Math 591 Lecure 28

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Recall: For G a Lie group, $\mathfrak{g} = T_e G$ is its Lie algebra. $\forall A \in \mathfrak{g}$, $A^\sharp$ is the left-invariant vector field on G determined by A, that is, $\forall g \in G$, $A^\sharp_g \stackrel{\text{def}}{=} (L_g)_{*,e}(A)$. In particular, $A^\sharp_e = A$.

Defn: $\forall A \in \mathfrak{g}$, $t \in \mathbb{R}$, $\exp t\,A$ is the integral curve of $A^\sharp$ at e, at time t.

Defn: $\exp : \mathfrak{g} \to G$
$A \mapsto \exp t\,A|_{t=1}$

Prop: exp is smooth.

Proof: Introduce the field $X \in \mathfrak{X}(G \times \mathfrak{g})$, defined by, $\forall (g, A) \in G \times \mathfrak{g}$,

$$X_{(g,A)} = (A^\sharp_g, 0) \in T_g G \times T_A \mathfrak{g} = T_{(g,A)}(G \times \mathfrak{g}) = T_g G \times \mathfrak{g}$$

X is smooth. Its flow, denoted F, is $F_t(g, A) = (F^A_t(g), A)$, where F^A is the flow of $A^\sharp$. Thus, the time-1 map is C^∞.

$$\begin{tikzcd} g \cong \{e\} \times \mathfrak{g} \arrow[r, hook] \arrow[rrr, bend right, "\exp"'] & G \times \mathfrak{g} \arrow[r, "F_1"] & G \times \mathfrak{g} \arrow[r, "\pi_G"] & G \end{tikzcd}$$

Because $F_1(e, A) = (\underbrace{F^A_1(e)}_{\exp(A)}, A)$. We conclude that exp is smooth. $\square$

Prop: With the same notation – $A \in \mathfrak{g}$, F^A the flow of $A^\sharp$ – $\forall g \in G$, $F^A_t(g) = g \cdot \exp t\,A$ (with $\cdot$ group multiplication).

Proof: $\frac{d}{dt}(g \cdot \exp t\,A) = \frac{d}{dt} L_g(\exp t\,A) = (L_g)_{*,\exp(A)}(A^\sharp_{\exp t\,A}) = A^\sharp_{g \cdot \exp t\,A}$.
This is precisely the ODE satisfied by $t \mapsto F^A_t(g)$. $\square$

Defn: If H, G are Lie groups, then a <u>Lie group morphism</u> from H to G is a smooth map $F : H \to G$ that is also a group homomorphism.

Prop: Let $F : H \to G$ be a Lie group homomorphism. Then

a) $\forall A \in \mathfrak{h} = T_e H$, $A^\sharp \in \mathfrak{X}(H)$ and $[\underbrace{F_{*,e}(A)}_{\in T_e G = \mathfrak{g}}]^\sharp \in \mathfrak{X}(G)$. These two fields are F-related.

b) $\forall A \in \mathfrak{h}$, $F(\underbrace{\exp}_{\text{in } H} A) = \underbrace{\exp}_{\text{in } G}(F_{*,e}(A))$.

c) $F_{*,e} : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra morphism. That is, $\forall A, B \in \mathfrak{h}$, $F_{*,e}([A, B]) = [F_{*,e}(A), F_{*,e}(B)]$ (with the appropriate Lie brackets).

Proof: (b) and (c) follow directly from (a).
Claim: $\forall h \in H$, the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "F"] \arrow[d, "L_h"'] & G \arrow[d, "L_{F(h)}"] \\ H \arrow[r, "F"'] & G \end{tikzcd}$$

Check: let $k \in H$. Then

$$(F \circ L_h)(k) = F(hk) = F(h)F(k) = L_{F(h)}(F(k))$$

because F is a group morphism. Thus, $\forall h$, $F_{*,h} \circ (L_h)_{*,e} = (L_{F(h)})_{*,F(h)} \circ F_{*,e}$ by the chain rule, and commutativity at the identity). We can apply this to any $A \in \mathfrak{h}$:

$$F_{*,h}(\underbrace{(L_h)_{*,e}(A)}_{=A^\sharp_h}) = (L_{F(h)})_{*,F(e)}(F_{*,e}(A)) = \underbrace{(L_{F(h)})_{*,e}(F_{*,e}(A))}_{=[F_{*,e}(A)]^\sharp{}_{F(h)}}$$

This *is* the fact that $A^\sharp$ and $[F_{*,e}(A)]^\sharp$ are F-related. $\square$

Now, back to subgroups.

Defn: A regular (or embedded) Lie subgroup of G is a regular submanifold that is also a subgroup.

Cor: If $H \subseteq G$ is a regular subgroup, then

(a) $\mathfrak{h} \subset \mathfrak{g}$ as a subspace is closed under $[\cdot, \cdot]$ of $\mathfrak{g}$.
(b) $\forall A \in \mathfrak{h}$, $\underbrace{\exp}_{\text{in } H} A = \underbrace{\exp}_{\text{in } G} A$.

Proof: Just apply the previous theorem to $F : H \hookrightarrow G$. $\square$

Observe: $\forall A, B \in \mathfrak{h}$, $[A,B]_{\mathfrak{h}} = [A,B]_{\mathfrak{g}}$, so $[\cdot, \cdot]_{\mathfrak{h}} = [\cdot, \cdot]_{\mathfrak{g}}\Big|_{\mathfrak{h}}$.

Defn: Given a Lie algebra $\mathfrak{g}$, a Lie-subalgebra is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under $[\cdot, \cdot]$.

Cor: If $H \subset G$ is a regular Lie subgroup, then $\mathfrak{h} = T_e H$ is a Lie-subalgebra of $\mathfrak{g}$.

Is the converse true? I.e., given G a Lie group, if $\mathfrak{h} \subset T_e G = \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, is there a Lie subgroup $H \subset G$ such that $T_e H = \mathfrak{h}$?
Answer: Yes, if we allow immersed submanifolds.

Thm: If $\mathfrak{h} \subset T_e G$ is a Lie subalgebra, then there is a Lie group H and an immersion $F : H \to G$ which is a group morphism and $F_{*,e} : T_e H \xrightarrow{\cong} \mathfrak{h}$.

Ex: An irrational line on the two-torus $S^1 \times S^1 = \mathbb{T}^2$,

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{T}^2 \\ t &\mapsto (e^{it}, e^{i\alpha t}) \end{aligned}$$

for α irrational, is a Lie group morphism and an immersion. $\mathrm{im}(F_{*,e})$ is a 1-dimensional subspace of $\mathfrak{t}^2 \cong \mathbb{R}^2$. This is always a subalgebra.

We're now done with Lie groups! Starting next week, we'll move on to differential forms.
Motivation: How do we integrate on manifolds and submanifolds?
(Think, for example, about surface integrals from calc 3.)