

# Math 635 Lecture 5

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Some stuff we can do with Riemannian metrics:

- A Riemannian metric on  $M$  allows us to define the length of a tangent vector, and the angle between two tangent vectors. For  $v, w \in T_p M$ ,  $\|v\| = \sqrt{\langle v, v \rangle_p}$ , and  $\langle v, w \rangle_p = \|v\| \|w\| \cos \theta$  defines  $\theta$ , the angle between  $v$  and  $w$ .
- A Riemannian metric on  $M$  allows us to define the length of a curve in  $M$ . Say  $\gamma : [a, b] \rightarrow M$ . We define its length

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{\gamma(t)}} dt$$

If  $\text{im} \gamma \subseteq U$ , a coordinate patch, with coordinates  $(x^1(t), \dots, x^k(t))$ , then

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t)} dt \quad g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \in C^\infty(U)$$

**Lemma:**  $L(\gamma)$  is invariant under re-parameterizations of  $\gamma$ . Consider  $[\alpha, \beta] \xrightarrow{\sim} [a, b] \xrightarrow{\gamma} M$ . Then

$$\int_\alpha^\beta \left\| \frac{d\gamma}{ds}(t(s)) \right\| ds = \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

Proof: The chain rule says  $\frac{d\gamma}{ds} = \frac{dt}{ds} \frac{d\gamma}{dt}$ . So  $\left\| \frac{d\gamma}{ds} \right\| = \left\| \frac{dt}{ds} \right\| \left\| \frac{d\gamma}{dt} \right\|$ . Now, use the change of variables formula for integrals.  $\square$

**Defn:** Let  $M$  be a connected Riemannian manifold. We define the Riemannian distance function by,  $\forall p, q \in M$ ,  
 $d(p, q) = \inf \{L(\gamma) \mid \gamma \text{ is a continuous curve, or "path", joining } p \text{ and } q \text{ that is piecewise smooth}\}.$

Note: We can replace this definition with just "smooth" – the definitions are equivalent. But that's harder to prove, and this definition will be useful later on.

Note that because  $M$  is connected, it's path connected. This means the set of lengths of curves connecting pairs of points is nonempty, so the infimum exists. And because  $L(\gamma) \geq 0$ ,  $d(p, q) \geq 0$ .

**Thm:**  $d$  is a metric, or distance function, i.e.,  $\forall p, q, r \in M$ ,

- (i)  $d(p, q) \geq 0$ , with  $d(p, q) = 0 \Leftrightarrow p = q$
- (ii)  $d(p, q) = d(q, p)$
- (iii)  $d(p, q) + d(q, r) \geq d(p, r)$

Proof: (This is only a partial proof)

- (i) If  $p = q$ , take a trivial/constant path.  $\dot{\gamma} \equiv 0$ , so  $L(\gamma) = 0$ . The converse remains to be shown: that  $d(p, q) = 0$  implies  $p = q$ . This will be a corollary of the "Gauss lemma", which we'll do later on.
- (ii) We never assumed reparameterizations couldn't reverse the direction of the curve. They can, which directly proves  $d(p, q) = d(q, p)$ .
- (iii) Among the paths joining  $p$  and  $r$  are paths that travel through  $q$ . Specifically, given any path from  $p$  to  $q$  and any path from  $q$  to  $r$ , we can concatenate them to get a path from  $p$  to  $r$ .

$\square$

Observe: The topology defined by  $d$  is the same as the given topology on  $M$ .

Sometimes, but not always, the infimum is attained, i.e., there exists a minimizing path. In fact, if such a path exists, it's always smooth.

**Ex:** Minimizing paths on the sphere are arcs of great circles – intersections of the sphere with hyperplanes through the origin.

**Ex:** Minimizing paths don't always exist! Consider  $M = \mathbb{R}^2 \setminus \{0\}$ . For  $p = (43, 0)$  and  $q = (-43, 0)$ ,  $d(p, q) = 86$ , but there's no path of that length between them (since you can't go through the origin).

Refer to Do Carmo, Chapter 1, §2 for more details.

## Volume Element of an Oriented Riemannian Manifold

Reminder: An orientation on an orientable manifold  $M$  is determined by a class of top-degree differential forms,  $\nu$ , with the defining property that  $\forall p \in M$  and any  $(v_1, \dots, v_n)$ , a positive basis of  $T_p M$ ,  $\nu_p(v_1, \dots, v_n) > 0$ .

In particular,  $\nu$  is nowhere-vanishing. Conversely, a nowhere-vanishing top-degree form can be used to define positive bases, and in turn, an orientation.

**Defn:** If  $M$  is a orientable Riemannian manifold, its volume form  $\nu$  is defined by the property that  $\forall p \in M$ ,  $\forall (v_1, \dots, v_n)$ , a positive, orthonormal basis of  $T_p M$ , one has  $\nu_p(v_1, \dots, v_n) = 1$ .

Observe: If this condition holds for some positive orthonormal basis, it holds for all positive orthonormal bases. The important calculation is as follows: Fix  $p \in M$ ,  $(v_1, \dots, v_n)$  a positive orthonormal basis of  $T_p M$ , and  $(e_1, \dots, e_n)$  any other ordered basis of  $T_p M$ . Then we can write each  $e_i = \sum_{\ell=1}^n a_i^\ell v_\ell$ . For any top-degree form  $\nu$ ,  $\nu_p(e_1, \dots, e_n) = \det(a_i^\ell) \nu(v_1, \dots, v_n)$ , so if  $(e_1, \dots, e_n)$  is also positive and orthonormal, then  $\det(a_i^\ell) = 1$ .  $\square$

## Computation of the Volume Form in Coordinates

Start with  $(x_1, \dots, x_n)$ , a positive coordinate system with domain  $U$ . (Recall that this means  $\forall p \in U$ ,  $(\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p)$  is a positive basis of  $T_p M$ .) Apply Gram-Schmidt to each basis (pointwise). We obtain vector fields  $v_1, \dots, v_n$  on  $U$  which are orthonormal at each point. And Gram-Schmidt shows the  $v$ 's are related to the partial derivatives by a smooth matrix, so  $\forall j, v_j \in \mathfrak{X}(U)$ . And, by possibly permuting the  $v_i$ 's, we can ensure it's a positive basis at every point.

In fact, let's write  $\frac{\partial}{\partial x^i} = \sum_{\ell} a_i^\ell v_\ell$ . Then

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i} \Big|_p, \frac{\partial}{\partial x^j} \Big|_p \right\rangle = \sum_{k, \ell} a_i^\ell a_j^k \underbrace{\langle v_k, v_\ell \rangle}_{=\delta_{k, \ell}} = \sum_k a_i^k a_j^k = AA^T \quad \text{for } A = (a_i^k)$$

So  $\det(g_{ij}) = \det(A)^2 > 0$ . On the other hand, with our Riemannian volume form  $\nu$ ,

$$\nu \left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) = \underbrace{\det(A) \nu(v_1, \dots, v_n)}_{=\sqrt{\det(g_{ij})}} = \sqrt{\det(g_{ij})}$$

So in coordinates,  $\nu = \sqrt{\det(g_{ij})} dx^1 \wedge \dots \wedge dx^n$ .

**Defn:** The volume of a subset  $U$  of a Riemannian manifold is  $\text{Vol}(U) = \int_U \nu$ , where  $\nu$  is the Riemannian volume form, if this integral is finite.

**Lemma:** For any coordinate system  $(y^1, \dots, y^n)$  on  $U$ , positive or not, the Riemannian integral

$$\int_U \sqrt{\det \left( \left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle \right)} \underbrace{dy^1 \cdots dy^n}_{\text{Riemann integral}}$$

is equal to  $\text{Vol}(U)$ .

This is true because the change of variables formula for a Riemann integral involves the absolute value of the Jacobian. So in the end, orientation is *not* needed to compute volumes of manifolds. In fact, we can even compute volumes of non-orientable manifolds! We generalize by using partitions of unity.