Math 591 Lecture 30

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

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Tensors

Defn: Let V be a finite-dimensional vector space. A tensor on V is an element of

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \otimes \underbrace{V \otimes \cdots \otimes V}_{m}$$

(where $\otimes$ is the tensor product).

Observe: This space is isomorphic to the space of multilinear maps $\underbrace{V \times \cdots \times V}_{\ell} \times \underbrace{V^* \times \cdots \times V^*}_{m} \to \mathbb{R}$, because $(V^*)^* \cong V$.

Last time, we defined the set of *alternating* multilinear functions $\bigwedge^k V^* \hookrightarrow \underbrace{V^* \otimes \cdots \otimes V^*}_{k}$.

Reminder: A basis for $\bigwedge^k V^*$: choose $\left\{\mathcal{E}^i\right\}_{1 \le i \le n}$ an ordered basis of V^*. For each $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, let $\mathcal{E}^I(v_1, \ldots, v_k) = \det(\mathcal{E}^{i_j}(v_\ell))_{(j,\ell)}$.

Prop: $\left\{\mathcal{E}^I : I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}, \#I = k\right\}$ is a basis of $\bigwedge^k V^*$.

In fact, $\forall \alpha \in \bigwedge^k V^*$, $\alpha = \sum_I{}' \alpha(e_{i_1}, \ldots, e_{i_k}) \mathcal{E}^I$, where $\{e_j\}$ is the basis of V dual to $\left\{\mathcal{E}^j\right\}$ (i.e., $\mathcal{E}^i(e_j) = \delta_{ij}$).

Note: the notation $\sum_I{}'$ means sum over increasing $I = \{i_1 < \cdots < i_k\}$.

Also, $\dim \bigwedge^k V^* = \binom{n}{k}$.

Defn: By convention, $\bigwedge^0 V^* = \mathbb{R}$.

Defn: If $\alpha \in \bigwedge^k V^*, \beta \in \bigwedge^\ell V^*$, we define the wedge product $\alpha \wedge \beta \in \bigwedge^{k+\ell} V^*$ by

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

Note: We can define $(\alpha \otimes \beta)(v_1, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+\ell})$, but this may not be alternating (in fact, it almost certainly isn't). But this can be skew symmetrized by forming the above sum (with appropriate normalization).

Note: An equivalent formula for the wedge product is

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in \mathrm{Sh}(k,\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

This is a much smaller sum, as it removes redundancies.

Recall: $\sigma \in \mathrm{Sh}(k, \ell)$ iff $\sigma \in S_{k+\ell}$, $\sigma(1) < \cdots < \sigma(k)$, $\sigma(k+1) < \cdots < \sigma(k+\ell)$.
Note: $\#\,\mathrm{Sh}(k,\ell) = \binom{k+\ell}{k} = \binom{k+\ell}{\ell}$.

Ex: Say $\alpha \in \bigwedge^2 V^*$, $\beta \in \bigwedge^2 V^*$. Then the elements of $\mathrm{Sh}(2,2)$ are:

1	2	3	4	sgn
1	2	3	4	+
1	3	2	4	−
1	4	2	3	−
2	3	1	4	+
2	4	1	3	+
3	4	1	2	+

So

$$\begin{aligned}(\alpha \wedge \beta)(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\beta(v_3, v_4) \\ - \alpha(v_1, v_3)\beta(v_2, v_4) \\ - \alpha(v_1, v_4)\beta(v_2, v_3) \\ + \alpha(v_2, v_3)\beta(v_1, v_4) \\ + \alpha(v_2, v_4)\beta(v_1, v_3) \\ + \alpha(v_3, v_4)\beta(v_1, v_2)\end{aligned}$$

Properties of the Wedge Product

- The wedge product is bilinear: $(\alpha_1 + \lambda\alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \lambda(\alpha_2 \wedge \beta)$.
- The wedge product is associative: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma$.
- The wedge product is anticommutative: for $\alpha \in \bigwedge^k, \beta \in \bigwedge^\ell$, $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$.
  This implies that even forms commute with any other form.
- If $\alpha^1, \dots, \alpha^k \in V^*$, then $(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \det\left(\alpha^i(v_j)\right)_{(i,j)}$.
  In particular, $\mathcal{E}^I = \mathcal{E}^{i_1} \wedge \cdots \wedge \mathcal{E}^{i_k}$.

Ex: Say $\alpha^1, \alpha^2 \in V^*$. Then $(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1)$.

Defn: $\left(\bigoplus_{k=0}^{n} \bigwedge^k V^*, \wedge\right)$ is the <u>exterior algebra</u> or <u>Grassmann algebra</u> of V.

Now, back to manifolds...

Defn: Let M be a smooth manifold. A <u>k-differential form</u> (or <u>k-form</u>) on M is an assignment $\forall p \in M$, $p \mapsto \alpha_p \in \bigwedge^k(T_p^*M)$.

Note: When $k = 1$, α is a 1-form as before. When $k = 0$, α is just an $\mathbb{R}$-valued function.

Local Expression

Let $(U, (x^1, \dots, x^n))$ be a coordinate chart. Then $\forall p \in U$, we get $\left\{ dx^i\big|_p \right\}_{i=1,\dots,n}$ a basis of T_p^*M, with corresponding dual basis $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ of T_pM. If α is a k-form, then $\forall p \in U$,

$$\alpha_p = \sum_I{}' a_I (dx^I)_p \qquad a_I(p) = \alpha_p\left(\left.\frac{\partial}{\partial x^{i_1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^{i_k}}\right|_p \right)$$

This defines functions $a_I : U \to \mathbb{R}$ for each $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}$.

Defn: A k-form α is <u>smooth</u> iff there exists a C^∞ atlas of M such that in each chart, each a_I is smooth.

Lemma: α is smooth iff for every chart in the smooth structure, each function a_I is smooth.

Proof: $\Leftarrow$ is trivial. For $\Rightarrow$, let $(y^1, \ldots, y^n) : V \to \mathbb{R}$ be an arbitrary coordinate chart. Let $p \in V$. By our assumption, there's a chart $(x^1, \ldots, x^n) : U \to \mathbb{R}$ near p such that $\alpha = \sum_I{}' a_I dx^I$, with $a_I \in C^\infty(U)$. Also, $\alpha = \sum_J{}' b_J dy^J$. We need to show $b_J \in C^\infty(U \cap V)$, but how? Well,

$$b_J = \alpha\left(\frac{\partial}{\partial y^{j_1}}, \ldots, \frac{\partial}{\partial y^{j_k}}\right) = \sum_I{}' a_I \underbrace{dx^I\left(\frac{\partial}{\partial y^{j_1}}, \ldots, \frac{\partial}{\partial y^{j_k}}\right)}$$

$$= \det\left(dx^{i_r}\left(\tfrac{\partial}{\partial y^{j_s}}\right)\right)_{(r,s)}$$

$$= \det\left(\tfrac{\partial x^{i_r}}{\partial y^{j_s}}\right)_{(r,s)}$$

The $\frac{\partial x^{i_r}}{\partial y^{j_s}}$ are smooth, so the whole determinant is smooth, so b_J is the sum of smooth functions. Thus, it's smooth. $\square$

From the bundle point-of-view, we can define $\bigwedge^k T^*M = \bigsqcup_{p \in M} \bigwedge^k (T_p^* M)$.

Prop: One can make $\bigwedge^k T^*M$ into a vector bundle, with trivializations given by the moving frames $\{dx^I\}$ associated to coordinates. Then C^∞ k-forms are C^∞ sections of this bundle.