Math 591 Lecture 34

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Lie Derivatives of Forms

Ex: Let $X = \langle F^1, \dots, F^n \rangle \in \mathfrak{X}(\mathbb{R}^n)$, $\mu = dx^1 \wedge \cdots \wedge dx^n$, and ϕ the flow of X. Then

$$\mathcal{L}_X(\mu) = \left.\frac{d}{dt}\phi_t^*\mu\right|_{t=0}$$

Well, $\phi_t^*\mu = \det(J(\phi_t)_*)\mu$, so

$$\det(J(\phi_t)_*)|_{t=0} = \operatorname{tr}\Bigg(\underbrace{\left.\frac{d}{dt}J(\phi_t)_*\right|_{t=0}}_{\star}\Bigg)\underbrace{\det(\phi_{t=0})}_{=1}$$

For $\star$, do $\frac{d}{dt}$ first, and then $\frac{\partial}{\partial x^i}$. And $\det(\phi_{t=0}) = 1$, because ϕ_t is the flow of X, so $\frac{d}{dt}$ is just X. Thus, we have

$$\operatorname{tr}\begin{pmatrix} — & \nabla F_1 & — \\ & \vdots & \\ — & \nabla F_n & — \end{pmatrix} = \sum_{j=1}^{n}\frac{\partial F_j}{\partial x^j} = \operatorname{div}(X)$$

We conclude that $\mathcal{L}_X\mu = (\operatorname{div} X)\mu$. $\square$

Thm: (Cartan's Magic Formula) $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X : \Omega^k \to \Omega^k$.

Proof: Let $P_X = \iota_X \circ d - d \circ \iota_X$. Then P_X has the following properties:
1) $\mathbb{R}$-linearity
2) It's a derivation w.r.t. $\wedge$: $P_X(\alpha \wedge \beta) = P_X(\alpha) \wedge \beta + \alpha \wedge P_X(\beta)$
3) It commutes with d
4) $f \in C^\infty(M) \Rightarrow P_X(f) = X(f)$
5) P_X is local

These properties belong to a unique operator. (Compute in coordinates, and by linearity, just use monomials.)

$$P_X(a\,dx^I) = P_X(a) \wedge dx^I + a \wedge P_X(dx^I) = X(a)\,dx^I + aP_X(dx^I)$$

We then expand $P_X(dx^I)$ with induction. $\square$

Ex: For $k = 2$:

$$P_X(dx^1 \wedge dx^2) = P_X(dx^1) \wedge dx^2 + dx^1 \wedge P(dx^2) = dX(x^1) \wedge dx^2 + dx^1 \wedge dX(x^2)$$

This is unique, and it just uses the five properties.

Applications of Cartan's Formula

Defn: A symplectic manifold is a pair (M, ω), with M a manifold, and $\omega \in \Omega^2(M)$ such that
1) $\forall p \in M, v \in T_pM \setminus \{0\}$, $\iota_v(\omega_p) = \omega_p(v, \cdot) : T_pM \to \mathbb{R}$ is nonzero.
2) $d\omega = 0$.

Question: What are the symmetries of a symplectic manifold (M, ω)? Specifically, are there one-parameter groups $\varphi_t : M \to M$ such that $\forall t$, $\varphi_t^*\omega = \omega$?
Use the Lie derivative: If $X \in \mathfrak{X}(M)$ is the generator of φ_t, then

$$\varphi_t^*\omega = \omega \Leftrightarrow \mathcal{L}_X\omega = 0 \Leftrightarrow \iota_X \underbrace{d\omega}_{=0} + d\iota_X\omega = 0 \Leftrightarrow d\iota_X\omega = 0$$

Defn: One particular class of such X's comes from the following: Take any function $H \in C^\infty(M)$, and define X by $\iota_X\omega = -dH$. By non-degeneracy of ω, X is unique! And,

$$\begin{aligned} T_pM &\to T_p^*M \\ v &\mapsto \iota_v\omega \end{aligned}$$

has no kernel, so it's a bijection. This X is called the <u>Hamilton field</u> of H.

Exer: Take $M = \mathbb{R}^{2n}$ with coordinates (x, p), where x and p are the standard coordinates in $\mathbb{R}^n$. Let

$$\omega = \sum_{i=1}^{n} dp^i \wedge dx^i \qquad H = \frac{1}{2}\,||p||^2 + V(x)$$

Compute the Hamilton field, and show that the integral curves satisfy $\ddot{x}(t) = -\nabla V(x(t))$. This is better known as Newton's second law!

We're now done with Lie derivatives.

Integration of Forms on Oriented Manifolds

First, we have to define orientation. Let V be an n-dimensional vector space. Let $\mathscr{B}(V)$ be the set of all ordered bases of V. For $e \in \mathscr{B}$, $e = (e_1, \ldots, e_n)$ is an ordered basis of V. Observe: $\forall e, f \in \mathscr{B}$, $\exists! M \in \mathrm{GL}(n, \mathbb{R})$ such that $\forall i$, $e_i = Mf_i$.

Defn: $e \sim f \Leftrightarrow \det M > 0$. We say e and f define the same orientation of V.

Check:
1) $\sim$ is an equivalence relation
2) $\mathscr{B}/\sim$ has two elements

Defn: An <u>orientation</u> of V is a choice of an equivalence class in $\mathscr{B}/\sim$. Bases in that equivalence class are said to be <u>positive</u>.

Alternatively, consider the set of nonzero top-degree forms, $\underbrace{\left(\bigwedge^n V\right)}_{\dim=1} \setminus \{0\}$. When we take away 0, any $\mu \in (\bigwedge^n V) \setminus \{0\}$ defines an orientation by: a basis e is positive iff $\mu(e) > 0$. The orientation defined by μ only depends on which connected component of $(\bigwedge^n V) \setminus \{0\}$ contains μ. Conversely, an orientation singles out one of the two components of $(\bigwedge^n V) \setminus \{0\}$.

Conclusion: An orientation is a choice of a connected component of $(\bigwedge^n V) \setminus \{0\}$.

Now, we move on to manifolds! Note: Not all manifolds are orientable (e.g. the Mobius band).

Defn: An <u>orientation</u> on M (if it exists) is a continuous choice of orientation of each tangent space.

Continuity means $\forall p \in M$, there exists a continuous moving frame $(E_1, \ldots, E_n)$ such that at every point q in the domain of E, $(E_1(q), \ldots, E_n(q))$ is a positive basis of T_qM.

Lemma: A connected manifold can have either two orientations, or it's non-orientable.

Proof: The idea is if the manifold is orientable, then consider orientations $\mathscr{O}_1$ and $\mathscr{O}_2$. Define $F : M \to \mathbb{R}$ by $f(p) = 1$ if $\mathscr{O}_1(p) = \mathscr{O}_2(p)$, and 0 otherwise. (Note: we don't really ever use this notation.) Then by the continuity of $\mathscr{O}_1$ and $\mathscr{O}_2$, f is continuous, so f is locally constant, so if M is connected, then f is constant. $\square$

Defn: Let M be an oriented manifold. Then a <u>positive atlas</u> on M is an atlas $\{(U_\alpha, \phi_\alpha)\}$ of M such that $\forall \alpha$, the moving frame $\left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right)$ is a positive frame.

Lemma:

1) The transition functions F and G of any two elements in a positive atlas satisfy $\det(J(F)) = 1 = \det(J(G))$.
2) An oriented manifold always has a positive atlas.

Proof: This is very tedious! Idea:

1) Recall that $J(F)$ is actually the change of basis matrix between $\left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right)$. 2) Start with any atlas. $\forall \alpha$, $\left\{\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right\}$ is either positive, or not. If it is positive, do nothing, and keep ϕ_α. If it's not positive, relable (switch) x^1 and x^2. Now it's positive!

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