Math 591 Lecture 32

Thomas Cohn

11/16/20

The Exterior Differential

Thm: Let M be a manifold. Then there exists a unique operator d

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k$$

s.t.

(1) d is $\mathbb{R}$-linear.
(2) $d : \Omega^0 \to \Omega^1$ is the usual differential of a function.
(3) $d^2 = 0$.
(4) The anti-derivation property: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. (This is also known as the super-symmetric version of Leibniz' rule.)
(5) If $F : M \to N$ is smooth, then $\forall \alpha \in \Omega^k N$, $d(F^*\alpha) = F^*(d\alpha)$. (This property is called "Naturality".)

Note: In theory, the d is a different operator for each Ω^i. But it would be inconvenient to call them different names.

Proof: We'll begin by proving this for $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, and then use coordinates. This works because the anti-derivation property implies locality.

For $\mathbb{R}^n$: If $\alpha = \sum_I a_I dx^I \in \Omega^k(\mathbb{R}^n)$, then the properties imply that we must have $d\alpha = \sum_I \sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$.
Use this as our definition of d on $\mathbb{R}^n$. Now, we must check the 5 properties. 1 and 4 follow directly from its definition. By linearlity, we can check this solely on "monomials".

Anti-derivation: Let $\alpha = a\, dx^I$, $\beta = b\, dx^J$, $a, b \in C^\infty$. Then

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(a\, dx^I \wedge b\, dx^J) \\
&= d(ab\, dx^I \wedge dx^J) \\
&= d(ab\, dx^{IJ}) \\
&= \sum_{j=1}^n \frac{\partial(ab)}{\partial x^j}\, dx^j \wedge dx^{IJ} \\
&= \sum_{j=1}^n \left[b\frac{\partial a}{\partial x^j} + a\frac{\partial b}{\partial x^j} \right] dx^j \wedge dx^{IJ} \\
&= \sum_{j=1}^n \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I \wedge (b\, dx^J) + a \sum_{j=1}^n \left(\frac{\partial b}{\partial x^j}\, dx^j \wedge dx^I \right) \wedge dx^J \\
&= \sum_{j=1}^n \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I \wedge (b\, dx^J) + (-1)^k a\, dx^I \sum_{j=1}^n \frac{\partial b}{\partial x^j}\, dx^j \wedge dx^I \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta
\end{aligned}$$

$\mathbf{d^2 = 0}$**:**

$$\begin{aligned}
d\left[d(a\,dx^I)\right] &= d\left[\sum_{j=1}^{n} \frac{\partial a}{\partial x^j}\,dx^j \wedge dx^I\right] \\
&\overset{(1)}{=} \sum_{j=1}^{n} d\left(\frac{\partial a}{\partial x^j}\right) \wedge dx^j \wedge dx^I \\
&= \underbrace{\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 a}{\partial x^\ell \partial x^j}\,dx^\ell \wedge dx^j\right)}_{\text{Observe: Terms with } j=\ell \text{ vanish.}} \wedge dx^I \\
&= \left(\sum_{1\le j<\ell\le n} \frac{\partial^2 a}{\partial x^\ell \partial x^j}\,dx^\ell \wedge dx^j + \frac{\partial^2 a}{\partial x^j \partial x^\ell}\,dx^j \wedge dx^\ell\right) \wedge dx^I \\
&= \left(\sum_{1\le j<\ell\le n} \left(\frac{\partial^2 a}{\partial x^\ell \partial x^j} - \frac{\partial^2}{\partial x^j \partial x^\ell}\right) dx^\ell \wedge dx^j\right) \wedge dx^I \\
&\overset{(2)}{=} 0
\end{aligned}$$

with (1) because $dx^j \wedge dx^I$ is constant, so $d(dx^j \wedge dx^I) = 0$, and (2) by Clairaut's Theorem, which tells us that $\frac{\partial^2 a}{\partial x^\ell \partial x^j} = \frac{\partial^2 a}{\partial x^j \partial x^\ell}$, so their difference is 0.

Naturality: Say $F : V \to U$ is C^∞, $V \subseteq \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$. Let $\alpha \in \Omega^k(U)$ of the form $\alpha = a\,dx^I$. We need to prove that $d(F^*\alpha) = F^*(d\alpha)$. Say $F = (F^1, \dots, F^n)$ with $F^j : V \to \mathbb{R}$. We proved that

$$F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}) = F^*(dx^I) = dF^{i_1} \wedge \cdots \wedge dF^{i_k} \overset{\text{def}}{=} dF^I$$

So $F^*\alpha = (\alpha \circ F)F^*(dx^I) = (\alpha \circ F)dF^I$, and thus,

$$d(F^*\alpha) = d(\alpha \circ F) \wedge dF^I + (\alpha \circ F)\underbrace{d(dF^I)}_{=0} = d(\alpha \circ F) \wedge dF^I$$

$$F^*(d\alpha) = F^*\left[\sum_{j=1}^{n} \frac{\partial a}{\partial x^j}\,dx^j \wedge dx^I\right] = \sum_{j=1}^{n} F^*\left(\frac{\partial a}{\partial x^j}\right) dx^j \wedge dx^I = \sum_{j=1}^{n}\left(\frac{\partial a}{\partial x^j} \circ F\right)\underbrace{\left(F^*(dx^j)\right)}_{dF^j} \wedge dF^I = d(\alpha \circ F) \wedge dF^I$$

So we conclude that $d(F^*\alpha) = F^*(d\alpha)$.

Now, we use coordinates to do it on a manifold.

Defn: Given $\alpha \in \Omega^k(M)$, $\phi : U \to \mathbb{R}^n$ a coordinate system, define $d\alpha \in \Omega^{k+1}(U)$ by $\phi^*\left[d(\phi^{-1})^*\alpha\right]$, where d is the Euclidean version, and $(\phi^{-1})^*\alpha \in \Omega^k(\phi(U))$.

Now, we need to prove that $d\alpha$ on U is independent of choice of coordinates. Suppose $\psi : V \to \mathbb{R}^n$ is another coordinate chart (WOLOG $U \cap V \neq \emptyset$). Call U the intersection of their domains. Write $\psi = F \circ \phi$

$$\begin{tikzcd}
 & U \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\phi(U) \arrow[rr, "F"'] & & \psi(U)
\end{tikzcd}$$

Then

$$\begin{aligned}
\psi^*\left[d(\psi^{-1})^*\alpha\right] &= \phi^* \circ F^*\left[d(\phi^{-1} \circ F^{-1})^*\alpha\right] \\
&= \phi^* \circ F^*\left[d((F^{-1})^* \circ (\phi^{-1})^*)\alpha\right] \\
&= \phi^* \circ F^* \circ (F^{-1})^*\left[d(\phi^{-1})^*\alpha\right] \\
&= \phi^*\left[d(\phi^{-1})^*\alpha\right]
\end{aligned}$$

As for showing uniqueness on manifolds, use that anti-derivation implies locality, as with vector fields. $\square$

Ex: $\mathbb{R}^3$

$$\begin{aligned} d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= (\partial_x f\,dx + \partial_y f\,dy + \partial_z f\,dz) + \cdots \\ &= \partial_y f\,dy \wedge dx + \partial_z f\,dz \wedge dx + \cdots \\ &= (\partial_x g - \partial_y f)\,dx \wedge dy + (\partial_x h - \partial_z f)\,dx \wedge dz + \cdots \end{aligned}$$

We get the components of the curl of the vector field $< f, g, h >$.

Defn: $\alpha \in \Omega^k(M)$ is closed if $d\alpha = 0$, and exact if $\exists \beta \in \Omega^{k-1}(M)$ such that $\alpha = d\beta$.

Note: $d^2 = 0$ means exact implies closed.

Defn: The kth DeRahm Cohomology $H^k(M) = \ker(d : \Omega^k \to \Omega^{k+1})/\mathrm{im}(d : \Omega^{k-1} \to \Omega^k)$ (with the vector space quotient).

Ex: If $M = S^1$, $d\theta$ dual to $\frac{\partial}{\partial \theta}$ is closed. (Any top-degree form is closed.)
Claim: $d\theta$ is not exact, and in fact, $H^1(S^1) = \mathbb{R}$.