# Math 591 Lecture 8

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9/18/20

**Defn:** Let $M, N$ be $C^\infty$ manifolds, and $F : M \to N$ a continuous map. Let $p \in M$. Then we say $F$ is smooth at $p$ iff there exist charts $(U, \phi)$ of $M$ and $(V, \psi)$ of $N$ s.t. $p \in U$, $F(p) \in V$, and

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^n$$

is $C^\infty$.

Observe: Since $F$ is continuous, $F^{-1}(V)$ is open, so $F^{-1}(V) \cap U$ is an open neighborhood of $p$. Thus, $\phi(F^{-1}(V) \cap U)$ is open in $\mathbb{R}^m$.

**Defn:** Let $M, N$ be $C^\infty$ manifolds, $F : M \to N$ continuous. Then $F$ is smooth iff $\forall p \in M$, $F$ is smooth at $p$.

**Lemma:** Let $M, N$ be $C^\infty$ manifolds, $F : M \to N$ continuous. Then $F$ is smooth iff there are atlases $\{(U_\alpha, \phi_\alpha)\}$ of $M$ and $\{(V_\beta, \psi_\beta)\}$ of $N$ s.t. $\forall \alpha, \beta$, $\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi_\alpha(F^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n$ is smooth. This, in turn, is true iff for any pair of atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, the previous condition holds.
Proof: (exercise)

The key outcome is that if a function is smooth according to one atlas, it's smooth according to all atlases.

**Ex:**

$$\begin{aligned} \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{GL}(n,\mathbb{R}) \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{GL}(n,\mathbb{R}) \\ g &\mapsto g^{-1} \end{aligned} \qquad \text{are smooth.}$$

$$\begin{aligned} O(n) \times O(n) &\to O(n) \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} O(n) &\to O(n) \\ g &\mapsto g^{-1} \end{aligned} \qquad \text{are smooth.}$$

**Defn:** A Lie group $G$ is a group which also has a $C^\infty$ structure s.t.

$$\begin{aligned} G \times G &\to G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are smooth.

## Tangent and Cotangent Spaces

We want to construct tangent vectors without requiring an ambient space!
Idea: Vectors in $\mathbb{R}^n$ define "directional" derivatives.

Pick $p \in U \overset{\text{open}}{\subseteq} \mathbb{R}^n$, and $v \in \mathbb{R}^n$. Then if $f : U \to \mathbb{R}$ is $C^\infty$, we can define $D_v f(p) = \nabla f(p) \cdot v$.

Remark: We can regard $D_v$ as an operator $C^\infty \ni f \mapsto D_v f(p) \in \mathbb{R}$. It has the following properties:
1) Linear over $\mathbb{R}$: $D_v(f + cg)(p) = D_v f(p) + c D_v g(p)$.
2) Leibniz' rule: $D_v(fg)(p) = f(p) D_v g(p) + D_v f(p) g(p)$.

This was all motivation. Now, for the formalization.

**Defn:** Let $M$ be a smooth manifold, $p \in M$. Then the space of germs of functions of $M$ at $p$ is

$$C_p^\infty(M) = \{(f : U \to \mathbb{R}, U) \mid U \subseteq M \text{ open }, p \in U, f \in C^\infty\} / \sim$$

where $(f,U) \sim (g,V) \Leftrightarrow \exists W \subseteq U \cap V$ s.t. $p \in W$ and $f|_W = g|_W$.
A germ at $p$ is an equivalence class $[f] = [(f,U)]$.

Notation: Given $(f,U)$ as above and $p \in U$, $[f]$ is the calss of $(f,U) \in C_p^\infty(M)$.

**Lemma:** $C_p^\infty(M)$ is an $\mathbb{R}$-vector space and a ring.

a) $[f] + c[g] \overset{\text{def}}{=} [f + cg]$

b) $[f] \cdot [g] \overset{\text{def}}{=} [fg]$ (defined by $fg|_{U \cap V} : U \cap V \to \mathbb{R}$)

EXER: Show the remaining properties.

**Defn:** A derivation on $M$ at $p$ is an $\mathbb{R}$-linear map $D : C_p^\infty \to \mathbb{R}$ s.t. $\forall [f], [g] \in C_p^\infty(M)$, $D([f]g]) = f(p)D[g] + g(p)D[f]$.

Observe: $f(p) = [f](p) \in \mathbb{R}$ is well defined by $[f]$.

**Defn:** The tangent space to $M$ at $p$ is $T_pM = \{\text{all derivations of } M \text{ at } p\}$.