Math 591 Lecture 3

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Group Actions

Defn: Let G be a group, X a set. A left action of G on X is a map

$$\begin{aligned} G \times X &\to X \\ (g,x) &\mapsto g \cdot x \end{aligned}$$

such that

a) if $e \in G$ is the identity, $\forall x \in X$, $e \cdot x = x$

b) $\forall g_1, g_2 \in G$, $\forall x \in X$, $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

In other words, if $\forall g \in G$, we define the map

$$\begin{aligned} L_g : X &\to X \\ x &\mapsto g \cdot x \end{aligned}$$

then $L_e = I_X$ and $L_{g_1 g_2} = L_{g_1} \circ L_{g_2}$.

Defn: Given a group action, if $x \in X$, the orbit of x is the set $G \cdot x = \{y \in X \mid \exists g \in G \text{ s.t.} g \cdot x = y\}$.

Lemma: The orbits partition X, i.e., $x \sim y$ iff $G \cdot x = G \cdot y$ is an equivalence relation.

Notation: X/G and $G \backslash X$ are both valid. We'll stick with $G \backslash X$. (This is the quotient space whose points are the orbits of points in X.)

Defn: Assume X is a topological space, and the group G acts on X (on the left). The action is by continuous maps iff $\forall G \in G$, $L_g : X \to X$ is continuous.

Observe that $\forall g, L_g$ is a homeomorphism, because $\exists g^{-1} \in G$, so $L_{g^{-1}}$ is continuous, and $L_g \circ L_{g^{-1}} = I_X = L_{g^{-1}} \circ L_g$.

Lemma: If G acts by continuous maps, the orbit relation is open.

Proof: Let $U \subseteq X$ be open. We need to show that saturation $\hat{U}$ of U is open.

$$\begin{aligned} \hat{U} &= \{x \in X \mid \exists y \in U \text{ s.t. } x \sim y\} \text{ ($\sim$ being the orbit relation)} \\ &= \{x \in X \mid \exists y \in U, g \in G \text{ s.t. } y = g \cdot x\} \end{aligned}$$

Thus,

$$\hat{U} = \bigcup_{g \in G} g \cdot U = \bigcup_{g \in G} \{g \cdot x \mid x \in U\} = \bigcup_{g \in G} L_g(U)$$

L_g is a homeomorphism, so it is an open map, so each $L_g(U)$ is open, so $\hat{U}$ is open. $\square$

Defn: A topological group is a group G with a topology s.t. the maps

$$\begin{aligned} G \times G &\to G \\ (g,k) &\mapsto gk \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous.

Aside: Later on, when we have a manifold, and these maps are smooth, then this is a Lie group.

Ex: $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$, the set of invertible $n \times n$ matrices.
In fact, this is an open subset, since it's described by $\mathrm{GL}(n,\mathbb{R}) = \{M \in \mathbb{R}^{n\times n} \mid \det M \neq 0\}$, i.e., $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\backslash\{0\})$. Because det is a continuous map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ and $\mathbb{R}\backslash\{0\}$ is open, we get that $\mathrm{GL}(n,\mathbb{R})$ is open.
Note that $\mathrm{GL}(n,\mathbb{R})$ is a topological group, with the induced topology. In fact, any subgroup of a topological group is naturally a topological group with respect to the subspace topology.

Ex: $O(n,\mathbb{R}) = \left\{g \in \mathrm{GL}(n,\mathbb{R}) \mid g^{-1} = g^T\right\}$.
$\mathrm{GL}(n,\mathbb{C}) \subseteq \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Note that $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{GL}(2n,\mathbb{R})$, since $\mathbb{C} \cong \mathbb{R}^2$.
$U(n) = \left\{g \in \mathrm{GL}(n,\mathbb{C}) \mid g^{-1} = \overline{g}^T\right\}$.

Defn: If G is a topological group acting on a topological space X, the action is <u>continuous</u> iff $G \times X \to X$ is a continuous map.

Lemma: A continuous action is an action by continuous maps. (I.e. $\forall g \in G$, $L_g : X \to X$ is continuous.)

Ex: $G = S^1 = \{z \in \mathbb{C} : |z| = 1\} = U(1)$ acts on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by $\lambda \in S^1$, $(z_1,\ldots,z_{n+1}) \in S^{2n+1}$, $\lambda \cdot (z_1,\ldots,z_{n+1}) = (\lambda z_1,\ldots,\lambda z_{n+1})$. This is a continuous action.

Question: Suppose G is a topological group acting on X. (So the orbit relation is open.) When is $G\backslash X$ Hausdorff?
Well, this is true iff the graph of the orbit relation is closed.

Define

$$\star \quad \begin{aligned} G \times X &\to X \times X \\ (g,x) &\mapsto (x, g\cdot x) \end{aligned}$$

This is a continuous map, whose image is the graph of the orbit relation.

Prop: If G and X are both compact, and X is Hausdorff, then $G\backslash X$ is Hausdorff.
Proof: The image of $\star$ is compact, and compact subsets of Hausdorff spaces are closed, so the orbit relation is closed. $\square$

Ex: $S^1 \times S^{2n+1} \to S^{2n+1}$ as above.
Then the proposition implies $\mathbb{CP}^n = S^1\backslash S^{2n+1}$ is Hausdorff and second-countable.
Note: $\mathbb{CP}^n \cong \left\{\text{1-dimensional subspaces of } \mathbb{C}^{n+1}\right\}$.