Math 591 Lecture 39

Thomas Cohn

12/16/20

De Rham Cohomology

Defn: Given M a manifold, $k \in \mathbb{N}$ (by our convention, $0 \in \mathbb{N}$), we define $Z^k(M) = \ker(d : \Omega^k \to \Omega^{k+1})$, the set of closed k-forms ("cocycles") and $B^k(M) = \operatorname{im}(d : \Omega^{k-1} \to \Omega^k)$ the set of exact k-forms ("coboundaries").

Defn: The <u>kth de Rham group</u> is the quotient $H^k \overset{\text{def}}{=} Z^k/B^k$. (In our case, this is a quotient vector space, but it can also be defined simply as a quotient group.)

Defn: $\beta_k = \dim H^k(M)$ is the <u>kth Betti number</u> of M.

Observe: If $k > \dim M$, then $\beta_k = 0$.

Observe: There is a "compact version" of this theory, for working with Ω_0^k, the set of compacly-supported k-forms.

Ex: $H^0 = \{f \in C^\infty \mid df = 0\}$ is the space of locally constant functions. Thus, β_0 is the number of connected components of M.

Ex: $M = \mathbb{R}$, $\beta_0 = 1$. What is β_1? Well,
$Z^1 = \Omega^1 = \{f\,dx \mid f \in C^\infty\}$.
$B^1 = \{dg = g'\,dx \mid g \in C^\infty\}$.
Every $f \in C^\infty$ has an antiderivative, so $H^1(\mathbb{R}) = \{0\}$, so $\beta_1 = 0$.

Prop: $H^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$

Ex: $M = \mathbb{R}$. Consider only forms with compactly-supported coefficients ($H_C^{\cdot}$). Then
$H^0 = \{f \in C^\infty(\mathbb{R}) \mid df = 0\} = \{0\}$.
To compute H^1, ask: Which functions $f \in C_0^\infty(\mathbb{R})$ have anti-derivatives that are of compact support? One can show that this is true iff $\int f = 0$.

Note: $\int : H_C^1 \to \mathbb{R}$ is an isomorphism, so $H_C^1(\mathbb{R}) \cong \mathbb{R}$.

Ex: $M = S^1$. M is connected, so $\beta_0 = 1$. What about β_1? Well,
$Z^1 = \Omega^1 = \{f\,d\theta \mid f : \mathbb{R} \to \mathbb{R} \text{ is } 2\pi\text{-periodic}\}$.
$B^1 = \big\{dg = g'\,d\theta \mid g \in C^\infty(S^1)\big\}$.
Question: Which 2π-periodic functions have 2π-periodic antiderivatives? We can figure this out using Fourier series. We want $f = dg$, so

$$f = \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \qquad \Rightarrow \qquad g = \frac{1}{i} \sum_{n \in \mathbb{Z}} \frac{a_n}{n} e^{in\theta}$$

So we need $a_0 = 0$. In fact, $[f\,d\theta] = [a_0\,d\theta]$, where $[\,\cdot\,]$ denotes the cohomology class. Thus, $H^1(S^1) = \mathbb{R}[d\theta] \cong \mathbb{R}$.

Observe: This generalizes greatly, to any compact manifold without boundary. It's known as "Hodge theory". Fourier series are replaced with the spectral of the Laplacian.

General Features

- Covariance: If $F : M \to N$ is C^∞, we can define, $\forall k$, $F^* : H^k(N) \to H^k(M)$ by $F^*[\omega] = [F^*\omega]$ (for ω $Z^k(N)$). This is well-defined because F^* and d commute.
- Ring Structure: The wedge product induces a "cup map" in cohomology:

$$\begin{aligned} H^k(M) \times H^\ell(M) &\to H^{k+\ell}(M) \\ ([\alpha], [\beta]) &\mapsto [\alpha \wedge \beta] \end{aligned}$$

  Check that this is well-defined:

  a) $d(\alpha \wedge \beta) = 0$ if $d\alpha = 0$ an $d\beta = 0$ by the product rule.
  b) $(\alpha + da) \wedge (\beta + db) = \alpha \wedge \beta + \cdots$. The remaining terms are each exact, so it's still the same cohomology class.

- F^* is a ring morphism.
- $(F \circ G)^* = G^* \circ F^*$.

Cor: Diffeomorphic manifolds have isomorphic cohomology.

Homotopy Equivalence

Defn: Two C^∞ functions $F, G : M \to N$ are homotopic (to each other) iff there is a smooth map $\Phi : M \times [0, 1] \to G$ s.t. $\forall p \in M$, $\Phi(p, 0) = F(p)$ and $\Phi(p, 1) = G(p)$. Φ is called a homotopy.

Some notation: $\forall t \in [0, 1]$, let $\iota_t : M \to M \times [0, 1]$ where $p \mapsto (p, t)$. Then $F = \Phi \circ \iota_0$, and $G = \Phi \circ \iota_1$.

Observe: In Tu's textbook, we use $\mathbb{R}$ instead of $[0, 1]$. The two definitions are equivalent, as we can easily extend Φ from $M \times [0, 1]$ to $M \times \mathbb{R}$ with a bump function, and we can simply restrict from $M \times \mathbb{R}$ to $M \times [0, 1]$.

Thm: Being homotopic is an equivalence relation in $C^\infty(M, N)$. We write $F \sim G$.

Ex: If $X \in \mathfrak{X}(M)$ is complete, then $\forall t$, ϕ_t, the flow's time-t map, is homotopic to the identity, ϕ_0.

Proof: For $t = 1$, use Φ. For $t = T$ (nonzero), use $\Phi(p, t) = \phi_{tT}(p)$. $\square$

Ex: Say $\mathrm{Id}, G : \mathbb{R}^n \to \mathbb{R}^n$, with $G(x) = 0$ a constant map. Then G and I are homotopic, with $\Phi(x, t) = (1 - t)x$ a homotopy.

Defn: A submanifold $S \subset M$ is a deformation retract of M iff $\exists \Phi : M \times [0, 1] \to M$ s.t.

a) $\Phi|_{\{t=0\}} = \mathrm{Id}_M$
b) $\Phi|_{\{t=1\}}$ maps M into S.
c) $\forall t \in [0, 1]$, $\forall p \in S$, $\Phi(p, t) = p$.

Such a Φ is called a deformation retraction.

Ex: $M = S^1 \times (-1, 1)$, $S = S^1 \times \{0\}$. S is a deformation retract of M.

Defn: $F : M \to N$ is a homotopy equivalence iff $\exists G : N \to M$ s.t. $G \circ F \sim \mathrm{Id}_M$ and $F \circ G \sim \mathrm{Id}_N$. If there's a homotopy equivalence $M \to N$, we say that M and N are homotopy equivalent, or that they "have the same homotopy type". We say G and F are homotopy inverses.

Prop: If $S \subset M$ is a deformation retract of M, then M and S are homotopy equivalent.

Proof: Let $\Phi : M \times [0, 1] \to M$ be a deformation retraction of M onto S. Define $F : M \to S$ to be $F = \Phi|^S_{M \times \{1\}} : M \to S$, and $G : S \hookrightarrow M$ to be the inclusion. $\square$

From the perspective of homotopy theory, $\mathbb{R}^n$ is the same as a point, the cylinder is the same as the circle. Dimension is not homotopy-invariant!

Back to de Rham Theory

Thm: (Homotopy Axiom) Let $F, G : M \to N$ be smooth maps, and homotopic to each other. Then $F^* = G^*$ in cohomology, i.e., $\forall k \in \mathbb{N}$, $F^*, G^* : H^k(N) \to H^k(M)$ are equal.

Proof: First, we'll prove this for $F = I$, and G, the time-1 map of a flow. Let $X \in \mathfrak{X}(M)$, and assume it's complete. Say $G = \varphi_1$. We need to show $\varphi_1^* : H^k(M) \to H^k(M)$ is the identity.

Let $[\omega] \in H^k(M)$, so $\omega \in Z^k$ $(d\omega = 0)$. We need to show $\exists \alpha \in \Omega^{k-1}(M)$ such that $\varphi_1^*\omega - \omega = d\alpha$.

Observe: $\frac{d}{dt}\varphi_t^*\omega = \mathcal{L}_X[\varphi_t^*\omega] = d(\iota_X \varphi_t^*\omega) + \iota_X d\varphi_t^*\omega$. We can commute d and φ_t^*, and use the fact that $d\omega = 0$, to see that $\frac{d}{dt}\varphi_t^*\omega = d(\iota_X\varphi_t^*\omega)$. Integrate both sides over $[0, 1]$ w.r.t t on each $\bigwedge^k T_pM$, $\forall p \in M$, and we obtain

$$\varphi_1^*\omega - \omega = \int_0^1 \frac{d}{dt}\varphi_t^*\omega \, dt = \int_0^1 (\iota_X\varphi_t^*\omega) \, dt = d \underbrace{\int_0^1 (\iota_X\varphi_t^*\omega) \, dt}_{\overset{\text{def}}{=}\alpha}$$

Now, for the general case, say $F, G : M \to N$ are homotopic. Then let $\Phi : M \times \mathbb{R} \to N$ be a homotopy between them. Let $X = \frac{\partial}{\partial t}$ on $M \times [0, 1]$. Then $\varphi_t(p, s) = (p, s + t)$. Thus, the following diagram commutes:

$$\begin{tikzcd}
 & M \times \mathbb{R} \arrow[r, "\Phi"] \arrow[d, "\varphi_1"] & N \\
M \arrow[ur, "\iota_0"] \arrow[dr, "\iota_1"'] \arrow[r, phantom, "\circlearrowleft"] & M \times \mathbb{R} \arrow[r, "\Phi"] & N
\end{tikzcd}$$

where $\iota_t(p) = (p, t)$. So we have $G = \Phi \circ \iota_1$ and $F = \Phi \circ \iota_0$.

Let $[\omega] \in H^k(N)$.

$$G^*[\omega] = (\iota_1^* \circ \Phi^*)[\omega] = ((\varphi_1 \circ \iota_0)^* \circ \Phi^*)[\omega] = (\iota_0^* \circ \varphi_1^* \circ \Phi^*)[\omega] = \iota_0^*(\underbrace{\varphi_1^*(\Phi^*[\omega])}_{\substack{=\Phi^*[\omega] \\ \text{because } \varphi_1^* = \text{Id}}}) = (\iota_0^* \circ \Phi^*)[\omega] = F^*[\omega]$$

□

Cor: If M and N are homotopy equivalent, then their cohomology is isomorphic.