## Math 635 Lecture 12

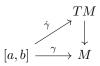
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Recall:

**Defn:**  $\gamma:[a,b]\to M$  (for M a Riemannian manifold) is a geodesic iff  $\frac{D}{dt}\dot{\gamma}=0$ .

Recall:  $\dot{\gamma}$  is the natural lift of  $\gamma$  along  $\gamma$ . We say  $\dot{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$ , so there's some ambiguity in the notation.



Review: In coordinates on  $U \subset M$ , we write  $\gamma(t) = (x^1(t), \dots, x^n(t))$ , with each  $x^i \in C^{\infty}([a, b], M)$ . Then  $\gamma$  is a geodesic iff  $\ddot{x}^k(t) = -\dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(\gamma(t))$ , where  $\Gamma^k_{ij}: U \to \mathbb{R}$  are the Christoffel symbols.

Observe: If  $\nabla$  is trivial, i.e., the "flat case", then  $\Gamma_{ij}^k = 0$ . So  $\ddot{x}^k = 0$ , and  $\forall k, x^k(t) = tv^k(0) + x^k(0)$ . See Do Carmo, Chapter 3, §2 for more details.

We want to rewrite the geodesic equations, locally, as a first order system in twice as many unknowns. We introduce  $v^1, \ldots, v^n$ , which we call the "velocities", such that  $v_k \stackrel{\text{def}}{=} \dot{x}^k$ , and  $\dot{v}^k = -\Gamma^k_{ij}(\gamma(t))v^iv^j$  are the "accelerations".

Note that time derivatives have been solved in all cases, so there is a unique solution (for a small time interval) given  $x^k(0)$  and  $v^k(0)$ , for k = 1, ..., n.

**Lemma:** (Do Carmo 2.3)  $\exists ! G \in \mathfrak{X}(TM)$  s.t. the integral curves of G are precisely of the form  $\dot{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$ , where  $\gamma$  is a geodesic. In other words, the integral curves of G are precisely the lifts to TM of geodesics on M. (Integral curves of G are locally solutions to the above system of differential equations.)

Proof: First, we'll prove local existence and uniqueness of G in coordinates. Let  $V \subset M$  be a coordinate neighborhood, with coordinates  $(x^1, \ldots, x^n)$ , inducing coordinates  $(x^1, \ldots, x^n, v^1, \ldots, v^n)$  on TV by  $v = \sum_{i=1}^n v^i \partial_{x^i}|_p$  for  $(p, v) \in TV$ . Then  $G = \sum_{i=1}^n a_i \partial_{x^i} + b_i \partial_{v^i}$ , for some  $a_i, b_i \in C^{\infty}(TV)$  (note that this is true for any vector field on TV).

Now, comparing with the system of differential equations, we can see that we must have  $a_i = v^i = \dot{x}^i$ ,  $\forall i$ . So  $G(x,v) = v^i \partial_{x^i} - \Gamma^k_{ij}(x) v^i v^j \partial_{v^k}$  iff the integral curves of V solve the system of equations.

Finally, local existence and uniqueness implies global existence and uniqueness by covering M with coordinate charts.  $\square$ 

Observe: The vector field G can be described using  $T^*M$  and its symplectic form, and  $TM \to T^*M$  by  $T_pM \to T_p^*M$  using  $\langle \ , \ \rangle_p$ . In the future, we'll also consider the Hamiltonian picture...

Now, we want to think about the flow of G on TM. Let  $X \in \mathfrak{X}(M)$ . Given any  $m \in M$ , there's a neighborhood  $\mathcal{U} \subseteq M$  of m,  $\delta > 0$ , and  $\varphi : (-\delta, \delta) \times \mathcal{U} \to M$  smooth such that  $\forall \mu \in \mathcal{U}, t \mapsto \varphi(t, \mu)$  is the integral curve of X s.t.  $\varphi(0, \mu) = X_{\mu}$ , and  $\forall t, \frac{d}{dt}\varphi(t, \mu) = X_{\varphi(t, \mu)}$ .

Now, apply this to  $\mathcal{M} = TM$ , X = G, and m = (p, 0) for  $p \in M$ . Then  $\exists \mathcal{U} \subseteq \mathcal{M}$  and  $\delta > 0$  as in the theorem. So we have  $\{(q, 0) \in TM : q \in M\} \cong M$ .

Claim:  $\exists V \subseteq M$ , a neighborhood of p, and  $\varepsilon > 0$  s.t.  $\{(q, v) \in TM \mid q \in V, ||v|| < \varepsilon\} \subseteq \mathcal{U}$ .

We get

$$(-\delta, \delta) \times \{(q, v) \mid q \in V, ||v|| < \varepsilon\} \xrightarrow{\varphi} TM \downarrow^{\pi}_{\gamma = \pi \circ \varphi} M$$

An important property of  $\gamma$  is that  $\forall (q,v), t \mapsto \gamma(t,q,v)$  is the unique geodesic s.t.  $\gamma(0,q,v) = q$  and  $\left. \frac{d}{dt} \gamma(t,q,v) \right|_{t=0} = v$ .

**Lemma:** By reparameterizing geodesics by a constant factor in time, one can show (keeping the notation from our previous discussion) that, for a > 0, then  $\gamma(t, q, av) = \gamma(at, q, v)$ , provided that both sies are defined.

Proof: Check that both sides are geodesics, with the same initial conditions. Then by uniqueness of geodesics, they're equivalent.  $\Box$