

# Math 591 Lecture 7

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9/16/20

Observe:

- (1)  $\dim O(n) = \dim(\text{ambient}) - \dim(\text{Symm}(n, \mathbb{R})) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ .
- (2)  $\ker F'(I) = \{M \in \text{Mat}(n, \mathbb{R}) \mid M + M^T = 0\} = \{\text{skew-symmetric matrices}\}$ . This is the tangent space to  $O(n)$  at  $I$ .

Similarly,  $U(n) = \{g \in \text{Mat}(n, \mathbb{C}) \mid g^{-1} = \bar{g}^T\}$  has a  $C^\infty$  structure, as well as  $SU(n) = \{g \in U \mid \det g = 1\}$ .  
 $(g \in U(n) \Rightarrow \det(g) \in S^1, \text{ i.e., } |\det g| = 1.)$

- (3)  $O(n)$ , in fact, has 2 connected components, as  $g \in O(n) \Rightarrow \underbrace{|\det g|}_{\in \mathbb{R}} = 1 \Rightarrow \det g = \pm 1$ .

**Defn:**  $SO(n) = \{g \in O(n) \mid \det g = 1\}$  is a subgroup of  $O(n)$ .  
 $O(n) = SO(n) \cup \{g \in O(n) \mid \det g = -1\}$ .

More examples:  $SL(n, \mathbb{R}) = \{g \in \text{GL}(n, \mathbb{R}) \mid \det g = 1\}$ .

Some facts:  $U(n)$  and  $O(n)$  are compact, whereas  $SL(n, \mathbb{R})$  is not.

More examples can be constructed by:

- Cartesian products: If  $M$  and  $N$  are  $C^\infty$  manifolds, then  $M \times N$  has a natural smooth structure. Charts on  $M \times N$  are just  $(U \times V, \phi \times \psi)$ , where  $(U, \phi)$  is a chart on  $M$  and  $(V, \psi)$  is a chart on  $N$ .  
 For example, the  $n$ th torus  $\underbrace{S^1 \times \cdots \times S^1}_n$ .
- Covering maps of  $C^\infty$  manifolds: Let  $M$  be a  $C^\infty$  manifold. A covering map on  $M$  is  $f : \tilde{M} \rightarrow M$  (with  $\tilde{M}$  a topological space) such that  $\forall p \in M, \exists U \ni p$  open s.t.  $F^{-1}(U) = \bigcup_{i \in I \text{ finite}} U_i$ , with  $U_i \subseteq \tilde{M}$  open s.t.  $\forall i, F|_{U_i} : U_i \xrightarrow{\cong} U$  is a homeomorphism.

Then:

**Thm:**  $\tilde{M}$  has a unique  $C^\infty$  manifold structure s.t.  $F$  is locally a diffeomorphism (isomorphism).

**Thm:**  $SO(n)$  has a double cover (a 2-to-1 covering space),  $\text{Spin}(n) \xrightarrow{2-1} SO(n)$ .  $\text{Spin}(n)$  has a group structure.

**Ex:** (of a covering map)

$$\begin{aligned} \mathbb{R} &\rightarrow S^1 \\ x &\mapsto e^{ix} \end{aligned}$$

**Defn:** Let  $M$  be a  $C^\infty$  manifold, and  $f : M \rightarrow \mathbb{R}, p \in M$ . Then  $f$  is  $C^\infty$  at  $p$  if there's a chart  $(U, \phi)$  of  $M$  such that  $p \in U$  and  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$  is  $C^\infty$ .

Observe: A chart  $(U, \phi)$  on a  $C^\infty$  manifold is also called a coordinate system. We'll often write  $\phi = (x^1, \dots, x^n)$ , where  $x^i : U \rightarrow \mathbb{R}$  is the  $i$ th component of  $\phi$ , i.e., a coordinate function.

Observe: In the definition above,  $f$  only needs to be defined in a neighborhood of  $p$ .

**Defn:** Let  $M$  be a  $C^\infty$  manifold,  $f : M \rightarrow \mathbb{R}$  is smooth iff  $\forall p \in M, f$  is smooth at  $p$ .

**Lemma:**  $f : M \rightarrow \mathbb{R}$  is smooth iff  $\forall (U, \phi)$  smooth chart of  $M$ ,  $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$  is smooth.

Proof: (see §6 for full details)

$\Leftarrow$  is immediate

$\Rightarrow$  is based on the fact that  $f$  smooth  $\Rightarrow \forall p \in M$ , there's a chart  $(V, \psi)$  around  $p$  s.t.  $f \circ \psi^{-1}$  is smooth.

**Ex:** Let  $M \subseteq \mathbb{R}^N$  be a local graph.  $M = F^{-1}(0)$ ,  $0$  is a regular value of  $F$ .

If  $\tilde{f} : \mathbb{R}^N \rightarrow \mathbb{R}$  is smooth, then  $f = \tilde{f}|_M : M \rightarrow \mathbb{R}$  is smooth.

Proof: There are charts on  $M$   $(U, \phi)$  s.t.  $\phi^{-1}(x') = (x', G(x'))$  after permuting coordinates ( $G$  is a graph function).  
Then  $(f \circ \phi^{-1})(x') = \tilde{f}(x', G(x'))$ , and this is  $C^\infty$ .  $\square$