Math 591 Lecture 6

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Many examples of C^∞ manifolds are produced by the implicit function theorem. Reminder:

Thm: (Implicit Function Theorem) (Theorem B2 in the textbook)
Let $U \subseteq \mathbb{R}^N$ open, $F : U \to \mathbb{R}^k$ C^∞, and $x_0 \in U$ s.t. $F(x_0) = 0$.
Split: for $x \in U$, write $x = (x', x'')$, where $x' \in \mathbb{R}^{N-k}$ and $x'' \in \mathbb{R}^k$. Accordingly, the Jacobian of F at x_0 splits:

$$F'(x_0) = (\underbrace{\tfrac{\partial F}{\partial x'}(x_0)}_{N-k} \vdots \underbrace{\tfrac{\partial F}{\partial x''}(x_0)}_{k})\}k$$

Assume $[\frac{\partial F}{\partial x''}(x_0)]$ $(k \times k)$ is invertible. Then there exist open sets A, B with $x_0' \in A \subseteq \mathbb{R}^{N-k}$, $x_0'' \in B \subseteq \mathbb{R}^k$ and $g : A \to B$ C^∞ s.t. $\{F^{-1}(0)\} \cap (A \times B) = \{(x', g(x') \mid x' \in A)\}$.

Application: Recall that given $F : U \to \mathbb{R}^k$, $U \subseteq \mathbb{R}^N$ open, zero is a regular value of F iff $\forall x \in F^{-1}(0$, $F'(x)$ has rank k.

Cor: If 0 is a regular value for F, then $F^{-1}(0) = M$ is locally a graph. Moreover, this structure of local graph gives M a C^∞ atlas, and therefore a smooth manifold structure.
(Note: we will define "submanifold", and then $F^{-1}(0)$ will be examples of submanifolds of $\mathbb{R}^N$.)
Proof/Explanation: Assume 0 is a regular value of F. Then $\forall x_0 \in F^{-1}(0) = M$,

$$F'(x_0) = \begin{pmatrix} — & \nabla f^1(x_0) & — \\ & \vdots & \\ — & \nabla f^k(x_0) & — \end{pmatrix}$$

(for $F = (f^1, \ldots, f^k)$). After permuting the indices among the x_i, without loss of generality $[\frac{\partial F}{\partial x''}(x_0)]_{k \times k}$ is non-degenerate. (Think of this as swapping the columns around so the block is invertible.)

The independent variables will depend on x_0. The number of independent variables is $N-k = \dim M$. k is the <u>codimension</u> of $M \subseteq \mathbb{R}^N$.

Statement about "Atlas"

Recall: A graph $\{(x', g(x')) \mid x' \in A\} = \Gamma$ has a global chart: just the projection onto the domain. $\Gamma \ni (x', x'') \mapsto x'$. Its inverse is $x' \mapsto (x', g(x'))$.

For two local representations of M as a local graph, transition functions are of the form $x' \mapsto (x', g(x')) \xrightarrow{\star} \mathbb{R}^{N-k}$. $\star$ is a projection onto an $N-k$-dimensional coordinate plane of $\mathbb{R}^N$. This is smooth, and a transition map.

Ex: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in $\mathbb{R}^3$ is the zero set of $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$.
(Check that 0 is a regular value.)

Ex: $O(n) = \{g \in \mathrm{Mat}(n, \mathbb{R}) \mid g^{-1} = g^T\}$. $\mathrm{Mat}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, so $O(n) \subseteq \mathbb{R}^{n^2}$.
In fact, $O(n) \subseteq \mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (and $\mathrm{GL}(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$).
Define

$$\begin{aligned} F : \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Symm}(n, \mathbb{R}) \\ g &\mapsto gg^T - I \end{aligned}$$

where $\mathrm{Symm}(n, \mathbb{R}) = \{g \in \mathrm{Mat}(n, \mathbb{R}) \mid g = g^T\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$, and I is the identity matrix. Note: We have to choose the codomain carefully so that 0 is a regular value.

So, $O(n) = F^{-1}(0)$.
Check: is 0 a regular value? To see if $F'(g)$, for $g \in O(n)$, has maximal rank, let $M \in \mathrm{Mat}(n, \mathbb{R})$. Then compute $\frac{d}{dt} F(g + tM)\big|_{t=0}$. (Compute this as a matrix to avoid $\mathbb{R}^{n^2}$ coords.) Then $(-\varepsilon, \varepsilon) \ni t \mapsto \mathrm{GL}(n, \mathbb{R}) \overset{F}{\mapsto} \mathrm{Symm}(n, \mathbb{R})$.

$$\frac{d}{dt} F(g + tM)\bigg|_{t=0} = \frac{d}{dt}(g + tM)(g + tM)^T\bigg|_{t=0} = \frac{d}{dt}\left(gg^T + t(Mg^T + gM^T) + t^2 MM^T\right)\bigg|_{t=0} = Mg^T + gM^T$$

Question: Is $\mathrm{Mat}(n, \mathbb{R}) \ni M \mapsto Mg^T + gM^T$ onto? Yes! (This is true iff $F'(g)$ has rank equal to the dimension of $\mathrm{Symm}(n, \mathbb{R})$. Let $S \in \mathrm{Symm}(n, \mathbb{R})$. What can M be?)
By the implicit function theorem, $O(n) \subseteq \mathbb{R}^{n^2}$ is locally a graph. So it has a natural C^∞ structure.