Math 591 Lecture 22

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Note: The lecture on 10/21 was devoted to review for the exam.

Recall: We defined, for $X, Y \in \mathfrak{X}(M)$, $[X, Y]$ by regarding X and Y as operators on $C^\infty(M)$. Then $[X, Y]$ is the commutator $X \circ Y - Y \circ X$.

Properties:
- $[\cdot, \cdot]$ is bilinear over $\mathbb{R}$.
- $[\cdot, \cdot]$ satisfies the Jacobi identity: $\forall X, Y, Z \in \mathfrak{X}(M)$, $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.
  Proof: Just compute! It's very straightforward, and only uses the fact that composition is associative.

Defn: A Lie algebra $\mathfrak{g}$ is a vector space (over $\mathbb{R}$), together with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which
- Is skew-symmetric: $\forall A, B \in \mathfrak{g}$, $[A, B] = -[B, A]$.
- Satisfies the Jacobi identity.

Ex: For any manifold M, $\mathfrak{g} = \mathfrak{X}(M)$ is a Lie algebra.

Ex: A finite dimensional example: Let $\mathfrak{g} = \{n \times n \text{ skew-symmetric real matrices}\} \cong T_I O(n)$, with $[A, B] = AB - BA$ (using matrix multiplication).
Check: $[A, B] \in \mathfrak{g}$. Well, $(AB - BA)^T = B^T A^T - A^T B^T = (-B)(-A) - (-A)(-B) = BA - AB = -(AB - BA)$.
We say $o(n) \overset{\text{def}}{=} \mathfrak{g}$.

At some point (next week), we will prove the following:
Claim: Given $A \in T_I O(n)$, define $A^\sharp \in \mathfrak{X}(O(n))$ by $\forall g \in O(n)$, $A^\sharp_g = gA$, and we claim that $A^\sharp_g \in T_g O(n)$.

To interpret the formula, let $\begin{aligned} L_g : O(n) &\to O(n) \\ k &\mapsto gk \end{aligned}$. Then $A^\sharp_g = (L_g)_{*,I}(A)$, i.e., $(L_g)_{*,I} : T_I O(n) \to T_g O(n)$.

Then $\forall A, B \in T_I O(n)$, $\underbrace{[A, B]^\sharp}_{\text{matrix commutator}} = \underbrace{[A^\sharp, B^\sharp]}_{\text{vector field commutator}}$. So $\sharp : o(n) \rightarrowtail \mathfrak{X}(O(n))$ is an injection, and $A^\sharp_I = A$.

So $o(n)$ appears as a (finite-dimensional) Lie subalgebra of $\mathfrak{X}(O(n))$.
There is more about this to come in the next chapter...

Now, back to the general study of vector fields. We've looked at vector fields as operators. Now, let's look at them as generators of dynamics.

Defn: Let $X \in \mathfrak{X}(M)$. An integral curve of X is a map

$$c : (a, b) \to M \qquad a < b; a, b \in \mathbb{R} \cup \{\pm\infty\}$$

such that $\forall t \in (a, b)$ "time", $\dot{c}(t) = X_{c(t)} \in T_{c(t)} M$.

Thm: (Existence and Uniqueness of Integral Curves)
Given $X \in \mathfrak{X}(M)$, $p \in M$, then there exists an integral curve of X, $c : (a,b) \to M$, with $a < 0 < b$, such that $c(o) = p$. Moreover, any two such curves agree on the intersection of their domains.

Proof: Reduce to the Euclidean case near p. Introduce local coordinates near p. WOLOG $\phi(p) = 0 \in \mathbb{R}^n$. We can write

$$X|_U = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x^i}$$

where the f_i are smooth functions on U. Then we have

$$\begin{tikzcd} (a,b) \arrow[r, "c"] \arrow[dr, "\tilde{c}"'] & U \arrow[d, "\phi"] \\ & \mathbb{R}^n \end{tikzcd}$$

with the unkown curve $\tilde{c}(t) = (x^1(t), \ldots, x^n(t))$, with each $x^i : (a,b) \to \mathbb{R}$ unknown. Note:

$$\begin{cases} \dot{c}(t) = X_{c(t)} \\ c(0) = p \end{cases} \quad \Leftrightarrow \quad \begin{cases} \forall i,\ x^i(t) = f_i(c(t)) \\ x^i(0) = 0 \end{cases}$$

Well, $f^i(c(t)) = \tilde{f}_i(x^1(t), \ldots, x^n(t))$, and these are real-valued functions, so $\tilde{x}^i(t) = \tilde{f}_i(x^1(t), \ldots, x^n(t))$. So we have a system of ordinary differential equations (ODEs), with an initial condition.
Well, the derivatives $\frac{dx^i(t)}{dt} = \dot{x}^i(t)$ are the left-hand side, so we can just quote the existence and uniqueness theorems from the theory of ordinary differential equations. Thus, we get the theorem in a coordinate chart. Now, consider a covering of c by overlapping coordinate charts... $\square$

Ex: Let $M = (\mathbb{R}_{>0})^2$ – the upper-left quadrant of $\mathbb{R}^2$, with standard coordinates (x,y).
Let $X = yx^2 \frac{\partial}{\partial x}$.

Well, $\dot{x} = yx^2$ and $\dot{y} = 0$, so $\forall t$, $y(t) = y(0)$. So $\frac{dx}{dt} = y(0)x^2$. We can solve this using separation of variables:

$$\begin{aligned} \frac{dx}{x^2} &= y(0)dt \\ \int x^{-2}\, dx &= \int y(0)\, dy \\ -\frac{1}{x(t)} + \frac{1}{x(0)} &= y(0)t \\ \frac{1}{x(t)} &= \frac{1}{x(0)} - yt = \frac{1 - y(0)x(0)t}{x(0)} \\ x(t) &= \frac{x(0)}{1 - x(0)y(0)t} \end{aligned}$$

And we also have $y(t) = y(0)$. Well, x will escape to infinity at the "escape time" $T = \frac{1}{x(0)y(0)}$. So the maximal domain of an integral curve with these initial conditions is $(-\infty, T)$.

This demonstrates a problem – if we want to look at maximal integral curves, the maximal domain may depend on the initial conditions.