# Math 591 Lecture 26

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## Lie Derivatives

The general notion of a Lie derivative with respect to a vector field $X$ is "pull back by $\theta_t$, the time $t$ map of the flow of $X$, and then differentiate with respect to $t$ at $t = 0$". We call this $\mathcal{L}_x$.

**Ex:** We can pull back functions, so we can have Lie derivatives of functions.
Given $X$, $\theta_t$, and $f \in C^\infty(M)$, $\phi_t^*(f)(p) = f(\phi_t(p))$ (pullback). Now differentiate:

$$\frac{d}{dt}\phi_t^*(f)(p)\bigg|_{t=0} = \frac{d}{dt}f(\phi_t(p))\bigg|_{t=0} = X(f)(p)$$

So $(\mathcal{L}_X f)(p) = X(f)(p) = df(X_p)$. So $\mathcal{L}_X f$ is again a function on $M$.

Recall: A 1-form $\alpha$ is an assignment to each $p \in M$, an element $\alpha_p \in T_p^*M$. I.e. $\alpha$ is a section of the cotangent bundle. In local coordinates, we can write $\alpha = \sum \alpha_i dx^i$, for $\alpha_i \in C^\infty(U)$.

**Ex:** 1-forms can be pulled back by any $F$, by $F^*(\alpha)(v) = \alpha(F_{*,\cdot}(v))$. So

$$(\mathcal{L}_X\alpha)(p) = \frac{d}{dt}\underbrace{\phi_t^*\alpha_{\phi_t(p)}}_{\substack{\in T_p^*M \\ \text{for each } t}}\Bigg|_{t=0}$$

Thus, $\mathcal{L}_X\alpha$ is again a 1-form on $M$.

Special case: $\alpha = df$ for some $f \in C^\infty(M)$. Then by the chain rule (1)

$$F^*(df)(v) = df(F_{*,\cdot}(v)) \overset{(1)}{=} d(f \circ F)(v) = d(F^*f)(v)$$

so $F^*(df) = d(F^*f)$.

So, in general,

$$\mathcal{L}_X(df) = \frac{d}{dt}\phi_t^*(df)\bigg|_{t=0} = \frac{d}{dt}d\phi_t^*(f)\bigg|_{t=0} \overset{(1)}{=} d\left(\frac{d}{dt}\phi_t^* f\bigg|_{t=0}\right) = d(df(X)) = d(X(f))$$

Where (1) is true because $\frac{d^2}{\partial t\partial x^i} = \frac{\partial^2}{\partial^i\partial t}$, so the differentiations commute.
In fact, in general, $\mathcal{L}_X(\alpha) = d(\alpha(X)) + \cdots$ (we'll fill in the other term later).

Now, back to vector fields...

Note: Vector fields can be pulled back by diffeomorphisms. This means if $X, Y \in \mathfrak{X}(M)$, $\mathcal{L}_X Y$ is defined by, $\forall p \in M$,

$$\mathcal{L}_X(Y)(p) = \underbrace{\frac{d}{dt}\left[(\phi_t)_{*,p}\right]^{-1}\left(Y_{\phi_t(p)}\right)}_{=V_t \in T_pM}\Bigg|_{t=0}$$

**Prop:** $\mathcal{L}_X Y = [X, Y]$.

Proof: Compute $V_t$ in coordinates $(U, (x^1, \ldots, x^n))$. Restrict to $V \subseteq U$ such that $\phi_t(V) \subseteq U$. Write $\phi_t = (\phi_t^1, \ldots, \phi_t^n)$. Then

$$J_t = \left(\tfrac{\partial \phi_t^i}{\partial x^j}(p)\right)_{(i,j)} \qquad Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x^i} \qquad X = \sum_{i=1}^n f_i \frac{\partial}{\partial x^i}$$

For a fixed $p$, let

$$\tilde{Y}_t = Y_{\phi_t(p)} = \begin{pmatrix} g_1(\phi_t(p)) \\ \vdots \\ g_n(\phi_t(p)) \end{pmatrix}$$

Then, let $\tilde{V}_t$ be the column vector with components of $V_t$. We have

$$\tilde{V}_t = J_t^{-1} \tilde{Y}_t$$

Differentiating with $\frac{d}{dt}$ on both sides, we get

$$\dot{\tilde{V}}_t = J_t^{-1} \dot{\tilde{Y}}_t - J_t^{-1} \dot{J}_t J_t^{-1} \tilde{Y}_t$$

which at $t = 0$, yields

$$\underbrace{\dot{\tilde{V}}_t \Big|_{t=0}}_{\substack{\text{components} \\ \text{of } \mathcal{L}_X Y}} = \begin{pmatrix} X(g_1)(p) \\ \vdots \\ X(g_n)(p) \end{pmatrix} - ?$$

Well,

$$\dot{J}_t = \left(\tfrac{\partial^2 \phi_t^i}{\partial t \partial x^j}(p)\right)_{(i,j)} = \left(\tfrac{\partial^2 \phi_t^i}{\partial x^j \partial t}(p)_{(i,j)}\right) = \left(\tfrac{\partial f_i}{\partial x^j}(p)\right)_{(i,j)}$$

since $\frac{\partial \phi_t^i}{\partial t}(p) = f_i$. Thus, we have

$$\dot{\tilde{V}}_t \Big|_{t=0} = \begin{pmatrix} X(g_1)(p) \\ \vdots \\ X(g_n)(p) \end{pmatrix} - \left(\tfrac{\partial f_i}{\partial x^j}(p)\right)_{(i,j)} \begin{pmatrix} g_1(p) \\ \vdots \\ g_n(p) \end{pmatrix}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_X Y &= \sum_{i=1}^n X(g_i) \frac{\partial}{\partial x^i} - \sum_{i,j=1}^n \frac{\partial f_i}{\partial x^j} g_j \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n X(g_i) \frac{\partial}{\partial x^i} - \sum_{i=1}^n Y(f_i) \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n (X(g_i) - Y(f_i)) \frac{\partial}{\partial x^i} \\
&= [X, Y]
\end{aligned}$$

□

**Thm:** Using the same notation as last time, $\forall t \in (-\varepsilon, \varepsilon)$, $V_t = \frac{d}{dt}(\phi_{t,*})^{-1}(Y_{\phi_t}(p)) = (\phi_{t,*})^{-1}[X, Y]_{\phi_t(p)}$.

Proof: We already have it for $t = 0$. Now, consider $V_{t+s}$, and use the group law/translation lemma. □

**Cor:** If $[X, Y] = 0$, then the flows of $X$ and $Y$ commute.