Math 591 Lecture 37

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

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Briefly, think back to integration of forms. Observe:

$$\int\limits_M \mu = \sum_\alpha \int (\phi_\alpha^{-1})^*(\chi_\alpha \mu)$$

We claimed (without proof) that this sum has finitely-many nonzero summands.

Proof: Let $K = \operatorname{supp}\mu$ (note that K must be compact), and $U_\alpha = \operatorname{supp}\chi_\alpha$. Note that $\{U_\alpha\}$ is locally finite: $\forall p \in K$, $\exists V_p$, a neighborhood of p, such that $\{\alpha \mid V_p \cap U_\alpha \neq \emptyset\}$ is finite. Also, $\{V_p\}_{p\in K}$ is naturally a cover of K, so by compactnes, $\exists p_1, \ldots, p_n \in K$ s.t. $\{V_{p_1}, \ldots, V_{p_n}\}$ is a cover of K. Finally,

$$\{\alpha \mid K \cap U_\alpha \neq \emptyset\} \subset \bigcup_{j=1}^n \underbrace{\{\alpha \mid V_{p_j} \cap U_\alpha \neq \emptyset\}}_{\text{finite}}$$

$\square$

Now, we return to working with manifolds with boundary...

The principle is that all definitions on manifold with boundary are exact analogues to the case where $\partial M = \emptyset$. The key difference is the model spaces are open subsets of $\mathbb{H}^n = \{x^n \geq 0\} \subseteq \mathbb{R}^n$, and transition maps are diffeomorphisms between open subsets of $\mathbb{H}^n$ (which carry boundary points to boundary points, and restrict to diffeomorphisms between open sets of $\partial\mathbb{H}^n \cong \mathbb{R}^{n-1}$).

Ex: Notions of smoothness. $f : M \to \mathbb{R}$ is smooth iff for any chart ϕ_α, $f_\alpha = f \circ \phi_\alpha^{-1} : U \to \mathbb{R}$ is smooth (with $U \overset{\text{open}}{\subseteq} \mathbb{H}^n$). In case $U \cap \partial\mathbb{H}^n \neq \emptyset$, smoothness of $f_\alpha : U \to \mathbb{R}$ means $\exists \tilde{f}_\alpha : \tilde{U}_\alpha \to \mathbb{R}$, a smooth extension of f_α to $\tilde{U}_\alpha \overset{\text{open}}{\subseteq} \mathbb{R}^n$. Then $U = \tilde{U}_\alpha \cap \mathbb{H}^n$.

Ex: Existence of partitions of unity, just as before.

Tangent Spaces

$\forall p \in \partial M$, T_pM is still n-dimensional, and is still spanned by $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$. $\frac{\partial}{\partial x^n}$ is well-defined on smooth functions, because such functions extend across the boundary, and last time, we showed the choice of extension doesn't matter.

Note: ∂M inherits a C^∞ manifold structure, by restricting charts of M.

$\forall p \in \partial M$, $T_p(\partial M) \overset{\iota_{*,p}}{\hookrightarrow} T_pM$, where $\iota : \partial M \to M$ is the inclusion. Identify $T_p(\partial M) \subset T_pM$ as a subspace – the image of $\iota_{*,p}$. It's a hyperplane, i.e., $T_p(\partial M)$ has codimension 1.

Note: $(T_pM) \setminus (T_p\partial M)$ has two components: "inward-pointing" vectors and "outward-pointing" vectors.

How do we characterize outward-pointing vectors? Well, say $p \in \partial M$. Let $\gamma : (-\varepsilon, 0] \to M$ smooth with $\gamma(0) = p$. Then define $\dot\gamma(0)$ to be outward pointing.

Orientation of Manifolds with Boundary

Once again, this is an exact analogue of manifolds without boundary. $\forall p \in M$, T_pM has dimension n, two orientations, etc.

Lemma: If M is orientable, then ∂M is orientable.

By convention, $\forall p \in \partial M$, a basis $(b_1, \ldots, b_{n-1})$ of $T_p(\partial M)$ is positive iff $(\nu, b_1, \ldots, b_{n-1})$ is a positive basis of T_pM, *for any outward-pointing* $\nu \in T_pM$. This defines the boundary's orientation.

Ex: Say $M = \mathbb{H}^n$, oriented such that $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ is positive. What is the boundary orientation?

For $n = 2$, $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$ is positive. So if ν is outward-pointing, $\left\{\nu, \frac{\partial}{\partial x^2}\right\}$ is positive, so $\left\{\frac{\partial}{\partial x^1}\right\}$ is positive on $\partial\mathbb{H}^2 \cong \mathbb{R}$.
For $n = 3$, is $(\vec{i}, \vec{j})$ positive? Look at $v = -\vec{k}$ – $(-\vec{k}, \vec{i}, \vec{j})$ is negative. So we claim that $\partial\mathbb{H}^n$ is $(-1)^n$ times the standard orientation of $\mathbb{R}^n$.

Stokes' Theorem

Thm: (Stokes' Theorem) Let M be a manifold (possibly with boundary). Let $\mu \in \Omega_0^{n-1}(M)$. Assume M is oriented; give ∂M the boundary orientation. Then

$$\int_{\partial M} \iota^*\mu = \int_M d\mu$$

(where $\iota : \partial M \hookrightarrow M$ is the inclusion map). One often omits the ι^*, so we say

$$\int_{\partial M} \mu = \int_M d\mu$$

(If $\partial M = \emptyset$, then $\int_M d\mu = 0$.)

Observe: $\iota^*\mu$ is a top degree form on the boundary.

Ex: $M = [a, b]$, $n = 1$, $\mu \in \Omega_0^0(M) \cong C^\infty(M)$. Then $\int_{\partial M} \mu = \mu(b) - \mu(a) = \int_M df$. The "$-$" sign comes because of the orientation of ∂M is outward-pointing.

Ex: $n = 2$. Then we get Green's Theorem:
$\mu = P\,dx + Q\,dy$. $d\mu = (Q_x - P_y)\,dx \wedge dy$.
The orientation on ∂M comes from the right-hand rule.

$$\int_{\partial M} P\,dx + Q\,dy = \iint_M (Q_x - P_y)\,dx \wedge dy$$

Exer: For $n = 3$, we get the usual Stokes' theorem.

Proof of Stokes' Theorem

First, assume it holds for $\mathbb{H}^n$. Then it follows for $\mu \in \Omega_0^{n-1}(M)$ if μ is supported in the domain of a chart ϕ:

$$\int_{\partial M} \mu = \int_{\partial\mathbb{H}^n} (\phi^{-1})^*\iota^*\mu \stackrel{(1)}{=} \int_{\mathbb{H}^n} d(\phi^{-1})^*\mu = \int_{\mathbb{H}^n} (\phi^{-1})^* d\mu = \int_M d\mu$$

with (1) by the definition of integrals, plus our assumption about μ. In general (still under the assumption that Stokes' theorem holds for $\mathbb{H}^n$), we use a partition of unity $\{\chi_j\}$ subordinate to an atlas.

$$\int_{\partial M} \iota^*\mu = \sum_j \int_{\partial M} \iota^*(\chi_j\mu) = \sum_j \int_M d(\chi_j\mu) = \sum_j \int_M d(\chi_j) \wedge \omega + \int_M \chi_j d\mu = \int_M d\mu + \int_m \Bigg(\underbrace{\sum_j d\chi_j}_{\substack{=d\sum_j \chi_j = 0 \\ \text{b/c } \sum_j \chi_j \equiv 1}} \Bigg) \wedge \mu = \int_M d\mu$$

So finally, it's enough to prove Stokes' Theorem for $M = \mathbb{H}^n$. Write

$$\mu = \sum_{i=1}^{n} a_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n, \quad a_i \in C^\infty(\mathbb{H}^n)$$

Then

$$d\mu = \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial x^j} (-1)^{j-1} \right) \underbrace{dx^1 \wedge \cdots \wedge dx^n}_{\text{standard volume form}}$$

Because we assume μ has compact support, $\operatorname{supp} \mu$ is bounded, so $\exists R > 0$ s.t.

$$\operatorname{supp} \mu \subseteq \underbrace{[-R, R] \times \cdots \times [-R, R]}_{n-1 \text{ times}} \times [0, R]$$

Thus,

$$\int\limits_{\mathbb{H}^n} d\mu = \sum_{i=1}^{n} (-1)^{i-1} \int\limits_{0}^{R} \int\limits_{-R}^{R} \cdots \int\limits_{-R}^{R} \frac{\partial a_i}{\partial x^j} \, dx^1 \cdots dx^n$$

So $\forall i$, first do $\int_{-R}^{R} \frac{\partial a_i}{\partial x^i} \, dx^i$. For $i = 1, \ldots, n-1$,

$$\int\limits_{-R}^{R} \frac{\partial a_i}{\partial x^i} \, dx^i = a(x^1, \ldots, R, \ldots, x^n) - a_i(x^1, \ldots, -R, \ldots, x^n) = 0 - 0 = 0 \qquad \int\limits_{0}^{R} \frac{\partial a_n}{\partial x^n} \, dx^n = -a_n(x^1, \ldots, x^{n-1}, R)$$

Thus,

$$\int\limits_{\mathbb{H}^n} d\mu = \underbrace{-(-1)^{n-1}}_{=(-1)^n} \int\limits_{\mathbb{R}^{n-1}} a_n(x^1, \ldots, x^{n-1}, 0) \, dx^1 \cdots dx^{n-1}$$

And on the other hand,

$$\int\limits_{\partial \mathbb{H}^n} \mu = \underbrace{(-1)^n}_{\substack{\uparrow \\ \text{boundary orientation}}} \int\limits_{\mathbb{R}^{n-1}} a_n(x^1, \ldots, x^{n-1}, 0) \, dx^1 \cdots dx^{n-1}$$

(with the 0 appearing in the a_n term because $\iota^*(dx^n) = 0$.)
$\square$