}$. This is the tangent space to $O(n)$ at I.

Similarly, $U(n) = \left\{g \in \text{Mat}(n,\mathbb{C}) \mid g^{-1} = \overline{g}^T\right\}$ has a C^∞ structure, as well as $SU(n) = \{g \in U \mid \det g = 1\}$.
($g \in U(n) \Rightarrow \det(g) \in S^1$, i.e., $|\det g| = 1$.)

(3) $O(n)$, in fact, has 2 connected components, as $g \in O(n) \Rightarrow \underbrace{|\det g|}_{\in\mathbb{R}} = 1 \Rightarrow \det g = \pm 1$.

Defn: $SO(n) = \{g \in O(n) \mid \det g = 1\}$ is a subgroup of $O(n)$.
$O(n) = SO(n) \cup \{g \in O(n) \mid \det g = -1\}$.

More examples: $SL(n,\mathbb{R}) = \{g \in \text{GL}(n,\mathbb{R}) \mid \det g = 1\}$.
Some facts: $U(n)$ and $O(n)$ are compact, whereas $SL(n,\mathbb{R})$ is not.

More examples can be constructed by:

- Cartesian products: If M and N are C^∞ manifolds, then $M \times N$ has a natural smooth structure. Charts on $M \times N$ are just $(U \times V, \phi \times \psi)$, where (U,ϕ) is a chart on M and (V,ψ) is a chart on N.
For example, the nth torus $\underbrace{S^1 \times \cdots \times S^1}_{n}$.
- Covering maps of C^∞ manifolds: Let M be a C^∞ manifold. A covering map on M is $f : \tilde{M} \to M$ (with $\tilde{M}$ a topological space) such that $\forall p \in M$, $\exists U \ni p$ open s.t. $F^{-1}(U) = \bigcup_{i \in I \text{ finite}} U_i$, with $U_i \subseteq \tilde{M}$ open s.t. $\forall i$, $F|_{U_i} : U_i \overset{\cong}{\to} V$ is a homeomorphism.

Then:

Thm: $\tilde{M}$ has a unique C^∞ manifold structure s.t. F is locally a diffeomorphism (isomorphism).

Thm: $SO(n)$ has a double cover (a 2-to-1 covering space), $\text{Spin}(n) \overset{2\text{-}1}{\to} SO(n)$. $\text{Spin}(n)$ has a group structure.

Ex: (of a covering map)

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto e^{ix} \end{aligned}$$

Defn: Let M be a C^∞ manifold, and $f : M \to \mathbb{R}$, $p \in M$. Then <u>f is C^∞ at p</u> if there's a chart (U,ϕ) of M such that $p \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞.

Observe: A chart (U,ϕ) one a C^∞ manifold is also called a coordinate system. We'll often write $\phi = (x^1, \ldots, x^n)$, where $x^i : U \to \mathbb{R}$ is the ith component of ϕ, i.e., a coordinate function.

Observe: In the definition above, f only needs to be defined in a neighborhood of p.

Defn: Let M be a C^∞ manifold, $f : M \to \mathbb{R}$ is <u>smooth</u> iff $\forall p \in M$, f is smooth at p.

Lemma: $f : M \to \mathbb{R}$ is smooth iff $\forall (U, \phi)$ smooth chart of M, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

Proof: (see §6 for full details)

$\Leftarrow$ is immediate

$\Rightarrow$ is based on the fact that f smooth $\Rightarrow \forall p \in M$, there's a chart (V, ψ) around p s.t. $f \circ \psi^{-1}$ is smooth.

Ex: Let $M \subseteq \mathbb{R}^N$ be a local graph. $M = F^{-1}(0)$, 0 is a regular value of F.

If $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ is smooth, then $f = \tilde{f}M : M \to \mathbb{R}$ is smooth.

Proof: There are charts on M (U, ϕ) s.t. $\phi^{-1}(x') = (x', G(x'))$ after permuting coordinates (G is a graph function).

Then $(f \circ \phi^{-1})(x') = \tilde{f}(x', G(x'))$, and this is C^∞. $\square$