Math 591 Lecture 1

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Transcribed by Thomas Cohn

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What is a differentiable manifold? It's a bit of a long answer. Here's something that's true, but not complete:

A differentiable manifold is a topological space that is
(a) Hausdorff (T_2)
(b) Second-Countable
(c) Locally Euclidean (Informally, this means every point has a neighborhood that is homeomorphic to a Euclidean space.)
(d) Has a C^∞ atlas (This is additional structure, not a topological property.)
Note that (a), (b), and (c) together define a topological manifold.

Hausdorff Spaces

Defn: A topological space X is Hausdorff (or T_2) if $\forall x, y \in X$ with $x \neq y$, there exist neighborhoods U of x and V of y such that $U \cap V = \emptyset$.

Ex: Any metric space is Hausdorff.
Proof: Fix distinct x, y. Let $r = \frac{1}{3}d(x, y) > 0$ (because $x \neq y$). Let $U = B(x, r)$, $V = B(y, r)$. Then by the triangle inequality, $U \cap V = \emptyset$. $\square$

Ex: The following topological space is *not* Hausdorff.
Let $X = (-\infty, 0) \cup \{A, B\} \cup (0, \infty)$ (with $A \neq B$), with the standard topology on $(-\infty, 0)$ and $(0, \infty)$, and neighborhoods of A and B contain $(-\varepsilon, 0)$ and $(0, \varepsilon)$.
This space is not Hausdorff, because you can't separate A and B.

Second-Countable Spaces

Defn: Let X be a topological space. A basis of X is a collection $\mathscr{B}$ of open sets such that every open set in X can be written as the (possibly infinite) union of elements in $\mathscr{B}$.

Defn: X is second-countable if there is a countable basis of X.

Ex: $X = \mathbb{R}^n$ (with the standard topology) is second countable.
Proof: Let $\mathscr{B} = \{B(q, r) \mid q \in \mathbb{Q}^n, r \in \mathbb{Q}\}$. We know $\mathscr{B}$ is countable, so we need to show $\mathscr{B}$ is a basis.
Let $U \subseteq \mathbb{R}^n$ be open. Let $\mathscr{B}_U = \{B \in \mathscr{B} \mid B \subseteq U\}$. We claim that $U = \bigcup_{B \in \mathscr{B}_U} B$.
Well, $\supseteq$ is trivial. To show $\subseteq$, let $x \in U$. Since U is open, $\exists r \in \mathbb{R}_{>0}$ s.t. $B(x, r) \subseteq U$. Let $q \in \mathbb{Q}^n$ s.t. $d(x, q) < \frac{r}{43}$. Let $\rho \in \mathbb{Q}$ such that $\frac{r}{43} \overset{(1)}{<} \rho \overset{(2)}{<} \frac{r}{42}$.

(1) implies that $B(q, \rho)$ contains x.
(2) implies (by the triangle inequality and the fact that $x \in B(q, \rho)$) that $B(q, \rho) \subseteq U$, so $B(q, \rho) \in \mathscr{B}_U$.

$\square$

Ex: The following topological space is *not* second-countable.
Let $X = \mathbb{R}$ with the discrete topology (i.e. singletons are open).
A basis must have every singleton, so it's obviously not second-countable.

Relationships of These Properties with "New Spaces from Old"

Product Spaces

Defn: Let X, Y be topological spaces. Then $X \times Y$ has a natural topology called the product topology, defined by the basis $\{U \times V \mid U \subseteq X, V \subseteq Y \text{ open}\}$.

Thm: (A.21, A.22) If X and Y are second-countable topological spaces, then so is $X \times Y$. If X and Y are T_2, then so is $X \times Y$.

Quotient Spaces

Next time...

Math 591 Lecture 2

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Quotient Spaces

Defn: Let X be a tangent space, $\sim$ an equivalence relation on X. Then $X/\sim$ is the set of equivalence classes of $\sim$, and

$$\begin{aligned} \pi : X &\to X/\sim \\ x &\mapsto [x] \end{aligned}$$

The quotient topology is defined by $W \subseteq X/\sim$ is open iff $\pi^{-1}(W) \subseteq X$ is open.

We just assume that this is a topology – is it?

Observe: π is continuous by definiion of the quotient topology. And the quotient topology is he finest topology on $X/\sim$ for which π is continuous.

Prop: Let Y be a topological space, $f : X/\sim \to Y$. Then f is continuous iff $f \circ \pi : X \to Y$ is continuous.

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "f\circ\pi"] & \\ X/\sim \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof: $\Rightarrow$ is trivial, because π is continuous.
$\Leftarrow$ is the real content of the proof (left as an exercise).

Observe: it's not true that X being Hausdorff implies $X/\sim$ being Hausdorff.
And it's not true that X being second countable implies $X/\sim$ being second countable.

Defn: An equivalence relation $\sim$ on a tangent space X is open iff $\pi : X \to X/\sim$ is an open map (i.e. $\forall U \subseteq X$ open, $\pi(U)$ is open).

Let's investigate: $\pi(U)$ is open, by definition, if $\pi^{-1}(\pi(U)) \subseteq X$ is open. So let $\hat{U} = \pi^{-1}(\pi(U)) = \{x \in X \mid \exists y \in U \text{ s.t. } x \sim y\}$.
To recap, $\sim$ is open $\Leftrightarrow \forall U \subseteq X$ open, $\hat{U}$ is open.

Thm: Let $\sim$ be an equivalence relation. Assume it is open, then
1) If X is second countable, then so is $X/\sim$.
2) $X/\sim$ is Hausdorff iff the graph of the relation, Γ, is closed in $X \times X$. $\Gamma = \{(x, y) \in X \times X \mid x \sim y\} \subseteq X \times X$.

Proof:
1) If $\sim$ is open, then for any basis $\mathscr{B} = \{B_j\}_{j\in J}$ of X, $\{\pi(B_j)\}_{j\in J}$ is a basis for $X/\sim$. $\square$
2) Γ is closed iff $\forall (x, y) \in (X \times X) \setminus \Gamma$, $\exists U, V \subseteq X$ open with $x \in U$, $y \in V$ s.t. $(U \times V) \cap \Gamma = \emptyset$. And this is true iff $\forall (u, v) \in U \times V$, $\pi(u) \neq \pi(v)$, which is true iff $\pi(U) \cap \pi(V) = \emptyset$. And $\pi(U)$ is a neighborhood of $\pi(x)$, $\pi(V)$ is a neighborhood of $\pi(y)$, because π is an open map.

Some remarks:

1. In general, if $X/\sim$ is Hausdorff, then for any point $p \in X/\sim$, $\{p\}$ is closed. Thus, $\pi^{-1}(p)$ (an equivalence class of $\sim$) is closed, because set complements ork nicely. So we conclude that a necessary condition for $X/\sim$ to be T_2 is all equivalence classes $[x] \subseteq X$ are closed. But this isn't sufficient!

2. If $P : X \to S$ is a surjective map...

Defn: $\forall x, y \in X$, $x \sim_P y \overset{\text{def}}{\Leftrightarrow} P(x) = P(y)$ $(\star)$ is an equivalence relation. If P is surjective, there's a natural idenification of $S \cong X/\sim_P$. If X, S are tangent spaces, P is called a quotient map iff $\star$ is a homeomorphism, i.e., iff $\forall W \subseteq S$, W is open iff $P^{-1}(W)$ is open.

Ex: $X = S^n \subseteq \mathbb{R}^{n+1}$, the n-dimensional sphere. Consider $\tau : S^n \to S^n$, where $x \mapsto -x$.
Observe that τ is continuous, and $\tau^2 = \mathrm{Id}$ (so $\tau = \tau^{-1}$).
Define the following equivalence relation: $\forall x, y \in S^n$, $x \sim y \Leftrightarrow x = y$ or $y = \tau(x)$.

Claim: $\sim$ is open.
Proof: Let $U \subseteq S^n$ be open. $\hat{U} = \pi^{-1}(\pi(U)) = U \cup \tau(U)$. So $\hat{U}$ is open because $\tau(U)$ is open because τ is a homeomorphism because τ is continuous and $\tau = \tau^{-1}$. Thus, $S^n/\sim$ is second-countable (because we know S^n is second-countable). Write

$$\begin{aligned}\Gamma &\subseteq S^n \times S^n \\ &= \{(x, y) \in S^n \times S^n \mid x \sim y\} \\ &= \{(x, y) \in S^n \times S^n \mid x = y \text{ or } y = \tau(x)\} \\ &= \underbrace{\{(x, x) \mid x \in S^n\}}_{\text{diagonal}} \cup \underbrace{\{(x, \tau(x)) \mid x \in S^n\}}_{\text{graph of } \tau}\end{aligned}$$

The graph of τ is closed.
Consider $F : S^n \times \{0, 1\} \to S^n$ (note: $\{0, 1\} = \mathbb{Z}_2$), where $(x, 0) \mapsto x$ and $(x, 1) \mapsto \tau(x)$. This is a group action.
So the graph of τ is the iamge of the map $S^n \to S^n \times S^n$, where $x \mapsto (x, \tau(x))$. This is a continuous map, and S^n is compact, so its image is compact, so its image is closed.
Thus, Γ is the finite union of closed sets, so Γ is closed, so $S^n/\sim$ is Hausdorff.
Note: $S^n/\sim = \mathbb{RP}^n$, the n-dimensional real projective space. This is isomorphic to the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$.

Math 591 Lecture 3

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Group Actions

Defn: Let G be a group, X a set. A left action of G on X is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that

a) if $e \in G$ is the identity, $\forall x \in X$, $e \cdot x = x$
b) $\forall g_1, g_2 \in G$, $\forall x \in X$, $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$.

In other words, if $\forall g \in G$, we define the map

$$\begin{aligned} L_g : X &\to X \\ x &\mapsto g \cdot x \end{aligned}$$

then $L_e = I_X$ and $L_{g_1 g_2} = L_{g_1} \circ L_{g_2}$.

Defn: Given a group action, if $x \in X$, the orbit of x is the set $G \cdot x = \{y \in X \mid \exists g \in G \text{ s.t.} g \cdot x = y\}$.

Lemma: The orbits partition X, i.e., $x \sim y$ iff $G \cdot x = G \cdot y$ is an equivalence relation.

Notation: X/G and $G \backslash X$ are both valid. We'll stick with $G \backslash X$. (This is the quotient space whose points are the orbits of points in X.)

Defn: Assume X is a topological space, and the group G acts on X (on the left). The action is by continuous maps iff $\forall G \in G$, $L_g : X \to X$ is continuous.

Observe that $\forall g, L_g$ is a homeomorphism, because $\exists g^{-1} \in G$, so $L_{g^{-1}}$ is continuous, and $L_g \circ L_{g^{-1}} = I_X = L_{g^{-1}} \circ L_g$.

Lemma: If G acts by continuous maps, the orbit relation is open.

Proof: Let $U \subseteq X$ be open. We need to show that saturation $\hat{U}$ of U is open.

$$\begin{aligned} \hat{U} &= \{x \in X \mid \exists y \in U \text{ s.t. } x \sim y\} \ (\sim \text{being the orbit relation}) \\ &= \{x \in X \mid \exists y \in U, g \in G \text{ s.t. } y = g \cdot x\} \end{aligned}$$

Thus,

$$\hat{U} = \bigcup_{g \in G} g \cdot U = \bigcup_{g \in G} \{g \cdot x \mid x \in U\} = \bigcup_{g \in G} L_g(U)$$

L_g is a homeomorphism, so it is an open map, so each $L_g(U)$ is open, so $\hat{U}$ is open. $\square$

Defn: A topological group is a group G with a topology s.t. the maps

$$\begin{aligned} G \times G &\to G \\ (g, k) &\mapsto gk \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are continuous.

Aside: Later on, when we have a manifold, and these maps are smooth, then this is a Lie group.

Ex: $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$, the set of invertible $n \times n$ matrices.
In fact, this is an open subset, since it's described by $\mathrm{GL}(n,\mathbb{R}) = \{M \in \mathbb{R}^{n\times n} \mid \det M \neq 0\}$, i.e., $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\backslash\{0\})$. Because det is a continuous map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ and $\mathbb{R}\backslash\{0\}$ is open, we get that $\mathrm{GL}(n,\mathbb{R})$ is open.
Note that $\mathrm{GL}(n,\mathbb{R})$ is a topological group, with the induced topology. In fact, any subgroup of a topological group is naturally a topological group with respect to the subspace topology.

Ex: $O(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid g^{-1} = g^T\}$.
$\mathrm{GL}(n,\mathbb{C}) \subseteq \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Note that $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathrm{GL}(2n,\mathbb{R})$, since $\mathbb{C} \cong \mathbb{R}^2$.
$U(n) = \{g \in \mathrm{GL}(n,\mathbb{C}) \mid g^{-1} = \overline{g}^T\}$.

Defn: If G is a topological group acting on a topological space X, the action is <u>continuous</u> iff $G \times X \to X$ is a continuous map.

Lemma: A continuous action is an action by continuous maps. (I.e. $\forall g \in G$, $L_g : X \to X$ is continuous.)

Ex: $G = S^1 = \{z \in \mathbb{C} : |z| = 1\} = U(1)$ acts on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by $\lambda \in S^1$, $(z_1, \ldots, z_{n+1}) \in S^{2n+1}$, $\lambda \cdot (z_1, \ldots, z_{n+1}) = (\lambda z_1, \ldots, \lambda z_{n+1})$. This is a continuous action.

Question: Suppose G is a topological group acting on X. (So the orbit relation is open.) When is $G\backslash X$ Hausdorff?
Well, this is true iff the graph of the orbit relation is closed.

Define

$$\star \quad \begin{aligned} G \times X &\to X \times X \\ (g,x) &\mapsto (x, g\cdot x) \end{aligned}$$

This is a continuous map, whose image is the graph of the orbit relation.

Prop: If G and X are both compact, and X is Hausdorff, then $G\backslash X$ is Hausdorff.
Proof: The image of $\star$ is compact, and compact subsets of Hausdorff spaces are closed, so the orbit relation is closed. $\square$

Ex: $S^1 \times S^{2n+1} \to S^{2n+1}$ as above.
Then the proposition implies $\mathbb{CP}^n = S^1\backslash S^{2n+1}$ is Hausdorff and second-countable.
Note: $\mathbb{CP}^n \cong \{\text{1-dimensional subspaces of } \mathbb{C}^{n+1}\}$.

Math 591 Lecture 4

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Defn: A space X is locally Euclidean iff every point in X has a neighborhood homeomorphic to $\mathbb{R}^n$, for some fixed n.

Defn: A topological manifold is a space that is locally Euclidean, Hausdorff, and second countable.

Thm: If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are homeomorphic nonempty open sets, then $m = n$. In other words, "dimension is topological".

The idea of this proof is to show that any open set in $\mathbb{R}$ can be covered by families of open sets with overlaps of at most 2 sets, any open set in $\mathbb{R}^2$ can be covered by families of open sets with overlaps of at most 3 sets, and so on.

Observe that in the definition of locally Euclidean, it's equivalent to ask that $\forall p \in X$, p has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Defn: Let M be a topological manifold. If $U \subseteq M$ is open, and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open set $\phi(U) \subseteq \mathbb{R}^n$, then the pair (U, ϕ) is a chart of M.

Defn: Let (U, ϕ) and (V, ψ) be charts, with $U \cap V \neq \emptyset$. The transition function (from ϕ to ψ) is a map

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap V)} : \phi(U \cap V) \to \psi(U \cap V)$$

Note: $\phi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^n$, because ϕ and ψ are homeomorphisms.
Note: Transition functions are automatically homeomorphisms.

Defn: Two charts of a topological manifold are C^∞-compatible (or just compatible) iff their transition functions are C^∞. That is,

$$\psi \circ \phi^{-1}\big|_{\phi(U\cap V)} \qquad \text{and} \qquad \phi \circ \psi^{-1}\big|_{\psi(U\cap V)}$$

are both C^∞ diffeomorphisms.

Defn: An atlas of a topological manifold M is a collection $\mathscr{A} = \{(U_i, \phi_i)\}_{i\in I}$ of charts s.t. $M = \bigcup_{i\in I} U_i$.

Preliminary "definition": An atlas of M s.t. $\forall i, j \in I$, the transition function $\phi_i \circ \phi_j^{-1}$ is C^∞ determines a differentiable structure on M. Note that the condition is vacuous if $U_i \cap U_j = \emptyset$.

Ex: Some topological manifolds and atlases satisfying the preliminary definition:

- A trivial example: $M \subseteq \mathbb{R}^n$ any open set, $\mathscr{A} = \{M \hookrightarrow \mathbb{R}^n \text{ (inclusion)}\}$.
- Let $A \subseteq \mathbb{R}^n$ be an open set, and $G : A \to \mathbb{R}^k$ a C^∞ map. Let M be the graph of G, i.e., $M = \{(x, G(x)) \in \mathbb{R}^{n+k} \mid x \in A\} \subseteq \mathbb{R}^{n+k}$ with the subspace topology. Then let $\mathscr{A} = \{\pi : M \to \mathbb{R}^n \mid \pi : (x, G(x)) \mapsto x\}$.
- Cases of $M \subseteq \mathbb{R}^N$ which are locally graphs. (Note: $\mathbb{R}^N$ is known as the "ambient space".)
  - S^1. Let

$$U_1 = \left\{(x, \sqrt{1-x^2}); x \in (-1,1)\right\}$$
$$U_2 = \left\{(y, \sqrt{1-y^2}); y \in (-1,1)\right\}$$
$$U_3 = \left\{(x, -\sqrt{1-x^2}); x \in (-1,1)\right\}$$
$$U_4 = \left\{(y, -\sqrt{1-y^2}); y \in (-1,1)\right\}$$
$$\mathscr{A} = \{(U_1, (x,y) \mapsto x), (U_2, (x,y) \mapsto y), (U_3, (x,y) \mapsto x), (U_4, (x,y) \mapsto y)\}$$

Let's explicitly compute a transition map. $\phi_1^{-1}(x) = (x, \sqrt{1-x^2})$, so $\phi_2 \circ \phi_1^{-1}(x) = \sqrt{1-x^2}$. Note: this is C^∞ on $(0,1)$.

– S^1 with a new atlas. Let $p = (u, v)$. Let $U_+ = \left\{S^1 \setminus \{(0,1)\}\right\}$ and $U_- = \left\{S^1 \setminus \{(1,0)\}\right\}$.
  Then let $\phi_+(p) = x = \frac{u}{1-v}$ and $\phi_-(p) = y = \frac{u}{1+v}$. Another atlas: $\mathscr{B} = \{(U_1, \phi_1), (U_2, \phi_2)\}$. We claim that ϕ_1 and ϕ_2 are C^∞-compatible.
  In fact, it turns out that $\mathscr{A} \cup \mathscr{B}$ consists of compatible charts. So $\mathscr{A}$ and $\mathscr{B}$ define the same differentiable structure on S^1.

Math 591 Lecture 5

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Defn: Let M be a topological manifold. A $\underline{C^\infty \text{ atlas}}$ on M, $\mathscr{A} = \{(U_i, \phi_i)\}_{i \in I}$, is a collection of charts on M such that $M = \bigcup_{i \in I} U_i$ and $\forall i, j \in I$, (U_i, ϕ_i) and (U_j, ϕ_j) are C^∞ compatible.

Last time, we considered $M = S^1 \subseteq \mathbb{R}^2$ with the subspace topology. We constructed two atlases, $\mathscr{A}$ and $\mathscr{B}$, and claimed every chart in $\mathscr{B}$ is compatible with every chart in $\mathscr{A}$, i.e., $\mathscr{A} \cup \mathscr{B}$ is a C^∞ atlas.

We want a definition of smooth structure on M that includes $\mathscr{A}$ and $\mathscr{B}$.

Defn: A smooth structure on a topological manifold M is a maximal smooth atlas, $\mathscr{M}$, i.e., a smooth atlas such that: If (U, ϕ) is any continuous chart on M s.t. $\forall (V, \psi) \in \mathscr{M}$, (U, ϕ) and (V, ψ) are compatible, then $(U, \phi) \in \mathscr{M}$.

Prop: If $\mathscr{A}$ is a smooth atlas on M, $\exists! \mathscr{M}$, a maximal smooth atlas, s.t. $\mathscr{A} \subseteq \mathscr{M}$.

Proof: Let $\mathscr{M} = \{(U, \phi)$ continuous chart s.t. $\forall (V, \psi) \in \mathscr{A}, (U, \phi)$ and (V, ψ) are compatible$\}$. We need to show $\mathscr{M}$ is a smooth atlas, and that it is maximal.

First, check that it's a smooth atlas. Clearly $\mathscr{A} \subseteq \mathscr{M}$. So charts in $\mathscr{M}$ cover M. We just need to show compatibility. Let (U_1, ϕ_1) and (U_2, ϕ_2) be charts in $\mathscr{M}$. We need to show they're compatible with each other. Let $p \in U_1 \cap U_2$, and show $\phi_2 \circ \phi_1^{-1}$ is C^∞ at $\phi_1(p)$. Well, $\exists (V, \psi) \in \mathscr{A}$ s.t. $p \in V$. And every (U_i, ϕ_i) is compatible with charts in $\mathscr{A}$, so $\phi_2 \circ \phi_1^{-1} = (\phi_2 \circ \psi^{-1}) \circ (\psi \circ \phi_1^{-1})$ on some neighborhood of $\phi_1(p)$. This is a smooth map, so we conclude $\mathscr{M}$ is a smooth atlas.

Proving it's maximal just uses topology, and is left as an exercise.

□

Defn: A differentiable manifold is a pair $(M, \mathscr{M})$, where M is a topological manifold, and $\mathscr{M}$ is a maximal C^∞ atlas.

Ex: $M = \mathbb{R}$, $\mathscr{M}$ is the maximal atlas containing $(\mathbb{R}, x \mapsto \sqrt[3]{x})$.
This is a different smooth structure on $\mathbb{R}$. However, the standard $\mathbb{R}$ and this $\mathbb{R}$ are isomorphic in the category of smooth manifolds: they're diffeomorphic! The isomorphism is $x \mapsto \sqrt[3]{x}$.

Question (very hard): Given a topological manifold M, are there non-isomorphic smooth structures on M? Yes, it can happen!

Consider $U \subseteq \mathbb{R}^N$ open (recall: $\mathbb{R}^N$ is called the ambient space), and $F : U \to \mathbb{R}^k$ smooth.
$F = (f^1, \ldots, f^k)$, with $f^i : U \to \mathbb{R}$ smooth.
Recall: If $x_0 \in U$, the Jacobian of F at x_0 is the matrix

$$J(F)(x_0) = F'(x_0) = \begin{pmatrix} — & \nabla f^1(x_0) & — \\ & \vdots & \\ — & \nabla f^k(x_0) & — \end{pmatrix}$$

Defn: $0 \in \mathbb{R}^k$ is a regular value of F iff $\forall x \in F^{-1}(0)$, $F'(x)$ has rank k. (I.e. it defines a surjective linear map onto $\mathbb{R}^k$.)

$$F' = \left.\underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f^k}{\partial x_1} & \cdots & \frac{\partial f^k}{\partial x_N} \end{pmatrix}}_{N}\right\} k$$

Thm: Assume $0 \in \mathbb{R}^k$ is a regular value of F. Then $\forall x_0 \in F^{-1}(0)$, there's some neighborhood W of x_0 in the ambient space s.t. $F^{-1}(0) \cap W$ is the graph of a function of $N-k$ variables into $\mathbb{R}^k$, up to a permutation of the coordinates $x_1, \ldots, x_N$. *This is just the implicit function theorem!*

Ex: An example of "permutation of coordinates".
Let $k=2$ and $N=5$ (so $N-k=3$). Let $A \subseteq \mathbb{R}^3$ open, $G=(g^1,g^2) : A \to \mathbb{R}^2$. Then

$$F^{-1}(0) \cap W = \left\{(x_1,x_2,x_3,x_4,x_5) \mid x_2 = g^1(x_1,x_3,x_4), x_5 = g^2(x_1,x_3,x_4)\right\}$$

Math 591 Lecture 6

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Many examples of C^∞ manifolds are produced by the implicit function theorem. Reminder:

Thm: (Implicit Function Theorem) (Theorem B2 in the textbook)
Let $U \subseteq \mathbb{R}^N$ open, $F : U \to \mathbb{R}^k$ C^∞, and $x_0 \in U$ s.t. $F(x_0) = 0$.
Split: for $x \in U$, write $x = (x', x'')$, where $x' \in \mathbb{R}^{N-k}$ and $x'' \in \mathbb{R}^k$. Accordingly, the Jacobian of F at x_0 splits:

$$F'(x_0) = (\underbrace{\tfrac{\partial F}{\partial x'}(x_0)}_{N-k} \,\vdots\, \underbrace{\tfrac{\partial F}{\partial x''}(x_0)}_{k})\}k$$

Assume $[\frac{\partial F}{\partial x''}(x_0)]$ $(k \times k)$ is invertible. Then there exist open sets A, B with $x_0' \in A \subseteq \mathbb{R}^{N-k}$, $x_0'' \in B \subseteq \mathbb{R}^k$ and $g : A \to B$ C^∞ s.t. $\left\{F^{-1}(0)\right\} \cap (A \times B) = \{(x', g(x') \mid x' \in A)\}$.

Application: Recall that given $F : U \to \mathbb{R}^k$, $U \subseteq \mathbb{R}^N$ open, zero is a regular value of F iff $\forall x \in F^{-1}(0)$, $F'(x)$ has rank k.

Cor: If 0 is a regular value for F, then $F^{-1}(0) = M$ is locally a graph. Moreover, this structure of local graph gives M a C^∞ atlas, and therefore a smooth manifold structure.
(Note: we will define "submanifold", and then $F^{-1}(0)$ will be examples of submanifolds of $\mathbb{R}^N$.)
Proof/Explanation: Assume 0 is a regular value of F. Then $\forall x_0 \in F^{-1}(0) = M$,

$$F'(x_0) = \begin{pmatrix} — & \nabla f^1(x_0) & — \\ & \vdots & \\ — & \nabla f^k(x_0) & — \end{pmatrix}$$

(for $F = (f^1, \ldots, f^k)$). After permuting the indices among the x_i, without loss of generality $[\frac{\partial F}{\partial x''}(x_0)]_{k \times k}$ is non-degenerate. (Think of this as swapping the columns around so the block is invertible.)

The independent variables will depend on x_0. The number of independent variables is $N - k = \dim M$. k is the <u>codimension</u> of $M \subseteq \mathbb{R}^N$.

Statement about "Atlas"

Recall: A graph $\{(x', g(x')) \mid x' \in A\} = \Gamma$ has a global chart: just the projection onto the domain. $\Gamma \ni (x', x'') \mapsto x'$. Its inverse is $x' \mapsto (x', g(x'))$.

For two local representations of M as a local graph, transition functions are of the form $x' \mapsto (x', g(x')) \xrightarrow{\star} \mathbb{R}^{N-k}$. $\star$ is a projection onto an $N - k$-dimensional coordinate plane of $\mathbb{R}^N$. This is smooth, and a transition map.

Ex: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in $\mathbb{R}^3$ is the zero set of $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$.
(Check that 0 is a regular value.)

Ex: $O(n) = \left\{g \in \mathrm{Mat}(n, \mathbb{R}) \mid g^{-1} = g^T\right\}$. $\mathrm{Mat}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, so $O(n) \subseteq \mathbb{R}^{n^2}$.
In fact, $O(n) \subseteq \mathrm{GL}(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (and $\mathrm{GL}(n, \mathbb{R})$ is open in $\mathbb{R}^{n^2}$).
Define

$$\begin{aligned} F : \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Symm}(n, \mathbb{R}) \\ g &\mapsto gg^T - I \end{aligned}$$

where $\mathrm{Symm}(n, \mathbb{R}) = \left\{g \in \mathrm{Mat}(n, \mathbb{R}) \mid g = g^T\right\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}$, and I is the identity matrix. Note: We have to choose the codomain carefully so that 0 is a regular value.

So, $O(n) = F^{-1}(0)$.
Check: is 0 a regular value? To see if $F'(g)$, for $g \in O(n)$, has maximal rank, let $M \in \mathrm{Mat}(n, \mathbb{R})$. Then compute $\frac{d}{dt}F(g+tM)\big|_{t=0}$. (Compute this as a matrix to avoid $\mathbb{R}^{n^2}$ coords.) Then $(-\varepsilon, \varepsilon) \ni t \mapsto \mathrm{GL}(n, \mathbb{R}) \overset{F}{\mapsto} \mathrm{Symm}(n, \mathbb{R})$.

$$\frac{d}{dt}F(g+tM)\bigg|_{t=0} = \frac{d}{dt}(g+tM)(g+tM)^T\bigg|_{t=0} = \frac{d}{dt}\left(gg^T + t(Mg^T + gM^T) + t^2MM^T\right)\bigg|_{t=0} = Mg^T + gM^T$$

Question: Is $\mathrm{Mat}(n, \mathbb{R}) \ni M \mapsto Mg^T + gM^T$ onto? Yes! (This is true iff $F'(g)$ has rank equal to the dimension of $\mathrm{Symm}(n, \mathbb{R})$. Let $S \in \mathrm{Symm}(n, \mathbb{R})$. What can M be?)
By the implicit function theorem, $O(n) \subseteq \mathbb{R}^{n^2}$ is locally a graph. So it has a natural C^∞ structure.

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Observe:

(1) $\dim O(n) = \dim(\text{ambient}) - \dim(\text{Symm}(n, \mathbb{R})) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

(2) $\ker F'(I) = \left\{M \in \text{Mat}(n, \mathbb{R}) \mid M + M^T = 0\right\} = \{\text{skew-symmetric matrices}\}$. This is the tangent space to $O(n)$ at I.

Similarly, $U(n) = \left\{g \in \text{Mat}(n, \mathbb{C}) \mid g^{-1} = \overline{g}^T\right\}$ has a C^∞ structure, as well as $SU(n) = \{g \in U \mid \det g = 1\}$. ($g \in U(n) \Rightarrow \det(g) \in S^1$, i.e., $|\det g| = 1$.)

(3) $O(n)$, in fact, has 2 connected components, as $g \in O(n) \Rightarrow \underbrace{|\det g|}_{\in \mathbb{R}} = 1 \Rightarrow \det g = \pm 1$.

Defn: $SO(n) = \{g \in O(n) \mid \det g = 1\}$ is a subgroup of $O(n)$.
$O(n) = SO(n) \cup \{g \in O(n) \mid \det g = -1\}$.

More examples: $SL(n, \mathbb{R}) = \{g \in \text{GL}(n, \mathbb{R}) \mid \det g = 1\}$.
Some facts: $U(n)$ and $O(n)$ are compact, whereas $SL(n, \mathbb{R})$ is not.

More examples can be constructed by:

- Cartesian products: If M and N are C^∞ manifolds, then $M \times N$ has a natural smooth structure. Charts on $M \times N$ are just $(U \times V, \phi \times \psi)$, where (U, ϕ) is a chart on M and (V, ψ) is a chart on N.
For example, the nth torus $\underbrace{S^1 \times \cdots \times S^1}_{n}$.
- Covering maps of C^∞ manifolds: Let M be a C^∞ manifold. A covering map on M is $f : \tilde{M} \to M$ (with $\tilde{M}$ a topological space) such that $\forall p \in M$, $\exists U \ni p$ open s.t. $F^{-1}(U) = \bigcup_{i \in I \text{ finite}} U_i$, with $U_i \subseteq \tilde{M}$ open s.t. $\forall i$, $F|_{U_i} : U_i \overset{\cong}{\to} V$ is a homeomorphism.

Then:

Thm: $\tilde{M}$ has a unique C^∞ manifold structure s.t. F is locally a diffeomorphism (isomorphism).

Thm: $SO(n)$ has a double cover (a 2-to-1 covering space), $\text{Spin}(n) \overset{\text{2-1}}{\to} SO(n)$. Spin$(n)$ has a group structure.

Ex: (of a covering map)

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto e^{ix} \end{aligned}$$

Defn: Let M be a C^∞ manifold, and $f : M \to \mathbb{R}$, $p \in M$. Then <u>f is C^∞ at p</u> if there's a chart (U, ϕ) of M such that $p \in U$ and $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞.

Observe: A chart (U, ϕ) one a C^∞ manifold is also called a coordinate system. We'll often write $\phi = (x^1, \ldots, x^n)$, where $x^i : U \to \mathbb{R}$ is the ith component of ϕ, i.e., a coordinate function.

Observe: In the definition above, f only needs to be defined in a neighborhood of p.

Defn: Let M be a C^∞ manifold, $f : M \to \mathbb{R}$ is <u>smooth</u> iff $\forall p \in M$, f is smooth at p.

Lemma: $f : M \to \mathbb{R}$ is smooth iff $\forall (U, \phi)$ smooth chart of M, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.
Proof: (see §6 for full details)
$\Leftarrow$ is immediate
$\Rightarrow$ is based on the fact that f smooth $\Rightarrow \forall p \in M$, there's a chart (V, ψ) around p s.t. $f \circ \psi^{-1}$ is smooth.

Ex: Let $M \subseteq \mathbb{R}^N$ be a local graph. $M = F^{-1}(0)$, 0 is a regular value of F.
If $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ is smooth, then $f = \tilde{f}M\Big|_{:} M \to \mathbb{R}$ is smooth.
Proof: There are charts on M (U, ϕ) s.t. $\phi^{-1}(x') = (x', G(x'))$ after permuting coordinates (G is a graph function).
Then $(f \circ \phi^{-1})(x') = \tilde{f}(x', G(x'))$, and this is C^∞. $\square$

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Defn: Let M, N be C^∞ manifolds, and $F : M \to N$ a continuous map. Let $p \in M$. Then we say F is <u>smooth</u> at p iff there exist charts (U, ϕ) of M and (V, ψ) of N s.t. $p \in U$, $F(p) \in V$, and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^n$$

is C^∞.

Observe: Since F is continuous, $F^{-1}(V)$ is open, so $F^{-1}(V) \cap U$ is an open neighborhood of p. Thus, $\phi(F^{-1}(V) \cap U)$ is open in $\mathbb{R}^m$.

Defn: Let M, N be C^∞ manifolds, $F : M \to N$ continuous. Then F is <u>smooth</u> iff $\forall p \in M$, F is smooth at p.

Lemma: Let M, N be C^∞ manifolds, $F : M \to N$ continuous. Then F is smooth iff there are atlases $\{(U_\alpha, \phi_\alpha)\}$ of M and $\{(V_\beta, \psi_\beta)\}$ of N s.t. $\forall \alpha, \beta$, $\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(F^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n$ is smooth. This, in turn, is true iff for any pair of atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, the previous condition holds.
Proof: (exercise)

The key outcome is that if a function is smooth according to one atlas, it's smooth according to all atlases.

Ex:

$$\begin{aligned} \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{GL}(n,\mathbb{R}) \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{GL}(n,\mathbb{R}) \\ g &\mapsto g^{-1} \end{aligned} \qquad \text{are smooth.}$$

$$\begin{aligned} O(n) \times O(n) &\to O(n) \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} O(n) &\to O(n) \\ g &\mapsto g^{-1} \end{aligned} \qquad \text{are smooth.}$$

Defn: A <u>Lie group</u> G is a group which also has a C^∞ structure s.t.

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth.

Tangent and Cotangent Spaces

We want to construct tangent vectors without requiring an ambient space!
Idea: Vectors in $\mathbb{R}^n$ define "directional" derivatives.

Pick $p \in U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, and $v \in \mathbb{R}^n$. Then if $f : U \to \mathbb{R}$ is C^∞, we can define $D_v f(p) = \nabla f(p) \cdot v$.

Remark: We can regard D_v as an operator $C^\infty \ni f \mapsto D_v f(p) \in \mathbb{R}$. It has the following properties:
1) Linear over $\mathbb{R}$: $D_v(f + cg)(p) = D_v f(p) + c D_v g(p)$.
2) Leibniz' rule: $D_v(fg)(p) = f(p) D_v g(p) + D_v f(p) g(p)$.

This was all motivation. Now, for the formalization.

Defn: Let M be a smooth manifold, $p \in M$. Then the space of germs of functions of M at p is

$$C_p^\infty(M) = \{(f : U \to \mathbb{R}, U) \mid U \subseteq M \text{ open }, p \in U, f \in C^\infty\} / \sim$$

where $(f, U) \sim (g, V) \Leftrightarrow \exists W \subseteq U \cap V$ s.t. $p \in W$ and $f|_W = g|_W$.
A germ at p is an equivalence class $[f] = [(f, U)]$.

Notation: Given (f, U) as above and $p \in U$, $[f]$ is the calss of $(f, U) \in C_p^\infty(M)$.

Lemma: $C_p^\infty(M)$ is an $\mathbb{R}$-vector space and a ring.

a) $[f] + c[g] \overset{\text{def}}{=} [f + cg]$

b) $[f] \cdot [g] \overset{\text{def}}{=} [fg]$ (defined by $fg|_{U \cap V} : U \cap V \to \mathbb{R}$)

EXER: Show the remaining properties.

Defn: A derivation on M at p is an $\mathbb{R}$-linear map $D : C_p^\infty \to \mathbb{R}$ s.t. $\forall [f], [g] \in C_p^\infty(M)$, $D([f]g]) = f(p)D[g] + g(p)D[f]$.

Observe: $f(p) = [f](p) \in \mathbb{R}$ is well defined by $[f]$.

Defn: The tangent space to M at p is $T_pM = \{\text{all derivations of } M \text{ at } p\}$.

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Ex: (of germs)

Let $p \in \mathbb{R}^n$. Then $C_p^\infty(\mathbb{R}^n) = \left\{(U, f) \mid p \in U \overset{\text{open}}{\subseteq} \mathbb{R}^n, f : U \to \mathbb{R}\ C^\infty\right\} / \sim$.

Observe: There is a well-defined map

$$\begin{aligned} C_p^\infty(\mathbb{R}^n) &\to \mathbb{R}[[r]], r = (r^1, \ldots, r^n) \\ [f] &\mapsto f(p) + \textstyle\sum_{j=1}^n (r^j - r_0^j)\frac{\partial f}{\partial r^j}(p) + \cdots \end{aligned}$$

where $\mathbb{R}[[r]]$ is the set of formal power series in the r^i variables, and $[f]$ maps to the Taylor series of f at $p = (r_0^1, \ldots, r_0^n)$. Why is this well defined? Well, if $[f] = [g]$, then f and g agree on a neighborhood of p.

Prop:

(1) This map is a surjection, i.e., any formal power series is the Taylor series of some smooth functions.

(2) This map is *not* injective, i.e., there exist C^∞ functions f defined near p s.t. $\forall \alpha$ multi-indices, $\frac{\partial^\alpha f}{\partial r^\alpha}(p) = 0$, but f is not zero near p.

This is just an FYI – we're not going to use this for a while.

Now, back to manifolds...
Let M be a C^∞ manifold, and $p \in M$. We defined $T_pM = \{\text{all derivations } D : C_p^\infty(M) \to \mathbb{R}\}$.

Ex: (of derivations)

Start with a curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ smooth s.t. $\gamma(t_0) = p$. Define

$$\begin{aligned} \dot\gamma(t_0) : C_p^\infty(M) &\to \mathbb{R} \\ [f] &\mapsto \frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=t_0} \end{aligned}$$

It's easy to check that $\dot\gamma(t_0)$ is a derivation (by calc III stuff). Note that this defines $\dot\gamma(t_0) \in T_{\gamma(t_0)}M$.
We will see today that *all* derivations are of this form.

Observe: In the case where $M \subseteq \mathbb{R}^N$ is a local graph, then $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M \hookrightarrow \mathbb{R}^N$ can be interepreted as a smooth curve in $\mathbb{R}^N$. $\dot\gamma(t_0)$ was defined in calc III as an element in $\mathbb{R}^N$. These definitions are consistent! But our definition doesn't need an ambient space.

Introducting Local Coordinates and Partial Derivatives

Let $p \in U \subseteq M$, with $\phi : U \to \mathbb{R}^N$ a chart. Then we use the notation $r^i : \mathbb{R}^N \to \mathbb{R}$ is the ith component/coordinate.
We say $x^i = r^i \circ \phi : U \to \mathbb{R}$, so we can write $\phi = (x^1, x^2, \ldots, x^N)$. (The x^is are defined on $U \subseteq M$.)

Defn: Given $f : U \to \mathbb{R}$ smooth, $p \in U$,

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial r^i}(f \circ \phi^{-1})[\phi(p)] \in \mathbb{R}$$

Some notation: we write $f_\phi \overset{\text{def}}{=} f \circ \phi^{-1}$. Observe that $\frac{\partial f}{\partial x^i} = \frac{\partial f_\phi}{\partial x^i} \circ \phi$.

Lemma: $\forall i \in \{1, \ldots, n\}$, the map

$$\left.\frac{\partial}{\partial x^i}\right|_p : C_p^\infty(M) \ni [f] \mapsto \frac{\partial f}{\partial x^i}(p)$$

is a derivation at p, and moreover,

$$\Phi \overset{\text{def}}{=} \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\}$$

is a basis (over $\mathbb{R}$) of T_pM.

Observe: $\left.\frac{\partial}{\partial x^i}\right|_p$ are velocities of curves. Let $\phi(p) = (r_0^1, \ldots, r_0^n)$. Then if $\gamma_i : t \mapsto \phi^{-1}(r_0^1, \ldots, r_0^i + t, \ldots, r_0^n)$ for $t \in (-\varepsilon, \varepsilon)$, we claim that $\dot{\gamma}(p) = \left.\frac{\partial}{\partial x^i}\right|_p$.

To actually prove that Φ is a basis, we need:

Thm: Let g be a C^∞ function defined in a neighborhood of a point $r_0 \in \mathbb{R}^n$. Then $\exists g_{ij}$, with $i, j \in \{1, \ldots, n\}$, that is smooth and defined near r_0, such that $\forall r \in \mathrm{dom}(g)$,

$$g(r) = \underbrace{g(0) + \sum_{j=1}^n (r^j - r_0^j) \frac{\partial g}{\partial r^j}}_{\text{First degree Taylor polynomial}} + \underbrace{\frac{1}{2} \sum_{i,j=1}^n (r^i - r_0^i)(r^j - r_0^j) \cdot g_{ij}(r)}_{\text{``An interesting way of writing the remainder''}}$$

(Moreover, $g_{ij}(r_0) = \frac{\partial^2 g}{\partial r^i \partial r^j}(r_0)$.)

Let $D \in T_pM$, $[f] \in C_p^\infty(M)$. Apply this to $g = f_\phi$. We claim that

$$D[f] = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) \cdot D([x^i])$$

This implies

$$D = \sum_{i=1}^n D([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p$$

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Review: Partial Derivatives

Given $p \in U \subseteq M$ and $\phi : U \to \mathbb{R}^n$, a coordinate chart, we can write $\phi = (x^1, \dots, x^n)$ where each $x^i : U \to \mathbb{R}$. Then we defined

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial f_\phi}{\partial r^i}(\phi(p))$$

where $f_\phi = f \circ \phi^{-1}$, and r^i is simply the coordinate in $\mathbb{R}^n$.

Ex: Take $M \subseteq \mathbb{R}^3$ defined as the graph of $G : A \to \mathbb{R}$ where A is an open subset of $\mathbb{R}^2$ and G is C^∞. Then we can write $M = \{(r^1, r^2, G(r^1, r^2)) \mid (r^1, r^2) \in A\}$. There is one chart: $U = M$, ϕ is projection onto the (r^1, r^2) plane. $\phi^{-1}(r^1, r^2) = (r^1, r^2, G(r^1, r^2))$.

Let $f : M \to \mathbb{R}$ be the restriction of some $\tilde{f} : \mathbb{R}^3 \to \mathbb{R}$ C^∞. Then

$$f_\phi({}^1, r^2) = (f \circ \phi^{-1})(r^1, r^2) = \tilde{f}(\underbrace{r^1, r^2, G(r^1, r^2)}_{p \in M})$$

Compute:

$$\frac{\partial f}{\partial x^1}(p) = \frac{\partial \tilde{f}}{\partial r^1}(p) + \frac{\partial G}{\partial r^1}(r^1, r^2)\frac{\partial \tilde{f}}{\partial r^3}(p)$$

Observe: $\frac{\partial f}{\partial x^1}(p) = \nabla \tilde{f} \cdot \dot{\gamma}$, for $\gamma(t) = (r^1 + t, r^2, G(r^1 + t, r^2)) \in M$, so $\dot{\gamma}(t)|_{t=0} = (1, 0, \frac{\partial g}{\partial r^1}(r^1, r^2))$.

Now, back to the theorem from last time:

Thm: If $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ is a chart, and $\phi = (x^1, \dots, x^n)$, then $\forall D \in T_pM$, one has

$$D = \sum_{j=1}^n D([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p, \quad [x^i] \in C_p^\infty(M)$$

Proof: It's based on the following observations:

- The set of derivations, T_pM, is an $\mathbb{R}$-vector space.
- For any constant function k, $\forall D \in T_pM$, $D[k] = 0$.
  Proof: $D([1]) = D([1^2]) = 2D([1])$ by the product rule, so $D([1]) = 0$. Then linearity implies $D[k] = 0$.
- $\forall D \in T_pM$, if $[f](p) = [g](p) = 0$, then $D([f][g]) = 0$.

We'll start with what we had last time.

$$f_\phi(r) = f(p) + \sum_{i=1}^n (r^i - r_0^i)\frac{\partial f_\phi}{\partial r^i}(r_0) + \frac{1}{2}\sum_{i,j=1}^n (r^i - r_0^i)(r^j - r_0^j) \cdot g_{ij}(r)$$

Composing with ϕ yields

$$f(r) = f(p) + \sum_{i=1}^n (x^i - x_0^i)\frac{\partial f}{\partial x^i}(p) + \frac{1}{2}\sum_{i,j=1}^n (x^i - x_0^i)(x^j - x_0^j) \cdot g_{ij}(x)$$

Apply D. Well, $f(p)$ is constant, so it vanishes. And the last term is second order, so based on the above observation, it vanishes as well. We're left with

$$D([f]) = \sum_{i=1}^n D([x^i - x_0^i])\frac{\partial f}{\partial x}^i(p) = \sum_{i=1}^n D([x^i])\frac{\partial f}{\partial x^i}(p)$$

Thus,

$$D = \sum_{i=1}^{n} D([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p$$

$\square$

Cor: It's easy to check that $\dfrac{\partial x^i}{\partial x^j} = \delta_{ij}$, so $\left\{ \left.\dfrac{\partial}{\partial x^i}\right|_p \right\}_{i=1}^{k}$ is a basis of T_pM over $\mathbb{R}$.

In summary,

- We are defining tangent vectors by T_pM, which is the set of derivations at p. We'll be changing our notation: $u, v, w, \ldots \in T_pM$.
- Representation of vectors in coordinates:

$$v = \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x^i}\right|_p \qquad (v_1, \ldots, v_n) \in \mathbb{R}^n$$

- Also, $v = \dot{\gamma}$ for some $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \to M$ C^∞, with $\gamma(t_0) = p$.
- Note for later: $p \neq q \Rightarrow T_pM \cap T_qM = \emptyset$.

Lemma: $\dot{\gamma}(t_0) = \displaystyle\sum_{i=1}^{n} \frac{\partial x^i}{\partial t}(t_0) \left.\frac{\partial}{\partial x}\right|_p$ if $\gamma(t) = (x^1(t), \ldots, x^n(t)) \in \mathbb{R}^n$.

Differentials of Functions

Defn: Let $p \in U \overset{\text{open}}{\subseteq} M$, $f : U \to \mathbb{R}$ C^∞. Then we define

$$\begin{aligned} df_p : T_pM &\to \mathbb{R} \\ v &\mapsto v[f] \end{aligned}$$

Notation: we say $T_pU \overset{\text{def}}{=} T_pM$.

Note: $df_p \in (T_pM)^*$, the dual of the tangent space.

Defn: $T_p^*M = (T_pM)^*$ is called the <u>cotangent space</u> of M at p. $df_p \in T_p^*M$.

Note: $df_p(v) = v[f]$.

In coordinates, we saw that if $\phi = (x^1, \ldots, x^n)$ and $\phi(p) = (x_0^1, \ldots, x_0^n) \in \mathbb{R}^n$, then

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - x_0^i) + O(2)$$

(With $O(2)$ denoting something that vanishes in the second order at p.) Then

$$v[f] = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v([x^i])$$

By definition, $v[x^i] = dx^i(v)$. We conclude that

$$df_p = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) dx_p^i$$

This is just like it was in Calc III!

Note that, in some ways, it makes more sense to first define T_p^*M, and then obtain T_pM as its dual.

Defn: $I_p = \left\{ [f] \in C_p^\infty(M) \mid [f](p) = 0 \right\}$, an ideal in the ring of germs.

$$I_p^2 = \left\{ \sum_{i,j} [f_i][g_j] : [f_i], [g_j] \in I_p \right\}$$

is the set of "$O(2)$ germs". We claim that $T_p^*M \cong I_p/I_p^2$. Then df is the class of $[f - f(p)] \in I_p/I_p^2$.

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Tangent Vectors

Last time, we proved that for $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ chart, $\phi = (x^1, \ldots, x^n)$, that $\forall v \in T_pM$, we can write

$$v = \sum_{i=1}^{n} v([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p$$

This is based on:

Thm: If $g : B \to \mathbb{R}$, with $B \subseteq \mathbb{R}^n$ being the open ball centered at the origin, then there exist $g_{ij} \in C^\infty(B)$ s.t. $\forall r \in B$,

$$g(r) = g(0) + \sum_{j=1}^{n} r^j \frac{\partial g}{\partial r^j}(0) + \frac{1}{2} \sum_{i,j=1}^{n} r^i r^j g_{ij}(r)$$

with $g_{ij}(0) = \frac{\partial^2 g}{\partial r^i \partial r^j}(0)$.

Proof: Start with $g(r) = g(0) + \int_0^1 \frac{d}{dt} g(tr)\, dt$. Then by the fundamental theorem of calculus, this is equal to

$$= g(0) + \int\limits_0^1 \sum_{j=1}^{n} r^j \frac{\partial g}{\partial r^j}(tr)\, dt = g(0) + \sum_{j=1}^{n} r^j \underbrace{\int\limits_0^1 \frac{\partial g}{\partial r^j}(tr)\, dt}_{g_j(r) \in C^\infty} = g(0) + \sum_{j=1}^{n} r^j g_j(r)$$

We can then repeat this argument with each g_j, so for each j, there are some $g_{ji} \in C^\infty$ s.t.

$$g_j(r) = g_j(0) + \sum_{i=1}^{n} r^i g_{ji}(r)$$

(The exact computation may be off here by a factor of 2, due to symmetry.)
Observe that

$$g_j(0) = \int\limits_0^1 \frac{\partial g}{\partial r^j}(0)\, dt = \frac{\partial g}{\partial r^j}(0)$$

Plugging the g_j's back in, we get

$$g(r) = g(0) + \sum_{j=1}^{n} r^j g_j(0) + \sum_{j=1}^{n} r^j \sum_{i=1}^{n} r^i g_{ji}(r) = g(0) + \sum_{j=1}^{n} r^j g_j(0) + \sum_{i,j=1}^{n} r^j r^i g_{ji}(r)$$

$\square$

Tangent Vectors and Curves

Let $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ chart, $\phi = (x^1, \ldots, x^n)$. Then let γ so that

$$\begin{tikzcd} (-\varepsilon,\varepsilon) \arrow[r, "\gamma"] \arrow[dr, "\phi\circ\gamma"'] & U \arrow[d, "\phi"] \arrow[dr, "f"] & \\ & \mathbb{R}^n \arrow[r, "f_\phi"'] & \mathbb{R} \end{tikzcd}$$

Previously, we defined $\dot\gamma(0) \in T_pM$ so that $\dot\gamma(0)([f]) = \frac{d}{dt}(f\circ\gamma)\big|_{t=0}$, where $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$.

Computation of $\dot\gamma(0)$ in coordinates:

Lemma: Let $(\phi\circ\gamma)(t) = (x^1(t), \ldots, x^n(t))$, defined by $x^i(t) : (-\varepsilon,\varepsilon) \to \mathbb{R}$. Then

$$\dot\gamma(0) = \sum_{j=1}^n \frac{dx^j(t)}{dt}\bigg|_{t=0} \frac{\partial}{\partial x^j}\bigg|_p$$

Proof: Let $f : U \to \mathbb{R}$. Then $f\circ\gamma = (f\circ\phi^{-1})\circ(\phi\circ\gamma) = f_\phi \circ (\phi\circ\gamma)$. Use the chain rule on the right-hand side. Then

$$\dot\gamma(0)[f] = \frac{d}{dt}(f\circ\gamma)(t)\bigg|_{t=0} = \sum_{j=1}^n \underbrace{\frac{\partial f_\phi}{\partial r^j}((\phi\circ\gamma)(t))}_{\frac{\partial f}{\partial x^j}(\gamma(t))} \frac{dx^j(t)}{dt}\bigg|_{t=0}$$

$\square$

Cor: Any $v \in T_pM$ is equal to $\dot\gamma(0)$ for some curve γ.
Proof: Choose a chart (U,ϕ) so that $\phi(p) = 0$. Then $v = \sum_{j=1}^n v_j \frac{\partial}{\partial x^j}\big|_p$, with each $v_j \in \mathbb{R}$. Define γ by $x^j(t) = tv_j$, $\forall j \in \{1, \ldots, n\}$, and letting this define $\phi\circ\gamma$.

$$\begin{tikzcd} (-\varepsilon,\varepsilon) \arrow[r, "\gamma"] \arrow[dr, "\phi\circ\gamma"'] & U \arrow[d, "\phi"] \\ & \mathbb{R}^n \end{tikzcd}$$

Then $\gamma(p) = \phi^{-1}(tv_1, \ldots, tv_n)$. $\square$

Smooth Maps Between Manifolds and Tangent Spaces

Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let $p \in M$, with $q = F(p) \in N$.

Observe: Given any $f : V \to \mathbb{R}$, $q \in V \overset{\text{open}}{\subseteq} N$, we have

$$M \xrightarrow{F} V \xrightarrow{f} \mathbb{R}$$

Defn: Consider $F^{-1}(V)$, an open neighborhood of p. $f\circ F : F^{-1}(V) \to \mathbb{R}$. This gives us a map

$$\begin{aligned} F^* : C_q^\infty(N) &\to C_p^\infty(M) \\ [f] &\mapsto [f\circ F] \end{aligned}$$

This is the pullback map on germs. Note that this is a ring morphism!

By duality, we can pushforward tangent vectors.

Defn: If $v \in T_pM$, we define the pushforward of v, $F_{*,p}(v) : C_q^\infty(N) \to \mathbb{R}$ by $F_{*,p}(v)([f]) = v(F^*([f])) \in \mathbb{R}$.

Claim: $F_{*,p}(v) \in T_qN$, i.e., $F_{*,p}(v)$ is also a derivation.
Rough proof: Recall that F^* is a ring morhpism. This, combined with the fact that v is a derivation, implies that $F_{*,p}(v)$ is a derivation. $\square$

Conclusion: We obtain $F_{*,p} : T_pM \to T_{F(p)}N$.

Defn: We can take its dual: $F_p^* : T_{F(p)}^*N \to T_p^*M$.

Lemma: F_p^* is linear.

Lemma: $F_{*,p}(\dot{\gamma}(0)) = \frac{d}{dt}(F \circ \gamma)\big|_{t=0}$.

This final lemma is very useful for computation!

Math 591 Lecture 12

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Recall: For $F : M \to N$ C^∞, and $p \in M$, we have

$$\begin{aligned} F_p^* : C^\infty_{F(p)}(N) &\to C^\infty_p(M) \\ [f] &\mapsto [f \circ F] \end{aligned}$$

By duality, we get $F_{*,p} : T_pM \to T_{F(p)}N$ defined so that, for $v \in T_pM$, $F_{*,p}(v)[f] = v[f \circ F]$. This is also called the differential.

F_p^* is a ring morphism, which maps the ideal

$$I_{F(p)} = \left\{[f] \in C_p^\infty \mid f(F(p)) = 0\right\}$$

into

$$I_p = \left\{[f] \in C_p^\infty \mid f(p) = 0\right\}$$

This induces a map

$$\begin{array}{ccc} I_{F(p)}/I_{F(p)}^2 & \xrightarrow{F_p^*} & I_p/I_p^2 \\ \wr\| & & \wr\| \\ T^*_{F(p)} & \xrightarrow{F_p^*} & T_p^*M \end{array}$$

Check that F_p^* is dual to $F_{p,*}$.

Thm: (Chain Rule) Let $M \overset{F}{\to} N \overset{G}{\to} O$ be smooth, and $p \in M$. Then $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$.

Proof: Let $[f] \in C^\infty_{G(F(p))}(O)$. Then $f \circ (G \circ F) = (f \circ G) \circ F$. Now pick $v \in T_pM$. Then

$$(G \circ F)_{*,p}(v)[f] = v(f \circ (G \circ F)) = v((f \circ G) \circ F) = F_{*,p}(v)(f \circ G) = G_{*,F(p)}(F_{*,p}(v))[f]$$

So $(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$. $\square$

Ex: Let $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ a coordinate chart. As usual, write $\phi = (x^1, \dots, x^n)$, with $x^i : U \to \mathbb{R}$. Say $\mathbb{R}^n = N$, a manifold with a single chart, the identity map $r = (r^1, \dots, r^n)$, $r^i : \mathbb{R}^n \to \mathbb{R}$.
Claim: $\phi_{*,p}(\frac{\partial}{\partial x^i}\big|_p) = \frac{\partial}{\partial r^i}\big|_{\phi(p)}$. In other words, partial derivatives in $\mathbb{R}^n$ correspond with standard basis vectors of T_pM, via the pushforward.

Proof: We can form $f_\phi = f \circ \phi^{-1}$. So by definition, $(\phi^{-1})_{*,\phi(p)}(\frac{\partial}{\partial r^i}) = \frac{\partial}{\partial x^i}$. By the chain ule, $(\phi^{-1})_{*,\phi(p)} = (\phi_{*,p})^{-1}$. $\square$

Differentials of Functions

Let $f : M \to N = \mathbb{R}$ (with a single chart, the identity map). Note that $\forall a \in \mathbb{R}$, we can identify $T_a\mathbb{R} \cong \mathbb{R}$ using $\frac{\partial}{\partial r}\big|_a$ as a basis of $T_a\mathbb{R}$.

Claim: Then $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$ is the same as $df_p : T_pM \to \mathbb{R}$ (defined as the class of $[f - f(p)] \in I_p$ in the quotient I_p/I_p^2).

Chain rule: $M \overset{F}{\to} N \overset{f}{\to} \mathbb{R}$, $v \in T_pM$ simply reads

$$(f \circ F)_{*,p}(v) = (f_{*,F(p)} \circ F_{*,p})(v) = df_p(F_{*,p}(v)) = F_p^*(df_p)(v)$$

by the duality between F_*^p and $F_{*,p}$.

Conclusion: The pullback map on differentials is the pushforward of the composition, i.e., $F_p^*(df_p) = (f \circ F)_{*,p} = d(f \circ F)_p$. So we frequently write $F_{*,p} = dF_p$.

Computation of $F_{*,p}$ in Coordinates

Let $F : M \to N$, $p \in M$. Let (V, ψ) be a coordinate chart near $F(p)$.

$$\begin{array}{ccc} p \in F^{-1}(V) \subseteq U \subseteq M & \xrightarrow{F\,:\,M \to N} & F(p) \in V \subseteq N \\ \Big\downarrow \phi = (x^1, \ldots, x^m) & & \Big\downarrow \psi = (y^1, \ldots, y^n) \\ \mathbb{R}^m & \xrightarrow{\tilde{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n} & \mathbb{R}^n \end{array}$$

Lemma: The matrix of $F_{*,p}$ in the ordered bases

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^m}\right|_p \right) \subseteq T_pM \qquad \text{and} \qquad \left(\left.\frac{\partial}{\partial y^1}\right|_{F(p)}, \ldots, \left.\frac{\partial}{\partial y^n}\right|_{F(p)} \right) \subseteq T_{F(p)}N$$

is simply $\left(\frac{\partial F^j}{\partial x^i}(p)\right)$, the Jacobian, where $F^j = y^j \circ F : U \to \mathbb{R}$, with $\psi \circ F = (F^1, \ldots, F^n)$.

Observe: This is the Jacobian of $\tilde{F}$ at $\phi(p)$ (in the Calc III sense).

Proof: We want to compute the component $F_{*,p}(\frac{\partial}{\partial x^i})$ with respect to $\frac{\partial}{\partial y^j}$. This is

$$F_{*,p}(\frac{\partial}{\partial x^i})([y^j]) = \frac{\partial}{\partial x^i}(y^j \circ F)\Big|_p = \frac{\partial}{\partial x^i}(F^j)\Big|_p$$

□

Defn: A C^∞ map $F : M \to N$ is a local diffeomorphism iff $\forall p \in M$, there are open neighborhoods U of p and V of $F(p)$, such that $F(U) = V$ and $F|_U^V : U \to V$ has a smooth inverse $(F|_U^V)^{-1} : V \to U$.

Ex: $F : S^n \to \mathbb{RP}^n$ is a local diffeomorphism, but not a global diffeomorphism.

Math 591 Lecture 13

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Let $F : M \to N$ C^∞, with $p \in M$. Last time, we defined $F_{*,p} = df_p : T_pM \to T_{F(p)}N$. Our first question today is: How do properties of $F_{*,p}$ reflect properties of F?

Thm: If $F_{*,p}$ is bijective (i.e. $\dim M = \dim N$), then F is a local diffeomorphism at p, i.e., there exist open neighborhoods U of p and V of $F(p)$ such that $F(U) = V$ and $F|_U^V : U \to V$ has a smooth inverse.

Proof: Start with coordinate charts (U, ϕ) near p and (V, ψ) near $F(p)$, so that $U \subseteq F^{-1}(V)$.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "m=\dim M \;\; (x^1,\dots,x^m)=\phi"'] & V \arrow[d, "\psi=(y^1,\dots,y^n) \;\; n=\dim N"] \\ \phi(U) \arrow[r, "\tilde{F}=\psi\circ F\circ\phi^{-1}"] & \psi(V) \end{tikzcd}$$

The matrix of $F_{*,p}$ is $\left(\frac{\partial F^i}{\partial x^j}(p)\right)$, where $F^i = y^i \circ F$ for $i \in \{1, \dots, n\}$. This matrix is the Jacobian of $\tilde{F}$. By assumption (that $m = n$), this matrix is invertible. So by the inverse function theorem in Euclidean space, by shrinking $\phi(U)$ and $\psi(V)$ if necessary, $\tilde{F}$ has a smooth inverse. (This is equivalent to shrinking U and V if necessary.)
So $(F|_U^V)^{-1} = \phi^{-1} \circ \tilde{F}^{-1} \circ \psi$. $\square$

Cor: $F : M \to N$ is a local diffeomorphism iff $\forall p \in M$, $F_{*,p}$ is bijective.

Proof: $\Rightarrow$ $\forall p \in M$, there are neighborhoods U of p and V of $F(p)$ such that $F|_U^V$ is a diffeomorphism. So $F_{*,p}$ has an inverse, $((F|_U^V)^{-1})_{*,p}$ by the chain rule.

$\Leftarrow$ We already showed this.

$\square$

Observe: We now have the notion of a *smooth* covering map.

Defn: $F : M \to N$ is a smooth covering map iff $\forall q \in N$, there is a neighborhood V of q s.t. $F^{-1}(V) = \bigsqcup_{i \in I} U_i$ s.t. $\forall i \in I$, $V = F(U_i)$ and $F|_{U_i}^V$ is a diffeomorphism. Such a V is said to be evenly covered.

Ex: $S^n \to \mathbb{RP}^n$.
The quotient map $S^n \to S^n/S^0 \cong \mathbb{RP}^n$ is a smooth covering map.

A smooth covering map is always a local diffeomorphism, but the converse is false.

Ex: Let

$$\begin{aligned} f : (0, 4\pi) &\to S^1 \subseteq \mathbb{C} \\ t &\mapsto e^{it} \end{aligned}$$

This is a local diffeomorphism, but not a covering map: $F^{-1}(1) = \{2\pi\}$, but every neighborhood of 1 is not evenly covered.

Defn: A smooth function $F : M \to N$ is called a diffeomorphism if it has a smooth inverse.

Defn: Let $F : M \to N$ be smooth.

a) A point $p \in M$ is a regular point of $F \Leftrightarrow F_{*,p}$ is onto.
b) F is a submersion $\Leftrightarrow \forall p \in M$, $F_{*,p}$ is onto.

Thm: (Normal Form for Submersions) Let $F : M \to N$ be a submersion. Then $\forall p \in M$, there are coordinate charts (U, ϕ) around p and (V, ψ) around $F(p)$ such that $U \subseteq F^{-1}(p)$ and $\tilde{F} = \psi \circ F \circ \phi^{-1}$ satisfies $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n)$.

Observe: $F_{*,p} : T_pM \to T_{F(p)}N$ surjective implies that $m \geq n$. Define $r' = (r^1, \ldots, r^n)$ and $r'' = (r^{n+1}, \ldots, r^m)$, so $(r^1, \ldots, r^m) = (r', r'')$. Then $\tilde{F}(r', r'') = r'$.

Cor: A submersion is an open map.

Preliminary Observation: (This is a corollary of the inverse function theorem.) Suppose $p \in U \overset{\text{open}}{\subseteq} M$, and $F : U \to \mathbb{R}^m$ $(m = \dim M)$ such that $F_{*,p}$ is bijective. Then we claim that (after shrinking U if necessary) (U, F) is a coordinate chart.

Proof: By the implicit function theorem, since we can shrink U, WOLOG $F : U \to F(U)$ is a diffeomorphism. So it's a continuous chart (homeomorphism), and by definition of C^∞, (U, F) is compatible with the smooth charts in an atlas. So (U, F) is in the C^∞ structure.

Math 591 Lecture 14

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Defn: Let $F : M \to N$ be smooth.

- F is a submersion at $p \in M \Leftrightarrow F_{*,p}$ is onto. We say p is a regular point of F.
- F is a submersion $\Leftrightarrow \forall p \in M$, F is a submersion at p.
- A critical point is a point which is not a regular point.
- $q \in N$ is a regular value $\Leftrightarrow \forall p \in F^{-1}(q)$, F is a submersion at p.

Observe: If $N = \mathbb{R}$, then either $F_{*,p} : T_pM \to T_{F(p)}\mathbb{R} \cong \mathbb{R}$ is onto, or it is zero. So our definition is consistent with that used in single-variable calculus.

Observe: If $q \in N$ satisfies $F^{-1}(q) = \emptyset$, then q is a regular value.

Thm: (Normal Form Theorem) Assume $F : M \to N$ is a submersion at $p \in M$. Let $m = \dim M$ and $n = \dim N$. Then there are coordinate charts (U, ϕ) around p and (V, ψ) around $F(p)$, with $U \subseteq F^{-1}(V)$, such that for $\tilde{F} = \psi \circ F \circ \phi^{-1}$, $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n)$.

$$\begin{array}{ccc} M & \xrightarrow{\quad F \quad} & N \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} \\ \mathbb{R}^m \supseteq \phi(U) & \xrightarrow{\tilde{F}} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Proof: Start with any coordinates $\phi = (x^1, \ldots, x^m)$, $\psi = (y^1, \ldots, y^n)$. Wrie $F = (F^1, \ldots, F^n)$, with each $F^i = y^i \circ F$. Then the Jacobian of F at p is

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^m}(p) \\ \vdots & & \vdots \\ \frac{\partial F^n}{\partial x^1}(p) & \cdots & \frac{\partial F^n}{\partial x^m}(p) \end{bmatrix}$$

We know J has full rank. So by permuting the x^j's, WOLOG $J : (M \ *)$ where M is $n \times n$ and full rank.
Define $\tilde{\phi} = (F^1, \ldots, F^n, x^{n+1}, \ldots, x^m)$. We claim that $\tilde{\phi} : U \to \mathbb{R}^m$ is a local diffeomorphism at p, and these are the desired coordinates.
Well, the Jacobian of $\tilde{\phi}$ in the x coordinates is

$$\begin{bmatrix} M & * \\ 0 & I_{m-n} \end{bmatrix}$$

because $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$. And this matrix is invertible because M is. $\square$

Defn: Let M be a manifold. Let $S \subseteq M$. Then S is a (regular) submanifold of M iff there is some $k \in \mathbb{N}$ s.t. $\forall p \in S$, there is a chart (U, ϕ) of M with $p \in U$, such that $S \cap U = \left\{ q \in U \mid x^j(q) = 0 \text{ for } j \in \{k+1, \ldots, m\} \right\}$. Such a coordinate system (U, ϕ) is adapted to the submanifold and defines an induced chart on S by $\phi_S : S \cap U \to \mathbb{R}^k$, where $\phi_S = (x^1, \ldots, x^k)\big|_{S \cap U}$.

Claim: $\phi_S : S \cap U \to \phi(S \cap U) \overset{\text{open}}{\subseteq} \mathbb{R}^k$ is a homeomorphism with respect to the subspace topology.

Claim: With the subspace topology and an atlas of adapted charts, S inherits a C^∞ structure. That is, any two charts on S arising from adapted coordinate systems are C^∞-compatible.

The idea of the proof is given two charts (U, ϕ) and (V, ψ), $\psi_S \circ (\phi_S)^{-1}$ is the restriction of $\psi \circ \phi^{-1}$, so the former is C^∞.

We also need to check the point-set topology conditions: Hausdorff and second-countable. But they follow directly from the subspace topology.

Can a chart on the submanifold be extended to a chart on the original manifold? Yes!
Observe: If we start with $(x^1, \ldots, x^k) : W \to \mathbb{R}^k$, a chart of S, then there is a chart ϕ on $U \overset{\text{open}}{\subseteq} M$ s.t. $W = U \cap S$ and $(x^1, \ldots, x^k) = \phi_S$.
Proof: Skip for now...

Prop: If $F : \mathbb{R}^N \to \mathbb{R}^{N-k}$ and 0 is a regular value of F, then $F^{-1}(0)$ is a regular submanifold of $\mathbb{R}^N$.

> Proof: WOLOG S is the graph of $G : A \to \mathbb{R}^{N-k}$, with $A \overset{\text{open}}{\subseteq} \mathbb{R}^k$. $S = \{(a, G(a)) \mid a \in A\}$.
> Define $x^i = x^i$ for $1 \leq i \leq k$, and $x^j = x^j - G^j(x^1, \ldots, x^k)$ for $k+1 \leq j \leq N$, with $G = (G^1, \ldots, G^{N-k})$. These are adapted coordinates! $\square$

The same proof, using the normal form theorem, gives:
Thm: If $q \in N$ is a regular value of $F : M \to N$, then $S = F^{-1}(q) \subseteq M$ is a regular submanifold.

Math 591 Lecture 15

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Thm: (Regular Value Theorem for Manifolds) Let M and N be manifolds, $F : M \to N$ C^∞, and $q \in N$ a regular value of F. Then $F^{-1}(q)$ is a regular submanifold of M.

Proof: Let $p \in F^{-1}(q)$. We want to show there are coordinates of M near p which are adapted to the preimage of $F^{-1}(q)$. Because q is a regular value, $F_{*,p} : T_pM \to T_qN$ is onto for any p. By the normal form for submersions, there are coordinates $(U, \phi = (x^1, \dots, x^m))$ near p and $(V, \psi = (y^1, \dots, y^n))$ near q, with $U \subseteq F^{-1}(V)$, such that $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n)$.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "\phi"] & V \arrow[d, "\psi"] \\ \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \end{tikzcd}$$

WOLOG assume $\psi(q) = 0$. Split ϕ, with $x' = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ and $x'' = (x^{n+1}, \dots, x^m) : U \to \mathbb{R}^{m-n}$. Then $F^{-1}(q) \cap U$ corresponds to $\tilde{F}^{-1}(0)$ by ϕ, i.e., $F^{-1}(q) \cap U = \{a \in U \mid x'(a) = 0\}$. Thus, (x'', x') are adapted coordinates to $F^{-1}(q) \cap U$. $\square$

Observe: (Keeping the notation of the proof) $x'' : F^{-1}(q) \cap U \to \mathbb{R}^{m-n}$ are coordinates on $F^{-1}(q) \cap U$. So $\dim F^{-1}(q) = m - n$. (Recall: $m \geq n$.)

Defn: The <u>codimension</u> of a submanifold is the dimension of the ambient space minus the dimension of the submanifold.

$\operatorname{codim} F^{-1}(q) = \dim M - \dim F^{-1}(q) = m - (m - n) = n$. This is the dimension of the target space.

Observe: $\forall p \in F^{-1}(q)$ (if q is a regular value), $T_p(F^{-1}(q)) \subseteq T_pM$ as a subspace. In fact, $T_p(F^{-1}(q))$ is the kernel of $F_{*,p}$.

A General Observation on Tangent Spaces of Submanifolds

Let $S \subseteq M$ be a submanifold, and $p \in S$. Then $T_pS \subseteq T_pM$ by: $\forall \gamma : (-\varepsilon, \varepsilon) \to S$ with $\gamma(0) = p$, we have

$$\begin{array}{ccc} \dot{\gamma}_S(0) & \xmapsto{\iota_{*,p}} & \dot{\gamma}_M(0) \\ \cup & & \cup \\ T_pS & & T_PM \end{array}$$

by using the differential of the inclusion $\iota : S \hookrightarrow M$. The inclusion in adapted coordinates is $x' \mapsto (x', 0)$. If $[f] \in C_p^\infty(M)$, $\dot{\gamma}_M(0)[f] = \dot{\gamma}_S(0)[f \circ \iota]$. Observe: $f \circ \iota$ is the restriction of f to S.

Conclusion: Tangent spaces of submanifolds are subspaces of the tangent spaces of the original manifold.

Defn: A map $F : M \to N$ is a <u>submersion</u> iff $\forall p \in M$, $F_{*,p}$ is onto.

Cor: If F is a submersion, then $\forall q \in N$, q is a regular value, so $F^{-1}(q)$ ("the fiber of f over q") if either empty or a codimension n submanifold of M.

Ex: Let $M = \mathbb{R}^2 \setminus S^1$, $N = \mathbb{R}$, $F : M \to N$ with $F(x, y) = x$. What are the fibers?

- For $q \in (-\infty, 1) \cup (1, \infty)$, $F^{-1}(q) = \mathbb{R}$.
- For $q \in (-1, 1)$, $F^{-1}(q) = (-\infty, -\sqrt{1 - q^2}) \cup (-\sqrt{1 - q^2}, \sqrt{1 - q^2}) \cup (\sqrt{1 - q^2}, \infty)$.
- For $q \in \{-1, 1\}$, $F^{-1}(q) = \mathbb{R} \setminus \{0\}$.

Note that in this example, some of the fibers are different topologically!

Ex: Let $M = S^3$, $F : S^3 \to \mathbb{RP}^1 \cong S^2$ (the Riemann Sphere). Then the fibers are all circles, and the map from S^3 to S^2 is called the Hopf fibration.

Defn: A C^∞ map $F : M \to N$ is a <u>fibration</u> with fiber Φ, where Φ is a manifold, iff there is an open covering $\{U_\alpha\}$ of N (called the <u>base</u>) and diffeomorphic maps $\chi_\alpha : F^{-1}(U_\alpha) \to U_\alpha \times \Phi$ (called <u>trivializations</u>) such that the diagram

$$\begin{tikzcd} F^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "F|_{F^{-1}(U_\alpha)}"'] & & U_\alpha \times \Phi \arrow[dl, "\pi \text{ projection}"] \\ & U_\alpha & \end{tikzcd}$$

commutes (i.e. all paths are the same). We say that F is a <u>fiber bundle</u> with <u>fiber</u> Φ.

Let's unpack what this means. Commutativity of the diagram means $\forall p \in F^{-1}(U_\alpha)$, $\chi_\alpha(p) = (F(p), \star)$ where $\star \in \Phi$. So $\forall q \in U_\alpha$, χ_α restricts to the fiber $F^{-1}(q)$, where $\chi_\alpha(p) \mapsto \star$.

Ex: The tangent bundle TM is a fiber bundle, with fiber $\mathbb{R}^m$ (with $m = \dim M$).

$$\begin{tikzcd} TM \arrow[d] \\ M \end{tikzcd}$$

Note: This has additional structure: $\Phi \cong \mathbb{R}^m$ is a vector space, the fibers are all vector spaces, and there exist trivializations that are linear on the fibers.

Math 591 Lecture 16

Professor Alejandro Uribe-Ahumada

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10/7/20

The Hopf Fibration

$$\begin{tikzcd}
\mathbb{C}^2 \supset S^3 \arrow[d, "\pi"] \\
\mathbb{RP}^1 \cong S^1 \backslash S^3
\end{tikzcd}$$

$\pi(z_0, z_1) = \left\{(e^{i\theta} z_0, e^{i\theta} z_1) \mid \theta \in \mathbb{R}/2\pi\mathbb{Z}\right\}$.
$\pi(z_0, z_1) = [z_0 : z_1]$.

Claim: This is a fiber bundle with fiber $\Phi = S^1$.

For the covering of $\mathbb{RP}^1$, we choose the same covering as used in the homework: $U_0 = \{[z_0 : z_1] \mid z_0 \neq 0\}$ and $U_1 = \{[z_0 : z_1] : z_1 \neq 0\}$.

Define $\mathscr{S}(\mathfrak{z}) = \frac{1}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then we have

$$\begin{tikzcd}
 & & S^3 \arrow[d, "\pi"] \\
\mathbb{R}^2 \cong \mathbb{C} & U_0 \arrow[l, "\cong"'] \arrow[ur, "\mathscr{S}"] \arrow[r, hook] & \mathbb{RP}^1 \\
\mathfrak{z} = \frac{z_1}{z_0} & {[z_0 : z_1]} \arrow[l, mapsto] &
\end{tikzcd}$$

Note: $\pi \circ \mathscr{S} = I_{U_0}$, since $\pi(\mathscr{S}(\mathfrak{z})) = \mathfrak{z}$.

Defn: If $\pi : M \to N$ is a fiber bundle, a <u>section</u> of π is a C^∞ map $\mathscr{S} : N \to M$ s.t. $\pi \circ \mathscr{S} = I_N$.

Observe: The Hopf fibration does not have a global section.

Observe: Local sections always exist, because they always exist for the trivial bundle $N \to N \times \Phi$: fix $\nu \in \Phi$, and map $p \mapsto (p, \nu)$.

More on the Hopf fibration... Define a trivialization

$$\begin{tikzcd}
\pi^{-1}(U_0) \arrow[rr, "\chi"] \arrow[dr] & & U_0 \times \Phi \arrow[dl] \\
 & U_0 &
\end{tikzcd}$$

with $\chi(\mathfrak{z}, e^{i\theta}) = e^{i\theta}\mathscr{S}(\mathfrak{z}) = \frac{e^{i\theta}}{\sqrt{1+|\mathfrak{z}|}}(1, \mathfrak{z})$. Then $\pi \circ \mathscr{S} = I_{U_0}$, and $\forall \mathfrak{z} \in U_0$, $\mathscr{S}(\mathfrak{z}) = \pi^{-1}(\mathfrak{z})$.

$\chi(z_0, z_1) = (\frac{z_1}{z_0}, \frac{z_0}{|z_0|})$ (note that $z_0 \neq 0$). Then $S^3 \cong \mathbb{R}^3 \cup \{\infty\} = \bigsqcup S^1$, an uncountable disjoint union.

Vector Bundles

Defn: $\pi : E \to B$ is a <u>vector bundle</u> with <u>base</u> B and <u>rank</u> $k \in \mathbb{N}$ iff

a) It's a fibration.
b) $\forall p \in B$, the fiber $\pi^{-1}(p)$ is a vector space.
c) There is an open covering $\{U_\alpha\}$ of B and all trivializations $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ restrict to linear maps on each fiber, i.e.,

$$\begin{tikzcd} \chi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \arrow[r, "\cong"] \arrow[rr, bend right, "\text{Linear Isomorphism}"'] & \{p\} \times \mathbb{R}^k \cong & \mathbb{R}^k \end{tikzcd}$$

Note: $\forall p \in B$, the zero section applied to $p \in B$ gives $0 \in \pi^{-1}(p)$.

Ex:

1) The tangent bundle $TB \to B$.
2) The cotangent bundle $T^*B = \bigsqcup_{p \in B} \{p\} \times T_p^*B \to B$.

Defn: If $S \subseteq M$ is a submanifold, then the <u>co-normal bundle of S</u> is $\mathcal{N} = \Big\{(p, \alpha) \in T^*M \mid \ \alpha|_{T_pS} = 0\Big\}$.

The co-normal bundle is a vector bundle.

Note: If we give T_pM a Euclidean inner product, we can identify

$$\begin{aligned} T_p^*M \ &\cong \ T_pM \\ \langle \cdot\,, v\rangle &\longleftarrow\!\!\shortmid \ v \end{aligned}$$

Claim: $\mathcal{N} \subset T^*M$ is a submanifold, and $\pi : \mathcal{N} \to S$ is a vector bundle of rank $\operatorname{codim} S = \dim M - \dim S$. The fiber of $\mathcal{N}$ over p is the annihilator of T_pS.

Math 591 Lecture 17

Professor Alejandro Uribe-Ahumada

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Recall that a submersion $F : M \to N$ satisfies $\forall p \in M$, $F_{*,p}$ is onto.

Fibrations are a special class of submersions. For any submersion, its fibers are $F^{-1}(q)$, $\forall q \in N$. $F^{-1}(q)$ is a regular submanifold by the regular value theorem. But not all fibers are diffeomorphic to each other. And $F^{-1}(q) = \emptyset$ is possible.

Also, the normal form theorem for submersions (also called "the submersion theorem") states that locally, a submersion is a map of the form $(x', x'') \mapsto x'$.

Immersions

Defn: A smooth map $F : M \to N$ is an immersion at $p \in M$ $\Leftrightarrow$ $F_{*,p}$ is injective. Note that this requires $\dim M \leq \dim N$.

Defn: F is an immersion iff $\forall p \in M$, F is an immersion at p.

Ex: Our model case is $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ with $r \mapsto (r, 0)$, where $k \leq n$.

Ex:

1) If $S \subset N$ is a regular submanifold, the inclusion $\iota : S \hookrightarrow N$ is an immersion, since $\forall p \in S$, $\iota_{*,p} : T_pS \hookrightarrow T_pN$.
2) Immersions don't have to be injecive. Consider the lemniscate $F : \mathbb{R} \to \mathbb{R}^2$. $\forall t \in \mathbb{R}$, $\dot{F}(t) \neq 0$, so F is an immersion. But it's not injective!
3) Restrict the domain of the lemniscate to obtain a one-to-one map whose image is a regular submanifold.
4) Can also restrict the domain of the lemniscate to obtain a one-to-one map whose image is not a regular submanifold. (I.e. it goes right up to the point where the curve would self intersect, but doesn't map to the intersection point more than once.)
5) Let $N = \mathbb{R}^2/\mathbb{Z}^2$ (where $\mathbb{Z}^2$ is the integer lattice). Define $\pi : \mathbb{R}^2 \to N$, and declare this to be a local diffeomorphism, giving us a smooth structure on N. This is the 2-Torus. Let $F : \mathbb{R} \to \mathbb{R}^2 \xrightarrow{\pi} N$. Fix $v = (a, b) \in \mathbb{R}^2 \setminus \{0\}$. Sy $F(t) = \pi(tv)$, $\forall t \in \mathbb{R}$. If a and b are rational, you get a periodic curve around the torus. These are all immersions. If a/b is irrational, then the image of F in $\mathbb{R}^2/\mathbb{Z}^2$ is dense! With the subspace topology, $F(\mathbb{R})$ is very much so *not* locally Euclidean.

Thm: (Normal Form for Immersions) Let $F : M \to N$ be an immersion at $p \in M$. Then there are coordinates $p \in U, \phi = (x^1, \ldots, x^m)$ of M and $f(p) \in V, \psi = (y^1, \ldots, y^n)$ of N such that $U \subset F^{-1}(V)$ and $\tilde{F}(r^1, \ldots, r^m) = (r^1, \ldots, r^n, \underbrace{0, \ldots, 0}_{n-m \text{ zeros}})$. We have

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\phi"] & V \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Proof: Next time...

Defn: An immersion $F : M \to N$ hich is a homeomorphism onto $F(M)$ with the subspace topology is called an embedding. This is a global property!

Observe: Embeddings are injective.

Math 591 Lecture 18

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Thm: (Normal Form for Immersions) Let $F : M \to N$ be an immersion at $p \in M$. Then there are coordinates $p \in U, \phi = (x^1, \dots, x^m)$ of M and $f(p) \in V, \psi = (y^1, \dots, y^n)$ of N such that $U \subset F^{-1}(V)$ and $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n, \underbrace{0, \dots, 0}_{n-m \text{ zeros}})$.

Proof: take any coordinates. We have

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\phi"] & V \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\tilde{F}"] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Write $F = (F^1, \dots, F^n)$, where each $F^i = y^i \circ F$.
The Jacobian of $\tilde{F}$ at $\phi(p)$ is $J = \left(\frac{\partial F^i}{\partial x^j}(p)\right)_{\substack{i \text{ rows} \\ j \text{ cols}}}$. By assumption, $\ker J = \{0\}$, so we can write

$$J = \begin{pmatrix} \mathcal{M}_{m \times m} \\ \star \end{pmatrix}_{\substack{m \le n \\ 1 \le i \le n \\ 1 \le j \le m}}$$

and we know J has max rank. Permute the y-coordinates to shuffle the rows, so that the top $m \times m$ minor $\mathcal{M}$ of J is non-degenerate. Thus, $\mathcal{M} = \left(\frac{\partial F^i}{\partial x^j}\right)_{\substack{1 \le i \le m \\ 1 \le j \le m}}$ gives us some $\tilde{\phi} = (F^1, \dots, F^m) : U \to \mathbb{R}^m$ which is a local diffeomorphism at p. By shrinking U if necessary, $\tilde{\phi} : U \to \mathbb{R}^m$ is a coordinate chart.

$$\begin{tikzcd} M \supseteq U \arrow[r, "F"] \arrow[d, "\tilde{\phi}"] & V \subseteq N \arrow[d, "\psi"] \\ \tilde{\phi}(U) \arrow[r, "\tilde{\tilde{F}}"] & \psi(V) \end{tikzcd}$$

where $\tilde{\tilde{F}}(\underbrace{r^1, \dots, r^m}_{r}) = (y^1(F \circ \tilde{\phi}(r)), \dots, y^n(F \circ \tilde{\phi}(r))) = (r^1, \dots, r^m, G^1(r), \dots, G^{n-m}(r))$. Now, we modify the y-coordinates:

$$\begin{cases} w^i = y^i & 1 \le i \le m \\ w^i = y^i - G^{i-m}(y^1, \dots, y^m) & m+1 \le i \le n \end{cases}$$

Both are invertible. The first m are trivially so, and the remainder are invertible because you can recover y^{i+m}.
Let $\tilde{\psi} = (w^1, \dots, w^n) : V \to \mathbb{R}^n$.

$$\begin{tikzcd} & V \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\ \mathbb{R}^n \arrow[rr, "\star"] & & \mathbb{R}^n \end{tikzcd}$$

with

$$\star : (r^1, \dots, r^n) \mapsto (\underbrace{r^1, \dots, r^m}_{r'}, \underbrace{r^{m+1} - G^1(r'), \dots, r^n - G^{m-n}(r')}_{r'' - G(r')})$$

This is a local diffeomorphism. So under the $\tilde{\phi}$ and $\tilde{\psi}$ coordinates, $(r^1, \dots, r^m) \xrightarrow{\tilde{\tilde{\tilde{F}}}} (r^1, \dots, r^m, 0, \dots, 0)$. $\square$

Recall: An embedding $F : M \to N$ is an immersion that is a homeomorphism onto its image, that is, $F|^{F(M)} : M \to F(M)$ is a homeomorphism. It has a continuous inverse.

Observe: Every embedding is also one-to-one.

Question: Assum F is a one-to-one immersion. Under what conditions is F an embedding?

Prop: An injective immersion is an embedding iff $F|^{F(M)} : M \to F(M)$ is an open (or closed) map w.r.t. the subspace topology. This is true iff $\forall U \subset M$ open, $\exists V \subset N$ open s.t. $F(U) = F(M) \cap V$ ($F(M) \cap V$) is a relatively open set).

Proof: The inverse of an open map is continuous iff the map is invertible. $\square$

Cor: If $F : M \to N$ is an injective immersion and M is compact, then F is an embedding.

Proof: F is a closed map – if $C \subset M$ is closed and M is compact, then C is compact, so $F(C)$ is compact. Thus, $F(C)$ is closed. $\square$

In fact...

Thm: If $F : M \to N$ is a proper (recall: the preimage of compact sets are compact) injective immersion, then F is an embedding.

"Proof": This is true because a proper continuous map into a locally compact space is closed.

Thm: The image of an embedding is a regular submanifold.

Proof: Let $F : M \to N$ be an embedding. Let $q = F(p) \in F(M)$ for $p \in M$. Use the immersion theorem to get coordinate $(x^1, \dots, x^m)$ and $(y^1, \dots, y^n)$ s.t. $\tilde{F}(r^1, \dots, r^m) = (r^1, \dots, r^n, \underbrace{0, \dots, 0}_{n-m})$.

Since F is an embedding, $F(U)$ is relatively open in $F(M)$. Thus, $F(U)$ is the intersection of $F(M)$ with some open set $\tilde{V}$ in N. Thus, $F(U) = \tilde{V} \cap F(M) \subseteq \left\{y^{m+1} = \dots = y^n = 0\right\}$. (Will finish next time.)

Math 591 Lecture 19

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Thm: The image of an embedding $F : M \to N$ is a regular submanifold of N.

Proof: Let $q \in F(M)$. We need to show that there are coordinates of N near q adapted to $F(M)$. Let $p \in M$ s.t. $F(p) = q$. By the immersion theorem, there are corodinates $(U, \phi = (x^1, \ldots, x^m))$ of M with $p \in U$, and $(V, \psi = (y^1, \ldots, y^n))$ of N with $q \in V$, with $U \subseteq F^{-1}(V)$, such that $\tilde{F}(I) = (I, 0)$ (with $m - n$ zeros).

$$\begin{array}{ccc} U & \xrightarrow{F} & V \\ \Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\ \mathbb{R}^m \supseteq \phi(U) & \xrightarrow{\tilde{F}} & \psi(V) \subseteq \mathbb{R}^n \end{array}$$

Well, $F(U) = \left\{w \in V \mid y^j(w) = 0, \forall j \in \{1, \ldots, n\}\right\}$. The point is that because F is an embedding (i.e. $F|^{F(M)} : M \to F(M)$ is a homeomorphism), then $F(U)$ is a relatively open set of $F(M)$. So there is an open set $W \subseteq N$ such that $F(U) = F(M) \cap W$. Therefore, $F(M) \cap W \left\{w^{-1}V \mid y^j(w) = 0, \forall j \in \{1, \ldots, n\}\right\}$. So ψ on $V \cap W$ is adapted to $F(M)$ at q. $\square$

We want a stronger statement for when an injective immersion is an embedding.

Prop: Let $F : M \to N$ be a continuous proper map between manifolds. Then F is closed.

Proof: Let $C \subseteq M$ be a closed set, and $q \in \overline{F(C)}$. We need to show $q \in F(C)$. Well, let $V \subseteq N$ be an open neighborhood of q such that $\overline{V}$ is compact. (We can do this because Euclidean spaces are locally compact.) Observe then that $q \in \overline{F(C) \cap \overline{W}}$. $F^{-1}(\overline{V})$ is compact, since F is proper, so $C \cap F^{-1}(\overline{V})$ is compact. Therefore, $F(C \cap F^{-1}(\overline{V}))$ is compact, so it's closed. But $F(C \cap F^{-1}(\overline{V})) = F(C) \cap \overline{V}$ (this is a set-theoretic fact). SO $F(C) \cap \overline{V}$ is closed. Since $q \in \overline{F(C) \cap \overline{V}}$, $q \in F(C) \cap \overline{V}$, so $q \in F(C)$. Thus, F is closed. $\square$

Vector Fields

Defn: Let M be a smooth manifold of dimension n. A <u>vector field on M</u> is a secion of $TM \overset{\pi}{\to} M$. That is, it's a map $X : M \to TM$ s.t. $\forall p \in M$, $X_p \in T_pM$.

This is a slight abuse of notation. Update: Write $X(p) = (p, X_p)$. The p is the same because X is a section $\pi \circ X = I_M$.

Basically, a vector field on a manifold assigns a tangent vector to every point in a manifold.

Defn: A vector field X on M is <u>smooth</u> (C^∞) iff it is smooth as a map between manifolds $X : M \to TM$.

What does this mean (concretely)?

Let's think about smooth sections of vector bundles in general (e.g. TM and T^*M). Let

$$\begin{array}{c} \mathcal{E} \\ \Big\downarrow{\scriptstyle\pi} \\ M \end{array}$$

be a C^∞ vector bundle of rank $\rho \in \mathbb{N}$. This means $\mathcal{E}$ is a manifold, π is C^∞ and onto, $\forall p \in M$, $\pi^{-1}(p)$ has the structure of a vector space over $\mathbb{R}$ of dimension ρ, and there exists a family $\{(U_\alpha, \chi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of M and $\forall \alpha$,

χ_α is a diffeomorphism, and we have the vector bundle trivialization

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^\rho \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

s.t. χ_α is linear on fibers, i.e., $\forall p \in U_\alpha$, we have

$$\chi_\alpha\big|_{\pi^{-1}(p)} : \pi^{-1}(p) \overset{\cong}{\to} \{p\} \times \mathbb{R}^\rho \cong \mathbb{R}^\rho$$

The mapping from $\pi^{-1}(p)$ to $\mathbb{R}^\rho$ is a linear isomorphism.

Ex: Let $\mathcal{E} = TM$. The trivializations are induced by coordinate charts. Let $U \subseteq M$, with $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ be a coordinate chart. Then $\forall p \in U, i \in \{1, \ldots, n\}$, $\frac{\partial}{\partial x^i}\big|_p \in T_pU = T_pM$. We have

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl] \\ & U & \end{tikzcd}$$

defined by

$$\chi^{-1}(p, v) = \left(p, \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p\right)$$

with $p \in U$ and $v = (v^1, \ldots, v^n) \in \mathbb{R}^n$.
We proved that $\forall p \in M$, $\left\{ \frac{\partial}{\partial x^i}\big|_p : i = 1, \ldots, m \right\}$ is a basis of T_pM. So χ^{-1} is invertible and a linear isomorphism of the fibers.
(The definition of the smooth structure on TM is such that χ as above is a diffeomorphism.)

Ex: $T^*M = \bigcup_{p \in M} \{p\} \times T_p^*M$
Again, a coordinate chart on M induces a trivialization.
Use: $\forall \alpha \in T_p^*M$, $\alpha = \sum_{i=1}^n \alpha_i \, dx^i\big|_p$, where $dx^i\big|_p$ is dual to $\frac{\partial}{\partial x^i}\big|_p$, and $\left\{ dx^i\big|_p \mid i = 1, \ldots, n \right\}$ is a basis of T_p^*M.

Back to

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

Let s be a section (i.e. $\pi \circ s = I_M$). When is s smooth? We will answer this question using trivializations. Given a trivialization χ,

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^p \arrow[dl, "\pi"] \\ & U \arrow[ul, bend right, "s|_U"'] \arrow[ur, bend right, "\chi \circ (s|_U) \overset{\text{def}}{=} s_\chi"'] & \end{tikzcd}$$

We have $\forall p \in U$, $s_\chi(p) = (p, \star)$, where $\star$ is given by a function $F : U \to \mathbb{R}^\rho$. F is just the projection onto $\mathbb{R}^\rho$ composed with s_χ. So $\forall p \in U$, $s_\chi(p) = (p, F(p))$.

Claim: $s|_U$ is smooth iff $F : U \to \mathbb{R}^\rho$ is smooth.

Proof: Next time...

Math 591 Lecture 20

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Smooth Sections of Vector Bundles

Start with a rank ρ vector bundle, with section s, i.e., $\pi \circ s = I_M$.

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

Let χ be a local trivialization

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\chi"] \arrow[dr] & & U \times \mathbb{R}^\rho \arrow[dl] \\ & U & \end{tikzcd}$$

with χ a diffeomorphism, and linear on each fiber. Then $s|_U : U \to \pi^{-1}(U)$ satisfies

$$\begin{aligned} \chi \circ (s|_U) : U &\to U \times \mathbb{R}^\rho \\ p &\mapsto (p, F(p)) \end{aligned}$$

where $F : U \to \mathbb{R}^\rho$. We write $F = (F^1, \ldots, F^\rho)$ with each $F^i : U \to \mathbb{R}$.

Lemma: (From last time) $s|_U$ is smooth iff $\forall i$, F^i is smooth.

Proof: $\Rightarrow$ is trivial.

$\Leftarrow$: It's a fact from analysis that F is C^∞ iff $\forall i$, F^i is C^∞. So $s|_U(p) = \chi^{-1}(p, F(p))$, which is smooth.

$\square$

Observe: The trivializationabove corresponds to a "moving frame" on U.

Defn: A <u>moving frame</u> on U i a collection of ρ smooth sections on U, $\{e_1, \ldots, e_\rho\}$, s.t. $\forall p \in U$, $\{e_1(p), \ldots, e_\rho(p)\}$ is a basis of the fiber $\pi^{-1}(p)$.

Ex: If $(U, \phi = (x^1, \ldots, x^n))$ is a coordinate chart, let $e_i(p) = \frac{\partial}{\partial x^i}\big|_p \in T_pM$. This defines a moving frame of TM on U.

Given a trivialization χ over U as above, how do we get a moving frame? Well, $\forall i \in \{1, \ldots, \rho\}, p \in U$, let $e_i(p) \overset{\text{def}}{=} \chi^{-1}(p, (0, \ldots, 1, \ldots 0))$ (with the 1 in the ith entry).

Observe: If $s|_U$ corresponds to $F = (F^1, \ldots, F^\rho) : U \to \mathbb{R}^\rho$, then $s|_U = \sum_{i=1}^\rho F^i e_i$, where the F^i are scalar-valued functions and the e_i are sections. So $\forall p \in U$, $s(p) \in \pi^{-1}(p)$, and $s(p) = \sum_{i=1}^\rho F^i(p) e_i(p)$ (using the vector space structure of $\pi^{-1}(p)$).

Conversely, we can also define a trivialization from a moving frame. (This is left as an exercise.)

Observe: If $C^\infty(M, \mathcal{E})$ is the space of C^∞ sections of $\mathcal{E} \to M$ vector bundles, then $C^\infty(M, \mathcal{E})$ is a module over $C^\infty(M)$. We can multiply a section s by a function $f \in C^\infty(M)$ fiber-wise, with $(fs)(p) = f(p)s(p)$.

Vector Fields

Let $\mathcal{E} = TM$.

Defn: $\mathfrak{X}$ is the set of all smooth vector fields on M.

$\forall X \in \mathfrak{X}$, with a coordinate system on U, $\exists a_i \in C^\infty(U)$ s.t. $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}$. X is C^∞ iff $\forall i, a_i \in C^\infty$.

Prop: Any $X \in \mathfrak{X}(M)$ defines an operator

$$\begin{aligned} C^\infty(M) &\to C^\infty(M) \\ f &\mapsto X(f) \end{aligned}$$

which

a) is $\mathbb{R}$-linear.

b) satisfies Leibniz' rule: $\forall f, g \in C^\infty(M)$, $X(fg) = fX(g) + gX(f)$.

(An aside: As a section, the value of X at $p \in M$ is denoted $X_p \in T_pM$.)

Proof: $X(f)(p) = X_p([f])$, where $[f]$ is the germ of f at p. Thus, X is a derivation on $C^\infty(M)$, because X_p is a derivation on $C_p^\infty(M)$ germs. $\square$

Defn: Such an operator is called a <u>derivation</u> of $C^\infty(M)$.

Prop: (1) The operator defined by $X \in \mathfrak{X}(M)$ is local, i.e., $\forall f \in C^\infty(M), U \subseteq M$ open such that $f|_U \equiv 0$, then $X(f)|_U \equiv 0$.

Observe: This "locality" characterizes differential operators.

Observe: In local coordinates, if $X = \sum_i a_i \frac{\partial}{\partial x^i}$, then $X(f)(p) = \sum_i a_i \frac{\partial f}{\partial x^i}(p)$.

Thm: (2) Any operator $D : C^\infty(M) \to C^\infty(M)$ that is a derivation is given by a vector field.

Thm: (3) The commutator of two derivations is a derivation.

Together, we have: If $X, Y \in \mathfrak{X}(M)$, then there is a vector field denoted $[X, Y] \in \mathfrak{X}(M)$ (said "X bracket Y" or "X commutator Y") such that $\forall f \in C^\infty(M)$, $[X, Y](f) = X(Y(f)) - Y(X(f))$.

Proof of (3): Define $[X, Y]$ as the operator commutator above. Clearly this is linear. Verify Leibniz' rule:

$$[X, Y](fg) = X(fY(g) + gY(f)) - Y(fX(g) + gX(f)) = \cdots = f[X, Y](g) + g[X, Y](f)$$

$\square$

In local coordinates, say $X = \sum_i a_i \frac{\partial}{\partial x^i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x^j}$. Then

$$[X, Y] = \sum_{ij} \left[a_i \frac{\partial}{\partial x^i}, b_j \frac{\partial}{\partial x^j} \right]$$

And

$$\begin{aligned} \left[a_i \frac{\partial}{\partial x^i}, b_j \frac{\partial}{\partial x^j} \right] &= a_i \frac{\partial}{\partial x^i}(b_j \frac{\partial f}{\partial x^j}) - b_j \frac{\partial}{\partial x^j}(a_i \frac{\partial f}{\partial x^i}) \\ &= \cancel{a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}} + a_i \frac{\partial b_j}{\partial x^j} \frac{\partial f}{\partial x^j} - \left(\cancel{b_j a_i \frac{\partial^2 f}{\partial x^j \partial x^i}} + b_j \frac{\partial a_i}{\partial x^j} \frac{\partial f}{\partial x^i} \right) \\ &= a_i \frac{\partial b_j}{\partial x^i} \frac{\partial f}{\partial x^j} - b_j \frac{\partial a_i}{\partial x^j} \frac{\partial f}{\partial x^i} \end{aligned}$$

This gives the commutator.

Math 591 Lecture 21

Professor Alejandro Uribe-Ahumada

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Last time, we showed that a smooth vector field $X \in \mathfrak{X}(M)$ defines a derivation

$$\begin{aligned} X : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto (p \mapsto X_p([f])) \end{aligned}$$

Here, we're thinking of X as an operator, i.e., $f \mapsto X(f)$. Note that $\forall p \in M$, $X_p \in T_pM$.

Prop: The commutator of any two derivations $C^\infty(M) \to C^\infty(M)$ is a derivation.
Proof: This is just an algebraic calculation.

Today, we'll prove the converse – that for any derivation D, there is a unique vector field $X \in \mathfrak{X}(M)$ such that $D = X$ (as an operator). So overall, we will have showed a one-to-one correspondence between derivations and vector fields. To do this, we need "bump functions".

Prop: Let $U \subseteq M$ open, $p \in U$. Then $\exists \chi \in C^\infty(M)$ s.t.

(1) $\mathrm{supp}(\chi) = \overline{\{q \in M : \chi(q) \neq 0\}} \subseteq U$ (and it is compact)
(2) $\exists V$ open with $p \in V$ such that $\chi|_V \equiv 1$.
Note: (1) implies that $\overline{V} \subseteq U$.

Defn: Such a χ is called a <u>bump function</u> at p.

Proof: It's enough to consider the case where $p = 0 \in \mathbb{R}^n$, as we can use a chart near p to define χ in some neighborhood of p, and then extend χ to be 0 outside that neighborhood.
Start with the case where $n = 1$ (i.e. $\mathbb{R}$). (See also §13 in the book.) Start with

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

We claim that f is C^∞ on $\mathbb{R}$. (This is because $\forall k \in \mathbb{N}$, $f^{(k)}(0)$ is defined.)

Note: f is a famous example of a non-analytic function.

Next, let $g(x) = \frac{f(x)}{f(x)+f(1-x)}$. Note: $\forall x \in \mathbb{R}$, $f(x) + f(1-x) \neq 0$, so g is well-defined, and C^∞. If $x \ge 1$, then $f(1-x) = 0$, so $g(x) = 1$. If $x \le 0$, $f(x) = 0$, so $g(x) = 0$.

Next, choose some $a, b \in \mathbb{R}_{>0}$ with $0 < a^2 < b^2$, and define $h(x) = g(\frac{x-a^2}{b^2-a^2})$. Then finally, take $\rho(x) = 1 - h(x^2)$. Then we have $\rho|_{[-a,a]} \equiv 1$, and $\rho|_{(-\infty,-b]\cup[b,\infty)} \equiv 0$, and ρ is C^∞.

For $\mathbb{R}^n$, let $\chi(x) = \rho(||x||^2)$. Then $\mathrm{supp}\,\chi$ is a subset of a ball around the origin, and χ restricted to a smaller ball is always 1. $\square$

Defn: $D : C^\infty(M) \to C^\infty(M)$ is a <u>local operator</u> if $\forall f, g \in C^\infty(M)$, $\forall U \overset{\text{open}}{\subseteq} M$, if $f|_U = g|_U$, then $D(f)|_U = D(g)|_U$.

Prop: A derivation $D : C^\infty(M) \to C^\infty(M)$ is a local operator.

Proof: By linearity of D, WOLOG $g \equiv 0$. Assume that $f|_U \equiv 0$, and let $p \in U$. Let $\chi \in C^\infty(M)$ be a bump function at p with $\mathrm{supp}(\chi) \subset U$. Note: $\chi \cdot f \equiv 0$ on M, so $D(\chi f) = 0$. Well, by the chain rule, $D(\chi f) = \chi D(f) + f D(\chi)$. If we evaluate at p, we have $f(p) = 0$ and $\chi(p) = 1$, so $0 = 0 + D(f)(p)$, so $D(f)(p) = 0$. Thus, $D(f)|_U \equiv 0$. $\square$

Note: One can show that every local (linear) operator is a differential operator.

Thm: Let $D : C^\infty(M) \to C^\infty(M)$ be a derivation. Then $\exists X \in \mathfrak{X}(M)$ such that $D = X$ (as an operator).

Proof: Let $p \in M$. To define $X_p \in T_pM$, pick some $[f] \in C_p^\infty(M)$. Let $f : U \to \mathbb{R}$ represent this germ. Let χ be a bump function at p with $\text{supp}(\chi) \subseteq U$. Define $\tilde{f} : M \to \mathbb{R}$ where $\tilde{f} = \chi f$, i.e.,

$$\tilde{f}(p) = \begin{cases} \chi(p)f(p) & p \in U \\ 0 & p \in M \setminus U \end{cases}$$

Observe that $\tilde{f} \in C^\infty(M)$, and since $\tilde{f}$ agrees with f in some open neighborhood V of p, it's an extension of $f|_V$. Define $X_p([f]) = D(\tilde{f})(p)$. We need to justify that this is well-defined – what if we changed our representation of $[f]$, or chose a different χ? Is the number $D(\tilde{f})(p)$ invariant with respect to these changes? Yes! Under the above changes, there's no effect on the germ $[\chi f] \in C_p^\infty(M)$, and we just proved that D is local.

Next, we need to show that X, as it's defined above, is smooth. Let $\phi = (x^1, \ldots, x^n)$ be any coordinate system on $U \subset M$. Then

$$X|_U = \sum_{j=1}^n X(x^j) \frac{\partial}{\partial x^j}$$

where $X(x^j)$ is a function on U. We need to check that each $X(x^j)$ is smooth. Again, we will use a bump function at $p \in U$. By definition, $X(x^j)(p) = D(\tilde{x}^j)(p)$, where $\tilde{x}^j = \chi \cdot x^j$ (extended by 0 outside of U). And by our assumption, $D(\tilde{x}^j) \in C^\infty(M)$.

We conclude that $X(x^j) \in C^\infty(M)$, so X is smooth. $\square$

Cor: If $X, Y \in \mathfrak{X}(M)$, then $[X, Y]$ (treating X and Y as operators) is itself a vector field.

Math 591 Lecture 22

Professor Alejandro Uribe-Ahumada

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Note: The lecture on 10/21 was devoted to review for the exam.

Recall: We defined, for $X, Y \in \mathfrak{X}(M)$, $[X, Y]$ by regarding X and Y as operators on $C^\infty(M)$. Then $[X, Y]$ is the commutator $X \circ Y - Y \circ X$.

Properties:
- $[\cdot, \cdot]$ is bilinear over $\mathbb{R}$.
- $[\cdot, \cdot]$ satisfies the Jacobi identity: $\forall X, Y, Z \in \mathfrak{X}(M)$, $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.
  Proof: Just compute! It's very straightforward, and only uses the fact that composition is associative.

Defn: A Lie algebra $\mathfrak{g}$ is a vector space (over $\mathbb{R}$), together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which
- Is skew-symmetric: $\forall A, B \in \mathfrak{g}$, $[A, B] = -[B, A]$.
- Satisfies the Jacobi identity.

Ex: For any manifold M, $\mathfrak{g} = \mathfrak{X}(M)$ is a Lie algebra.

Ex: A finite dimensional example: Let $\mathfrak{g} = \{n \times n \text{ skew-symmetric real matrices}\} \cong T_I O(n)$, with $[A, B] = AB - BA$ (using matrix multiplication).
Check: $[A, B] \in \mathfrak{g}$. Well, $(AB - BA)^T = B^T A^T - A^T B^T = (-B)(-A) - (-A)(-B) = BA - AB = -(AB - BA)$.
We say $o(n) \overset{\text{def}}{=} \mathfrak{g}$.

At some point (next week), we will prove the following:
Claim: Given $A \in T_I O(n)$, define $A^\sharp \in \mathfrak{X}(O(n))$ by $\forall g \in O(n)$, $A^\sharp_g = gA$, and we claim that $A^\sharp_g \in T_g O(n)$.

To interpret the formula, let $L_g : \begin{array}{c} O(n) \to O(n) \\ k \mapsto gk \end{array}$. Then $A^\sharp_g = (L_g)_{*,I}(A)$, i.e., $(L_g)_{*,I} : T_I O(n) \to T_g O(n)$.

Then $\forall A, B \in T_I O(n)$, $\underbrace{[A, B]^\sharp}_{\text{matrix commutator}} = \underbrace{[A^\sharp, B^\sharp]}_{\text{vector field commutator}}$. So $\sharp : o(n) \rightarrowtail \mathfrak{X}(O(n))$ is an injection, and $A^\sharp_I = A$.

So $o(n)$ appears as a (finite-dimensional) Lie subalgebra of $\mathfrak{X}(O(n))$.
There is more about this to come in the next chapter...

Now, back to the general study of vector fields. We've looked at vector fields as operators. Now, let's look at them as generators of dynamics.

Defn: Let $X \in \mathfrak{X}(M)$. An integral curve of X is a map

$$c : (a, b) \to M \qquad a < b; a, b \in \mathbb{R} \cup \{\pm\infty\}$$

such that $\forall t \in (a, b)$ "time", $\dot{c}(t) = X_{c(t)} \in T_{c(t)} M$.

Thm: (Existence and Uniqueness of Integral Curves)
Given $X \in \mathfrak{X}(M)$, $p \in M$, then there exists an integral curve of X, $c : (a, b) \to M$, with $a < 0 < b$, such that $c(o) = p$. Moreover, any two such curves agree on the intersection of their domains.

Proof: Reduce to the Euclidean case near p. Introduce local coordinates near p. WOLOG $\phi(p) = 0 \in \mathbb{R}^n$. We can write

$$X|_U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}$$

where the f_i are smooth functions on U. Then we have

$$\begin{tikzcd} (a,b) \arrow[r, "c"] \arrow[dr, "\tilde{c}"'] & U \arrow[d, "\phi"] \\ & \mathbb{R}^n \end{tikzcd}$$

with the unkown curve $\tilde{c}(t) = (x^1(t), \ldots, x^n(t))$, with each $x^i : (a, b) \to \mathbb{R}$ unknown. Note:

$$\left\{ \begin{array}{l} \dot{c}(t) = X_{c(t)} \\ c(0) = p \end{array} \right. \qquad \Leftrightarrow \qquad \left\{ \begin{array}{l} \forall i,\ x^i(t) = f_i(c(t)) \\ x^i(0) = 0 \end{array} \right.$$

Well, $f^i(c(t)) = \tilde{f}_i(x^1(t), \ldots, x^n(t))$, and these are real-valued functions, so $\tilde{x}^i(t) = \tilde{f}_i(x^1(t), \ldots, x^n(t))$. So we have a system of ordinary differential equations (ODEs), with an initial condition.
Well, the derivatives $\frac{dx^i(t)}{dt} = \dot{x}^i(t)$ are the left-hand side, so we can just quote the existence and uniqueness theorems from the theory of ordinary differential equations. Thus, we get the theorem in a coordinate chart. Now, consider a covering of c by overlapping coordinate charts... □

Ex: Let $M = (\mathbb{R}_{>0})^2$ – the upper-left quadrant of $\mathbb{R}^2$, with standard coordinates (x, y).
Let $X = yx^2 \frac{\partial}{\partial x}$.

Well, $\dot{x} = yx^2$ and $\dot{y} = 0$, so $\forall t$, $y(t) = y(0)$. So $\frac{dx}{dt} = y(0)x^2$. We can solve this using separation of variables:

$$\begin{aligned} \frac{dx}{x^2} &= y(0)dt \\ \int x^{-2}\, dx &= \int y(0)\, dy \\ -\frac{1}{x(t)} + \frac{1}{x(0)} &= y(0)t \\ \frac{1}{x(t)} &= \frac{1}{x(0)} - yt = \frac{1 - y(0)x(0)t}{x(0)} \\ x(t) &= \frac{x(0)}{1 - x(0)y(0)t} \end{aligned}$$

And we also have $y(t) = y(0)$. Well, x will escape to infinity at the "escape time" $T = \frac{1}{x(0)y(0)}$. So the maximal domain of an integral curve with these initial conditions is $(-\infty, T)$.

This demonstrates a problem – if we want to look at maximal integral curves, the maximal domain may depend on the initial conditions.

Math 591 Lecture 23

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Uniqueness of Integral Curves

Last time, given $X \in \mathfrak{X}(M)$, we defined integral curves of X, and proved *local* existence and uniqueness by reducing to the Euclidean case and using theory from ordinary differential equations. Today, we'll start with global uniqueness.

Lemma: Assume that $c_1, c_2 : (\alpha, \beta) \to M$ (with $\alpha < 0 < \beta$) are integral curves of X, and $c_1(0) = c_2(0) = p$. Then $\forall t \in (\alpha, \beta)$, $c_1(t) = c_2(t)$.

Proof: Assume not. Then $S \overset{\text{def}}{=} \{t \in (\alpha, \beta) \mid t > 0, c_1(t) \neq c_2(t)\} \neq \emptyset$. Well, this set is bounded below and nonempty, so let $\tau = \inf S$.

We claim that $c_1(\tau) = c_2(\tau)$ – assume not. Then because M is Hasudorff, there exist neighborhoods U_1 around $c_1(\tau)$ and U_2 around $c_2(\tau)$ such that $U_1 \cap U_2 = \emptyset$. By the continuity of c_1 and c_2, $\exists t < \tau$ such that $c_1(t) \neq c_2(t)$. Thus, τ is not the infimum of S. Oops! This is a contradiction, so we must have $c_1(\tau) = c_2(\tau)$.

Now, we use local uniqueness of integral curves of X. Let c be an integral curve with initial condition $c(\tau) = c_1(\tau) = c_2(\tau)$. By local uniqueness, c must agree with c_1 and c_2 on a neighborhood of τ, so $\tau < \inf S$. Oops!

Therefore, we must have $c_1(t) = c_2(t)$, $\forall t \in (\alpha, \beta)$. $\square$

Cor: Given $X \in \mathfrak{X}(M)$ and $p \in M$, there is an interval $(\alpha(p), \beta(p))$ containing 0 (possibly with $\alpha(p) = -\infty$ and/or $\beta(p) = +\infty$), and an integral curve $c : (\alpha(p), \beta(p)) \to M$ of X with $c(0) = p$, such that for any other integral curve $\tilde{c} : I \to M$ (with I and open interval) with $\tilde{c}(0) = p$, one has $I \subset (\alpha(p), \beta(p))$ and on I, $c|_I = \tilde{c}$.

Proof: Let $\mathcal{I}$ be the set of intervals which are domains for some integral curve c of X with $c(0) = p$. Then we have $(\alpha(p), \beta(p)) = \bigcup_{I \in \mathcal{I}} I$. Any wo $c_1 : I_1 \to M$, $c_2 : I_2 \to M$ agree on their overlap, $I_1 \cap I_2$, so they define an integral curve on their union, $I_1 \cup I_2$. Doing this for all $I \in \mathcal{I}$ gives us the desired integral curve. $\square$

Defn: Such an integral curve is the unique maximal integral curve of X through p at $t = 0$.

We can refer to a chapter on vector fields in the book by Boothby, but it's a little too detailed.

Defn: Given $X \in \mathfrak{X}(M)$, define $\mathcal{W} = \{(t, p) \in \mathbb{R} \times M \mid t \in (\alpha(p), \beta(p))\} \subseteq \mathbb{R} \times M$. This is the domain of a map $\phi : \mathcal{W} \to M$, which takes a pair (t, p) to he unique maximal integral curve of X with initial condition p at time t. In other words, $\forall (t, p) \in \mathcal{W}$, $\phi(0, p) = p$ and $\frac{\partial \phi}{\partial t}(t, p) = X_{\phi(t,p)}$. ϕ is called the flow of X.

Thm: $\mathcal{W} \subseteq \mathbb{R} \times M$ is open, and ϕ is a smooth map (of t, p).

This general theorem is rather challenging to prove, but the local version (which is included in our textbook) is sufficient for our purposes.

Thm: Let $X \in \mathfrak{X}(M)$, $p \in M$. Then there exists a neighborhood V of p, $\varepsilon > 0$, and a smooth map $\phi : (-\varepsilon, \varepsilon) \times V \to M$ such that

- $\phi(0, q) = q$, $\forall q \in V$
- $\frac{\partial \phi}{\partial t}(t, q) = X_{\phi(t,q)}$. (Note: This is the velocity of the curve $t \mapsto \phi(t, q)$ at time t. That is, $t \mapsto \phi(t, q)$ is an integral curve of X with initial condition q.)

Proof: Just quote Calc IV/diffeq. In case $M = \mathbb{R}^n$, this is a theorem. Then just use local coordinates to reduce any manifold to the Euclidean case. $\square$

Note: This isn't as fancy as the previous (global) version, but it's enough for our purposes. $(-\varepsilon, \varepsilon) \times V$ is sometimes referred to as a "flow box". $\mathcal{W}$ may be quite complicated, but the flow boxes are always easy to work with.

Main points:

1. ϕ is C^∞ in (t, p). We refer to this as "smooth dependence on initial conditions".
2. $\varepsilon > 0$ can be uniform on V.

Notation: It is standard to write $\phi(t, p) = \phi_t(p)$. This emphasizes, in the local flow theorem, that $\forall t \in (-\varepsilon, \varepsilon)$, we can think of the map $\phi_t : V \to M$. This is called the "time t map". Think about it as moving every point in V by time t along their respective integral curves. In other words, it takes the blob V to a new blob $\phi_t(V)$.

Thm: Given $X \in \mathfrak{X}(M)$, $p \in M$. If $t, s, t + s \in (\alpha(p), \beta(p))$, then

$$\phi_t(\phi_s(p)) = \phi_{t+s}(p) = \phi_{s+t}(p) = \phi_s(\phi_t(p)).$$

This is known as the "1-parameter group".

Proof: Fix s, and consider the curves $t \mapsto \phi_{t+s}(p)$ and $t \mapsto \phi_t(\phi_s(p))$. Both are integral curves of X, with the same initial conditions (check that this is true), so by uniqueness, they're the same curve. The rest follows from commutativity of addition. $\square$

So altogether, if there are no domain issues,

$$\phi_t \circ \phi_s = \phi_{t+s} = \phi_{s+t} = \phi_s \circ \phi_t.$$

For complete fields, this shows that $t \mapsto \phi_t$ is a map $\mathbb{R} \to \{\text{All diffeomorphisms } M \to M\} \stackrel{\text{def}}{=} \text{Diff}(M)$, and this map is a group morphism from $(\mathbb{R}, +) \to \text{Diff}(M)$!

Math 591 Lecture 24

Professor Alejandro Uribe-Ahumada

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Recall: If $X \in \mathfrak{X}(M)$, $p \in M$, then there exists a neighborhood V of p, $\varepsilon > 0$, and a function $\phi : (-\varepsilon, \varepsilon) \times V \to M$ such that $\forall q \in V$, $t \mapsto \phi(t, q)$ is an integral curve of X with $\phi(0, p) = p$.

Notation: $\phi(t, q) = \phi_t(q)$. So $\forall t \in (-\varepsilon, \varepsilon)$, we can think of $\phi_t : V \to M$ (a "time t map").

Notation: Fix $X \in \mathfrak{X}(M)$. Then $\forall p \in M$, $(\alpha(p), \beta(p))$ is the domain of the (unique) maximal integral curve of X through p.

$\forall t \in \mathbb{R}$, let $M_t = \{p \in M \mid t \in (\alpha(p), \beta(p))\}$. M_t is the set of points whose integral curves are defined at time t. This is a little fussy for our purposes, since most of the vector fields we care about are complete.

From last time, $\mathcal{W} = \{(p, t) \in M \times \mathbb{R} \mid p \in M_t\}$. Recall that $\mathcal{W}$ is open in $M \times \mathbb{R}$. The map $\phi : \mathcal{W} \to M$ is the global flow of X.

Observe: For our purposes, we don't need the global theory as much. We'll concentrate on:

a) Local flows $\phi : (-\varepsilon, \varepsilon) \times V \to M$ (uniform time)
b) Complete fields, i.e., those $X \in \mathfrak{X}(M)$ for which $\forall p \in M$, $(\alpha(p), \beta(p)) = \mathbb{R}$.

Recall the example from last friday.

Ex: $X = yx^2 \frac{\partial}{\partial x}$. Then $x(t) = \frac{x(0)}{1 - x(0)y(0)t}$ and $y(t) = y(0)$. This is not a complete vector field. $\phi_t(x, y) = (\frac{x}{1 - xyt}, y)$.

Ex: (From physics) Let $M = \mathbb{R}^2$ with coordinates (x, p). Let $\dot{x} = p$ and $\dot{p} = 0$. Then the integral curves of $x = p\frac{\partial}{\partial x} + 0\frac{\partial}{\partial p}$ are of the form $x(t) = tp + x(0)$, $p(t) = p(0)$. Thus,

$$\phi_t(x, p) = (x + tp, p) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

This is a linear shear. Note that the integral curves are just horizontal lines.

Prop: If $t, s, t + s \in (\alpha(p), \beta(p))$, then $\phi_{t+s}(p) = \phi_t(\phi_s(p))$.

We call this our group law. This means, where defined, $\phi_{t+s} = \phi_t \circ \phi_s$, $\forall t, s \in \mathbb{R}$.

Let's amplify this idea. Suppose the vector field is complete. From the group law, $\forall t \in \mathbb{R}$, $\phi_t \circ \phi_{-t} = I$, i.e., $\phi_t : M \to M$ is a diffeomorphism with inverse $(\phi_t)^{-1} = \phi_{-t}$.

Also, the mapping $\mathbb{R} \to \mathrm{Diff}(M)$ is a group morphism from $(\mathbb{R}, +)$ to $(\mathrm{Diff}(M), \circ)$. In this case, ϕ is called a one-parameter group of diffeomorphisms.

More generally, one can consider smooth maps $\phi : \mathbb{R} \times M \to M$, and define $\phi_t(p) = \phi(t, p)$. Then $\{\phi_t\}_{t \in \mathbb{R}}$ is a smooth one-parameter family of maps $M \to M$. The ones which satisfy the group law $\phi_{t+s} = \phi_t \circ \phi_s$ correspond precisely to vector fields. Specifically, let X_p be the velocity at $t = 0$ of the integral curve $t \mapsto \phi_t(p)$.

Defn: X is the <u>infinitesimal generator</u> of the 1-parameter subgroup ϕ_t.

Lemma: (Translation Lemma) Let ϕ be the 1-parameter group (flow) generated by $X \in \mathfrak{X}(M)$. Then $\forall s \in \mathbb{R}, p \in M$, $t \mapsto \phi_{t+s}(p)$ is the integral curve of X through $\phi_s(p)$.

Proof: Use the calc 1 chain rule and the group law. $\phi_{t+s}(p) = \phi_t(\phi_s(p))$. Now differentiate both sides with respect to t. $\square$

Thm: If M is compact, any $X \in \mathfrak{X}(M)$ is complete, i.e., all maximal integral curves of X have domain $\mathbb{R}$.

We're not quite ready to prove this yet, but we will soon.

Lemma: (Uniform Time Lemma) For any M, for any $X \in \mathfrak{X}(M)$, if $\exists \varepsilon > 0$ such that all maximal integral curves' domains contain $(-\varepsilon, \varepsilon)$, then X is complete.

Proof: Assume X is not complete. THen $\exists p \in M$ s.t. $\beta(p) < \infty$ (the argument would follow identically if instead $\alpha(p) > -\infty$). Let $t_0 \in \mathbb{R}$ s.t. $\beta(p) - \varepsilon < t_0 < \beta(p)$, and consider the curve

$$c(t) = \begin{cases} \phi_t(p) & \alpha(p) < t < t_0 \\ \phi_{t-t_0}(\phi_{t_0}(p)) & -\varepsilon < t - t_0 < \varepsilon \end{cases}$$

Then this is an integral curve of X, with $c(0) = p$, and it is defined for all t such that $t_0 - \varepsilon < t < \varepsilon + t_0$. Note that $\varepsilon + t_0 > \beta(p)$. Since it is defined for all $t \in (\alpha(p), \varepsilon + t_0)$, this is a contradiction, as $(\alpha(p), \beta(p))$ is the maximal domain of an integral curve through p. $\square$

Math 591 Lecture 25

Professor Alejandro Uribe-Ahumada

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Thm: If M is compact, then every $X \in \mathfrak{X}(M)$ is complete.

Proof: We know (by the existence of local flows) that $\forall p \in M$, there exists a neighborhood V of p, with $\varepsilon_p > 0$, such that the flow of X is defined on $(-\varepsilon_p, \varepsilon_p) \times V_p \to M$. We can extract a finite subcover of $\{V_p \mid p \in M\}$, say $\{V_{p_1}, \ldots, V_{p_k}\}$. Let $\varepsilon = \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_k}\}$. Then the flow of X is defined on $(-\varepsilon, \varepsilon) \times M$, $\forall p$. So by the uniform time lemma, X is complete. □

Question: How do vector fields relate with smooth maps? The answer is "not well".

In general, vector fields cannot be pushed forward or pulled back. Let $F : M \to N$ smooth. We can certainly push forward single tangent vectors: $F_{*,p} : T_pM \to T_{F(p)}N$. But F may not be injective or surjective. If F is not injective, we have $F(p_1) = F(p_2)$ for some $p_1, p_2 \in M$. If F is not injective, there's a $q \in N$ such that $F(p) \neq q$, $\forall p \in M$. In either case, it's not clear what the vector field should be at that point.

Observe: If F is a diffeomorphism, thne given $X \in \mathfrak{X}(M)$, we *can* define $(F_*X)_q$ by $\forall q \in N$, $(F_*X)_q = F_{*,F^{-1}(q)}(X_{F^{-1}(q)})$.

Defn: Given $F : M \to N$, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, we say X and Y are <u>F-related</u> iff $\forall p \in M$, $F_{*,p}(X_p) = Y_{F(p)}$.

Ex: Let $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x$. Take $X \in \mathfrak{X}(\mathbb{R}^2)$ to be $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$, $f, g \in C^\infty(\mathbb{R}^2)$. When is X F-related to some $Y \in \mathfrak{X}(\mathbb{R})$?

Well, $F_{*,(x,y)}(X_{(x,y)}) = f(x, y)\frac{\partial}{\partial x}$. So only when f doesn't depend on y.

Prop: Let $F : M \to N$ be smooth, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. X and Y are F-related iff (1) $\forall c$, an integral curve of X, $F \circ c$ is an integral curve of Y (if there are no domain issues) iff (2) the following diagram commutes:

$$\begin{tikzcd}
C^\infty(M) \arrow[d, "X"] & C^\infty(N) \arrow[l, "F^*"'] \arrow[d, "Y"] \\
C^\infty(M) & C^\infty(N) \arrow[l, "F^*"']
\end{tikzcd}$$

where $F^*(g) = g \circ F$, i.e., $\forall g \in C^\infty(N)$, $X(g \circ F) = Y(g) \circ F$.

Proof: (1) essentially follows directly from the definition, and uniqueness of integral curves.

For (2), recall how fields act as operators.

$$X(g \circ F)(p) = d(g \circ F)(X_p) = dg(F_{*,p}(X_p))$$

$$(Y(g) \circ F)(p) = Y(g)(F(p)) = dg(Y_{F(p)})$$

These are equal $\forall g$ iff $F_{*,p}(X_p) = Y_{F(p)}$, which is precisely the condition that X and Y are F-related. □

Prop: Given $F : M \to N$ smooth, $X_1, X_2 \in \mathfrak{X}(M)$, $Y_1, Y_2 \in \mathfrak{X}(N)$, if X_1 is F-related to Y_1 and X_2 is F-related to Y_2, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

Proof: Left as an exercise. Use condition (2) from the previous proposition.

Cor: If $X_1, X_2 \in \mathfrak{X}(G)$, G a Lie group, and X_1 and X_2 are left-invariant, then $[X_1, X_2]$ is left-invariant.

Proof: $\forall g \in G$, we know X_1 and X_2 are L_g-related to themselves. Therefore, $[X_1, X_2]$ is L_g-related to itself, i.e.,

$[X_1, X_2]$ is left-invariant. $\square$

Question: Given $X, Y \in \mathfrak{X}(M)$, we defined $[X, Y]$ regarding X and Y as operators. What is the dynamical interpretation/meaning of the commutator?

Thm: Let $p \in M$, let ϕ be the flow of X. Form the curve in T_pM:

$$(-\varepsilon, \varepsilon) \ni t \mapsto (d(\phi_t)_p)^{-1}(Y_{\phi_t(p)}) \stackrel{\text{def}}{=} v_t$$

(with $d(\phi_t)_p : T_pM \to T_{\phi_t(p)}M$). Then $\frac{d}{dt}v_t = d(\phi_t)_p^{-1}([X, Y]_{\phi_t(p)})$ (the derivative of a curve in a vector space). At $t = 0$, $\frac{d}{dt}v_t = [X, Y]_p$.

Proof: Next time.

Cor: If $[X, Y] \equiv 0$ (everywhere), then $\forall t, s$, $\phi_t \circ \phi_s = \phi_s \circ \phi_t$, where ϕ is the flow of X and ψ is the flow of Y.

$$\begin{array}{ccc} p & \xrightarrow{\phi} & \phi_t(p) \\ \downarrow \psi & & \downarrow \psi \\ \psi_s(p) & \longrightarrow & \phi_t(\psi_s(p)) \ = \ \psi_s(\phi_t(p)) \end{array}$$

Proof: The assumption $[X, Y] = 0$ implies the curves v_t from above are constant. So $\forall t$, $d(\phi_t)^{-1}(Y_{\phi_t(p)}) = Y_p$. Thus, Y is ϕ_t-related to itself, $\forall t$. So ϕ_t maps integral curves of Y to integral curves of Y. This is equivalent to the commutativity we're trying to show. $\square$

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Lie Derivatives

The general notion of a Lie derivative with respect to a vector field X is "pull back by θ_t, the time t map of the flow of X, and then differentiate with respect to t at $t = 0$". We call this $\mathcal{L}_x$.

Ex: We can pull back functions, so we can have Lie derivatives of functions.
Given X, θ_t, and $f \in C^\infty(M)$, $\phi_t^*(f)(p) = f(\phi_t(p))$ (pullback). Now differentiate:

$$\frac{d}{dt}\phi_t^*(f)(p)\Big|_{t=0} = \frac{d}{dt}f(\phi_t(p))\Big|_{t=0} = X(f)(p)$$

So $(\mathcal{L}_X f)(p) = X(f)(p) = df(X_p)$. So $\mathcal{L}_X f$ is again a function on M.

Recall: A 1-form α is an assignment to each $p \in M$, an element $\alpha_p \in T_p^*M$. I.e. α is a section of the cotangent bundle. In local coordinates, we can write $\alpha = \sum \alpha_i dx^i$, for $\alpha_i \in C^\infty(U)$.

Ex: 1-forms can be pulled back by any F, by $F^*(\alpha)(v) = \alpha(F_{*,\cdot}(v))$. So

$$(\mathcal{L}_X\alpha)(p) = \frac{d}{dt}\underbrace{\phi_t^*\alpha_{\phi_t(p)}}_{\substack{\in T_p^*M \\ \text{for each } t}}\Bigg|_{t=0}$$

Thus, $\mathcal{L}_X\alpha$ is again a 1-form on M.

Special case: $\alpha = df$ for some $f \in C^\infty(M)$. Then by the chain rule (1)

$$F^*(df)(v) = df(F_{*,\cdot}(v)) \overset{(1)}{=} d(f \circ F)(v) = d(F^*f)(v)$$

so $F^*(df) = d(F^*f)$.

So, in general,

$$\mathcal{L}_X(df) = \frac{d}{dt}\phi_t^*(df)\Big|_{t=0} = \frac{d}{dt}d\phi_t^*(f)\Big|_{t=0} \overset{(1)}{=} d\left(\frac{d}{dt}\phi_t^*f\Big|_{t=0}\right) = d(df(X)) = d(X(f))$$

Where (1) is true because $\frac{d^2}{\partial t\partial x^i} = \frac{\partial^2}{\partial^i\partial t}$, so the differentiations commute.
In fact, in general, $\mathcal{L}_X(\alpha) = d(\alpha(X)) + \cdots$ (we'll fill in the other term later).

Now, back to vector fields...

Note: Vector fields can be pulled back by diffeomorphisms. This means if $X, Y \in \mathfrak{X}(M)$, $\mathcal{L}_XY$ is defined by, $\forall p \in M$,

$$\mathcal{L}_X(Y)(p) = \underbrace{\frac{d}{dt}\left[(\phi_t)_{*,p}\right]^{-1}(Y_{\phi_t(p)})}_{=V_t \in T_pM}\Bigg|_{t=0}$$

Prop: $\mathcal{L}_X Y = [X,Y]$.

Proof: Compute V_t in coordinates $(U,(x^1,\dots,x^n))$. Restrict to $V \subseteq U$ such that $\phi_t(V) \subseteq U$. Write $\phi_t = (\phi_t^1,\dots,\phi_t^n)$. Then

$$J_t = \left(\frac{\partial \phi_t^i}{\partial x^j}(p)\right)_{(i,j)} \quad Y = \sum_{i=1}^n g_i \frac{\partial}{\partial x^i} \quad X = \sum_{i=1}^n f_i \frac{\partial}{\partial x^i}$$

For a fixed p, let

$$\tilde{Y}_t = Y_{\phi_t(p)} = \begin{pmatrix} g_1(\phi_t(p)) \\ \vdots \\ g_n(\phi_t(p)) \end{pmatrix}$$

Then, let $\tilde{V}_t$ be the column vector with components of V_t. We have

$$\tilde{V}_t = J_t^{-1}\tilde{Y}_t$$

Differentiating with $\frac{d}{dt}$ on both sides, we get

$$\dot{\tilde{V}}_t = J_t^{-1}\dot{\tilde{Y}}_t - J_t^{-1}\dot{J}_t J_t^{-1}\tilde{Y}_t$$

which at $t=0$, yields

$$\underbrace{\dot{\tilde{V}}_t\Big|_{t=0}}_{\substack{\text{components} \\ \text{of } \mathcal{L}_X Y}} = \begin{pmatrix} X(g_1)(p) \\ \vdots \\ X(g_n)(p) \end{pmatrix} - ?$$

Well,

$$\dot{J}_t = \left(\frac{\partial^2 \phi_t^i}{\partial t \partial x^j}(p)\right)_{(i,j)} = \left(\frac{\partial^2 \phi_t^i}{\partial x^j \partial t}(p)_{(i,j)}\right) = \left(\frac{\partial f_i}{\partial x^j}(p)\right)_{(i,j)}$$

since $\frac{\partial \phi_t^i}{\partial t}(p) = f_i$. Thus, we have

$$\dot{\tilde{V}}_t\Big|_{t=0} = \begin{pmatrix} X(g_1)(p) \\ \vdots \\ X(g_n)(p) \end{pmatrix} - \left(\frac{\partial f_i}{\partial x^j}(p)\right)_{(i,j)} \begin{pmatrix} g_1(p) \\ \vdots \\ g_n(p) \end{pmatrix}$$

Therefore,

$$\begin{aligned}
\mathcal{L}_X Y &= \sum_{i=1}^n X(g_i)\frac{\partial}{\partial x^i} - \sum_{i,j=1}^n \frac{\partial f_i}{\partial x^j} g_j \frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n X(g_i)\frac{\partial}{\partial x^i} - \sum_{i=1}^n Y(f_i)\frac{\partial}{\partial x^i} \\
&= \sum_{i=1}^n (X(g_i) - Y(f_i))\frac{\partial}{\partial x^i} \\
&= [X,Y]
\end{aligned}$$

□

Thm: Using the same notation as last time, $\forall t \in (-\varepsilon,\varepsilon)$, $V_t = \frac{d}{dt}(\phi_{t,*})^{-1}(Y_{\phi_t}(p)) = (\phi_{t,*})^{-1}[X,Y]_{\phi_t(p)}$.

Proof: We already have it for $t=0$. Now, consider V_{t+s}, and use the group law/translation lemma. □

Cor: If $[X,Y]=0$, then the flows of X and Y commute.

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Lie Groups and Their Algebras

Reminder/Review: Given G a Lie group, $\forall g \in G$, the map $L_g : G \to G$ where $L_g(k) = gk$. $X \in \mathfrak{X}(M)$ is left-invariant iff $\forall g \in G$, X is L_g-related to itself.

Prop: (HW 8 Problem 4) There is a bijective linear correspondence between $\mathfrak{g} = T_eG$, the Lie algebra, and the set of left-invariant fields on G, where $T_eG \ni A \mapsto A^\sharp \in \mathfrak{X}(G)$. $A^\sharp$ is defined by $\forall g \in G$, $A^\sharp_g = (L_g)_{*,e}(A)$. $A^\sharp$ is smooth.

Observe: $\forall X, Y \in \mathfrak{X}(G)$ left-invariant, $[X, Y]$ is also left-invariant, because being related by L_g preserves commutators.

Defn: Under this correspondence, we can define the <u>bracket of fields</u>

$$\begin{aligned}[\,\cdot\,,\,\cdot\,] &: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \\ (A, B) &\mapsto [A, B] \overset{\text{def}}{=} \left[A^\sharp, B^\sharp\right]_e\end{aligned}$$

Defn: $(g, [\,\cdot\,,\,\cdot\,]$ is the <u>Lie algebra</u> of G.

$[\,\cdot\,,\,\cdot\,]$ is $\mathbb{R}$-bilinear and satisfies the Jacobi identity.

The Exponential Map

Notation: $\forall A \in \mathfrak{g}$, let F^A be the flow of $A^\sharp$.

Defn: $\forall A \in \mathfrak{g}$, the <u>exponential map</u> is defined to be $\exp t\,A \overset{\text{def}}{=} F^A_t(e)$.

Prop: Given $A \in \mathfrak{g}$:

(1) $\exp t\,A$ is defined $\forall t \in \mathbb{R}$.
(2) $\exp(t + s)\,A = (\exp t\,A) \cdot (\exp s\,A)$, $\forall s, t \in \mathbb{R}$ (with $\cdot$ being group multiplication).

Proof (2): Assume $t + s$ is small. Then

$$\begin{aligned}\exp\,(t + s)\,A = F^A_{t+s}(e) = F^A_t(F^A_s(e)) \\ (\exp t\,A) \cdot (\exp s\,A) = L_{\exp t\,A}(\exp s\,A)\end{aligned}$$

So $L_{\exp t\,A}$ maps integral curves of $A^\sharp$ to integral curves of $A^\sharp$, because $A^\sharp$ is $L_{\exp t\,A}$-related to itself. Thus, the map $s \mapsto L_{\exp t\,A}(\exp s\,A)$ is the integral cuve of $A^\sharp$ through $\exp t\,A$, so it must agree with $F^A_s(\exp t\,A)$. This proves (2) for small s, t. $\square$

Proof (1): Well, we know $\exists \varepsilon > 0$ s.t. $\exp t\,A$ is defined for $t \in (-\varepsilon, \varepsilon)$. So we'll make use of the fact that $\exp(t + s)\,A = (\exp t\,A) \cdot (\exp s\,A)$. Note: the right-hand side is defined for $t + s \in (-\varepsilon, \varepsilon)$, so extend the left-hand side to $t + s \in (-2\varepsilon, 2\varepsilon)$. This is somewhat sketchy, but it works. Then, we just have to check that this extension is an integral curve of $A^\sharp$, and it must agree with $\exp(t + s)\,A$. Now, we have $\exp t\,A$ defined for $t \in (-2\varepsilon, 2\varepsilon)$. Repeat ad nauseum... $\square$

Cor: (2) $\Rightarrow \exp t\,A, \exp s\,A \in G$ commute.

Ex: $G = \mathrm{GL}(n,\mathbb{R}) \overset{\text{open}}{\subseteq} \mathbb{R}^{n^2}$. $\mathfrak{g} = \mathrm{gl}(n,\mathbb{R}) = \mathbb{R}^{n^2}$, the set of $n \times n$ real matrices. Then

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n$$

We need to check that this series converges absolutely (i.e. for some matrix norm). Well, $||AB|| \leq ||A||\,||B||$, and $\frac{d}{dt}(\exp t\, A) = A \exp t\, A = (\exp t\, A)A = A^\sharp A$.
(Claim: $\forall g \in \mathrm{GL}(n,\mathbb{R})$, $L_g(A) = A^\sharp g$. Proof: $L_g : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is linear, so its differential is itself, i.e., $(L_g)_{*,e} = L_g$.)

Defn: $\exp : \mathfrak{g} \to G$ is defined by $\exp(A) \overset{\text{def}}{=} \exp(t)\, A|_{t=1}$.

Prop: $(\exp)_{*,0} : T_0\mathfrak{g} \to \mathfrak{g}$ is the identity map $\mathfrak{g} \to \mathfrak{g}$, so exp is a local diffeomorphism at $0 \in \mathfrak{g}$.

Proof:

$$(\exp)_{*,0}(A) \overset{(1)}{=} \frac{d}{dt} \exp t\, A \bigg|_{t=0} = A^\sharp_e = A$$

where (1) holds by using the curve $t \mapsto tA$, in $\mathfrak{g}$ adapted to $(0, A)$. $\square$

Prop: $\forall A \in \mathfrak{g}$, $A^\sharp$ is complete.

Proof: $\forall g \in G$, $L_g(\exp t\, A) = g \cdot \exp t\, A$ is the integral curve of $A^\sharp$ starting at g. $\square$

Subgroups (Part 1)

Defn: A regular (or closed, or embedded) subgroup H of G is a regular submanifold that is also a subgroup. It follows directly that H is a lie group in its own right, and $\mathfrak{h} = T_eH \hookrightarrow \mathfrak{g} = T_eG$.

Prop: $\mathfrak{h}$ is closed under $[\,\cdot\,,\,\cdot\,]$ of $\mathfrak{g}$. This means, $\forall A, B \in \mathfrak{h}$, $\left[A^\sharp, B^\sharp\right]$ is tangent to H, and $\left[A^\sharp, B^\sharp\right]_e \in \mathfrak{h}$.

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Recall: For G a Lie group, $\mathfrak{g} = T_eG$ is its Lie algebra. $\forall A \in \mathfrak{g}$, $A^\sharp$ is the left-invariant vector field on G determined by A, that is, $\forall g \in G$, $A^\sharp_g \overset{\text{def}}{=} (L_g)_{*,e}(A)$. In particular, $A^\sharp_e = A$.

Defn: $\forall A \in \mathfrak{g}$, $t \in \mathbb{R}$, $\exp t\, A$ is the integral curve of $A^\sharp$ at e, at time t.

Defn: $\exp : \mathfrak{g} \to G$

$$A \mapsto \exp t\, A|_{t=1}$$

Prop: exp is smooth.

Proof: Introduce the field $X \in \mathfrak{X}(G \times \mathfrak{g})$, defined by, $\forall (g, A) \in G \times \mathfrak{g}$,

$$X_{(g,A)} = (A^\sharp_g, 0) \in T_gG \times T_A\mathfrak{g} = T_{(g,A)}(G \times \mathfrak{g}) = T_gG \times \mathfrak{g}$$

X is smooth. Its flow, denoted F, is $F_t(g, A) = (F^A_t(g), A)$, where F^A is the flow of $A^\sharp$. Thus, the time-1 map is C^∞.

$$\begin{tikzcd} g \cong \{e\} \times \mathfrak{g} \arrow[r, hook] \arrow[rrr, bend right, "\exp"'] & G \times \mathfrak{g} \arrow[r, "F_1"] & G \times \mathfrak{g} \arrow[r, "\pi_G"] & G \end{tikzcd}$$

Because $F_1(e, A) = (\underbrace{F^A_1(e)}_{\exp(A)}, A)$. We conclude that exp is smooth. $\square$

Prop: With the same notation – $A \in \mathfrak{g}$, F^A the flow of $A^\sharp$ – $\forall g \in G$, $F^A_t(g) = g \cdot \exp t\, A$ (with $\cdot$ group multiplication).

Proof: $\frac{d}{dt}(g \cdot \exp t\, A) = \frac{d}{dt} L_g(\exp t\, A) = (L_g)_{*,\exp(A)}(A^\sharp_{\exp t\, A}) = A^\sharp_{g \cdot \exp t\, A}$.
This is precisely the ODE satisfied by $t \mapsto F^A_t(g)$. $\square$

Defn: If H, G are Lie groups, then a <u>Lie group morphism</u> from H to G is a smooth map $F : H \to G$ that is also a group homomorphism.

Prop: Let $F : H \to G$ be a Lie group homomorphism. Then

a) $\forall A \in \mathfrak{h} = T_eH$, $A^\sharp \in \mathfrak{X}(H)$ and $[\underbrace{F_{*,e}(A)}_{\in T_eG = \mathfrak{g}}]^\sharp \in \mathfrak{X}(G)$. These two fields are F-related.

b) $\forall A \in \mathfrak{h}$, $F(\underbrace{\exp}_{\text{in } H} A) = \underbrace{\exp}_{\text{in } G}(F_{*,e}(A))$.

c) $F_{*,e} : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra morphism. That is, $\forall A, B \in \mathfrak{h}$, $F_{*,e}([A, B]) = [F_{*,e}(A), F_{*,e}(B)]$ (with the appropriate Lie brackets).

Proof: (b) and (c) follow directly from (a).
Claim: $\forall h \in H$, the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, "F"] \arrow[d, "L_h"'] & G \arrow[d, "L_{F(h)}"] \\ H \arrow[r, "F"'] & G \end{tikzcd}$$

Check: let $k \in H$. Then

$$(F \circ L_h)(k) = F(hk) = F(h)F(k) = L_{F(h)}(F(k))$$

because F is a group morphism. Thus, $\forall h$, $F_{*,h} \circ (L_h)_{*,e} = (L_{F(h)})_{*,F(h)} \circ F_{*,e}$ by the chain rule, and commutativity at the identity). We can apply this to any $A \in \mathfrak{h}$:

$$F_{*,h}(\underbrace{(L_h)_{*,e}(A)}_{=A^\sharp_h}) = (L_{F(h)})_{*,F(e)}(F_{*,e}(A)) = \underbrace{(L_{F(h)})_{*,e}(F_{*,e}(A))}_{=[F_{*,e}(A)]^\sharp{}_{F(h)}}$$

This *is* the fact that $A^\sharp$ and $[F_{*,e}(A)]^\sharp$ are F-related. $\square$

Now, back to subgroups.

Defn: A <u>regular</u> (or <u>embedded</u>) <u>Lie subgroup</u> of G is a regular submanifold that is also a subgroup.

Cor: If $H \subseteq G$ is a regular subgroup, then

(a) $\mathfrak{h} \subset \mathfrak{g}$ as a subspace is closed under $[\,\cdot\,,\,\cdot\,]$ of $\mathfrak{g}$.
(b) $\forall A \in \mathfrak{h}$, $\underbrace{\exp}_{\text{in } H} A = \underbrace{\exp}_{\text{in } G} A$.

Proof: Just apply the previous theorem to $F : H \hookrightarrow G$. $\square$

Observe: $\forall A, B \in \mathfrak{h}$, $[A,B]_{\mathfrak{h}} = [A,B]_{\mathfrak{g}}$, so $[\,\cdot\,,\,\cdot\,]_{\mathfrak{h}} = [\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}\big|_{\mathfrak{h}}$.

Defn: Given a Lie algebra $\mathfrak{g}$, a <u>Lie-subalgebra</u> is a subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under $[\,\cdot\,,\,\cdot\,]$.

Cor: If $H \subset G$ is a regular Lie subgroup, then $\mathfrak{h} = T_e H$ is a Lie-subalgebra of $\mathfrak{g}$.

Is the converse true? I.e., given G a Lie group, if $\mathfrak{h} \subset T_e G = \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$, is there a Lie subgroup $H \subset G$ such that $T_e H = \mathfrak{h}$?
Answer: Yes, if we allow immersed submanifolds.

Thm: If $\mathfrak{h} \subset T_e G$ is a Lie subalgebra, then there is a Lie group H and an immersion $F : H \to G$ which is a group morphism and $F_{*,e} : T_e H \xrightarrow{\cong} \mathfrak{h}$.

Ex: An irrational line on the two-torus $S^1 \times S^1 = \mathbb{T}^2$,

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{T}^2 \\ t &\mapsto (e^{it}, e^{i\alpha t}) \end{aligned}$$

for α irrational, is a Lie group morphism and an immersion. $\mathrm{im}(F_{*,e})$ is a 1-dimensional subspace of $\mathfrak{t}^2 \cong \mathbb{R}^2$. This is always a subalgebra.

We're now done with Lie groups! Starting next week, we'll move on to differential forms.
Motivation: How do we integrate on manifolds and submanifolds?
(Think, for example, about surface integrals from calc 3.)

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Motivation: Integration

How do we integrate on a manifold? Start with calc 3. Let $U \subseteq \mathbb{R}^n$ be an open set. Then we can change variables with

$$\int_U f(x)\,dx = \int_V f(x(y)) \underbrace{\det\left(\frac{\partial x}{\partial y}\right)}_{\text{if positive}} dy$$

for $x = x(y)$.

Now, for the general interpretation: let $\begin{array}{l} F : V \to U \\ y \mapsto x = x(y) \end{array}$ be a diffeomorphism. Then $\left(\frac{\partial x}{\partial y}\right)$ is the Jacobian of F. We'll interpret this as the pullback of the RHS integral by $F - F^*(dx^1, \ldots, dx^n) =?$. But what does det mean in general? It's an alternating, multilinear function, so let's work with that.

Defn: Let V be a n-dimensional vector space, and $k \in \mathbb{N}$. A <u>k-covector</u> or <u>k-form</u> on V is a multilinear map $\alpha : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$ that is alternating (i.e. if you swap two elements, the sign flips).

Observe: Let $\sigma \in S_k$ (the symmetric group). Given $v_1, \ldots, v_k$, α alternating, we have

$$\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \underbrace{(-1)^\sigma}_{=\operatorname{sgn}(\sigma)} \alpha(v_1, \ldots, v_k)$$

Observe: If $\{v_1, \ldots, v_k\}$ is linearly dependent, and α is alternating, then $\alpha(v_1, \ldots, v_k) = 0$.
Proof: Say $v_1 = \lambda_2 v_2 + \cdots + \lambda_k v_k$. Then

$$\alpha(v_1, \ldots, v_k) = \alpha\left(\sum_{i=2}^k \lambda_i v_i, v_2, \ldots, v_k\right) = \sum_{i=2}^k \lambda_i \alpha(v_i, v_2, \ldots, v_i, \ldots, v_k) = 0$$

Defn: $\bigwedge^k V^*$ is the set of alternating $\mathbb{R}$-multilinear functions on V^k. We say that k is the <u>degree</u>.

Observe: k-forms can be pulled back by linear maps.

Defn: If $F : V \to W$ is linear, and $\alpha \in \bigwedge^k W^*$, $v_1, \ldots, v_k \in V$, then

$$(F^*\alpha)(v_1, \ldots, v_k) = \alpha(F(v_1), \ldots, F(v_k))$$

and $F^*\alpha \in \bigwedge^k V^*$.

Defn: Let $(\mathcal{E}^1, \ldots, \mathcal{E}^n)$ be an ordered basis of V^*. Let $A = \{a_1 < \cdots < a_k\} \subset \{1, \ldots, n\}$. Define $\mathcal{E}^A : V \times \cdots \times V \to \mathbb{R}$ by

$$\mathcal{E}^A(v_1, \ldots, v_k) = \det\left(\mathcal{E}^{a_i}(v_j)\right)_{(i,j)}$$

Ex: Say $V = \mathbb{R}^3$ with the standard basis. Then $\mathcal{E}^{13}((x_1, x_2, x_3), (y_1, y_2, y_3)) = \det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix}$.

Prop: For a given V and k, and an ordered basis $(\mathcal{E}^1, \ldots, \mathcal{E}^n)$ of V^*, the set $\left\{\mathcal{E}^I : I \subset \{1, \ldots, n\}, \#I = k\right\}$ is a basis of $\bigwedge^k V^*$. In particular, $\dim \bigwedge^k V^* = \binom{n}{k}$.

Observe: If $k > n$, $\bigwedge^k V^* = \{0\}$. If $k = n$, $\dim \bigwedge^k V^* = 1$. If $k = 1$, $\bigwedge^1 V^* = V^*$.

As a warmup, let $k = 2$¿ Let $\{e_1, \dots, e_n\}$ be a basis of V, and $\{\mathcal{E}^1, \dots, \mathcal{E}^n\}$ the corresonding dual basis of V^*. Note that $\mathcal{E}^i(e_j) = \delta_{ij}$.
Say $\alpha \in \bigwedge^2 V^*$, $v_1 = \sum_{a=1}^n v_1^a e_a$, $v_2 = \sum_{b=1}^n v_2^b e_b$. Then

$$\begin{aligned}
\alpha(v_1, v_2) &= \alpha\left(\sum_{a=1}^n v_1^a e_a, \sum_{b=1}^n v_2^b e_b\right) \\
&= \sum_{a=1}^n v_1^a \alpha\left(e_a, \sum_{b=1}^n v_2^b e_b\right) \\
&= \sum_{a=1}^n \sum_{b=1}^n v_1^a v_2^b \alpha(e_a, e_b) \\
&= \sum_{1 \le a < b \le n} v_1^a v_2^b \alpha(e_a, e_b) + v_1^b v_2^a \alpha(e_b, e_a) \\
&= \sum_{1 \le a < b \le n} \underbrace{(v_1^a v_2^b - v_1^b v_2^a)}_{= \begin{vmatrix} v_1^a & v_2^a \\ v_1^b & v_2^b) \end{vmatrix} = \mathcal{E}^{ab}(v_1, v_2)} \alpha(e_a, e_b) \\
&= \sum_{I = \{a < b\}} \alpha(e_a, e_b) \mathcal{E}^I(v_1, v_2)
\end{aligned}$$

In general, $\alpha = \displaystyle\sum_{1 \le a < b \le n} \alpha(e_a, e_b) \mathcal{E}^{ab}$.

Now, for general $k \in \mathbb{N}$, fix $\alpha \in \bigwedge^k V^*$. For $i = 1, \dots, k$, let $v_i = \sum_{a=1}^n v_i^a e_a$. Then

$$\begin{aligned}
\alpha(v_1, \dots, v_k) &= \alpha\left(\sum_{a_1=1}^n v_1^{a_1} e_{a_1}, \dots, \sum_{a_k}^n v_k^{a_k} e_{a_k}\right) \\
&= \sum_{a_1, \dots, a_k = 1}^n \left(\prod_{j=1}^k v_j^{a_j}\right) \alpha(e_{a_1}, \dots, e_{a_k}) \\
&= \sum_{1 \le a_1 < \cdots < a_k \le n} \underbrace{\left(\sum_{\sigma \in S_k} \left(\prod_{j=1}^k v_j^{\sigma(a_i)}\right) (-1)^\sigma\right)}_{= \det\left(v_j^{a_i}\right)_{(i,j)} = \mathcal{E}^A(v_1, \dots, v_k),\, A = \{a_1, \dots, a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \\
&= \sum_{A = \{a_1 < \cdots < a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \mathcal{E}^A(v_1, \dots, v_k)
\end{aligned}$$

So in general, $\alpha = \displaystyle\sum_{A = \{a_1, \cdots < a_k\}} \alpha(e_{a_1}, \dots, e_{a_k}) \mathcal{E}^A$.

The Wedge Product

We want to take a k-form and an ℓ-form and make a $k + \ell$-form.

Defn: For $\alpha \in \bigwedge^k V^*$, $\beta \in \bigwedge^\ell V^*$, define the wedge product of α and β, $\alpha \wedge \beta \in \bigwedge^{k+\ell} V^*$, by

$$\begin{aligned}
(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) &\stackrel{\text{def}}{=} \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\
&= \sum_{\sigma \in \mathrm{Sh}(k,\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})
\end{aligned}$$

Defn: $\mathrm{Sh}(k, \ell)$ is the set of $k - \ell$ shuffles, which are permutations $\sigma \in S(k + \ell)$ such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma_{k+1} < \cdots < \sigma_{k+\ell}$.

The Skew-Symmetrizer

Given $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^\ell V^*$, we can define

$$(\alpha \otimes \beta)(v_1, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+\ell})$$

As a map, $\alpha \otimes \beta : V^{k+\ell} \to \mathbb{R}$ is $k+\ell$-multilinear, but not alternating/skew-symmetric.

Defn: The skew-symmetrizer of a multilinear map $f : V^m \to \mathbb{R}$ is defined by

$$A(f)(v_1, \ldots, v_m) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma f(v_{\sigma(1)}, \ldots, v_{\sigma(m)})$$

Lemma: $A(f)$ is skew-symmetric/alternating, and if f is already skew-symmetric, $A(f) = f$.

Proof: Let $\tau \in S_m$. Then

$$\begin{aligned} A(f)(v_{\tau(1)}, \ldots, v_{\tau(m)}) &= \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma f(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(m))}) \\ &\overset{(1)}{=} \frac{1}{m!} \sum_{\mu \in S_m} \underbrace{(-1)^{\mu\tau}}_{=(-1)^\tau(-1)^\mu} f(v_{\mu(1)}, \ldots, v_{\mu(m)}) \\ &= (-1)^\tau A(f) \end{aligned}$$

with (1) true because if $\mu = \sigma\tau$, then $\sigma = \mu\tau^{-1}$. $\square$

Thus, we have $\alpha \wedge \beta = \frac{(k+\ell)!}{k!\ell!} A(\alpha \otimes \beta)$.

Ex: $k = \ell = 1$. Then $\mathrm{Sh}(1,1) = S_2$, so

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

Ex: $k = 1$, $\ell = 2$. Then the elements of S_3 are

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\in \mathrm{Sh}(1,2)$?	sgn
1	2	3	Yes	$+$
1	3	2	No	
2	1	3	Yes	$-$
2	3	1	No	
3	1	2	Yes	$+$
3	2	1	No	

so $\mathrm{Sh}(1,2) = \{(1\ 2\ 3), (2\ 1\ 3), (3\ 1\ 2)\}$, so

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1)\beta(v_2, v_3) - \alpha(v_2)\beta(v_1, v_3) + \alpha(v_3)\beta(v_1, v_2)$$

If $\beta = \gamma \wedge \delta$, then we get

$$\begin{aligned} (\alpha \wedge (\gamma \wedge \delta))(v_1, v_2, v_3) =& \alpha(v_1)(\gamma(v_2)\delta(v_3) - \gamma(v_3)\delta(v_2)) \\ &- \alpha(v_2)(\gamma(v_1)\delta(v_3) - \gamma(v_3)\delta(v_1)) \\ &+ \alpha(v_3)(\gamma(v_1)\delta(v_2) - \gamma(v_2)\delta(v_1)) \\ =& \begin{vmatrix} \alpha(v_1) & \alpha(v_2) & \alpha(v_3) \\ \gamma(v_1) & \gamma(v_2) & \gamma(v_3) \\ \delta(v_1) & \delta(v_2) & \delta(v_3) \end{vmatrix} \end{aligned}$$

Lemma: $\mathcal{E}^I \wedge \mathcal{E}^J = \mathcal{E}^{IJ}$.

Lemma: $\mathcal{E}^I = \mathcal{E}^{I_1} \wedge \cdots \wedge \mathcal{E}^{I_k}$ ($\wedge$ is associative).

Prop: The wedge product is

- Bilinear
- Associative
- Anti-commutative: $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$, for $\alpha \in \bigwedge^k$, $\beta \in \bigwedge^\ell$
- If $F : V \to W$, $\alpha \in \bigwedge^k W^*$, $\beta \in \bigwedge^\ell W^*$, then $F^*(\alpha \wedge \beta) = (F^\alpha) \wedge (F^*\beta)$

Math 591 Lecture 30

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Tensors

Defn: Let V be a finite-dimensional vector space. A tensor on V is an element of

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \otimes \underbrace{V \otimes \cdots \otimes V}_{m}$$

(where $\otimes$ is the tensor product).

Observe: This space is isomorphic to the space of multilinear maps $\underbrace{V \times \cdots \times V}_{\ell} \times \underbrace{V^* \times \cdots \times V^*}_{m} \to \mathbb{R}$, because $(V^*)^* \cong V$.

Last time, we defined the set of *alternating* multilinear functions $\bigwedge^k V^* \hookrightarrow \underbrace{V^* \otimes \cdots \otimes V^*}_{k}$.

Reminder: A basis for $\bigwedge^k V^*$: choose $\left\{\mathcal{E}^i\right\}_{1 \le i \le n}$ an ordered basis of V^*. For each $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$, let $\mathcal{E}^I(v_1, \ldots, v_k) = \det(\mathcal{E}^{i_j}(v_\ell))_{(j,\ell)}$.

Prop: $\left\{\mathcal{E}^I : I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}, \#I = k\right\}$ is a basis of $\bigwedge^k V^*$.

In fact, $\forall \alpha \in \bigwedge^k V^*$, $\alpha = \sum_I{}' \alpha(e_{i_1}, \ldots, e_{i_k})\mathcal{E}^I$, where $\{e_j\}$ is the basis of V dual to $\left\{\mathcal{E}^j\right\}$ (i.e., $\mathcal{E}^i(e_j) = \delta_{ij}$).

Note: the notation $\sum_I{}'$ means sum over increasing $I = \{i_1 < \cdots < i_k\}$.

Also, $\dim \bigwedge^k V^* = \binom{n}{k}$.

Defn: By convention, $\bigwedge^0 V^* = \mathbb{R}$.

Defn: If $\alpha \in \bigwedge^k V^*, \beta \in \bigwedge^\ell V^*$, we define the wedge product $\alpha \wedge \beta \in \bigwedge^{k+\ell} V^*$ by

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

Note: We can define $(\alpha \otimes \beta)(v_1, \ldots, v_{k+\ell}) = \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+\ell})$, but this may not be alternating (in fact, it almost certainly isn't). But this can be skew symmetrized by forming the above sum (with appropriate normalization).

Note: An equivalent formula for the wedge product is

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in \mathrm{Sh}(k,\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

This is a much smaller sum, as it removes redundancies.

Recall: $\sigma \in \mathrm{Sh}(k, \ell)$ iff $\sigma \in S_{k+\ell}$, $\sigma(1) < \cdots < \sigma(k)$, $\sigma(k+1) < \cdots < \sigma(k+\ell)$.
Note: $\#\,\mathrm{Sh}(k, \ell) = \binom{k+\ell}{k} = \binom{k+\ell}{\ell}$.

Ex: Say $\alpha \in \bigwedge^2 V^*$, $\beta \in \bigwedge^2 V^*$. Then the elements of $\mathrm{Sh}(2,2)$ are:

1	2	3	4	sgn
1	2	3	4	$+$
1	3	2	4	$-$
1	4	2	3	$-$
2	3	1	4	$+$
2	4	1	3	$+$
3	4	1	2	$+$

So

$$\begin{aligned}(\alpha \wedge \beta)(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\beta(v_3, v_4) \\ - \alpha(v_1, v_3)\beta(v_2, v_4) \\ - \alpha(v_1, v_4)\beta(v_2, v_3) \\ + \alpha(v_2, v_3)\beta(v_1, v_4) \\ + \alpha(v_2, v_4)\beta(v_1, v_3) \\ + \alpha(v_3, v_4)\beta(v_1, v_2)\end{aligned}$$

Properties of the Wedge Product

- The wedge product is bilinear: $(\alpha_1 + \lambda\alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \lambda(\alpha_2 \wedge \beta)$.
- The wedge product is associative: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge \beta \wedge \gamma$.
- The wedge product is anticommutative: for $\alpha \in \bigwedge^k, \beta \in \bigwedge^\ell$, $\alpha \wedge \beta = (-1)^{k\ell}\beta \wedge \alpha$. This implies that even forms commute with any other form.
- If $\alpha^1, \dots, \alpha^k \in V^*$, then $(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \det\left(\alpha^i(v_j)\right)_{(i,j)}$. In particular, $\mathcal{E}^I = \mathcal{E}^{i_1} \wedge \cdots \wedge \mathcal{E}^{i_k}$.

Ex: Say $\alpha^1, \alpha^2 \in V^*$. Then $(\alpha^1 \wedge \alpha^2)(v_1, v_2) = \alpha^1(v_1)\alpha^2(v_2) - \alpha^1(v_2)\alpha^2(v_1)$.

Defn: $\left(\bigoplus_{k=0}^{n} \bigwedge^k V^*, \wedge\right)$ is the <u>exterior algebra</u> or <u>Grassmann algebra</u> of V.

Now, back to manifolds...

Defn: Let M be a smooth manifold. A <u>k-differential form</u> (or <u>k-form</u>) on M is an assignment $\forall p \in M$, $p \mapsto \alpha_p \in \bigwedge^k(T_p^*M)$.

Note: When $k = 1$, α is a 1-form as before. When $k = 0$, α is just an $\mathbb{R}$-valued function.

Local Expression

Let $(U, (x^1, \dots, x^n))$ be a coordinate chart. Then $\forall p \in U$, we get $\left\{ dx^i\big|_p \right\}_{i=1,\dots,n}$ a basis of T_p^*M, with corresponding dual basis $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}$ of T_pM. If α is a k-form, then $\forall p \in U$,

$$\alpha_p = \sum_I{}' a_I (dx^I)_p \qquad a_I(p) = \alpha_p\left(\left.\frac{\partial}{\partial x^{i_1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^{i_k}}\right|_p \right)$$

This defines functions $a_I : U \to \mathbb{R}$ for each $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}$.

Defn: A k-form α is <u>smooth</u> iff there exists a C^∞ atlas of M such that in each chart, each a_I is smooth.

Lemma: α is smooth iff for every chart in the smooth structure, each function a_I is smooth.

Proof: $\Leftarrow$ is trivial. For $\Rightarrow$, let $(y^1, \ldots, y^n) : V \to \mathbb{R}$ be an arbitrary coordinate chart. Let $p \in V$. By our assumption, there's a chart $(x^1, \ldots, x^n) : U \to \mathbb{R}$ near p such that $\alpha = \sum_I{}' a_I dx^I$, with $a_I \in C^\infty(U)$. Also, $\alpha = \sum_J{}' b_J dy^J$. We need to show $b_J \in C^\infty(U \cap V)$, but how? Well,

$$b_J = \alpha\left(\frac{\partial}{\partial y^{j_1}}, \ldots, \frac{\partial}{\partial y^{j_k}}\right) = \sum_I{}' a_I \underbrace{dx^I\left(\frac{\partial}{\partial y^{j_1}}, \ldots, \frac{\partial}{\partial y^{j_k}}\right)}$$

$$= \det\left(dx^{i_r}\left(\tfrac{\partial}{\partial y^{j_s}}\right)\right)_{(r,s)}$$

$$= \det\left(\tfrac{\partial x^{i_r}}{\partial y^{j_s}}\right)_{(r,s)}$$

The $\frac{\partial x^{i_r}}{\partial y^{j_s}}$ are smooth, so the whole determinant is smooth, so b_J is the sum of smooth functions. Thus, it's smooth. $\square$

From the bundle point-of-view, we can define $\bigwedge^k T^*M = \bigsqcup_{p\in M} \bigwedge^k (T_p^*M)$.

Prop: One can make $\bigwedge^k T^*M$ into a vector bundle, with trivializations given by the moving frames $\{dx^I\}$ associated to coordinates. Then C^∞ k-forms are C^∞ sections of this bundle.

Math 591 Lecture 31

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To do today:
- Review differential forms
- Pullbacks
- Exterior derivatives

Let M be a manifold. Last time, we defined a smooth k-form α on M as an assignment $M \ni p \mapsto \alpha_p \in \bigwedge^k T_p^*M$.

In local coordinates $(x^1, \dots, x^n)$, $\alpha = \sum_I{}' a_I dx^I$, $I = \{i_1 < \cdots < i_k\} \subseteq \{1, \dots, n\}$.

Smoothness: $\forall I$, for any coordinate chart, the a_I are C^∞.

Defn: $\Omega^k(M) \overset{\text{def}}{=} \{\text{all } C^\infty\ k\text{-forms}\}$.

Ex: On $\mathbb{R}^n$: volume form $dx^1 \wedge \cdots \wedge dx^n$.
On $\mathbb{R}^3$: $\Omega^1 = \{\alpha = f dx + g dy + h dz\}$.
$\Omega^2 = \{\alpha = f dx \wedge dy + g dy \wedge dz + h dx \wedge dz\}$.

Defn: Take $M \subseteq \mathbb{R}^3$ a surface such that ther is a smooth unit normal vector field $\vec{n}$ on M. Define a 2-form σ on M by $\forall p \in M$, $v, w \in T_pM \subset \mathbb{R}^3$, $\sigma_p(v, w) = \det(v, w, \vec{n}_p)$. So $\sigma_p(v, w)$ is the area of the parallelogram spanned by v, w, and $\vec{n}_p$. σ is called the area form of M, for the given $\vec{n}$ (orientation).

Pull-backs of Differential Forms

Defn: Let $F : N \to M$ be smooth, and $\alpha \in \Omega^k(M)$, We define the pullback of α by F, $(F^*\alpha) \in \Omega^k(N)$, by $\forall p \in N$, $v_1, \dots, v_k \in T_pN$,

$$(F^*\alpha)_p(v_1, \dots, v_k) = \alpha_{F(p)}(F_{*,p}(v_1), \dots, F_{*,p}(v_k))$$

Lemma:
1. $F^*\alpha$ is C^∞.
2. $(F \circ G)^*\alpha = G^*(F^*\alpha)$ (the chain rule).
3. $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$.

Observe: $\Omega^0(M) = C^\infty(M)$. If f is a 0-form on M, then $F^*f = f \circ F$.

Pullbacks in Coordinates

Given

$$\mathbb{R}^n \xleftarrow{(y^1,\dots,y^n)} V \subset N \xrightarrow{F} M \supset U \xrightarrow{(x^1,\dots,x^m)} \mathbb{R}^m$$

and $\alpha = \sum_I{}' a_I dx^I$ with the $a_I \in C^\infty(U)$, then

$$F^*(\alpha) = \sum_I{}'(a_I \circ F)F^*(dx^I) = \sum_I{}'(a_I \circ F)F^*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \sum_I{}'(a_I \circ F)(F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}))$$

Lemma: Let $F^i = x^i \circ F : V \to \mathbb{R}$ for each i. Then $F^*(dx^i) = dF^i$, the differential of F^i.

Proof: First, introduce some shorthand notation: $\partial_{y^j} = \frac{\partial}{\partial y^j}$. Now,

$$F^*(dx^i)(\partial_{y^j}) = dx^i(F_*\partial_{y^j}) \overset{(1)}{=} dx^i \sum_{\ell=1}^{m} \frac{\partial F^\ell}{\partial y^j}\partial_{x^\ell} = F^*(dx^i)(\partial_{y^j}) = \frac{\partial F^i}{\partial y^j}$$

with (1) because $F' = \left(\frac{\partial F^\ell}{\partial y^j}\right)_{(j,\ell)}$ is the matrix of F_* in $(\partial_{y^j}), (\partial_{x^i})$. Thus,

$$F^*(dx^i) = \sum_{j=1}^{n} \frac{\partial F^i}{\partial y^j} dy^j = dF^i$$

□

Now, back to the main computation:

$$F^*(\alpha) = \cdots = \sum_I{}' (a_I \circ F)(dF^{i_1} \wedge \cdots \wedge dF^{i_k})$$

Observe: The right hand side is a smooth form. There's a special case for $k = m = n$:

Prop: $F^*(dx^1 \wedge \cdots \wedge dx^n) = \det \underbrace{\left(\frac{\partial F^i}{\partial y^j}\right)_{(j,i)}}_{=F'=J(F)} (dy^1 \wedge \cdots \wedge dy^n)$

Proof: Well, the left hand side is

$$dF^1 \wedge \cdots \wedge dF^n = \underbrace{\left(\sum_{j_1=1}^{n} \frac{\partial F^1}{\partial y^{j_1}} dy^{j_1}\right)}_{dF^1} \wedge \cdots \wedge \underbrace{\left(\sum_{j_n=1}^{n} \frac{\partial F^n}{\partial y^{j_n}} dy^{j_n}\right)}_{dF^n} = \sum_{j_1,\ldots,j_n=1}^{n} \left(\prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{j_i}}\right)(dy^{j_1} \wedge \cdots \wedge dy^{j_n})$$

Observe that the terms of the sum with $j_a = j_b$ with $a \neq b$ vanish, so the sum is really over all orderings of $\{1, \ldots, n\}$.

$$dF^1 \wedge \cdots \wedge dF^n = \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{\sigma(i)}}\right) \underbrace{\left(dy^{\sigma(1)}, \ldots, dy^{\sigma(n)}\right)}_{=\underbrace{(-1)^\sigma}_{\mathrm{sgn}(\sigma)} dy^1 \wedge \cdots \wedge dy^n} = \underbrace{\left(\sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} \frac{\partial F^i}{\partial y^{\sigma(i)}}\right)}_{\det F'} (dy^1 \wedge \cdots \wedge dy^n)$$

□

Cor: If $\alpha = f dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, $f \in C_0^\infty(U)$ (i.e. f has compact support), and we define

$$\int \alpha = \underbrace{\int f dx^1 \wedge \cdots \wedge dx^n}_{\text{Riemann Integral}}$$

Then $\int F^*\alpha = \int \alpha$ by the change of variables formula, provided that $\det(F') > 0$.

Proof: $F^*\alpha = (f \circ F)\det(F')dy^1 \wedge \cdots \wedge dy^n$. □

The Exterior Differential

Thm: Let M be a manifold. Then there exists a unique operator d

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k$$

s.t.

(1) d is $\mathbb{R}$-linear.
(2) $d : \Omega^0 \to \Omega^1$ is the usual differential of a function.
(3) $d^2 = 0$.
(4) The anti-derivation property: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. (This is also known as the super-symmetric version of Leibniz' rule.)
(5) If $F : M \to N$ is smooth, then $\forall \alpha \in \Omega^k N$, $d(F^* \alpha) = F^*(d\alpha)$. (This property is called "Naturality".)

Proof: Next time...

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The Exterior Differential

Thm: Let M be a manifold. Then there exists a unique operator d

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{k-1} \xrightarrow{d} \Omega^k$$

s.t.

(1) d is $\mathbb{R}$-linear.
(2) $d : \Omega^0 \to \Omega^1$ is the usual differential of a function.
(3) $d^2 = 0$.
(4) The anti-derivation property: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. (This is also known as the super-symmetric version of Leibniz' rule.)
(5) If $F : M \to N$ is smooth, then $\forall \alpha \in \Omega^k N$, $d(F^* \alpha) = F^*(d\alpha)$. (This property is called "Naturality".)

Note: In theory, the d is a different operator for each Ω^i. But it would be inconvenient to call them different names.

Proof: We'll begin by proving this for $U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, and then use coordinates. This works because the anti-derivation property implies locality.

For $\mathbb{R}^n$: If $\alpha = \displaystyle\sum_I a_I dx^I \in \Omega^k(\mathbb{R}^n)$, then the properties imply that we must have $d\alpha = \displaystyle\sum_I \sum_{j=1}^n \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$.
Use this as our definition of d on $\mathbb{R}^n$. Now, we must check the 5 properties. 1 and 4 follow directly from its definition. By linearlity, we can check this solely on "monomials".

Anti-derivation: Let $\alpha = a\, dx^I$, $\beta = b\, dx^J$, $a, b \in C^\infty$. Then

$$\begin{aligned}
d(\alpha \wedge \beta) &= d(a\, dx^I \wedge b\, dx^J) \\
&= d(ab\, dx^I \wedge dx^J) \\
&= d(ab\, dx^{IJ}) \\
&= \sum_{j=1}^n \frac{\partial(ab)}{\partial x^j}\, dx^j \wedge dx^{IJ} \\
&= \sum_{j=1}^n \left[b \frac{\partial a}{\partial x^j} + a \frac{\partial b}{\partial x^j} \right] dx^j \wedge dx^{IJ} \\
&= \sum_{j=1}^n \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I \wedge (b\, dx^J) + a \sum_{j=1}^n \left(\frac{\partial b}{\partial x^j}\, dx^j \wedge dx^I \right) \wedge dx^J \\
&= \sum_{j=1}^n \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I \wedge (b\, dx^J) + (-1)^k a\, dx^I \sum_{j=1}^n \frac{\partial b}{\partial x^j}\, dx^j \wedge dx^I \\
&= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta
\end{aligned}$$

$\mathbf{d^2 = 0}$:

$$\begin{aligned}
d\left[d(a\,dx^I)\right] &= d\left[\sum_{j=1}^{n} \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I\right] \\
&\overset{(1)}{=} \sum_{j=1}^{n} d\left(\frac{\partial a}{\partial x^j}\right) \wedge dx^j \wedge dx^I \\
&= \underbrace{\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n} \frac{\partial^2 a}{\partial x^\ell \partial x^j}\, dx^\ell \wedge dx^j\right)}_{\text{Observe: Terms with } j=\ell \text{ vanish.}} \wedge dx^I \\
&= \left(\sum_{1\le j<\ell\le n} \frac{\partial^2 a}{\partial x^\ell \partial x^j}\, dx^\ell \wedge dx^j + \frac{\partial^2 a}{\partial x^j \partial x^\ell}\, dx^j \wedge dx^\ell\right) \wedge dx^I \\
&= \left(\sum_{1\le j<\ell\le n} \left(\frac{\partial^2 a}{\partial x^\ell \partial x^j} - \frac{\partial^2}{\partial x^j \partial x^\ell}\right) dx^\ell \wedge dx^j\right) \wedge dx^I \\
&\overset{(2)}{=} 0
\end{aligned}$$

with (1) because $dx^j \wedge dx^I$ is constant, so $d(dx^j \wedge dx^I) = 0$, and (2) by Clairaut's Theorem, which tells us that $\frac{\partial^2 a}{\partial x^\ell \partial x^j} = \frac{\partial^2 a}{\partial x^j \partial x^\ell}$, so their difference is 0.

Naturality: Say $F : V \to U$ is C^∞, $V \subseteq \mathbb{R}^m$, $U \subseteq \mathbb{R}^n$. Let $\alpha \in \Omega^k(U)$ of the form $\alpha = a\,dx^I$. We need to prove that $d(F^*\alpha) = F^*(d\alpha)$. Say $F = (F^1, \dots, F^n)$ with $F^j : V \to \mathbb{R}$. We proved that

$$F^*(dx^{i_1}) \wedge \cdots \wedge F^*(dx^{i_k}) = F^*(dx^I) = dF^{i_1} \wedge \cdots \wedge dF^{i_k} \overset{\text{def}}{=} dF^I$$

So $F^*\alpha = (\alpha \circ F)F^*(dx^I) = (\alpha \circ F)dF^I$, and thus,

$$d(F^*\alpha) = d(\alpha \circ F) \wedge dF^I + (\alpha \circ F)\underbrace{d(dF^I)}_{=0} = d(\alpha \circ F) \wedge dF^I$$

$$F^*(d\alpha) = F^*\left[\sum_{j=1}^{n} \frac{\partial a}{\partial x^j}\, dx^j \wedge dx^I\right] = \sum_{j=1}^{n} F^*\left(\frac{\partial a}{\partial x^j}\right) dx^j \wedge dx^I = \sum_{j=1}^{n} \left(\frac{\partial a}{\partial x^j} \circ F\right) \underbrace{(F^*(dx^j))}_{dF^j} \wedge dF^I = d(\alpha \circ F) \wedge dF^I$$

So we conclude that $d(F^*\alpha) = F^*(d\alpha)$.

Now, we use coordinates to do it on a manifold.

Defn: Given $\alpha \in \Omega^k(M)$, $\phi : U \to \mathbb{R}^n$ a coordinate system, define $d\alpha \in \Omega^{k+1}(U)$ by $\phi^*\left[d(\phi^{-1})^*\alpha\right]$, where d is the Euclidean version, and $(\phi^{-1})^*\alpha \in \Omega^k(\phi(U))$.

Now, we need to prove that $d\alpha$ on U is independent of choice of coordinates. Suppose $\psi : V \to \mathbb{R}^n$ is another coordinate chart (WOLOG $U \cap V \neq \emptyset$). Call U the intersection of their domains. Write $\psi = F \circ \phi$

$$\begin{tikzcd}
 & U \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
\phi(U) \arrow[rr, "F"'] & & \psi(U)
\end{tikzcd}$$

Then

$$\begin{aligned}
\psi^*\left[d(\psi^{-1})^*\alpha\right] &= \phi^* \circ F^*\left[d(\phi^{-1} \circ F^{-1})^*\alpha\right] \\
&= \phi^* \circ F^*\left[d((F^{-1})^* \circ (\phi^{-1})^*)\alpha\right] \\
&= \phi^* \circ F^* \circ (F^{-1})^*\left[d(\phi^{-1})^*\alpha\right] \\
&= \phi^*\left[d(\phi^{-1})^*\alpha\right]
\end{aligned}$$

As for showing uniqueness on manifolds, use that anti-derivation implies locality, as with vector fields. $\square$

Ex: $\mathbb{R}^3$

$$\begin{aligned} d(f\,dx + g\,dy + h\,dz) &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= (\partial_x f\,dx + \partial_y f\,dy + \partial_z f\,dz) + \cdots \\ &= \partial_y f\,dy \wedge dx + \partial_z f\,dz \wedge dx + \cdots \\ &= (\partial_x g - \partial_y f)\,dx \wedge dy + (\partial_x h - \partial_z f)\,dx \wedge dz + \cdots \end{aligned}$$

We get the components of the curl of the vector field $< f, g, h >$.

Defn: $\alpha \in \Omega^k(M)$ is closed if $d\alpha = 0$, and exact if $\exists \beta \in \Omega^{k-1}(M)$ such that $\alpha = d\beta$.

Note: $d^2 = 0$ means exact implies closed.

Defn: The kth DeRahm Cohomology $H^k(M) = \ker(d : \Omega^k \to \Omega^{k+1})/\mathrm{im}(d : \Omega^{k-1} \to \Omega^k)$ (with the vector space quotient).

Ex: If $M = S^1$, $d\theta$ dual to $\frac{\partial}{\partial \theta}$ is closed. (Any top-degree form is closed.)
Claim: $d\theta$ is not exact, and in fact, $H^1(S^1) = \mathbb{R}$.

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Global formulae for d

First, we want to consider the pairing of forms and vector fields. If $\alpha \in \Omega^k$ and $X_1, \ldots, X_k \in \mathfrak{X}(M)$, then $\alpha(X_1, \ldots, X_k) : M \to \mathbb{R}$ is a function on M.

Prop: (1) If $\alpha \in \Omega^1(M)$, $Y_0, Y_1 \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, Y_1) = Y_0(\alpha(Y_1)) - Y_1(\alpha(Y_0)) - \alpha([Y_0, Y_1])$$

Proof: WOLOG assume $\alpha = f\,dg$. Then $d\alpha = df \wedge dg$. Thus,

$$(d\alpha)(Y_0, Y_1) = df(Y_0)dg(Y_1) - df(Y_1)dg(Y_0) = Y_0(f)Y_1(g) - Y_1(f)Y_0(g)$$

Well, the RHS of this is

$$Y_0(fY_1(g)) - Y_1(fY_0(g)) - f\left[Y_0, Y_1\right]g = (Y_0 f)(Y_1 g) + \cancel{fY_0Y_1g} - (Y_1 f Y_0 g + \cancel{fY_1Y_0g}) - f(\cancel{Y_0Y_1g} - \cancel{Y_1Y_0g})$$

$\square$

Prop: (2) If $\alpha \in \Omega^2(M)$, $Y_0, Y_1, Y_2 \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, Y_1, Y_2) = Y_0\alpha(Y_1, Y_2) - Y_1\alpha(Y_0, Y_2) + Y_2\alpha(Y_0, Y_1) - \alpha(\left[Y_0, Y_1\right], Y_2) + \alpha(\left[Y_0, Y_2\right], Y_1) - \alpha(\left[Y_1, Y_2\right], Y_0)$$

In general, if $\alpha \in \Omega^k(M)$, $Y_0, \ldots, Y_k \in \mathfrak{X}(M)$, then

$$(d\alpha)(Y_0, \ldots, Y_k) = \sum_{j=0}^{k}(-1)^j Y_j \alpha(Y_0, \ldots, \widehat{Y_j}, \ldots, Y_k) + \sum_{0 \le i < j \le k}(-1)^{i+j}\alpha(\left[Y_i, Y_j\right], Y_0, \ldots, \widehat{Y_i}, \ldots, \widehat{Y_j}, \ldots, Y_k)$$

where $\widehat{\cdot}$ indicates that we omit that term.

Refer to Proposition 20.14 in Tu/Proposition 14.32 in Lee.

Another way to think of this is, gven $\alpha \in \Omega^k(M)$, $X_1, \ldots, X_k \in \mathfrak{X}(M)$, and $p \in M$, we have $(X_i)_p \in T_pM$, so

$$\alpha_p : \underbrace{T_pM \times \cdots \times T_pM}_{k} = (T_pM)^k \to \mathbb{R}$$
$$p \mapsto \alpha_p((X_1)_p, \ldots, (X_k)_p) \in \mathbb{R}$$

Lie Derivatives of Forms

Defn: Let $X \in \mathfrak{X}(M)$, $\alpha \in \Omega^k(M)$, and $p \in M$. Let φ be the local flow of X. Form the curve $(-\varepsilon, \varepsilon) \ni t \mapsto (\varphi_t^*\alpha)_p \in \bigwedge^k(T_p^*M)$. (Note that $\bigwedge^k(T_p^*M)$ is independent of t.) Then we defined

$$(\mathcal{L}_X\alpha)_p = \frac{d}{dt}(\varphi_t^*\alpha)_p\bigg|_{t=0}$$

Observe: If $k = 0$, then α is a function on M, so

$$(\mathcal{L}_X\alpha)_p = \frac{d}{dt}(\alpha \circ \varphi_t)_p\bigg|_{t=0} = \frac{d}{dt}(\alpha(\varphi_t(p)))\bigg|_{t=0} = \dot{\varphi}_t\bigg|_{t=0}(\alpha)$$

Thus, $(\mathcal{L}_X\alpha)_p = X(\alpha)(p)$ if α is a function.

First Properties

Prop:

a) The expression $\mathcal{L}_X(\alpha)$ is $\mathbb{R}$-linear w.r.t X and α.
b) $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$ – $\mathcal{L}_X$ is a derivation.
c) $\mathcal{L}_X(d\alpha) = d\mathcal{L}_X(\alpha)$.
d) If $F : M \to M$ is a diffeomorphism, then $\mathcal{L}_X(F^*\alpha) = F^*\left(\mathcal{L}_{F_*(X)}\alpha\right)$.

Proof: (b) holds because $\varphi_t^*(\alpha \wedge \beta) = (\varphi_t^*\alpha) \wedge (\varphi_t^*\beta)$. We just need to show the right hand side satisfies he usual product rule with respect to $\frac{d}{dt}$.
(c) holds because $\varphi_t^*(d\alpha) = d(\varphi_t^*\alpha)$, $\forall t$. This uses the fact that d and $\frac{d}{dt}$ commute, i.e., $\frac{\partial^2}{\partial t \partial x^j} = \frac{\partial^2}{\partial x^j \partial t}$.
(d) holds because for $p \in M$, $v_1, \ldots, v_k \in T_pM$,

$$\mathcal{L}_X(F^*\alpha)_p(v_1, \ldots, v_k) = \frac{d}{dt}\underbrace{\varphi_t^*(F^*\alpha)_p}_{(\varphi_t^* \circ F^*)(\alpha) = (F \circ \varphi_t)^*(\alpha)}(v_1, \ldots, v_k)\Big|_{t=0} = \frac{d}{dt}\alpha((F \circ \varphi_t)_{*,p})(v_1), \ldots, (F \circ \varphi_t)_{*,p}(v_k)\Big|_{t=0}$$

Observe: $t \mapsto F \circ \varphi_t$ is the integral curve of $F_*(X)$ at $F(p)$. So

$$\mathcal{L}_X(F^*\alpha)_p(v_1, \ldots, v_k) = \cdots = (\mathcal{L}_{F_*X}\alpha)(F_{*,p}(v_1), \ldots, F_{*,p}(v_k)) = F^*(\mathcal{L}_{F_*(X)}\alpha)(v_1, \ldots, v_k)$$

□

Cor: If $\mathcal{L}_X\alpha = 0$ (everywhere), then $\varphi_t^*\alpha = \alpha$, $\forall t$.

Proof: We will show $\forall p \in M$, the curve $t \mapsto (\varphi_t^*\alpha)_p \in \bigwedge^k T_p^*M$ is constant.

Proof: $\forall s$,

$$\frac{d}{dt}\varphi_t^*\alpha\Big|_{t=s} = \frac{d}{dt}(\varphi_{t+s}^*\alpha)\Big|_{t=0} = \frac{d}{dt}\varphi_t^*(\varphi_s^*(\alpha))\Big|_{t=0} = \mathcal{L}_X(\varphi_s^*\alpha) = \varphi_s^*\mathcal{L}_{\underbrace{(\varphi_s)_*(X)}_{=X}}\alpha = \varphi_s^*\underbrace{\mathcal{L}_X\alpha}_{=0} = 0$$

□

Towards Cartan's Formula

We need interior multiplication/contraction.

Defn: Given $\alpha \in \Omega^k$, $X \in \mathfrak{X}(M)$, we define $\iota_X\alpha \in \Omega^{k-1}$ by, $\forall p \in M$ and $v_1, \ldots, v_{k-1} \in T_pM$,

$$(\iota_X\alpha)_p(v_1, \ldots, v_{k-1}) = \alpha_p(X_p, v_1, \ldots, v_k)$$

Thm: (Cartan's Magic Formula) $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d : \Omega^k \to \Omega^k$.

Picture:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1} \arrow[r, "d"] & \Omega^k \arrow[r, "d"] \arrow[dl, "\iota_X"'] \arrow[d, Rightarrow, "\mathcal{L}_X"] & \Omega^{k+1} \arrow[r, "d"] \arrow[dl, "\iota_X"] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1} \arrow[r, "d"] & \Omega^k \arrow[r, "d"] & \Omega^{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

where $\mathcal{L}_X$ is the "sum of two paths".

On Friday, we'll show that when $\alpha = dx \wedge dy$ and $M = \mathbb{R}^2$, then $\mathcal{L}_X(\alpha) = (\operatorname{div} X)\alpha$, in the calc III sense.

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Lie Derivatives of Forms

Ex: Let $X = \langle F^1, \dots, F^n \rangle \in \mathfrak{X}(\mathbb{R}^n)$, $\mu = dx^1 \wedge \cdots \wedge dx^n$, and ϕ the flow of X. Then

$$\mathcal{L}_X(\mu) = \left. \frac{d}{dt}\phi_t^*\mu \right|_{t=0}$$

Well, $\phi_t^*\mu = \det(J(\phi_t)_*)\mu$, so

$$\det(J(\phi_t)_*)|_{t=0} = \operatorname{tr}\left(\underbrace{\left.\frac{d}{dt}J(\phi_t)_*\right|_{t=0}}_{\star} \right) \underbrace{\det(\phi_{t=0})}_{=1}$$

For $\star$, do $\frac{d}{dt}$ first, and then $\frac{\partial}{\partial x^i}$. And $\det(\phi_{t=0}) = 1$, because ϕ_t is the flow of X, so $\frac{d}{dt}$ is just X. Thus, we have

$$\operatorname{tr}\begin{pmatrix} \text{—} & \nabla F_1 & \text{—} \\ & \vdots & \\ \text{—} & \nabla F_n & \text{—} \end{pmatrix} = \sum_{j=1}^n \frac{\partial F_j}{\partial x^j} = \operatorname{div}(X)$$

We conclude that $\mathcal{L}_X\mu = (\operatorname{div} X)\mu$. $\square$

Thm: (Cartan's Magic Formula) $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X : \Omega^k \to \Omega^k$.

Proof: Let $P_X = \iota_X \circ d - d \circ \iota_X$. Then P_X has the following properties:
1) $\mathbb{R}$-linearity
2) It's a derivation w.r.t. $\wedge$: $P_X(\alpha \wedge \beta) = P_X(\alpha) \wedge \beta + \alpha \wedge P_X(\beta)$
3) It commutes with d
4) $f \in C^\infty(M) \Rightarrow P_X(f) = X(f)$
5) P_X is local

These properties belong to a unique operator. (Compute in coordinates, and by linearity, just use monomials.)

$$P_X(a\, dx^I) = P_X(a) \wedge dx^I + a \wedge P_X(dx^I) = X(a)\, dx^I + aP_X(dx^I)$$

We then expand $P_X(dx^I)$ with induction. $\square$

Ex: For $k = 2$:

$$P_X(dx^1 \wedge dx^2) = P_X(dx^1) \wedge dx^2 + dx^1 \wedge P(dx^2) = dX(x^1) \wedge dx^2 + dx^1 \wedge dX(x^2)$$

This is unique, and it just uses the five properties.

Applications of Cartan's Formula

Defn: A <u>symplectic manifold</u> is a pair (M, ω), with M a manifold, and $\omega \in \Omega^2(M)$ such that
1) $\forall p \in M, v \in T_pM \setminus \{0\}$, $\iota_v(\omega_p) = \omega_p(v, \cdot) : T_pM \to \mathbb{R}$ is nonzero.
2) $d\omega = 0$.

Question: What are the symmetries of a symplectic manifold (M, ω)? Specifically, are there one-parameter groups $\varphi_t : M \to M$ such that $\forall t$, $\varphi_t^*\omega = \omega$?
Use the Lie derivative: If $X \in \mathfrak{X}(M)$ is the generator of φ_t, then

$$\varphi_t^*\omega = \omega \Leftrightarrow \mathcal{L}_X\omega = 0 \Leftrightarrow \iota_X \underbrace{d\omega}_{=0} + d\iota_X\omega = 0 \Leftrightarrow d\iota_X\omega = 0$$

Defn: One particular class of such X's comes from the following: Take any function $H \in C^\infty(M)$, and define X by $\iota_X\omega = -dH$. By non-degeneracy of ω, X is unique! And,

$$\begin{aligned} T_pM &\to T_p^*M \\ v &\mapsto \iota_v\omega \end{aligned}$$

has no kernel, so it's a bijection. This X is called the <u>Hamilton field</u> of H.

Exer: Take $M = \mathbb{R}^{2n}$ with coordinates (x,p), where x and p are the standard coordinates in $\mathbb{R}^n$. Let

$$\omega = \sum_{i=1}^n dp^i \wedge dx^i \qquad H = \frac{1}{2}\,||p||^2 + V(x)$$

Compute the Hamilton field, and show that the integral curves satisfy $\ddot{x}(t) = -\nabla V(x(t))$. This is better known as Newton's second law!

We're now done with Lie derivatives.

Integration of Forms on Oriented Manifolds

First, we have to define orientation. Let V be an n-dimensional vector space. Let $\mathscr{B}(V)$ be the set of all ordered bases of V. For $e \in \mathscr{B}$, $e = (e_1, \dots, e_n)$ is an ordered basis of V. Observe: $\forall e, f \in \mathscr{B}$, $\exists! M \in \mathrm{GL}(n,\mathbb{R})$ such that $\forall i$, $e_i = Mf_i$.

Defn: $e \sim f \Leftrightarrow \det M > 0$. We say e and f define the same orientation of V.

Check:
1) $\sim$ is an equivalence relation
2) $\mathscr{B}/\sim$ has two elements

Defn: An <u>orientation</u> of V is a choice of an equivalence class in $\mathscr{B}/\sim$. Bases in that equivalence class are said to be <u>positive</u>.

Alternatively, consider the set of nonzero top-degree forms, $\underbrace{\left(\bigwedge^n V\right)}_{\dim=1} \setminus \{0\}$. When we take away 0, any $\mu \in (\bigwedge^n V) \setminus \{0\}$ defines an orientation by: a basis e is positive iff $\mu(e) > 0$. The orientation defined by μ only depends on which connected component of $(\bigwedge^n V) \setminus \{0\}$ contains μ. Conversely, an orientation singles out one of the two components of $(\bigwedge^n V) \setminus \{0\}$.

Conclusion: An orientation is a choice of a connected component of $(\bigwedge^n V) \setminus \{0\}$.

Now, we move on to manifolds! Note: Not all manifolds are orientable (e.g. the Mobius band).

Defn: An <u>orientation</u> on M (if it exists) is a continuous choice of orientation of each tangent space.

Continuity means $\forall p \in M$, there exists a continuous moving frame $(E_1, \dots, E_n)$ such that at every point q in the domain of E, $(E_1(q), \dots, E_n(q))$ is a positive basis of T_qM.

Lemma: A connected manifold can have either two orientations, or it's non-orientable.

Proof: The idea is if the manifold is orientable, then consider orientations $\mathscr{O}_1$ and $\mathscr{O}_2$. Define $F : M \to \mathbb{R}$ by $f(p) = 1$ if $\mathscr{O}_1(p) = \mathscr{O}_2(p)$, and 0 otherwise. (Note: we don't really ever use this notation.) Then by the continuity of $\mathscr{O}_1$ and $\mathscr{O}_2$, f is continuous, so f is locally constant, so if M is connected, then f is constant. $\square$

Defn: Let M be an oriented manifold. Then a <u>positive atlas</u> on M is an atlas $\{(U_\alpha, \phi_\alpha)\}$ of M such that $\forall \alpha$, the moving frame $\left(\frac{\partial}{\partial x_\alpha^1}, \dots, \frac{\partial}{\partial x_\alpha^n}\right)$ is a positive frame.

Lemma:

1) The transition functions F and G of any two elements in a positive atlas satisfy $\det(J(F)) = 1 = \det(J(G))$.
2) An oriented manifold always has a positive atlas.

Proof: This is very tedious! Idea:
1) Recall that $J(F)$ is actually the change of basis matrix between $\left(\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right)$. 2) Start with any atlas. $\forall \alpha$, $\left\{\frac{\partial}{\partial x_\alpha^1}, \ldots, \frac{\partial}{\partial x_\alpha^n}\right\}$ is either positive, or not. If it is positive, do nothing, and keep ϕ_α. If it's not positive, relable (switch) x^1 and x^2. Now it's positive!
$\square$

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Final remarks on orientation:

Recall: If M is oriented, $\exists\{(U_\alpha, \varphi_\alpha)\}$, a positive atlas of M. This means all transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ have the determinant of their Jacobian positive at every point, and the coordinate frames are positive.

Conversely, if M is an atlas satisfying he above property, then one can define an orientation of M by requiring the coordinate frames are positive.

In general, to show a manifold is orientable, exhibit such an atlas.

Ex: Check: The atlas of $\mathbb{RP}^n$ used in homework has this condition, so it is orientable.

Observe: If $S \subset M$ is a codim-1 submanifold, and M is oriented, and there exists a continuous field of normal vectors on S,

$$S \ni p \mapsto \vec{n}_p \in T_pM \quad \text{s.t.} \quad T_pM = T_pS \oplus \mathbb{R}\vec{n}_p$$

then S is orientable, and the convention for its orientation is: a basis $\{b_1, \ldots, bn-1\}$ of T_pS is positive iff $\{\vec{n}_p, b_1, \ldots, b_{n-1}\}$ is positive w.r.t. M.
TL;DR, put the normal vector first.

Ex: We can embed the Klein bottle in a dim-3 manifold M s.t. there exists a continuous $\vec{n}$, but M is non-orientable.

Partitions of Unity

This is a very technical, but very useful tool. We begin with point-set topology.

Defn: An indexed covering (not necessarily open) $\{S_\alpha\}_{\alpha\in A}$ of (a manifold) M (doesn't have to be a manifold) (with $S_\alpha \subset M$) is said to be locally finite iff every $p \in M$ has a neighborhood U s.t. $\{\alpha \in A \mid S_\alpha \cap U \neq \emptyset\}$ is finite. That is, every p is in only finitely-many S_α.

Thm: (Thm 1.15 in Lee) Any topological manifold is paracompact: every open cover $\{U_\alpha\}_{\alpha\in A}$ has a countable, locally-finite refinement $\{V_i\}_{i\in\mathbb{N}}$. That is, $\forall i \in \mathbb{N}$, V_i is open, and $\exists\alpha \in A$ s.t. $V_i \subset U_\alpha$ and M is covered by $\{V_i\}_{i\in\mathbb{N}}$.

Proof: This proof is long and complex, but it only uses point-set topology. This is the first time we're using the fact that M is second-countable!

Observe: If $\mathscr{B}$ is any basis of M, the V_i can be chosen to be in $\mathscr{B}$.

Defn: Let M be a smooth manifold. A partition of unity on M is an indexed family $\{\chi_\alpha\}_{\alpha\in A}$ of C^∞ functions on M s.t.

(1) $\{\text{supp}(\chi_\alpha)\}_{\alpha\in A}$ is a locally finite cover of M.
(2) $\forall p \in M$, $\sum_{\alpha\in A} \chi_\alpha(p) = 1 \in \mathbb{R}$. (Note that this is a finite sum by (1).)

Thm: Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then $\exists\{\chi_\alpha\}_{\alpha\in A}$, a partition of unity, that is subordinate to $\{U_\alpha\}_{\alpha\in A}$, i.e., $\forall\alpha \in A$, $\text{supp}\,\chi_\alpha \subseteq U_\alpha$.

Proof: We'll use paracompactness. (It may be easier to start by just thinking of a compact manifold.)
Let $\mathscr{B}$ be the set of normal coordinate balls in M; we define $B \subset M$ to be a normal coordinate ball in M iff there's a chart (U, ϕ) s.t. $\overline{B} \subset U$ and $\phi(B) = B_r(0) \subset \mathbb{R}^n$, the ball of radius r centered at 0 in $\mathbb{R}^n$, and also $\exists r' > r$ s.t.

$\overline{B_r(0)} \subset B_{r'}(0) \subset \phi(U)$.

We claim that $\mathscr{B}$ is a basis of the topology of M. Let $\{U_\alpha\}_{\alpha \in A}$ be any open cover. Use the theorem on paracompactness: $\exists \{B_i\}_{i\in\mathbb{N}}$, a locally-finite refinement, and $\forall i \in \mathbb{N}$, B_i is a normal coordinate ball. $\forall i \in \mathbb{N}$, let

$$\phi_i(B_i) = B_{r_i}(0) \subset \overline{B_{r_i}(0)} \subset B_{r_i'}(0)$$

and H_i be a function on $\operatorname{Im}(\phi_i)$ such that $H_i : \operatorname{Im}(\phi_i) \to \mathbb{R}$ is smooth, with

- $H_i > 0$ on $B_{r_i}(0)$
- $H_i = 0$ on $B_{r_i}(0)^\complement$

Thus, $\operatorname{supp} H_i = \overline{B_{r_i}(0)}$.

Define $\psi_i \in C^\infty(M)$ by $\psi_i = H_i \circ \phi_i$ on $\operatorname{dom} \phi_i$, and 0 everywhere else. Then $\operatorname{supp} \psi_i = \overline{B_i} \subset M$. We claim that $\left\{\overline{B_i}\right\}_{i\in\mathbb{N}}$. Observe that $\forall p$, $\sum_{i\in\mathbb{N}} \psi_i(p) > 0$, because $\{B_i\}_{i\in\mathbb{N}}$ forms a cover of M, and $\psi_i|_{B_i} > 0$.

Now, define

$$f_i = \frac{1}{\displaystyle\sum_{j\in\mathbb{N}} \psi_j} \psi_i$$

so that $\{\operatorname{supp} f_i\}_{i\in\mathbb{N}} = \left\{\overline{B_i}\right\}$ is locally finite, and $\sum_{i\in\mathbb{N}} f_i = 1$, $\forall p \in M$. Then, we just have to fix it so that he indexing sets are the same as $\{U_\alpha\}_{\alpha\in A}$ by $\forall i \in \mathbb{N}$, pick $\alpha(i) \in A$ such that $B_i \subset U_{\alpha(i)}$ and $\forall \alpha \in A$, let

$$\chi_\alpha = \sum_{\substack{i \text{ s.t.} \\ \alpha(i) = \alpha}} f_i$$

(Note that $\chi_\alpha = 0$ if the sum is empty.)

We claim that $\left\{\operatorname{supp} \chi_{\alpha(i)}\right\}_{i\in\mathbb{N}}$ is still locally finite. This follows from $\{\operatorname{supp} f_i\}_{i\in\mathbb{N}}$ being locally finite. $\square$

There are many applications of partitions of unity!

Ex: Existence of C^∞ sections of any vector bundle.

Say $E \to M$ is a vector bundle of rank r. Then there exist $\{(U_\alpha, f_\alpha)\}$ local trivializations:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "f_\alpha"] \arrow[dr] & & U_\alpha \times \mathbb{R}^n \arrow[dl] \\
& U_\alpha \arrow[ul, bend left, "s_\alpha"] \arrow[ur, bend right, "\text{pick any section } s"'] &
\end{tikzcd}$$

Let $\{\chi_\alpha\}$ be a partition of unity on M subordinate to $\{U_\alpha\}$. Then let $s = \sum_{\alpha\in A} \chi_\alpha \cdot s_\alpha$ (we interpret $\chi_\alpha \cdot s_\alpha$ as a C^∞ section on M).

The main application of partitions of unity is integrating forms.

Defn: Let M be an oriented n-dimensional manifold. Let $\mu \in \Omega_0^n(M)$ be a top degree form with compact support. Let $\{\phi_\alpha\}$ be a positive atlas, and $\{\chi_\alpha\}$ a subordinate partition of unity (i.e. $\operatorname{supp} \chi_\alpha \subseteq \operatorname{supp} \phi_\alpha$, $\forall \alpha$). Then we define

$$\int_M \mu = \sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$$

We have to check that the right hand side is independent of choice of coordinates. We'll do this next time...

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Continuing from last time, W.T.S. $\sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$ is independent of choice of coordinates and partition of unity. Say $\left\{\tilde{\phi}_\beta\right\}, \{\tilde{\chi}_\beta\}$ is another choice. Then $\forall \alpha$,

$$\chi_\alpha \mu = \sum_\beta \tilde{\chi}_\beta \chi_\alpha \mu \qquad \sum_\beta \tilde{\chi}_\beta = 1$$

Thus,

$$\int (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \sum_\beta \int (\phi_\alpha^{-1})^*(\underbrace{\tilde{\chi}_\beta \chi_\alpha \mu}_{\text{supp in } (\operatorname{dom} \tilde{\phi}_\beta) \cap (\operatorname{dom} \phi_\alpha)})$$

So $\forall \beta$,

$$\int (\phi_\alpha^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu) = \int (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu)$$

because of the invariance of integrals of top-degree forms in Euclidean space, under orientation-preserving diffeomorphisms. (In this case, the transition function is that orientation-preserving diffeomorphism.) Now, back to the full term in the integral:

$$\sum_\alpha (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \sum_{\alpha,\beta} (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \chi_\alpha \mu)$$

$\forall \beta$, sum over α first, and use the fact that $\sum_\alpha \chi_\alpha = 1$. This yields

$$\sum_\alpha (\phi_\alpha^{-1})^*(\chi_\alpha \mu) = \cdots = \sum_\beta (\tilde{\phi}_\beta^{-1})^*(\tilde{\chi}_\beta \mu)$$

$\square$

Observe: In practice, don't use partitions of unity. Use parameterizations that partition M (or at least, $\operatorname{supp} \mu \subset M$), up to sets of measure 0.

Ex: On a torus, pullback μ to $\mathbb{R}^n$ and integrate over a fundamental domain.

Defn: Let $S \subset M$ be an oriented k-dimensional submanifold. Define

$$\int_S : \Omega_0^k(M) \to \mathbb{R}$$
$$\alpha \mapsto \int_S \iota^* \alpha$$

where $\iota : S \hookrightarrow M$ is the inclusion map. In general, we omit the ι^*, and just say

$$\int_S \alpha \overset{\text{def}}{=} \int_S \iota^* \alpha$$

Manifolds with Boundary

These are needed for Stokes' theorem. We begin with some preliminary definitions...

Defn: Let $S, T \subset \mathbb{R}^n$ (with no assumptions of their properties). S and T are diffeomorphic iff $\exists U, V \overset{\text{open}}{\subset} \mathbb{R}^n$ s.t. $S \subset U$, $T \subset V$, and $\exists F : U \to V$ a diffeomorphism s.t. $F(S) = T$. We will say $F|_S^T : S \to T$ is a diffeomorphism between the sets.

Lemma: If $U \subset \mathbb{R}^n$ is open and diffeomorphic to $T \subset \mathbb{R}^n$, then T is open.

Proof: By definition, $\exists V, \tilde{U}$ open with $T \subset V$ and $U \subset \tilde{U}$, and $\exists F : \tilde{U} \to V$ diffeomorphic such that $F(U) = T$. Then $F|_U : U \to \mathbb{R}^n$ is an open map, so T is open. $\square$

Defn: $\mathbb{H}^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Ex: $B_r(0) \cap \mathbb{H}^n$. A diffeomorphism on this set must map interior points to interior points, so it maps $B_r(0) \cap \{x^n = 0\}$ to the boundary of its image.

In fact, the same works for any relatively open sets of $\mathbb{H}^n$! And such diffeomorphisms restrict to diffeomorphisms of $W \cap \partial\mathbb{H}^n$ onto $F(W) \cap \partial\mathbb{H}^n$ (for any W relatively open).

Defn: A topological manifold with boundary M is a second-countable, Hausdorff topological space that is local homeomorphic to relatively open sets in $\mathbb{H}^n$.

Note: This includes ordinary open sets in $\mathbb{R}^n$, and the sets with boundary.

Defn: A C^∞ atlas on a topological manifold with boundary is an atlas such that all transition functions are C^∞.

Defn: A C^∞ manifold with boundary is a topological manifold with boundary, together with a maximal C^∞ atlas.

Defn: Let M be a smooth manifold with boundary. Then the boundary ∂M is the set of points $p \in M$ which have a nearby chart ϕ such that $\phi(p)$ is in the boundary of $\mathbb{H}^n$.

Note that this is true for one coordinate chart iff it's true for all coordinate charts.

Lemma: ∂M inherits a C^∞ manifold structure (without boundary) by restricting charts of M defined near boundary points.

Lemma: $\partial(\partial M) = \emptyset$.

Tangent Spaces

They're defined as before!

Defn: Given $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^k$ is smooth iff $\exists U \subset \mathbb{R}^n$ open with $S \subset U$, and $\exists \tilde{f} : U \to \mathbb{R}^k$ smooth such that $\tilde{f}\Big|_S \equiv f$. We call $\tilde{f}$ a smooth extension.

So, for f smooth, $f \circ \phi^{-1} : W \to \mathbb{R}$, $W \subset \mathbb{H}^n$ must be C^∞ for all charts $(\phi^{-1}(W), \phi)$. This gives us germs: $\forall p \in M$, $C_p^\infty(M)$. We define T_pM to be the set of all derivations on $C_p^\infty(M)$.

But what happens at the boundary? If $p \in \partial M$, what is T_pM?

Claim: Well, if $p \in \partial\mathbb{H}^n$, $T_p\mathbb{H}^n$ is still spanned by $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$.

The question is how do we define $\frac{\partial}{\partial x^n}$, and how does it act? If $f \in C^\infty$ near p, then $\exists \tilde{f}$, an extension of f to an open set of $\mathbb{R}^n$. Then define $\frac{\partial f}{\partial x^n}(p) \overset{\text{def}}{=} \frac{\partial \tilde{f}}{\partial x^n}(p)$.

Claim: The RHS is independent of choice of extension. Well,

$$\frac{\partial \tilde{f}}{\partial x^n}(p) = \lim_{h \to \infty} \frac{\tilde{f}(0, \dots, h) - \tilde{f}(0, \dots, 0)}{h}$$

This limit will always be the same, since the limit always exists (by definition of the extension).

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Briefly, think back to integration of forms. Observe:

$$\int_M \mu = \sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$$

We claimed (without proof) that this sum has finitely-many nonzero summands.

Proof: Let $K = \operatorname{supp}\mu$ (note that K must be compact), and $U_\alpha = \operatorname{supp}\chi_\alpha$. Note that $\{U_\alpha\}$ is locally finite: $\forall p \in K$, $\exists V_p$, a neighborhood of p, such that $\{\alpha \mid V_p \cap U_\alpha \neq \emptyset\}$ is finite. Also, $\{V_p\}_{p\in K}$ is naturally a cover of K, so by compactnes, $\exists p_1, \ldots, p_n \in K$ s.t. $\{V_{p_1}, \ldots, V_{p_n}\}$ is a cover of K. Finally,

$$\{\alpha \mid K \cap U_\alpha \neq \emptyset\} \subset \bigcup_{j=1}^n \underbrace{\{\alpha \mid V_{p_j} \cap U_\alpha \neq \emptyset\}}_{\text{finite}}$$

□

Now, we return to working with manifolds with boundary...

The principle is that all definitions on manifold with boundary are exact analogues to the case where $\partial M = \emptyset$. The key difference is the model spaces are open subsets of $\mathbb{H}^n = \{x^n \geq 0\} \subseteq \mathbb{R}^n$, and transition maps are diffeomorphisms between open subsets of $\mathbb{H}^n$ (which carry boundary points to boundary points, and restrict to diffeomorphisms between open sets of $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$).

Ex: Notions of smoothness. $f : M \to \mathbb{R}$ is smooth iff for any chart ϕ_α, $f_\alpha = f \circ \phi_\alpha^{-1} : U \to \mathbb{R}$ is smooth (with $U \overset{\text{open}}{\subseteq} \mathbb{H}^n$). In case $U \cap \partial\mathbb{H}^n \neq \emptyset$, smoothness of $f_\alpha : U \to \mathbb{R}$ means $\exists \tilde{f}_\alpha : \tilde{U}_\alpha \to \mathbb{R}$, a smooth extension of f_α to $\tilde{U}_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^n$. Then $U = \tilde{U}_\alpha \cap \mathbb{H}^n$.

Ex: Existence of partitions of unity, just as before.

Tangent Spaces

$\forall p \in \partial M$, T_pM is still n-dimensional, and is still spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$. $\frac{\partial}{\partial x^n}$ is well-defined on smooth functions, because such functions extend across the boundary, and last time, we showed the choice of extension doesn't matter.

Note: ∂M inherits a C^∞ manifold structure, by restricting charts of M.

$\forall p \in \partial M$, $T_p(\partial M) \overset{\iota_{*,p}}{\hookrightarrow} T_pM$, where $\iota : \partial M \to M$ is the inclusion. Identify $T_p(\partial M) \subset T_pM$ as a subspace – the image of $\iota_{*,p}$. It's a hyperplane, i.e., $T_p(\partial M)$ has codimension 1.

Note: $(T_pM) \setminus (T_p\partial M)$ has two components: "inward-pointing" vectors and "outward-pointing" vectors.

How do we characterize outward-pointing vectors? Well, say $p \in \partial M$. Let $\gamma : (-\varepsilon, 0] \to M$ smooth with $\gamma(0) = p$. Then define $\dot{\gamma}(0)$ to be outward pointing.

Orientation of Manifolds with Boundary

Once again, this is an exact analogue of manifolds without boundary. $\forall p \in M$, T_pM has dimension n, two orientations, etc.

Lemma: If M is orientable, then ∂M is orientable.

By convention, $\forall p \in \partial M$, a basis $(b_1, \ldots, b_{n-1})$ of $T_p(\partial M)$ is positive iff $(\nu, b_1, \ldots, b_{n-1})$ is a positive basis of T_pM, *for any outward-pointing* $\nu \in T_pM$. This defines the boundary's orientation.

Ex: Say $M = \mathbb{H}^n$, oriented such that $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ is positive. What is the boundary orientation?

For $n = 2$, $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ is positive. So if ν is outward-pointing, $\left\{\nu, \frac{\partial}{\partial x^2}\right\}$ is positive, so $\left\{\frac{\partial}{\partial x^1}\right\}$ is positive on $\partial\mathbb{H}^2 \cong \mathbb{R}$.
For $n = 3$, is $(\vec{i}, \vec{j})$ positive? Look at $v = -\vec{k}$ – $(-\vec{k}, \vec{i}, \vec{j})$ is negative. So we claim that $\partial\mathbb{H}^n$ is $(-1)^n$ times the standard orientation of $\mathbb{R}^n$.

Stokes' Theorem

Thm: (Stokes' Theorem) Let M be a manifold (possibly with boundary). Let $\mu \in \Omega_0^{n-1}(M)$. Assume M is oriented; give ∂M the boundary orientation. Then

$$\int\limits_{\partial M} \iota^*\mu = \int\limits_M d\mu$$

(where $\iota : \partial M \hookrightarrow M$ is the inclusion map). One often omits the ι^*, so we say

$$\int\limits_{\partial M} \mu = \int\limits_M d\mu$$

(If $\partial M = \emptyset$, then $\int_M d\mu = 0$.)

Observe: $\iota^*\mu$ is a top degree form on the boundary.

Ex: $M = [a, b]$, $n = 1$, $\mu \in \Omega_0^0(M) \cong C^\infty(M)$. Then $\int_{\partial M} \mu = \mu(b) - \mu(a) = \int_M df$. The "$-$" sign comes because of the orientation of ∂M is outward-pointing.

Ex: $n = 2$. Then we get Green's Theorem:
$\mu = P\,dx + Q\,dy$. $d\mu = (Q_x - P_y)\,dx \wedge dy$.
The orientation on ∂M comes from the right-hand rule.

$$\int\limits_{\partial M} P\,dx + Q\,dy = \iint\limits_M (Q_x - P_y)\,dx \wedge dy$$

Exer: For $n = 3$, we get the usual Stokes' theorem.

Proof of Stokes' Theorem

First, assume it holds for $\mathbb{H}^n$. Then it follows for $\mu \in \Omega_0^{n-1}(M)$ if μ is supported in the domain of a chart ϕ:

$$\int\limits_{\partial M} \mu = \int\limits_{\partial\mathbb{H}^n} (\phi^{-1})^*\iota^*\mu \overset{(1)}{=} \int\limits_{\mathbb{H}^n} d(\phi^{-1})^*\mu = \int\limits_{\mathbb{H}^n} (\phi^{-1})^* d\mu = \int\limits_M d\mu$$

with (1) by the definition of integrals, plus our assumption about μ. In general (still under the assumption that Stokes' theorem holds for $\mathbb{H}^n$), we use a partition of unity $\{\chi_j\}$ subordinate to an atlas.

$$\int\limits_{\partial M} \iota^*\mu = \sum_j \int\limits_{\partial M} \iota^*(\chi_j\mu) = \sum_j \int\limits_M d(\chi_j\mu) = \sum_j \int\limits_M d(\chi_j) \wedge \omega + \int\limits_M \chi_j d\mu = \int\limits_M d\mu + \int\limits_m \Bigg(\underbrace{\sum_j d\chi_j}_{\substack{=d\sum_j \chi_j = 0 \\ \text{b/c } \sum_j \chi_j \equiv 1}} \Bigg) \wedge \mu = \int\limits_M d\mu$$

So finally, it's enough to prove Stokes' Theorem for $M = \mathbb{H}^n$. Write

$$\mu = \sum_{i=1}^{n} a_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad a_i \in C^\infty(\mathbb{H}^n)$$

Then

$$d\mu = \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x^j} (-1)^{j-1} \right) \underbrace{dx^1 \wedge \cdots \wedge dx^n}_{\text{standard volume form}}$$

Because we assume μ has compact support, $\operatorname{supp} \mu$ is bounded, so $\exists R > 0$ s.t.

$$\operatorname{supp} \mu \subseteq \underbrace{[-R, R] \times \cdots \times [-R, R]}_{n-1 \text{ times}} \times [0, R]$$

Thus,

$$\int_{\mathbb{H}^n} d\mu = \sum_{i=1}^{n} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial a_i}{\partial x^j} \, dx^1 \cdots dx^n$$

So $\forall i$, first do $\int_{-R}^R \frac{\partial a_i}{\partial x^i} \, dx^i$. For $i = 1, \ldots, n-1$,

$$\int_{-R}^{R} \frac{\partial a_i}{\partial x^i} \, dx^i = a(x^1, \ldots, R, \ldots, x^n) - a_i(x^1, \ldots, -R, \ldots, x^n) = 0 - 0 = 0 \qquad \int_0^R \frac{\partial a_n}{\partial x^n} \, dx^n = -a_n(x^1, \ldots, x^{n-1}, R)$$

Thus,

$$\int_{\mathbb{H}^n} d\mu = \underbrace{-(-1)^{n-1}}_{=(-1)^n} \int_{\mathbb{R}^{n-1}} a_n(x^1, \ldots, x^{n-1}, 0) \, dx^1 \cdots dx^{n-1}$$

And on the other hand,

$$\int_{\partial \mathbb{H}^n} \mu = \underbrace{(-1)^n}_{\substack{\uparrow \\ \text{boundary orientation}}} \int_{\mathbb{R}^{n-1}} a_n(x^1, \ldots, x^{n-1}, 0) \, dx^1 \cdots dx^{n-1}$$

(with the 0 appearing in the a_n term because $\iota^*(dx^n) = 0$.)

$\square$

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Sard's Theorem (with an Application)

As a preliminary, we have to talk about sets of measure 0.

Defn: Informally speaking, $S \subset \mathbb{R}^n$ has <u>measure zero</u> iff $\forall \varepsilon > 0$, S can be covered by countably many n-cubes of total volume less than ε.

Prop: If S has measure 0, and $F : S \to \mathbb{R}^m$ is smooth, then $F(S)$ has measure 0.

Proof: Based on the fact that C^∞ functions are Lipschitz on compact sets. I.e., $||F(p) - F(q)|| < C\,||p - q||$ for some constant $C \in \mathbb{R}_{>0}$.

Defn: A subset $S \subset M$ has <u>measure zero</u> iff $\forall (U, \phi)$, a coordinate chart, the set $\phi(U \cap S) \subseteq \mathbb{R}^n$ has measure zero.

Prop: Equivalently, S can be covered by countably many charts $\{(U_i, \phi_i)\}$ s.t. $\forall i$, $\phi_i(U_i \cap S)$ has measure zero.

Thm: (Sard's Theorem) If $F : M \to N$ is smooth, the set of crtical values of F has measure 0.

Reminder: $q \in N$ is a regular value iff $\forall q \in F^{-1}(p)$, $F_{*,p}$ is surjective.
$q \in N$ is a critical value iff q is not a regular value.

Note: If $q \notin \mathrm{Im}(F)$, then q is a regular value.

Cor: The set of regular values of F is dense in N. (It's the complement of a set of measure zero.) In particular, if $F : M \to N$ is smooth, and $\dim M < \dim N$, then the only regular values are $N \setminus \mathrm{Im}(F)$, so we conclude that $N \setminus \mathrm{Im}(F)$ is dense, and $\mathrm{Im}(F)$ has measure zero. In particular, submanifolds of nonzero codimension have measure zero.

(Recall: A set S is dense if $\forall U$ open, $U \cap S \neq \emptyset$.)

The Embedding Theorem

Thm: (Whitney Embedding Theorem) Let M be an n-dimensional manifold. Then M can be embedded in $\mathbb{R}^{2n+1}$ and immersed in $\mathbb{R}^{2n}$. (This is the weak version.)

Thm: M can be embedded in $\mathbb{R}^{2n}$. (This is the strong version.)

Proof of the weak version: We start by embedding M into some $\mathbb{R}^N$. Then we successively project $M \subset \mathbb{R}^N$ onto "good hyperplanes". The first step is to cover M with an atlas $\{(U_i, \phi_i)\}_{i=1,\ldots,k}$. Let $\{\chi_i\}_{i=1,\ldots,k}$ be a subordinate partition of unity: $\forall i$, $\operatorname{supp}\chi_i \subset U_i$.

Now, define

$$\forall i,\ \psi_i : M \to \mathbb{R}^n \qquad\qquad F : M \to \mathbb{R}^{kn+k \overset{\mathrm{def}}{=} N}$$

$$p \mapsto \begin{cases} \chi_i(p)\phi_i(p) & p \in U_i \\ 0 & p \notin U_i \end{cases} \qquad\qquad p \mapsto (\psi_1(p), \ldots, \psi_k(p), \chi_1(p), \ldots, \chi_k(p))$$

We claim that F is injective, and an immersion. Assume $p, q \in M$ s.t. $F(p) = F(q)$.
Then $\exists i$ s.t. $\chi_i(p) = \chi_i(q) \neq 0$. Thus, $p, q \in U_i$. So $F(p) = F(q) \Rightarrow \psi_i(p) = \psi_i(q) \Rightarrow \phi_i(p) = \phi_i(q) \Rightarrow p = q$.

Now, to show that F is an immersion. Assume $v \in T_pM$ s.t. $F_{*,p}(v) = 0$. Again, choose i s.t. $\chi_i(p) \neq 0$, so $p \in U_i$. Then

$$0 = (\psi_i)_{*,p}(v) = d\chi_i(v)\phi_i(p) + \chi_i(p)d\phi_p(v)$$

since $\psi_i = \chi_i \cdot \phi_i$. But also $d\chi_i(v) = 0$, so $d(\phi_i)_p(v) = 0$, so we must have $v = 0$ (as ϕ_i is a diffeomorphism). $\square$

The next step is lowering the dimension. Let $\mathbb{P} = \left\{\ell \subseteq \mathbb{R}^N \text{ subspaces of dimension } 1\right\}$. $\forall \ell \in \mathbb{P}$, let $\pi_\ell : \mathbb{R}^N \to \ell^\perp$, where $\ell^\perp$ is the orthogonal complement (and a hyperplane). We claim that $\exists \ell \in \mathbb{P}$ s.t. $\pi_\ell|_M : M \to \ell^\perp$ is an embedding, provided that $N > 2n + 1$.

Well, we need to find an ℓ s.t. $\pi_\ell|_M$ is injective and an immersion. Consider

$$\begin{aligned} G : M \times M \setminus \{(p,p) \mid p \in M\} &\to \mathbb{P} \\ \underbrace{(p,q)}_{p,q\in\mathbb{R}^N} &\mapsto \mathbb{R}\underbrace{(p-q)}_{\neq 0} \end{aligned}$$

G is smooth. If $\dim(M \times M) < \dim(P)$, then $\mathbb{P} \setminus \mathrm{Im}(G)$ is dense, so if we pick $\ell \in \mathbb{P} \setminus \mathrm{Im}(G)$, then $\pi_\ell|_M$ is injective. Because $dim(M \times M) = 2n$, and we need $2n < N - 1$, we require $N > 2n + 1$.

Now, to ensure $\pi_\ell|_M$ is an immersion, let

$$\begin{aligned} H : SM \stackrel{\text{def}}{=} \{(p,v) \in TM \mid \ ||v|| = 1\} &\to \mathbb{P} \\ (p,v) &\mapsto \mathbb{R}v \end{aligned}$$

We claim that if $\ell \in \mathbb{P} \setminus \mathrm{Im}(H)$, then $\pi_\ell|_M$ is an immersion. We perform a similar dimension count as before, and we get to $N > 2n$. $\square$

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De Rham Cohomology

Defn: Given M a manifold, $k \in \mathbb{N}$ (by our convention, $0 \in \mathbb{N}$), we define $Z^k(M) = \ker(d : \Omega^k \to \Omega^{k+1})$, the set of closed k-forms ("cocycles") and $B^k(M) = \operatorname{im}(d : \Omega^{k-1} \to \Omega^k)$ the set of exact k-forms ("coboundaries").

Defn: The kth de Rham group is the quotient $H^k \stackrel{\text{def}}{=} Z^k/B^k$. (In our case, this is a quotient vector space, but it can also be defined simply as a quotient group.)

Defn: $\beta_k = \dim H^k(M)$ is the kth Betti number of M.

Observe: If $k > \dim M$, then $\beta_k = 0$.

Observe: There is a "compact version" of this theory, for working with Ω_0^k, the set of compacly-supported k-forms.

Ex: $H^0 = \{f \in C^\infty \mid df = 0\}$ is the space of locally constant functions. Thus, β_0 is the number of connected components of M.

Ex: $M = \mathbb{R}$, $\beta_0 = 1$. What is β_1? Well,
$Z^1 = \Omega^1 = \{f\, dx \mid f \in C^\infty\}$.
$B^1 = \{dg = g'\, dx \mid g \in C^\infty\}$.
Every $f \in C^\infty$ has an antiderivative, so $H^1(\mathbb{R}) = \{0\}$, so $\beta_1 = 0$.

Prop: $H^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0 \end{cases}$

Ex: $M = \mathbb{R}$. Consider only forms with compactly-supported coefficients ($H_C^\cdot$). Then
$H^0 = \{f \in C^\infty(\mathbb{R}) \mid df = 0\} = \{0\}$.
To compute H^1, ask: Which functions $f \in C_0^\infty(\mathbb{R})$ have anti-derivatives that are of compact support? One can show that this is true iff $\int f = 0$.

Note: $\int : H_C^1 \to \mathbb{R}$ is an isomorphism, so $H_C^1(\mathbb{R}) \cong \mathbb{R}$.

Ex: $M = S^1$. M is connected, so $\beta_0 = 1$. What about β_1? Well,
$Z^1 = \Omega^1 = \{f\, d\theta \mid f : \mathbb{R} \to \mathbb{R} \text{ is } 2\pi\text{-periodic}\}$.
$B^1 = \left\{dg = g'\, d\theta \mid g \in C^\infty(S^1)\right\}$.
Question: Which 2π-periodic functions have 2π-periodic antiderivatives? We can figure this out using Fourier series. We want $f = dg$, so

$$f = \sum_{n \in \mathbb{Z}} a_n e^{in\theta} \qquad \Rightarrow \qquad g = \frac{1}{i} \sum_{n \in \mathbb{Z}} \frac{a_n}{n} e^{in\theta}$$

So we need $a_0 = 0$. In fact, $[f\, d\theta] = [a_0\, d\theta]$, where $[\,\cdot\,]$ denotes the cohomology class. Thus, $H^1(S^1) = \mathbb{R}[d\theta] \cong \mathbb{R}$.

Observe: This generalizes greatly, to any compact manifold without boundary. It's known as "Hodge theory". Fourier series are replaced with the spectral of the Laplacian.

General Features

- Covariance: If $F : M \to N$ is C^∞, we can define, $\forall k$, $F^* : H^k(N) \to H^k(M)$ by $F^*[\omega] = [F^*\omega]$ (for ω $Z^k(N)$). This is well-defined because F^* and d commute.
- Ring Structure: The wedge product induces a "cup map" in cohomology:

$$\begin{aligned} H^k(M) \times H^\ell(M) &\to H^{k+\ell}(M) \\ ([\alpha], [\beta]) &\mapsto [\alpha \wedge \beta] \end{aligned}$$

  Check that this is well-defined:

  a) $d(\alpha \wedge \beta) = 0$ if $d\alpha = 0$ an $d\beta = 0$ by the product rule.
  b) $(\alpha + da) \wedge (\beta + db) = \alpha \wedge \beta + \cdots$. The remaining terms are each exact, so it's still the same cohomology class.

- F^* is a ring morphism.
- $(F \circ G)^* = G^* \circ F^*$.

Cor: Diffeomorphic manifolds have isomorphic cohomology.

Homotopy Equivalence

Defn: Two C^∞ functions $F, G : M \to N$ are <u>homotopic</u> (to each other) iff there is a smooth map $\Phi : M \times [0,1] \to G$ s.t. $\forall p \in M$, $\Phi(p, 0) = F(p)$ and $\Phi(p,1) = G(p)$. Φ is called a <u>homotopy</u>.

Some notation: $\forall t \in [0,1]$, let $\iota_t : M \to M \times [0,1]$ where $p \mapsto (p,t)$. Then $F = \Phi \circ \iota_0$, and $G = \Phi \circ \iota_1$.

Observe: In Tu's textbook, we use $\mathbb{R}$ instead of $[0,1]$. The two definitions are equivalent, as we can easily extend Φ from $M \times [0,1]$ to $M \times \mathbb{R}$ with a bump function, and we can simply restrict from $M \times \mathbb{R}$ to $M \times [0,1]$.

Thm: Being homotopic is an equivalence relation in $C^\infty(M, N)$. We write $F \sim G$.

Ex: If $X \in \mathfrak{X}(M)$ is complete, then $\forall t$, ϕ_t, the flow's time-t map, is homotopic to the identity, ϕ_0.

Proof: For $t = 1$, use Φ. For $t = T$ (nonzero), use $\Phi(p,t) = \phi_{tT}(p)$. $\square$

Ex: Say $\mathrm{Id}, G : \mathbb{R}^n \to \mathbb{R}^n$, with $G(x) = 0$ a constant map. Then G and I are homotopic, with $\Phi(x,t) = (1-t)x$ a homotopy.

Defn: A submanifold $S \subset M$ is a <u>deformation retract</u> of M iff $\exists \Phi : M \times [0,1] \to M$ s.t.

a) $\Phi|_{\{t=0\}} = \mathrm{Id}_M$
b) $\Phi|_{\{t=1\}}$ maps M into S.
c) $\forall t \in [0,1]$, $\forall p \in S$, $\Phi(p,t) = p$.

Such a Φ is called a <u>deformation retraction</u>.

Ex: $M = S^1 \times (-1, 1)$, $S = S^1 \times \{0\}$. S is a deformation retract of M.

Defn: $F : M \to N$ is a <u>homotopy equivalence</u> iff $\exists G : N \to M$ s.t. $G \circ F \sim \mathrm{Id}_M$ and $F \circ G \sim \mathrm{Id}_N$. If there's a homotopy equivalence $M \to N$, we say that M and N are <u>homotopy equivalent</u>, or that they "have the same homotopy type". We say G and F are <u>homotopy inverses</u>.

Prop: If $S \subset M$ is a deformation retract of M, then M and S are homotopy equivalent.

Proof: Let $\Phi : M \times [0,1] \to M$ be a deformation retraction of M onto S. Define $F : M \to S$ to be $F = \Phi|^S_{M \times \{1\}} : M \to S$, and $G : S \hookrightarrow M$ to be the inclusion. $\square$

From the perspective of homotopy theory, $\mathbb{R}^n$ is the same as a point, the cylinder is the same as the circle. Dimension is not homotopy-invariant!

Back to de Rham Theory

Thm: (Homotopy Axiom) Let $F, G : M \to N$ be smooth maps, and homotopic to each other. Then $F^* = G^*$ in cohomology, i.e., $\forall k \in \mathbb{N}$, $F^*, G^* : H^k(N) \to H^k(M)$ are equal.

Proof: First, we'll prove this for $F = I$, and G, the time-1 map of a flow. Let $X \in \mathfrak{X}(M)$, and assume it's complete. Say $G = \varphi_1$. We need to show $\varphi_1^* : H^k(M) \to H^k(M)$ is the identity.

Let $[\omega] \in H^k(M)$, so $\omega \in Z^k$ $(d\omega = 0)$. We need to show $\exists \alpha \in \Omega^{k-1}(M)$ such that $\varphi_1^*\omega - \omega = d\alpha$.

Observe: $\frac{d}{dt}\varphi_t^*\omega = \mathcal{L}_X[\varphi_t^*\omega] = d(\iota_X\varphi_t^*\omega) + \iota_X d\varphi_t^*\omega$. We can commute d and φ_t^*, and use the fact that $d\omega = 0$, to see that $\frac{d}{dt}\varphi_t^*\omega = d(\iota_X\varphi_t^*\omega)$. Integrate both sides over $[0, 1]$ w.r.t t on each $\bigwedge^k T_pM$, $\forall p \in M$, and we obtain

$$\varphi_1^*\omega - \omega = \int_0^1 \frac{d}{dt}\varphi_t^*\omega \, dt = \int_0^1 (\iota_X\varphi_t^*\omega)\, dt = d \underbrace{\int_0^1 (\iota_X\varphi_t^*\omega)\, dt}_{\overset{\text{def}}{=}\alpha}$$

Now, for the general case, say $F, G : M \to N$ are homotopic. Then let $\Phi : M \times \mathbb{R} \to N$ be a homotopy between them. Let $X = \frac{\partial}{\partial t}$ on $M \times [0, 1]$. Then $\varphi_t(p, s) = (p, s + t)$. Thus, the following diagram commutes:

$$\begin{tikzcd} & M \times \mathbb{R} \arrow[r, "\Phi"] \arrow[d, "\varphi_1"] & N \\ M \arrow[ur, hook, "\iota_0"] \arrow[dr, hook, "\iota_1"'] \arrow[r, phantom, "\circlearrowleft"] & M \times \mathbb{R} \arrow[r, "\Phi"] & N \end{tikzcd}$$

where $\iota_t(p) = (p, t)$. So we have $G = \Phi \circ \iota_1$ and $F = \Phi \circ \iota_0$.

Let $[\omega] \in H^k(N)$.

$$G^*[\omega] = (\iota_1^* \circ \Phi^*)[\omega] = ((\varphi_1 \circ \iota_0)^* \circ \Phi^*)[\omega] = (\iota_0^* \circ \varphi_1^* \circ \Phi^*)[\omega] = \iota_0^*(\underbrace{\varphi_1^*(\Phi^*[\omega])}_{\substack{=\Phi^*[\omega] \\ \text{because } \varphi_1^* = \mathrm{Id}}}) = (\iota_0^* \circ \Phi^*)[\omega] = F^*[\omega]$$

□

Cor: If M and N are homotopy equivalent, then their cohomology is isomorphic.

Math 591 Lecture 40

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We'll begin with a very brief look at the algebra behind cohomology.

Defn: A cochain complex $\mathcal{A}$ of vector spaces is a sequence of linear maps

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

s.t. $d \circ d = 0$ (whenever defined).

Defn: The cohomology of a cochain complex $\mathcal{A}$ is, $\forall k \in \mathbb{N}$, $H^k(\mathcal{A}) = Z^k(\mathcal{A})/B^k(\mathcal{A})$, where $Z^k(\mathcal{A}) = \ker(d)$ (with $d : A^k \to A^{k+1}$) and $B^k(\mathcal{A}) = \operatorname{im}(d)$ (with $d : A^{k-1} \to A^k$).

Defn: If $\mathcal{A}$ and $\mathcal{B}$ are cochain complexes, a map $f : \mathcal{A} \to \mathcal{B}$ between them is a sequence: $\forall k \in \mathbb{N}$, we have $f_k : A^k \to B^k$ s.t. $d \circ f^k = f^k \circ d$. I.e., the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & A^k \arrow[r, "d"] \arrow[d, "f^k"] & A^{k+1} \arrow[r, "d"] \arrow[d, "f^{k+1}"] & \cdots \\
\cdots \arrow[r, "d"] & B^k \arrow[r, "d"] & B^{k+1} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

Lemma: Such an $f : \mathcal{A} \to \mathcal{B}$ induces $f^\sharp : H^k(\mathcal{A}) \to H^k(\mathcal{B})$ by $f^\sharp[a] = [f(a)]$ for any $a \in Z^k(\mathcal{A})$, and $f^\sharp$ is well-defined.

Observe:

a) $(f \circ g)^\sharp = f^\sharp \circ g^\sharp$.

b) For de Rham theory, if $F : M \to N$ is C^∞, then we get $f : \Omega^*(N) \to \Omega^*(M)$ ($\Omega^*(N)$ is the de Rham complex of N), where $\forall \alpha \in \Omega^k(N)$, $f(\alpha) = F^*\alpha$.

Homotopies between Maps of Cochain Complexes

Defn: Say $f, g : \mathcal{A} \to \mathcal{B}$. A (chain) homotopy (operator) between them is a sequence of maps: $\forall k$, $h : A^k \to B^{k-1}$ s.t. the following diagram commutes:

$$\begin{tikzcd}
 & \cdots \arrow[r, "d"] & A^k \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, "f-g"] & A^{k+1} \arrow[r, "d"] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r, "d"] & B^{k-1} \arrow[r, "d"] & B^k \arrow[r, "d"] & \cdots &
\end{tikzcd}$$

That is, $h \circ d + d \circ h = f - g$.

Lemma: If there exists a homotopy between f and g, then $f^\sharp = g^\sharp$.

Last time, we showed that for $X \in \mathfrak{X}(M)$, with φ the flow of X (which we assume to be complete), then $\forall \omega \in \Omega^k(M)$, $\frac{d}{dt}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_X\omega = \varphi_t^*(\iota_X d\omega + d\iota_X\omega)$. Thus, $\varphi_1^*\omega - \omega = \int_0^1 \varphi_t^*(\iota_X d\omega + d\iota_X\omega)\,dt$.

Check: If we define $h(\omega) = \int_0^1 \varphi_t^*(\iota_X\omega)\,dt \in \Omega^{k-1}(M)$, hen the above formula shows that h is a chain homotopy between φ_1^* and the identity map.

Mayer-Vietoris Sequence

Motivation: How can we compute $H^*(S^2)$?

Well, we can describe S^2 as the union of U and V, where U and V are diffeomorphic to the open disk, and their intersection is diffeomoprhic to the cylinder $S^1 \times (-1, 1)$. Can we say anything about $H^*(S^2)$ in terms of $H^*(U)$, $H^*(V)$, and $H^*(U \cap V)$?

Hypothesis: In general, for U, V open with $M = U \cup V$, we have

$$\begin{tikzcd} & U \arrow[dr, hook] & \\ U \cap V \arrow[ur, hook] \arrow[dr, hook] & & M \\ & V \arrow[ur, hook] & \end{tikzcd}$$

We can then form, $\forall k \in \mathbb{N}$,

$$\begin{array}{c} 0 \longrightarrow \Omega^k(M) \xrightarrow{f} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{g} \Omega^k(U \cap V) \longrightarrow 0 \\ (\alpha, \beta) \longmapsto (\alpha - \beta)|_{U \cap V} \end{array}$$

where f is the pullback/restriction.

Lemma: $\forall k \in \mathbb{N}$, this is an exact sequence, i.e., the image of each map is the kernel of the next one. (This is true iff it's a complex with zero cohomology).

Proof: We have exactness at $\Omega^k(M)$ iff f is injecive. This is true because $M = U \cup V$, and U and V are both open.

We have exactness at $\Omega^k(U) \oplus \Omega^k(V)$ iff $\text{im}(f) = \ker(g)$. Well, $\text{im}(f)$ is the set of restrictions of globally-defined forms, so we're still okay.

We have exactness at $\Omega^k(U \cap V)$ iff g is surjective. Let $\omega \in \Omega^k(U \cap V)$. We need to show $\exists \alpha \in \Omega^k(U), \beta \in \Omega^k(V)$ s.t. $(\alpha - \beta)|_{U \cap V} = \omega$. Let $\{\chi_U, \chi_V\}$ be a subordinate partition of unity to $\{U, V\}$. Define

$$\alpha(p) \stackrel{\text{def}}{=} \begin{cases} \chi_V \omega & p \in U \cap V \\ 0 & p \in U \setminus V \end{cases} \qquad \beta(p) \stackrel{\text{def}}{=} \begin{cases} -\chi_U \omega & p \in U \cap V \\ 0 & p \in V \setminus U \end{cases}$$

Then $(\alpha - \beta)|_{U \cap V} = \chi_V \omega + \chi_I \omega|_{U \cap V} = \omega$.

□

Observe: f and g are cochain maps – they commute with d!

Lemma: (Zig-Zag Lemma) Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be cochain complexes, and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ cochain maps, s.t. $\forall k \in \mathbb{N}$,

$$0 \longrightarrow A^k \xrightarrow{f} B^k \xrightarrow{g} C^k \longrightarrow 0$$

is exact. Then $\forall k$, $\exists \delta_k : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, a linear map referred to as the connecting morphism, s.t. the following sequence is exact:

$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{A}) \xrightarrow{f^\sharp} H^0(\mathcal{B}) \xrightarrow{g^\sharp} H^0(\mathcal{C}) \\ \hookrightarrow \cdots \\ \\ \cdots \\ \hookrightarrow H^k(\mathcal{A}) \xrightarrow{f^\sharp} H^k(\mathcal{B}) \xrightarrow{g^\sharp} H^k(\mathcal{C}) \\ \hookrightarrow H^{k+1}(\mathcal{A}) \xrightarrow{f^\sharp} H^{k+1}(\mathcal{B}) \xrightarrow{g^\sharp} H^{k+1}(\mathcal{C}) \\ \hookrightarrow \cdots \end{array}$$

Observe: This applied to the case $M = U \cup V$ is precisely the Mayer-Vietoris sequence.

Sketch of the proof:

1. Check exactness at $H^k(\mathcal{B})$:

$$H^k(\mathcal{A}) \xrightarrow{f^\sharp} H^k(\mathcal{B}) \xrightarrow{g^\sharp} H^k(\mathcal{C})$$

   We need to show $\mathrm{im}(f^\sharp) = \ker(g^\sharp)$. Well, we know $0 = (g \circ f)^\sharp = g^\sharp \circ f^\sharp$, so $\mathrm{im}(f^\sharp) \subseteq \ker(g^\sharp)$. For the reverse inclusion, let $[\beta] \in \ker(g^\sharp)$, so $\beta \in Z^k(\mathcal{B})$. We rely on the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A^k \arrow[r, "f"] & B^k \arrow[r, "g"] & C^k \arrow[r] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[r, "f"] \arrow[u, "d"] & B^{k-1} \arrow[r, "g"] \arrow[u, "d"] & C^{k-1} \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

   Assume that $g^\sharp[\beta] = 0$, i.e., $\exists c \in C^{k-1}$ s.t. $g(\beta) = dc$. Then $\exists b \in B^{k-1}$ s.t. $g(b) = c$.
   Thus, $g(\beta) = dc = dg(b) = gd(b)$. This means $g(\beta - db) = 0$, so $\exists a \in A^k$ s.t. $f(a) = \beta - db$, so $\beta = db + f(a)$.
   We need to show $[\beta] \in \mathrm{im} f^\sharp$, so we need to have $da = 0$. Well, $0 = df(a) = fda$. Because f is injective, we must have $da = 0$. We conclude that $\beta = db + f(a)$ and $da = 0$, so $[\beta] = [f(a)] = f^\sharp[a]$.

2. Check existence of δ:

$$\begin{tikzcd}
0 \arrow[r] & A^k \arrow[r] \arrow[d] & B^k \arrow[r] \arrow[d] & C^k \arrow[r] \arrow[d] \arrow[dll, dashed] & 0 \\
0 \arrow[r] & A^{k+1} \arrow[r] & B^{k+1} \arrow[r] & C^{k+1} \arrow[r] & 0
\end{tikzcd}$$

   Let $c \in Z^k(\mathcal{C})$, so $c \in C^k$, $dc = 0$. Then $\exists b \in B^k$ s.t. $g(b) = c$. So $0 = dc = dg(b) = g(db)$. Thus, $db \in \ker(g) = \mathrm{im}(f)$, so $\exists a \in A^{k+1}$ s.t. $f(a) = db$. In summary, $c = g(b)$ and $db = f(a)$. We claim:

   (i) $da = 0$.
   (ii) $[a] \in H^{k+1}(\mathcal{A})$ depends only on $[c]$.

   So we define $\delta([c]) = [a]$. Check:

   (i) $fda = df(a) = ddb = 0$. f is injective, so $da = 0$.
   (ii) This just requires more diagram chasing.

Cor: (Mayer-Vietoris Sequence) If $M = U \cup V$, there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(M) \arrow[r, "f^\sharp"] & H^0(U) \oplus H^0(V) \arrow[r, "g^\sharp"] & H^0(U \cap V) \arrow[dll, out=-10, in=170] \\
 & H^1(M) \arrow[r, "f^\sharp"] & H^1(U) \oplus H^1(V) \arrow[r, "g^\sharp"] & H^1(U \cap V) \arrow[dll, out=-10, in=170] \\
 & \cdots & &
\end{tikzcd}$$

with $f^\sharp$ and $g^\sharp$ given as above.

Application: $H^k(S^n) = \begin{cases} \mathbb{R} & k \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$ We can prove this using induction on n. For example, for $n = 2$,

$$\begin{tikzcd}
 & S^2 & U \sqcup V & U \cap V \\
H^0 & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \arrow[ddll, dashed] \\
 & x \arrow[r, mapsto] & (x, x) & \\
 & & (x, y) \arrow[r, mapsto] & x - y \\
H^1 & H^1(S^2) \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[dll, dashed] \\
H^2 & H^2(S^2) \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

We have the exact sequence $0 \longrightarrow \mathbb{R} \longrightarrow H^2(S^2) \longrightarrow 0$, so the mapping from $\mathbb{R}$ to $H^2(S^2)$ must be injective and surjective, so $H^2(S^2) = \mathbb{R}$. As for $H^1(S^2)$, the map $(x, y) \mapsto x - y$ is surjective, so the map into $H^1(S^2)$ must be the zero map. By exactness at $H^1(S^2)$, we must have the kernel of the map from $H^1(S^2)$ to 0 also be 0, so we must have $H^1(S^2) = 0$. Then, our inductive step uses the fact that the "equator" $U \cap V$ is homotopy equivalent to S^{n-1}.

Another example is the 2-torus, T^2. We can cut the torus in half to get two components U, V, each of which is diffeomorphic to the cylinder, which in turn is homotopy equivalent to S^1. The $U \cap V$ is the disjoint union of 2 cylinders. We then have

$$\begin{array}{cccccc} & T^2 & & U \sqcup V & & U \cap V \\ H^0 & \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\ H^1 & H^1(T^2) & \longrightarrow & \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \oplus \mathbb{R} \\ H^2 & H^2(T^2) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Exer: Show that $H^k(T^2) = \begin{cases} \mathbb{R} & k \in \{0, 2\} \\ \mathbb{R}^2 & k = 1 \end{cases}$, and that $H^1(T^2)$ is generated by $[dx^1]$ and $[dx^2]$.

Thm: If M is a compact, oriented, connected manifold (with $m = \dim M$), then

$$\int_M : H^k(M) \to \mathbb{R}$$

is an isomorphism, so $H^m(M) \cong \mathbb{R}$.