# Math 591 Lecture 11

Thomas Cohn

9/25/20

## Tangent Vectors

Last time, we proved that for $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ chart, $\phi = (x^1, \ldots, x^n)$, that $\forall v \in T_pM$, we can write

$$v = \sum_{i=1}^{n} v([x^i]) \left.\frac{\partial}{\partial x^i}\right|_p$$

This is based on:
**Thm:** If $g : B \to \mathbb{R}$, with $B \subseteq \mathbb{R}^n$ being the open ball centered at the origin, then there exist $g_{ij} \in C^\infty(B)$ s.t. $\forall r \in B$,

$$g(r) = g(0) + \sum_{j=1}^{n} r^j \frac{\partial g}{\partial r^j}(0) + \frac{1}{2} \sum_{i,j=1}^{n} r^i r^j g_{ij}(r)$$

with $g_{ij}(0) = \frac{\partial^2 g}{\partial r^i \partial r^j}(0)$.

Proof: Start with $g(r) = g(0) + \int_0^1 \frac{d}{dt} g(tr)\, dt$. Then by the fundamental theorem of calculus, this is equal to

$$= g(0) + \int_0^1 \sum_{j=1}^{n} r^j \frac{\partial g}{\partial r^j}(tr)\, dt = g(0) + \sum_{j=1}^{n} r^j \underbrace{\int_0^1 \frac{\partial g}{\partial r^j}(tr)\, dt}_{g_j(r) \in C^\infty} = g(0) + \sum_{j=1}^{n} r^j g_j(r)$$

We can then repeat this argument with each $g_j$, so for each $j$, there are some $g_{ji} \in C^\infty$ s.t.

$$g_j(r) = g_j(0) + \sum_{i=1}^{n} r^i g_{ji}(r)$$

(The exact computation may be off here by a factor of 2, due to symmetry.)
Observe that

$$g_j(0) = \int_0^1 \frac{\partial g}{\partial r^j}(0)\, dt = \frac{\partial g}{\partial r^j}(0)$$

Plugging the $g_j$'s back in, we get

$$g(r) = g(0) + \sum_{j=1}^{n} r^j g_j(0) + \sum_{j=1}^{n} r^j \sum_{i=1}^{n} r^i g_{ji}(r) = g(0) + \sum_{j=1}^{n} r^j g_j(0) + \sum_{i,j=1}^{n} r^j r^i g_{ji}(r)$$

$\square$

## Tangent Vectors and Curves

Let $p \in U \overset{\text{open}}{\subseteq} M$, $\phi : U \to \mathbb{R}^n$ chart, $\phi = (x^1, \dots, x^n)$. Then let $\gamma$ so that

$$\begin{tikzcd}
(-\varepsilon, \varepsilon) \arrow[r, "\gamma"] \arrow[dr, "\phi\circ\gamma"'] & U \arrow[d, "\phi"] \arrow[dr, "f"] & \\
& \mathbb{R}^n \arrow[r, "f_\psi"'] & \mathbb{R}
\end{tikzcd}$$

Previously, we defined $\dot\gamma(0) \in T_pM$ so that $\dot\gamma(0)([f]) = \frac{d}{dt}(f \circ \gamma)\big|_{t=0}$, where $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$.

Computation of $\dot\gamma(0)$ in coordinates:

**Lemma:** Let $(\phi \circ \gamma)(t) = (x^1(t), \dots, x^n(t))$, defined by $x^i(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}$. Then

$$\dot\gamma(0) = \sum_{j=1}^n \frac{dx^j(t)}{dt}\bigg|_{t=0} \frac{\partial}{\partial x^j}\bigg|_p$$

> Proof: Let $f : U \to \mathbb{R}$. Then $f \circ \gamma = (f \circ \phi^{-1}) \circ (\phi \circ \gamma) = f_\phi \circ (\phi \circ \gamma)$. Use the chain rule on the right-hand side. Then
>
> $$\dot\gamma(0)[f] = \frac{d}{dt}(f \circ \gamma)(t)\bigg|_{t=0} = \sum_{j=1}^n \underbrace{\frac{\partial f_\phi}{\partial r^j}((\phi \circ \gamma)(t))}_{\frac{\partial f}{\partial x^j}(\gamma(t))} \frac{dx^j(t)}{dt}\bigg|_{t=0}$$
>
> $\square$

**Cor:** Any $v \in T_pM$ is equal to $\dot\gamma(0)$ for some curve $\gamma$.
Proof: Choose a chart $(U, \phi)$ so that $\phi(p) = 0$. Then $v = \sum_{j=1}^n v_j \frac{\partial}{\partial x^j}\big|_p$, with each $v_j \in \mathbb{R}$. Define $\gamma$ by $x^j(t) = tv_j$, $\forall j \in \{1, \dots, n\}$, and letting this define $\phi \circ \gamma$.

$$\begin{tikzcd}
(-\varepsilon, \varepsilon) \arrow[r, "\gamma"] \arrow[dr, "\phi\circ\gamma"'] & U \arrow[d, "\phi"] \\
& \mathbb{R}^n
\end{tikzcd}$$

Then $\gamma(p) = \phi^{-1}(tv_1, \dots, tv_n)$. $\square$

## Smooth Maps Between Manifolds and Tangent Spaces

Let $F : M \to N$ be a smooth map between smooth manifolds $M$ and $N$. Let $p \in M$, with $q = F(p) \in N$.

Observe: Given any $f : V \to \mathbb{R}$, $q \in V \overset{\text{open}}{\subseteq} N$, we have

$$M \xrightarrow{F} V \xrightarrow{f} \mathbb{R}$$

**Defn:** Consider $F^{-1}(V)$, an open neighborhood of $p$. $f \circ F : F^{-1}(V) \to \mathbb{R}$. This gives us a map

$$\begin{aligned} F^* : C_q^\infty(N) &\to C_p^\infty(M) \\ [f] &\mapsto [f \circ F] \end{aligned}$$

This is the <u>pullback map on germs</u>. Note that this is a ring morphism!

By duality, we can pushforward tangent vectors.

**Defn:** If $v \in T_pM$, we define the <u>pushforward</u> of $v$, $F_{*,p}(v) : C_q^\infty(N) \to \mathbb{R}$ by $F_{*,p}(v)([f]) = v(F^*([f])) \in \mathbb{R}$.

Claim: $F_{*,p}(v) \in T_qN$, i.e., $F_{*,p}(v)$ is also a derivation.
Rough proof: Recall that $F^*$ is a ring morhpism. This, combined with the fact that $v$ is a derivation, implies that $F_{*,p}(v)$ is a derivation. $\square$

Conclusion: We obtain $F_{*,p} : T_pM \to T_{F(p)}N$.

**Defn:** We can take its dual: $F_p^* : T_{F(p)}^*N \to T_p^*M$.

**Lemma:** $F_p^*$ is linear.

**Lemma:** $F_{*,p}(\dot{\gamma}(0)) = \frac{d}{dt}(F \circ \gamma)\big|_{t=0}$.

This final lemma is very useful for computation!