

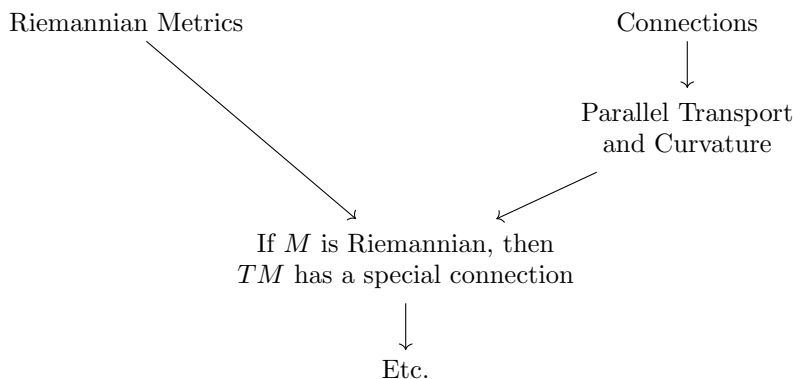
# Math 635 Lecture 6

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A brief roadmap for the next few weeks:



First, some motivation... (See also Lee Riemannian Geometry, Chapter 4)

Start with  $\gamma : (a, b) \rightarrow \mathbb{R}^n$ . Let  $Y = \sum_i f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$ , a smooth vector field in  $\mathbb{R}^n$ . We can compute  $\frac{d}{dt}Y(\gamma(t))|_{t=t_0}$  in  $\mathbb{R}^n$ . But what's really happening? Well, we're computing

$$\left. \frac{d}{dt}Y(\gamma(t)) \right|_{t=t_0} = \lim_{h \rightarrow 0} \frac{Y(\gamma(t_0 + h)) - Y(\gamma(t_0))}{h}$$

But we can only do this in  $\mathbb{R}^n$ , not on manifolds in general! Strictly speaking,  $Y(\gamma(t_0 + h)) \in T_{\gamma(t_0+h)}\mathbb{R}^n$  and  $Y_{\gamma(t_0)} \in T_{\gamma(t_0)}\mathbb{R}^n$ . We can take their difference because we're identifying all tangent spaces of  $\mathbb{R}^n$  with each other, using translations of  $\mathbb{R}^n$ . And  $\{\text{translations of } \mathbb{R}^n\} \cong \mathbb{R}^n$  as a vector space. In other words, we can translate vectors in  $\mathbb{R}^n$  "parallel to themselves".

For  $Y = \sum_i f^i \frac{\partial}{\partial x^i}$ , we get a formula:

$$\left. \frac{d}{dt}Y(\gamma(t)) \right|_{t=t_0} = \sum_{i=1}^n df_{\gamma(t_0)}^i(\dot{\gamma}(t_0)) \frac{\partial}{\partial x^i} \stackrel{\text{def}}{=} (\bar{\nabla}_{\dot{\gamma}(t_0)} Y)(\gamma(t_0))$$

(Note that  $\frac{\partial}{\partial x^i}$  is a constant frame on  $\mathbb{R}^n$ , so we can use  $\frac{\partial}{\partial x^i}$  and  $\frac{\partial}{\partial x^i}|_p$  interchangeably).

**Defn:** If  $Y = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$ ,  $p \in \mathbb{R}^n$ , and  $v \in T_p M$ , we define

$$(\bar{\nabla}_v Y)(p) \stackrel{\text{def}}{=} \sum_{i=1}^n df_p^i(v) \frac{\partial}{\partial x^i}$$

We can think of  $(\bar{\nabla}_v Y)(p)$  as a vector, which only depends on the values of  $Y$  along a curve  $\gamma$  as above.

The question remains: Is there something analogous to this on manifolds? It may look a bit like a Lie derivative, but note that  $\bar{\nabla}$  is **not** a Lie derivative!

Recall: Given  $X, Y \in \mathfrak{X}(M)$ , we can define  $\mathcal{L}_X Y$  using the flow  $\varphi$  of  $X$ :

$$(\mathcal{L}_X Y)(p) = \lim_{t \rightarrow 0} \frac{(\varphi_{-t})_{*, \varphi_t(p)}(X_{\varphi_t(p)}) - X_p}{t}$$

In this case, we need  $X$  as a vector field, whereas above, we just need a vector. And  $\mathcal{L}_X Y$  is dependent on  $X$ , but there are infinitely many vector fields  $X$  s.t.  $X_p = v$  (with  $v$  as above).

In fact, we need some additional structure on the manifold, because we cannot natural identify  $T_p M$  with  $T_q N$ , when  $p \neq q$ . This additional structure is called a connection.

**Defn:** Let  $\mathcal{E} \rightarrow M$  be a vector bundle. A connection on  $\mathcal{E}$  is an operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{E}) &\rightarrow \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

that satisfies:

- 1)  $\forall X, Y \in \mathfrak{X}(M), \forall s \in \Gamma(\mathcal{E}), \nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
- 2)  $\forall f \in C^\infty(M), \nabla_{fX} s = f \nabla_X s$
- 3)  $\nabla_X (fs) = f \nabla_X s + X(f)s = f \nabla_X s + df(X)s$

Because of properties 1 and 2, we say that a connection is “linear in  $X$  over  $C^\infty(M)$ ”.

Note that although our definition above uses vector fields, we will show that this dependence is pointwise.

**Ex:**  $\nabla = \bar{\nabla}$  on  $\mathcal{E} = T\mathbb{R}^n$ .

**Prop:** If  $\nabla$  is a connection on  $\mathcal{E} \rightarrow M$ , then  $\forall X \in \mathfrak{X}(M), s \in \Gamma(\mathcal{E}), p \in M, (\nabla_X s)(p) \in \mathcal{E}_p = \pi^{-1}(p)$  only depends on  $X_p$  and the values of  $s$  in an arbitrarily small open neighborhood of  $p$ .

Proof: Let  $U$  be a neighborhood of  $p$ ;  $\chi$  a bump function supported on  $U$ , with  $\chi \equiv 1$  on  $V \subseteq^{\text{open}} U$ , a smaller open neighborhood of  $p$ . Consider  $\nabla_X (\chi s) = X(\chi)s + \chi \nabla_X s$ . Evaluate at  $p$ : the right hand side is just  $(\nabla_X s)(p)$  because  $\chi \equiv 1$  on  $V$  and  $X(\chi) \equiv 0$  on  $V$  (because  $\chi$  is constant on  $V$ , and  $X$  is a derivation). The computation of  $(\nabla_X s)(p)$  can be localized to, say, a coordinate neighborhood of  $p$ .

Let  $X = \sum_i a^i \frac{\partial}{\partial x^i}$  in local coordinates. Then, by  $C^\infty$  linearity of  $\nabla$ ,

$$(\nabla_X s)(p) = \sum_i a^i(p) (\nabla_{\frac{\partial}{\partial x^i}} s)(p)$$

If  $X(p) = 0$  (which is true iff  $\forall j, a^j(p) = 0$ ), then  $(\nabla_X s)(p) = 0$ . So if  $X(p) = \tilde{X}(p)$ , then  $(\nabla_X s)(p) = (\nabla_{\tilde{X}} s)(p)$ .  $\square$

Observe: Given  $\nabla$  and  $s$ ,  $(\nabla_X s)(p) \in \mathcal{E}_p$  depends only on  $X(p)$ , and does so linearly! So  $\nabla$  and  $s$  define a map

$$\begin{aligned} T_p M &\rightarrow \mathcal{E}_p \\ v &\mapsto (\nabla_v s)(p) \end{aligned}$$

which is itself an element of  $T_p^* M \otimes \mathcal{E}_p$ . Therefore,  $\nabla$  can be thought of as an operator  $\nabla : \Gamma(\mathcal{E}) \rightarrow \mathcal{E}(T^* M \otimes \mathcal{E})$ , whose image is “ $\mathcal{E}$ -valued differential forms”.

## Local Expression of a $\nabla$

Let  $\mathcal{E} \rightarrow M$  be a vector bundle, with connection  $\nabla$ ,  $U \subseteq^{\text{open}} M$ , and  $(E_1, \dots, E_r)$  a moving frame of  $\mathcal{E}$  over  $U$ . That is,  $\forall j, E_j \in \Gamma(\mathcal{E}|_U)$ , and at each  $p \in U$ ,  $(E_1(p), \dots, E_r(p))$  is a basis of  $\mathcal{E}_p$ . So if  $s \in \Gamma(\mathcal{E}|_U)$ , then  $\exists f^i \in C^\infty(U)$  s.t.  $s = \sum_j f^j E_j$ . So  $\forall X \in \mathfrak{X}(U)$ , we get

$$\nabla_X s = \sum_{j=1}^r f^j \nabla_X E_j + X(f^j) E_j$$

But

$$\nabla_X E_j = \sum_{i=1}^r \theta_j^i(X) E_i$$

By the discussion above,  $\forall i, j, \theta_j^i \in \Omega^1(U)$ , a one-form. So we can define  $\vartheta = (\theta_j^i)$ , an  $r \times r$  matrix of one-forms on  $U$ , depending on the moving frame  $(E_1, \dots, E_r)$ . In fact, this  $\vartheta$  determines  $\nabla$  on  $U$ !