Math 591 Lecture 40

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We'll begin with a very brief look at the algebra behind cohomology.

Defn: A cochain complex $\mathcal{A}$ of vector spaces is a sequence of linear maps

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

s.t. $d \circ d = 0$ (whenever defined).

Defn: The cohomology of a cochain complex $\mathcal{A}$ is, $\forall k \in \mathbb{N}$, $H^k(\mathcal{A}) = Z^k(\mathcal{A})/B^k(\mathcal{A})$, where $Z^k(\mathcal{A}) = \ker(d)$ (with $d : A^k \to A^{k+1}$) and $B^k(\mathcal{A}) = \mathrm{im}(d)$ (with $d : A^{k-1} \to A^k$).

Defn: If $\mathcal{A}$ and $\mathcal{B}$ are cochain complexes, a map $f : \mathcal{A} \to \mathcal{B}$ between them is a sequence: $\forall k \in \mathbb{N}$, we have $f_k : A^k \to B^k$ s.t. $d \circ f^k = f^k \circ d$. I.e., the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^k \arrow[r, "d"] \arrow[d, "f^k"] & A^{k+1} \arrow[r, "d"] \arrow[d, "f^{k+1}"] & \cdots \\
\cdots \arrow[r, "d"] & B^k \arrow[r, "d"] & B^{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Lemma: Such an $f : \mathcal{A} \to \mathcal{B}$ induces $f^\sharp : H^k(\mathcal{A}) \to H^k(\mathcal{B})$ by $f^\sharp[a] = [f(a)]$ for any $a \in Z^k(\mathcal{A})$, and $f^\sharp$ is well-defined.

Observe:
a) $(f \circ g)^\sharp = f^\sharp \circ g^\sharp$.
b) For de Rham theory, if $F : M \to N$ is C^∞, then we get $f : \Omega^*(N) \to \Omega^*(M)$ ($\Omega^*(N)$ is the de Rham complex of N), where $\forall \alpha \in \Omega^k(N)$, $f(\alpha) = F^*\alpha$.

Homotopies between Maps of Cochain Complexes

Defn: Say $f, g : \mathcal{A} \to \mathcal{B}$. A (chain) homotopy (operator) between them is a sequence of maps: $\forall k$, $h : A^k \to B^{k-1}$ s.t. the following diagram commutes:

$$\begin{tikzcd}
 & \cdots \arrow[r, "d"] & A^k \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, "f-g"] & A^{k+1} \arrow[r, "d"] \arrow[dl, "h"] & \cdots \\
\cdots \arrow[r, "d"] & B^{k-1} \arrow[r, "d"] & B^k \arrow[r, "d"] & \cdots &
\end{tikzcd}$$

That is, $h \circ d + d \circ h = f - g$.

Lemma: If there exists a homotopy between f and g, then $f^\sharp = g^\sharp$.

Last time, we showed that for $X \in \mathfrak{X}(M)$, with φ the flow of X (which we assume to be complete), then $\forall \omega \in \Omega^k(M)$, $\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_X\omega = \varphi_t^*(\iota_X d\omega + d\iota_X\omega)$. Thus, $\varphi_1^*\omega - \omega = \int_0^1 \varphi_t^*(\iota_X d\omega + d\iota_X\omega)\,dt$.

Check: If we define $h(\omega) = \int_0^1 \varphi_t^*(\iota_X\omega)\,dt \in \Omega^{k-1}(M)$, hen the above formula shows that h is a chain homotopy between φ_1^* and the identity map.

Mayer-Vietoris Sequence

Motivation: How can we compute $H^*(S^2)$?

Well, we can describe S^2 as the union of U and V, where U and V are diffeomorphic to the open disk, and their intersection is diffeomoprhic to the cylinder $S^1 \times (-1,1)$. Can we say anything about $H^*(S^2)$ in terms of $H^*(U)$, $H^*(V)$, and $H^*(U \cap V)$?

Hypothesis: In general, for U, V open with $M = U \cup V$, we have

$$\begin{tikzcd} & U \arrow[dr, hook] & \\ U \cap V \arrow[ur, hook] \arrow[dr, hook] & & M \\ & V \arrow[ur, hook] & \end{tikzcd}$$

We can then form, $\forall k \in \mathbb{N}$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^k(M) & \xrightarrow{f} & \Omega^k(U) \oplus \Omega^k(V) & \xrightarrow{g} & \Omega^k(U \cap V) & \longrightarrow & 0 \\ & & & & (\alpha, \beta) & \longmapsto & (\alpha - \beta)|_{U \cap V} & & \end{array}$$

where f is the pullback/restriction.

Lemma: $\forall k \in \mathbb{N}$, this is an exact sequence, i.e., the image of each map is the kernel of the next one. (This is true iff it's a complex with zero cohomology).

Proof: We have exactness at $\Omega^k(M)$ iff f is injecive. This is true because $M = U \cup V$, and U and V are both open.

We have exactness at $\Omega^k(U) \oplus \Omega^k(V)$ iff $\mathrm{im}(f) = \ker(g)$. Well, $\mathrm{im}(f)$ is the set of restrictions of globally-defined forms, so we're still okay.

We have exactness at $\Omega^k(U \cap V)$ iff g is surjective. Let $\omega \in \Omega^k(U \cap V)$. We need to show $\exists \alpha \in \Omega^k(U), \beta \in \Omega^k(V)$ s.t. $(\alpha - \beta)|_{U \cap V} = \omega$. Let $\{\chi_U, \chi_V\}$ be a subordinate partition of unity to $\{U, V\}$. Define

$$\alpha(p) \stackrel{\text{def}}{=} \begin{cases} \chi_V \omega & p \in U \cap V \\ 0 & p \in U \setminus V \end{cases} \qquad \beta(p) \stackrel{\text{def}}{=} \begin{cases} -\chi_U \omega & p \in U \cap V \\ 0 & p \in V \setminus U \end{cases}$$

Then $(\alpha - \beta)|_{U \cap V} = \chi_V \omega + \chi_I \omega|_{U \cap V} = \omega$.

$\square$

Observe: f and g are cochain maps – they commute with d!

Lemma: (Zig-Zag Lemma) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be cochain complexes, and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ cochain maps, s.t. $\forall k \in \mathbb{N}$,

$$0 \longrightarrow A^k \xrightarrow{f} B^k \xrightarrow{g} C^k \longrightarrow 0$$

is exact. Then $\forall k$, $\exists \delta_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, a linear map referred to as the connecting morphism, s.t. the following sequence is exact:

$$0 \longrightarrow H^0(\mathcal{A}) \xrightarrow{f^\sharp} H^0(\mathcal{B}) \xrightarrow{g^\sharp} H^0(\mathcal{C}) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^k(\mathcal{A}) \xrightarrow{f^\sharp} H^k(\mathcal{B}) \xrightarrow{g^\sharp} H^k(\mathcal{C}) \longrightarrow H^{k+1}(\mathcal{A}) \xrightarrow{f^\sharp} H^{k+1}(\mathcal{B}) \xrightarrow{g^\sharp} H^{k+1}(\mathcal{C}) \longrightarrow \cdots$$

Observe: This applied to the case $M = U \cup V$ is precisely the Mayer-Vietoris sequence.

Sketch of the proof:

1. Check exactness at $H^k(\mathcal{B})$:

$$H^k(\mathcal{A}) \xrightarrow{f^\sharp} H^k(\mathcal{B}) \xrightarrow{g^\sharp} H^k(\mathcal{C})$$

   We need to show $\mathrm{im}(f^\sharp) = \ker(g^\sharp)$. Well, we know $0 = (g \circ f)^\sharp = g^\sharp \circ f^\sharp$, so $\mathrm{im}(f^\sharp) \subseteq \ker(g^\sharp)$. For the reverse inclusion, let $[\beta] \in \ker(g^\sharp)$, so $\beta \in Z^k(\mathcal{B})$. We rely on the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^k \arrow[r, "f"] & B^k \arrow[r, "g"] & C^k \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[r, "f"] \arrow[u, "d"] & B^{k-1} \arrow[r, "g"] \arrow[u, "d"] & C^{k-1} \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

   Assume that $g^\sharp[\beta] = 0$, i.e., $\exists c \in C^{k-1}$ s.t. $g(\beta) = dc$. Then $\exists b \in B^{k-1}$ s.t. $g(b) = c$.
   Thus, $g(\beta) = dc = dg(b) = gd(b)$. This means $g(\beta - db) = 0$, so $\exists a \in A^k$ s.t. $f(a) = \beta - db$, so $\beta = db + f(a)$.
   We need to show $[\beta] \in \mathrm{im} f^\sharp$, so we need to have $da = 0$. Well, $0 = df(a) = fda$. Because f is injective, we must have $da = 0$. We conclude that $\beta = db + f(a)$ and $da = 0$, so $[\beta] = [f(a)] = f^\sharp[a]$.

2. Check existence of δ:

$$\begin{tikzcd}
0 \arrow[r] & A^k \arrow[r] \arrow[d] & B^k \arrow[r] \arrow[d] & C^k \arrow[r] \arrow[d] \arrow[dll, dashed] & 0 \\
0 \arrow[r] & A^{k+1} \arrow[r] & B^{k+1} \arrow[r] & C^{k+1} \arrow[r] & 0
\end{tikzcd}$$

   Let $c \in Z^k(\mathcal{C})$, so $c \in C^k$, $dc = 0$. Then $\exists b \in B^k$ s.t. $g(b) = c$. So $0 = dc = dg(b) = g(db)$. Thus, $db \in \ker(g) = \mathrm{im}(f)$, so $\exists a \in A^{k+1}$ s.t. $f(a) = db$. In summary, $c = g(b)$ and $db = f(a)$. We claim:

   (i) $da = 0$.
   (ii) $[a] \in H^{k+1}(\mathcal{A})$ depends only on $[c]$.

   So we define $\delta([c]) = [a]$. Check:

   (i) $fda = df(a) = ddb = 0$. f is injective, so $da = 0$.
   (ii) This just requires more diagram chasing.

Cor: (Mayer-Vietoris Sequence) If $M = U \cup V$, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(M) \arrow[r, "f^\sharp"] & H^0(U) \oplus H^0(V) \arrow[r, "g^\sharp"] & H^0(U \cap V) \arrow[dll, rounded corners] \\
 & H^1(M) \arrow[r, "f^\sharp"] & H^1(U) \oplus H^1(V) \arrow[r, "g^\sharp"] & H^1(U \cap V) \arrow[dll, rounded corners] \\
 & \cdots & &
\end{tikzcd}$$

with $f^\sharp$ and $g^\sharp$ given as above.

Application: $H^k(S^n) = \begin{cases} \mathbb{R} & k \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$ We can prove this using induction on n. For example, for $n = 2$,

$$\begin{tikzcd}
 & S^2 & U \sqcup V & U \cap V \\
H^0 & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \arrow[ddll, dashed, rounded corners] \\
 & x \arrow[r, mapsto] & (x, x) & \\
 & & (x, y) \arrow[r, mapsto] & x - y \\
H^1 & H^1(S^2) \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[dll, dashed, rounded corners] \\
H^2 & H^2(S^2) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We have the exact sequence $0 \longrightarrow \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0$, so the mapping from $\mathbb{R}$ to $H^2(S^2)$ must be injective and surjective, so $H^2(S^2) = \mathbb{R}$. As for $H^1(S^2)$, the map $(x, y) \mapsto x - y$ is surjective, so the map into $H^1(S^2)$ must be the zero map. By exactness at $H^1(S^2)$, we must have the kernel of the map from $H^1(S^2)$ to 0 also be 0, so we must have $H^1(S^2) = 0$. Then, our inductive step uses the fact that the "equator" $U \cap V$ is homotopy equivalent to S^{n-1}.

Another example is the 2-torus, T^2. We can cut the torus in half to get two components U, V, each of which is diffeomorphic to the cylinder, which in turn is homotopy equivalent to S^1. The $U \cap V$ is the disjoint union of 2 cylinders. We then have

$$
\begin{tikzcd}
 & T^2 & U \sqcup V & U \cap V \\
H^0 & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, dashed] \\
H^1 & H^1(T^2) \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, dashed] \\
H^2 & H^2(T^2) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Exer: Show that $H^k(T^2) = \begin{cases} \mathbb{R} & k \in \{0, 2\} \\ \mathbb{R}^2 & k = 1 \end{cases}$, and that $H^1(T^2)$ is generated by $[dx^1]$ and $[dx^2]$.

Thm: If M is a compact, oriented, connected manifold (with $m = \dim M$), then

$$\int_M : H^k(M) \to \mathbb{R}$$

is an isomorphism, so $H^m(M) \cong \mathbb{R}$.