

Math 591 Lecture 2

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Quotient Spaces

Defn: Let X be a topological space, \sim an equivalence relation on X . Then X/\sim is the set of equivalence classes of \sim , and

$$\begin{aligned}\pi : X &\rightarrow X/\sim \\ x &\mapsto [x]\end{aligned}$$

The quotient topology is defined by $W \subseteq X/\sim$ is open iff $\pi^{-1}(W) \subseteq X$ is open.

We just assume that this is a topology – is it?

Observe: π is continuous by definition of the quotient topology. And the quotient topology is the finest topology on X/\sim for which π is continuous.

Prop: Let Y be a topological space, $f : X/\sim \rightarrow Y$. Then f is continuous iff $f \circ \pi : X \rightarrow Y$ is continuous.

$$\begin{array}{ccc} X & & \\ \downarrow \pi & \searrow f \circ \pi & \\ X/\sim & \xrightarrow{f} & Y \end{array}$$

Proof: \Rightarrow is trivial, because π is continuous.

\Leftarrow is the real content of the proof (left as an exercise).

Observe: it's not true that X being Hausdorff implies X/\sim being Hausdorff.

And it's not true that X being second countable implies X/\sim being second countable.

Defn: An equivalence relation \sim on a topological space X is open iff $\pi : X \rightarrow X/\sim$ is an open map (i.e. $\forall U \subseteq X$ open, $\pi(U)$ is open).

Let's investigate: $\pi(U)$ is open, by definition, if $\pi^{-1}(\pi(U)) \subseteq X$ is open. So let $\hat{U} = \pi^{-1}(\pi(U)) = \{x \in X \mid \exists y \in U \text{ s.t. } x \sim y\}$. To recap, \sim is open $\Leftrightarrow \forall U \subseteq X$ open, \hat{U} is open.

Thm: Let \sim be an equivalence relation. Assume it is open, then

- 1) If X is second countable, then so is X/\sim .
- 2) X/\sim is Hausdorff iff the graph of the relation, Γ , is closed in $X \times X$. $\Gamma = \{(x, y) \in X \times X \mid x \sim y\} \subseteq X \times X$.

Proof:

- 1) If \sim is open, then for any basis $\mathcal{B} = \{B_j\}_{j \in J}$ of X , $\{\pi(B_j)\}_{j \in J}$ is a basis for X/\sim . \square
- 2) Γ is closed iff $\forall (x, y) \in (X \times X) \setminus \Gamma$, $\exists U, V \subseteq X$ open with $x \in U$, $y \in V$ s.t. $(U \times V) \cap \Gamma = \emptyset$. And this is true iff $\forall (u, v) \in U \times V$, $\pi(u) \neq \pi(v)$, which is true iff $\pi(U) \cap \pi(V) = \emptyset$. And $\pi(U)$ is a neighborhood of $\pi(x)$, $\pi(V)$ is a neighborhood of $\pi(y)$, because π is an open map.

Some remarks:

1. In general, if X/\sim is Hausdorff, then for any point $p \in X/\sim$, $\{p\}$ is closed. Thus, $\pi^{-1}(p)$ (an equivalence class of \sim) is closed, because set complements work nicely. So we conclude that a necessary condition for X/\sim to be T_2 is all equivalence classes $[x] \subseteq X$ are closed. But this isn't sufficient!
2. If $P : X \rightarrow S$ is a surjective map...

Defn: $\forall x, y \in X, x \sim_P y \stackrel{\text{def}}{\iff} P(x) = P(y)$ (\star) is an equivalence relation. If P is surjective, there's a natural identification of $S \cong X / \sim_P$. If X, S are tangent spaces, P is called a quotient map iff \star is a homeomorphism, i.e., iff $\forall W \subseteq S, W$ is open iff $P^{-1}(W)$ is open.

Ex: $X = S^n \subseteq \mathbb{R}^{n+1}$, the n -dimensional sphere. Consider $\tau : S^n \rightarrow S^n$, where $x \mapsto -x$. Observe that τ is continuous, and $\tau^2 = \text{Id}$ (so $\tau = \tau^{-1}$). Define the following equivalence relation: $\forall x, y \in S^n, x \sim y \iff x = y$ or $y = \tau(x)$.

Claim: \sim is open.

Proof: Let $U \subseteq S^n$ be open. $\hat{U} = \pi^{-1}(\pi(U)) = U \cup \tau(U)$. So \hat{U} is open because $\tau(U)$ is open because τ is a homeomorphism because τ is continuous and $\tau = \tau^{-1}$. Thus, S^n / \sim is second-countable (because we know S^n is second-countable). Write

$$\begin{aligned} \Gamma &\subseteq S^n \times S^n \\ &= \{(x, y) \in S^n \times S^n \mid x \sim y\} \\ &= \{(x, y) \in S^n \times S^n \mid x = y \text{ or } y = \tau(x)\} \\ &= \underbrace{\{(x, x) \mid x \in S^n\}}_{\text{diagonal}} \cup \underbrace{\{(x, \tau(x)) \mid x \in S^n\}}_{\text{graph of } \tau} \end{aligned}$$

The graph of τ is closed.

Consider $F : S^n \times \{0, 1\} \rightarrow S^n$ (note: $\{0, 1\} = \mathbb{Z}_2$), where $(x, 0) \mapsto x$ and $(x, 1) \mapsto \tau(x)$. This is a group action.

So the graph of τ is the image of the map $S^n \rightarrow S^n \times S^n$, where $x \mapsto (x, \tau(x))$. This is a continuous map, and S^n is compact, so its image is compact, so its image is closed.

Thus, Γ is the finite union of closed sets, so Γ is closed, so S^n / \sim is Hausdorff.

Note: $S^n / \sim = \mathbb{RP}^n$, the n -dimensional real projective space. This is isomorphic to the set of 1-dimensional subspaces of \mathbb{R}^{n+1} .