Math 635 Lecture 7

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Review: Connections on Vector Bundles

Given $\mathcal{E} \to M$ a vector bundle, a connection is an operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

with universal quantifiers $\forall X, Y \in \mathfrak{X}(M)$, $s, t \in \Gamma(\mathcal{E})$, and $f \in C^\infty(M)$:

(1) $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
(2) $\nabla_{fX} s = f \nabla_X s$
(3) $\nabla_X (s + t) = \nabla_X s + \nabla_X t$
(4) $\nabla_X (fs) = f \nabla_X s + X(f) s$

Properties (1) and (2) together are written as ∇ is linear in X over $C^\infty(M)$. Note that property (4) implies that $\forall c \in \mathbb{R}$, $\nabla_X (cs) = c \nabla_X s$.

Last time, we saw that $(\nabla_X s)(p)$ only depends on $X_p \in T_p M$ and $s|_U$ for any (arbitrarily small) neighborhood U of p. In other words, if we fix a section $s \in \Gamma(\mathcal{E})$, $\nabla_\cdot s$ is a 1-form that takes values in sections. And if we fix $X \in \mathfrak{X}(M)$, then $\nabla_X \cdot : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is a derivation.

Ex: Say $\mathcal{E} = M \times \mathbb{R}^r \to M$. (This is locally the general case.) Then $\Gamma(M \times \mathbb{R}^r) \ni s \leftrightarrow \vec{f} : M \to \mathbb{R}^r$ by $\forall p \in M$, $s(p) = (p, \vec{f}(p))$. This is the same as having a global moving frame $\Gamma(M \times \mathbb{R}^r) \ni E_i \leftrightarrow \vec{f_i}(p) = (0, \ldots, 1, \ldots, 0)$ (with a 1 in the ith entry), for $i = 1, \ldots, r$, because $s = \sum_{i=1}^r f^i E_i$ for $f^i \in C^\infty(M, \mathbb{R})$, where $\vec{f} = (f^1, \ldots, f^r)$.

Suppose we have a connection ∇. Last time, we defined $\vartheta = (\theta_i^j)$, the connection matrix associated with $(E_1, \ldots, E_r)$, by $\forall X$, $\forall j$,

$$\nabla_X E_j = \sum_{i=1}^r \theta_i^j(X) E_j$$

$\forall i, j$, $\theta_i^j \in \Omega^1(M)$ (a C^∞ one-form). Then:

$$\nabla_X s = \sum_{i=1}^r \nabla_X (f^i E_i) = \sum_{i=1}^r X(f^i) E_i + \sum_{i=1}^r f^i \nabla_X E_i = \sum_{i=1}^r X(f^i) E_i + \sum_{i,j=1}^r f^i \theta_i^j(X) E_j$$

The corresponding vector-valued function on M is

$$\nabla_X \vec{f} = \left(X(f^1) + \sum_{i=1}^r \theta_i^1(X) f^i, \ldots, X(f^r) + \sum_{i=1}^r \theta_i^r(X) f^i \right) = \left(X(f^1), \ldots, X(f^r) \right) + \left(\sum_{i=1}^r \theta_i^1(X) f^i, \ldots, \sum_{i=1}^r \theta_i^r(X) f^i \right)$$

So in vector/matrix notation, with $\vec{f}$ as a column vector, we write

$$\nabla_X \vec{f} = d\vec{f}(X) + \vartheta(X) \vec{f}$$

where $\vartheta(X) = (\theta_i^j(X))$ with lower index i being the columns, and upper index j being the rows.

Conversely, here, ϑ can be any $r \times r$ matrix of one-forms, and this can be used to define a connection on the trivial bundle!

Observe: The previous calculation is valid locally, given some moving frame $(E_1, \ldots, E_r)$ of $\mathcal{E} \to M$ on $U \subseteq M$. Suppose $(F_1, \ldots, F_r)$ is another moving frame on U. Then $\forall i$, $F_i = \sum_j a_i^j E_j$, where the matrix $A = (a_i^j)$ is invertible at each $p \in U$, and $\forall i, j,\ a_i^j \in C^\infty(U)$. Given ∇, we get ϑ, the connection matrix corresponding to the E_j's, and $\tilde{\vartheta}$, the connection matrix corresponding to the F_j's.

Exer: Check that $\tilde{\vartheta} = A^{-1}dA + A^{-1}\vartheta A$.

A special feature of the case where $\mathcal{E} = TM$ ($r = n = \dim M$). Consider again a moving frame $(E_1, \ldots, E_n)$. In this case, we can write $X = \sum_k a^k E_k$. In the generic case,

$$\nabla_X E_i = \sum_j \theta_i^j(X) E_j = \sum_{j,k} a^k \underbrace{\theta_i^j(E_k)}_{\substack{\text{Christoffel}\\ \text{Symbols}}} E_j$$

Defn: $\forall i, j, k,\ \Gamma_{ki}^j \stackrel{\text{def}}{=} \theta_i^j(E_k) \in C^\infty(U)$ are the <u>Christoffel symbols</u>.

Note that $\forall i, k$, we get $\nabla_{E_k} E_i = \sum_j \Gamma_{ki}^j E_j$. So Γ_{ki}^j determines ϑ, and therefore ∇ on U.

Now, back to the general case: vector bundle $\mathcal{E} \to M$ with connection ∇. We want to look at "parallelism" and "parallel transport".

Defn: Let $\gamma : [a, b] \to M$, $s \in \Gamma(\mathcal{E})$. Then s is said to be <u>covariant constant</u>, or <u>parallel</u>, iff $\forall t \in [a, b]$, $(\nabla_{\dot{\gamma}(t)} s)(\gamma(t)) = 0$.

Let's analyze this equation... It will turn out to be a system of ordinary differential equations!

Let $(E_1, \ldots, E_r)$ be a moving frame on U, a neighborhood of $\gamma(t)$. For the time being, just assume $\mathrm{Im}(\gamma) \subseteq U$. We can write $s = \sum_i f^i E_i$, for some $f^i \in C^\infty(U)$. Let ϑ be the connection matrix w.r.t. the E_i's.
Introduce $\vec{f} = (f^1, \ldots, f^r) : U \to \mathbb{R}^r$. We saw that $\nabla_{\dot{\gamma}} \vec{f} = d\vec{f}(\dot{\gamma}) + \vartheta(\dot{\gamma})\vec{f}$. Define $f^i(t) = f^i(\gamma(t))$, so here, $\vec{f}(t) = \vec{f}(\gamma(t))$. Then

$$d\vec{f}(\dot{\gamma}) = \frac{d}{dt}\vec{f} \qquad \text{and} \qquad \frac{d}{dt}\vec{f} + \vartheta(\dot{\gamma})\vec{f} = \vec{0}$$

(taking $\vec{f}$ to be the column vector of the $f^i(t)$'s, and $\vartheta(\dot{\gamma})$ being a t-dependent matrix).