Math 635 Lecture 20

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Review: Derivation of the first variation formula. Suppose $\gamma \in \Omega_{pq}^a$, $\gamma : [0,a] \to M$, with $\gamma(0) = p$ and $\gamma(a) = q$. Let

$$f : \underset{s}{(-\varepsilon, \varepsilon)} \times \underset{t}{[0,a]} \to M$$

be a proper variation of γ, which is C^∞ on rectangles $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$, where $0 = t_0 < t_1 < \cdots < t_N = a$ is a partition. We have our energy formula

$$E(s) = \frac{1}{2} \int_0^a ||\partial_t f(s,t)||^2 \, dt$$

The key step is computing

$$\frac{dE}{ds} = \int_0^a \left\langle \frac{D}{ds} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt = \int_0^a \left\langle \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt = -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt} \partial f t \right\rangle dt + \text{(boundary terms)}$$

So really, we're intergating on each segment $[t_i, t_{i+1}]$:

$$\int_{t_i}^{t_{i+1}} \frac{d}{dt} \langle \partial_s f, \partial_t f \rangle \, dt \bigg|_{s=0} = \left\langle V(t_{i+1}), \dot{\gamma}(t_{i+1}^-) - \dot{\gamma}(t_i^+) \right\rangle$$

where $V = \partial_s f|_{s=0} \in \Gamma_\gamma(TM)$ is the variation field. When we sum over i, we get

$$\left\langle V(t_i), \dot{\gamma}(t_i^-) - \dot{\gamma}(t_i^+) \right\rangle = \langle V(t_i), \Delta\dot{\gamma}(t_i) \rangle$$

Cor: If γ is such that, for all proper variations of γ, $\frac{dE}{ds}(s = 0) = 0$, then γ is a geodesic. (And the converse is true as well.)

Proof: Choose $V(t)$ as follows

$$V(t) = \begin{cases} \frac{D}{dt}\dot{\gamma}(t) & t \neq t_i, \forall i \\ \Delta\dot{\gamma}(t_i) & t = t_i \end{cases}$$

Then we get

$$0 = \frac{dE}{ds}(0) = \int_0^a \left|\left| \frac{D}{dt} \dot{\gamma}(t) \right|\right|^2 dt + \sum ||\Delta\dot{\gamma}(t)||^2$$

This is the case iff $\left|\left|\frac{D}{dt}\dot{\gamma}(t)\right|\right| \equiv 0$, and $\forall i$, $\Delta\dot{\gamma}(t_i) = 0$. Thus, γ is a geodesic. $\square$

Observe that one can replace the "energy" functional $E : \Omega_{pq}^a \to \mathbb{R}$ with other functionals.

Ex: $V \in C^\infty(M)$, $\mathscr{L} : \Omega_{pq}^a \to \mathbb{R}$
$\gamma \mapsto \int_0^a \frac{1}{2} ||\dot{\gamma}||^2 - V(\gamma(t)) \, dt$
We call this functional the Lagrangian.
Question: Which curves satisfy $\frac{d\mathscr{L}}{ds}(0) = 0$ for all variations? The answer is curves that follow Newton's second law, $\frac{D}{dt}\dot{\gamma} = -\nabla V(\gamma(t))$.

Ex: Given a particle rolling from a point p to a point q in a vertical plane under the influence of gravity, what curve will minimize the time it takes? The answer is the brachistochrone curve.

Now, we examine the second variation. Let $\gamma \in \Omega^a_{pq}$ be a geodesic, and $E : \Omega^a_{pq} \to \mathbb{R}$. Let f be a proper C^∞ variation of γ (i.e. no jumps). Then compute $\frac{d^2}{ds^2} E(s)\Big|_{s=0}$. Well,

$$\frac{d}{ds} E(f_s) = -\int_0^a \left\langle \partial_s f, \frac{D}{dt} \partial_t f \right\rangle \, dt$$

So

$$\frac{d^2}{ds^2} E(s) = -\cancel{\int_0^a \left\langle \frac{D}{ds} \partial_s f, \frac{D}{dt} \partial_t f \right\rangle} \, dt - \int_0^a \left\langle \partial_s f, \frac{D}{ds}\frac{D}{dt} \partial_t f \right\rangle \, dt$$

where the first term is eliminated because $\frac{D}{dt}\partial_t f$ vanishes at $s = 0$, because γ is a geodesic.

Lemma: $\left[\frac{D}{ds}, \frac{D}{dt}\right] = \mathcal{R}(\partial_s f, \partial_t f)$ as an operator acting on vector fields V along f. (Recall: $\mathcal{R}$ is the curvature of ∇.)

$$\begin{tikzcd} & TM \arrow[d] \\ f : (-\varepsilon, \varepsilon) \times [0, a] \arrow[ur, "V"] \arrow[r] & M \end{tikzcd}$$

Recall that $\mathcal{R}(\partial_s f, \partial_t f)_{f(s,t)} : T_{f(s,t)}M \to T_{f(s,t)}M$. So the lemma really says that

$$\frac{D}{ds}\frac{D}{dt} V - \frac{D}{dt}\frac{D}{ds} V = \mathcal{R}(\partial_s f, \partial_t f)(V)$$

So why is the lemma true? Well, assume for simplicity that f is an embedding away from p and q. We can extend $\partial_s f$ and $\partial_t f$ to fields X and Y (respectively) on M. Then $\frac{D}{ds}$ " $=$ " ∇_X and $\frac{D}{dt}$ " $=$ " ∇_Y, and by the definition of $\mathcal{R}$,

$$[\nabla_X, \nabla_Y] = \mathcal{R}(X, Y) + \nabla_{[X,Y]}$$

But $[X, Y]|_{\operatorname{Im} f} = 0$, because $X = \partial_s f$ and $Y = \partial_t f$ on $\operatorname{Im} f$ Thus,

$$\begin{aligned} \frac{d^2}{ds^2} E(s)\Big|_{s=0} &= -\int_0^a \left\langle V, \left(\frac{D}{dt}\frac{D}{ds} + \mathcal{R}(\partial_s f, \partial_t f) \right) \partial_t f \Big|_{s=0} \right\rangle dt \\ &= -\int_0^a \left\langle V, \frac{D}{dt}\frac{D}{ds} V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt \\ &= -\int_0^a \left\langle V, \frac{D^2}{dt^2} V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt \end{aligned}$$

Thus, we conclude that

$$\frac{d^2 E}{ds^2}\Big|_{s=0} = -\int_0^a \left\langle V, \frac{D^2}{dt^2} V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt$$

Observe: This is quadratic in V. But of course this is true, since it's a Hessian!

$$\langle V, \mathcal{R}(V, \dot\gamma)(\dot\gamma) \rangle \sim \underbrace{\langle \mathcal{R}(V, W)(W), V \rangle}_{\text{scalar, related to ``sectional curvature''}}$$

We can think of $\frac{D^2}{dt^2} V + \mathcal{R}(V, \dot\gamma)(\dot\gamma)$ as an operator on $V \in \Gamma_\gamma(TM)$ called the "Jacobi operator". Elements of its kernel are called "Jacobi fields".