Math 635 Lecture 37

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4/16/21

Recall from last time: Let M be a compact, oriented, Riemannian manifold. Then the Laplacian on forms $\Delta : \Omega^k \to \Omega^k$ is $\Delta = \delta d + d\delta$. (Note that these are technically different d's and different δ's, because d increases the degree and δ reduces it.) And we have the deRham complex

$$\cdots \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{k-1} \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^k \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{k+1} \underset{\delta}{\overset{d}{\rightleftarrows}} \cdots$$

where $d^* = \delta$, i.e., $\forall \alpha \in \Omega^{k-1}, \beta \in \Omega^k$, $\langle d\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \delta\beta\rangle_{\ell^2}$.

Lemma:

(1) $\delta^2 = 0$ (because $\delta^2 = (d^2)^*$)
(2) $\Delta^* = \Delta$: $\forall \alpha, \beta \in \Omega^k$, $\langle \Delta\,\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \Delta\,\beta\rangle_{\ell^2}$
(3) $[\Delta, d] = 0$, $[\Delta, \delta] = 0$
(4) $\Delta\,\alpha = 0$ iff $d\alpha = 0$ and $\delta\alpha = 0$

Proof (3): $(\delta d + d\delta)d = \cancel{\delta d d} + d\delta d$ and $d(\delta d + d\delta) = d\delta d + \cancel{dd\delta} = d\delta d$. So $[\Delta, d] = 0$. (Identical proof for $[\Delta, \delta] = 0$.) $\square$
(4): $\Leftarrow$ is obvious. For $\Rightarrow$, note that $\Delta\,\alpha = 0$ implies

$$0 = \langle \Delta\,\alpha, \alpha\rangle = \langle \delta d\alpha, \alpha\rangle + \langle d\delta\alpha, \alpha\rangle = \langle d\alpha, d\alpha\rangle + \langle \delta\alpha, \delta\alpha\rangle = ||d\alpha||_{\ell^2}^2 + ||\delta\alpha||_{\ell^2}^2$$

So $d\alpha = 0$ and $\delta\alpha = 0$. $\square$

General Things about Linear Differential Operators

Let $U \subseteq \mathbb{R}^n$, consider $C^\infty(U, \mathbb{C})$.

Defn: A differential operator P on $C^\infty(U, \mathbb{C})$ is of the form $\forall f \in C^\infty(U, \mathbb{C})$,

$$P(f) = \sum_\alpha c_\alpha(x)(D^\alpha f)(x) \quad \alpha \text{ multi-index with } |\alpha| \leq n \qquad D^\alpha = \frac{1}{i}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \frac{1}{i}\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

(here, we're using $i = \sqrt{-1}$). Note that P is local – $P(f)(x)$ only depends on f in a neighborhood of x.

On manifolds, on $C^\infty(M, \mathbb{C})$, linear differential operators are generated as a ring by multiplying by functions. For vector bundles/systems, we have $P : C^\infty(U, \mathbb{C}^r) \to C^\infty(U, \mathbb{C}^s)$, where $P = (P_{ij})_{s\times r}$, and P_{ij} is a scalar differential operator.

In the manifold setting, suppose we have two bundles $\mathcal{E}$ and $\mathcal{F}$:

$$\begin{tikzcd} \mathcal{E} \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & M & \end{tikzcd}$$

Consider $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$. Under local trivializations on the same $U \subseteq M$, we have $\Gamma(\mathcal{E}|_U) \cong C^\infty(U, \mathbb{C}^r) \cong \Gamma(\mathcal{F}|_U)$. So P is locally the Euclidean case.

We already know a bunch of examples!

Ex: $d : \Omega^k \to \Omega^{k+1}$ and $\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E} \otimes T^*M)$ are differential operators of order 1.

The main invariant associated to a differential operator P is called its symbol. The symbol captures the top degree part of the operator. For an operator on $C^\infty(U, \mathbb{C})$, think of computing: take $x_0 \in U, \xi_0 \in T^*_{x_0}U$ (i.e. $(x_0, \xi_0) \in T^*U$). Pick χ, ρ functions, both $C_0^\infty(U, \mathbb{R})$ such that $\chi \equiv 1$ near x_0 and $d\rho_{x_0} = \xi_0$. Consider $P(\chi e^{i\tau\rho})(x_0)$ for $\tau \gg 1$. We get

$$P(\chi e^{i\tau\rho})(x_0) = \sum_{|\alpha| = m \overset{\text{def}}{=} \deg P} c_\alpha(x_0) D^\alpha(\chi e^{i\tau\rho}) + \text{lower degree terms}$$

Look for the highest power of τ on the right-hand side.

$$\tau^m e^{i\tau\rho(x_0)} \sum_{|\alpha|=m} c_\alpha(x_0) \underbrace{(\nabla\rho(x_0))^\alpha}_{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}} + \text{lower order terms in } \tau$$

Now, forget the τ and forget the exponential.

Defn: The <u>symbol</u> of a differential operator P is

$$\sigma_P(x_0, \xi_0) = \sum_{|\alpha|=m} c_\alpha(x_0)\xi^\alpha = \lim_{\tau \to \infty} e^{-i\tau\rho(x_0)} \frac{1}{\tau^m} P(\chi e^{i\tau\rho})(x_0)$$

Conclusion: On manifolds, $P : C^\infty(M, \mathbb{C})\circlearrowleft$, $\sigma_P : T^*M \to \mathbb{C}$ is well-defined. Specifically, it's a homogeneous polynomial in χ of degree M on each fiber.

Ex: Let $X \in \mathfrak{X}(M)$. Then $\sigma_{\mathcal{L}_X} : T^*M \to \mathbb{C}$ is $\sigma_{\mathcal{L}_X}(p, \xi) = i\langle \xi, X_p \rangle$.

Ex: $\Delta : C^\infty(M)\circlearrowleft$, $\sigma_\Delta : T^*M \to \mathbb{C}$ is $\sigma_\Delta(p, \xi) = g^{ij}(p)\xi_i\xi_j = ||\xi||^2 \,\text{Id}$.
(Recall: this is from the definition of Δ in coordinates.)

If $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$,

$$\begin{tikzcd} \mathcal{E} \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & M & \end{tikzcd}$$

Then for $(p, \xi) \in T^*M$, $\sigma_P(p, \xi) : \mathcal{E}_p \to \mathcal{F}_p$ is a linear map between the fibers.

Prop: If P and Q are differential operators such that $P \circ Q$ makes sense, then $\sigma_{P \circ Q} = \sigma_P \circ \sigma_Q$.

Defn: P is an <u>elliptic operator</u> iff $\forall (p, \xi)$ with $\xi \neq 0$, $\sigma_P(p, \xi)$ is invertible.

Ex: $P = \Delta$ is elliptic. $\sigma_\Delta = ||\xi||^2 \,\text{Id}$, so it's invertible everywhere. Δ has an approximate inverse G – "$\sigma_G(p, \xi) = \frac{\text{Id}}{||\xi||^2}$".
$\Delta \circ G - I$ and $G \circ \Delta - I$ are smoothing operators – they're very small.

We're skipping a lot of stuff here, but...

Thm: (Spectral Theorem of the Laplacian) Consider $\Delta : \Omega^k(M) \to \Omega^k(M)$, where M is a compact, oriented, Riemannian manifold. Then

(1) $\mathcal{U}^k = \ker \Delta$ has finite dimension.
(2) There's an orthonormal basis (in the ℓ^2 sense) of Ω^k, $\{\alpha_j\}$, and $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to +\infty$ s.t. $\Delta\,\alpha_j = \lambda_j\alpha_j$. In other words, we can think of Δ as an infinite matrix, that can be diagonalized by α_j and λ_j. We'll write $\forall \alpha \in \Omega^k$, $\alpha = \alpha^H + \sum_{\lambda > 0 \text{ distinct}} \alpha_\lambda$. α^H is the harmonic piece – $\Delta\,\alpha^H = 0$, $\Delta\,\alpha_\lambda = \lambda\alpha_\lambda$. We define Green's operator $G(\alpha) = \sum_{\lambda > 0} \frac{1}{\lambda}\alpha_\lambda$. So $(I - \Delta \circ G)(\alpha) = G^H$.

Ex: For Δ on $C^\infty(S^2)$, we have $\lambda = k(k+1)$ with multiplicity $2k + 1$ (for $k \in \mathbb{Z}_{\geq 0}$).