Math 635 Lecture 2

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Lemma: Let M be a C^∞ manifold. Assume $\forall p \in M$, we have a vector space $\mathcal{E}_p$ (of dimension r). Let $\mathcal{E} = \bigsqcup_{p \in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \to M$ the natural projection, where $\mathcal{E}_p \mapsto p \in M$. Assume we're given $\{U_\alpha\}$, a cover of M, plus bijections χ_α such that $\forall \alpha$, the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl] \\
& U_\alpha &
\end{tikzcd}$$

i.e. $\chi_\alpha(\mathcal{E}_p) = \{p\} \times \mathbb{R}^r$.

Observe that this gives us $\forall \alpha, \beta$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$, a map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$, by:

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\chi_\alpha"'] \arrow[dr, "\chi_\beta"] & \\
(U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[rr, "\chi_\beta \circ \chi_\alpha^{-1}"] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\
(p, v) \arrow[rr, mapsto] & & (p, \tau_{\alpha,\beta}(p) v) \\
p \in U_\alpha \cap U_\beta & & \\
v \in \mathbb{R}^r & &
\end{tikzcd}$$

This mapping is a "change of trivialization", like a transition map for vector bundles. The matrix of $\tau_{\alpha,\beta}$ depends on p, but it's always a linear map.

If, $\forall \alpha, \beta$, $\tau_{\alpha,\beta}$ is C^∞, then there is a unique topology on $\mathcal{E}$, and a unique differentiable structure, that makes $\pi : \mathcal{E} \to M$ a smooth vector bundle, and each χ_α is a diffeomorphism, i.e., a smooth local trivialization.

Observe: On triple intersections, $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall p$, one has $\tau_{\alpha,\beta}(p)\tau_{\beta,\gamma}(p) = \tau_{\alpha,\gamma}(p)$. This is called a "cocycle condition".

Now, imageine starting with a cover $\{U_\alpha\}$ and C^∞ maps $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ satisfying the cocycle condition. Then, if we choose, $\forall p$, $\mathcal{E}_p = \mathbb{R}^r$, and $\chi_\alpha = I_n$, then we get a vector bundle $\mathcal{E} \to M$. So $\{U_\alpha; \tau_{\alpha,\beta}\}$, called a Čech cocycle, is all we need to put together a vector bundle.

Now, we apply the lemma to construct new bundles from old ones.

Ex: Given $\mathcal{E}', \mathcal{E}'' \to M$ vector bundles, with ranks r' and r'', respectively, define, $\forall p \in M$, $(\mathcal{E}' \oplus \mathcal{E}'')_p = \mathcal{E}'_p \oplus \mathcal{E}''_p$. Consider a cover $\{U_\alpha\}$ of M, and local trivializations for both $\mathcal{E}'$ and $\mathcal{E}''$. We get, $\forall \alpha$,

$$\chi'_\alpha : (\pi')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r'}$$
$$\chi''_\alpha : (\pi'')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r''}$$

Now define $\pi : \mathcal{E}' \oplus \mathcal{E}'' \to M$, and

$$\begin{aligned}
\chi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''}) \\
(p, (v', v'')) &\mapsto (p, (\chi'_\alpha(v'), \chi''_\alpha(v''))) \\
v' &\in \mathcal{E}'_p \\
v'' &\in \mathcal{E}''_p
\end{aligned}$$

We have to check that the $\tau_{\alpha,\beta}$ are smooth. Note that $\tau_{\alpha,\beta}(p)$ will be a block diagonal matrix, and the two diagonal blocks vary smoothly, since χ_α and χ_β are diffeomorphisms. Thus, $\mathcal{E}' \oplus \mathcal{E}'' \to M$ is a natural bundle. This is called the <u>Whitney direct sum</u> or <u>Whitney sum</u>.

Observe:

- $\mathrm{rank}(\mathcal{E}' \oplus \mathcal{E}'') = \mathrm{rank}(\mathcal{E}') + \mathrm{rank}(\mathcal{E}'')$
- $\dim(\mathcal{E}' \oplus \mathcal{E}'') = r' + r'' + n$, where n is the dimension of the total space.

Similarly, we can define $\mathcal{E}' \otimes \mathcal{E}'' \to M$ with fibers $(\mathcal{E}' \otimes \mathcal{E}'')_p = \mathcal{E}'_p \otimes \mathcal{E}''_p$. $\mathrm{rank}(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r''$, and $\dim(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r'' + n$.

Review of Tensor Products

Defn: Let V, W be finite-dimensional vector spaces. Their tensor product $V \otimes W$ is the free vector space over $V \times W$, modulo an equivalence relation, $\sim$. The free vector space over $V \times W$ is the set of all formal finite linear combinations of pairs $(v, w) \in V \times W$, and $\sim$ is defined such that

$$\begin{cases} (v_1 + v_2, w) \sim (v_1, w) + (v_2, w) \\ (\lambda v, w) \sim \lambda(v, w) \end{cases}$$

Notation: $\forall (v, w) \in V \times W$, $v \otimes w = [(v, w)]$.

Claim: If $(e_1, \ldots, e_k)$, $(f_1, \ldots, f_\ell)$ are bases of V, W, then $\{e_i \otimes f_j \mid i \in \{1, \ldots, k\}, j \in \{1, \ldots, \ell\}\}$ is a basis of $V \otimes W$.

Cor: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

The universal property of $V \otimes W$: We have a bilinear map

$$\begin{aligned} V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

It's "universal" in that for any bilinear map, there's a unique linear map such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[rr, "(v,w)\mapsto v\otimes w"] \arrow[dr, "\text{bilinear map}"'] & & V \otimes W \arrow[dl, "\text{linear map}"] \\ & Z & \end{tikzcd}$$

There are other realizations of $V \otimes W$.

$$V \otimes W \cong \mathrm{Hom}(V^*, W) \qquad \text{where} \qquad v \otimes w \mapsto \begin{pmatrix} V^* & \to & W \\ \alpha & \mapsto & \alpha(v)w \end{pmatrix}$$

This is completely natural – we don't need to choose a basis.

Note: This isomorphism, as it's currently written, actually doesn't work! In an exercise, we prove that most tensors in $V \otimes W$ cannot be written as $v \otimes w$ for any $v \in V, w \in W$ (i.e. most tensors are not "pure tensors"). Rather, they have to be written as linear combinations of pure tensors. So it would appear that this isomoprhism doesn't work. Fortunately, this map will work if we instead consider linear combinations of pure tensors.

Cor: $V^* \otimes V^* \cong \mathrm{Hom}((V^*)^*, V^*) = \mathrm{Hom}(V, V^*) \overset{\text{claim}}{\cong} \mathrm{Bil}(V \times V, \mathbb{R})$, by

$$\begin{aligned} \mathrm{Hom}(V, V^*) \;&\cong\; \mathrm{Bil}(V \times V, \mathbb{R}) \\ (f : V \to V^*) &\mapsto \begin{pmatrix} V \times V \to \mathbb{R} \\ (v_1, v_2) \mapsto (f(v_1))(v_2) \end{pmatrix} \end{aligned}$$

(Note that $f(v_1) \in V^*$.)

Thus, $V \otimes V \cong \mathrm{Bil}(V^* \times V^*, \mathbb{R})$. This is the realization we will use!

Note: When taking multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \cong \{\text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}\}$$

Now, we put everything together, and define tensor bundles.

Defn: Given a smooth manifold M, $\forall p \in M$, we define

$$T^{(k,\ell)}(T_pM) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{\ell}$$

k is called the <u>contravariant degree</u>, and ℓ is called the <u>covariant degree</u>. Using the lemma from before, we get <u>tensor bundles</u>

$$\bigsqcup_{p \in M} T^{(k,\ell)}(T_pM) = T^{(k,\ell)}(TM) \to M$$

Exer: Compute the rank of this bundle.

In coordinate $(x^1, \ldots, x^n)$ on $U \overset{\text{open}}{\subseteq} M$, we have a moving frame of $T^{(k,\ell)}(TM)$:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \;\middle|\; i_a, j_b \in \{1, \ldots, n\} \right\}$$

Defn: Any smooth section of a tensor bundle is called a <u>tensor</u>.

Note that any tensor is a combination of these basis elements with C^∞ functions as coefficients:

$$\sum A^{i_1 \cdots i_k}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

Defn: A <u>Riemannian metric</u> is a symmetric, positive-definite $(0,2)$ tensor $g_p : T_pM \times T_pM \to \mathbb{R}$.