Math 591 Lecture 18

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Thm: (Normal Form for Immersions) Let $F : M \to N$ be an immersion at $p \in M$. Then there are coordinates $p \in U, \phi = (x^1, \dots, x^m)$ of M and $f(p) \in V, \psi = (y^1, \dots, y^n)$ of N such that $U \subset F^{-1}(V)$ and $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n, \underbrace{0, \dots, 0}_{n-m \text{ zeros}})$.

Proof: take any coordinates. We have

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\phi"] & V \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Write $F = (F^1, \dots, F^n)$, where each $F^i = y^i \circ F$.
The Jacobian of $\tilde{F}$ at $\phi(p)$ is $J = \left(\frac{\partial F^i}{\partial x^j}(p)\right)_{\substack{i \text{ rows} \\ j \text{ cols}}}$. By assumption, $\ker J = \{0\}$, so we can write

$$J = \begin{pmatrix} \mathcal{M}_{m\times m} \\ \star \end{pmatrix}_{\substack{m \le n \\ 1 \le i \le n \\ 1 \le j \le m}}$$

and we know J has max rank. Permute the y-coordinates to shuffle the rows, so that the top $m \times m$ minor $\mathcal{M}$ of J is non-degenerate. Thus, $\mathcal{M} = \left(\frac{\partial F^i}{\partial x^j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ gives us some $\tilde{\phi} = (F^1, \dots, F^m) : U \to \mathbb{R}^m$ which is a local diffeomorphism at p. By shrinking U if necessary, $\tilde{\tilde{\phi}} : U \to \mathbb{R}^m$ is a coordinate chart.

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\tilde{\phi}"] & V \subseteq N \arrow[d, "\psi"] \\ \tilde{\phi}(U) \arrow[r, "\tilde{\tilde{F}}"] & \psi(V) \end{tikzcd}$$

where $\tilde{\tilde{F}}(\underbrace{r^1, \dots, r^m}_{r}) = (y^1(F \circ \tilde{\phi}(r)), \dots, y^n(F \circ \tilde{\phi}(r))) = (r^1, \dots, r^m, G^1(r), \dots, G^{n-m}(r))$. Now, we modify the y-coordinates:

$$\begin{cases} w^i = y^i & 1 \le i \le m \\ w^i = y^i - G^{i-m}(y^1, \dots, y^m) & m+1 \le i \le n \end{cases}$$

Both are invertible. The first m are trivially so, and the remainder are invertible because you can recover y^{i+m}.
Let $\tilde{\psi} = (w^1, \dots, w^n) : V \to \mathbb{R}^n$.

$$\begin{tikzcd} & V \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\ \mathbb{R}^n \arrow[rr, "\star"] & & \mathbb{R}^n \end{tikzcd}$$

with

$$\star : (r^1, \dots, r^n) \mapsto (\underbrace{r^1, \dots, r^m}_{r'}, \underbrace{r^{m+1} - G^1(r'), \dots, r^n - G^{m-n}(r')}_{r'' - G(r')})$$

This is a local diffeomorphism. So under the $\tilde{\phi}$ and $\tilde{\psi}$ coordinates, $(r^1, \dots, r^m) \xrightarrow{\tilde{\tilde{F}}} (r^1, \dots, r^m, 0, \dots, 0)$. $\square$

Recall: An embedding $F : M \to N$ is an immersion that is a homeomorphism onto its image, that is, $F|^{F(M)} : M \to F(M)$ is a homeomorphism. It has a continuous inverse.

Observe: Every embedding is also one-to-one.

Question: Assum F is a one-to-one immersion. Under what conditions is F an embedding?

Prop: An injective immersion is an embedding iff $F|^{F(M)} : M \to F(M)$ is an open (or closed) map w.r.t. the subspace topology. This is true iff $\forall U \subset M$ open, $\exists V \subset N$ open s.t. $F(U) = F(M) \cap V$ ($F(M) \cap V$) is a relatively open set).

Proof: The inverse of an open map is continuous iff the map is invertible. $\square$

Cor: If $F : M \to N$ is an injective immersion and M is compact, then F is an embedding.

Proof: F is a closed map – if $C \subset M$ is closed and M is compact, then C is compact, so $F(C)$ is compact. Thus, $F(C)$ is closed. $\square$

In fact...

Thm: If $F : M \to N$ is a proper (recall: the preimage of compact sets are compact) injective immersion, then F is an embedding.

"Proof": This is true because a proper continuous map into a locally compact space is closed.

Thm: The image of an embedding is a regular submanifold.

Proof: Let $F : M \to N$ be an embedding. Let $q = F(p) \in F(M)$ for $p \in M$. Use the immersion theorem to get coordinate $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ s.t. $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n, \underbrace{0, \dots, 0}_{n-m})$.

Since F is an embedding, $F(U)$ is relatively open in $F(M)$. Thus, $F(U)$ is the intersection of $F(M)$ with some open set $\tilde{V}$ in N. Thus, $F(U) = \tilde{V} \cap F(M) \subseteq \{y^{m+1} = \dots = y^n = 0\}$. (Will finish next time.)