Math 635 Lecture 23

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Ricci Curvature

For a Riemannian manifold, the full Riemannian curvature at a point $p \in M$ is $\mathcal{R}_p \in V^* \otimes V^* \otimes V^* \otimes V$, where $V = T_pM$. The Ricci Tensor is a partial trace of $\mathcal{R}$.

Defn: $\forall p \in M$, $\forall u, v \in T_pM$, we define the Ricci tensor by

$$\mathrm{Ric}_p(u,v) = \mathrm{tr}\begin{bmatrix} T_pM & \to T_pM \\ w & \mapsto \mathcal{R}_p(u,v)w \end{bmatrix}$$

Note: The Do Carmo definition of the Ricci tensor has the same convention as ours. It deals with the minus sign by putting w in the second slot of $\mathcal{R}$.

Note that the trace of an arbitrary linear operator $F : V \to V$ can be defined without coordinates as follows. Think of $\mathrm{Hom}(V,V) \cong V^* \otimes V$. The evaluation map $(\alpha, v) \mapsto \alpha(v)$ for $\alpha \in V^*$, $v \in V$ is bilinear, so it corresponds to

$$\begin{aligned} \mathrm{tr} : V^* \otimes V &\to \mathbb{R} \\ F &\mapsto \mathrm{tr}(F) \end{aligned}$$

So Ric_p is bilinear.

We want to compute $\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R}$ in terms of components of $\mathcal{R}$ with respect to an arbitrary frame. Let $e_1, \ldots, e_n$ be a local frame of TM. Then $\mathcal{R}(e_i, e_j)e_k = \mathcal{R}_{ijk}^{\ell} e_\ell$.

Defn:

$$\begin{aligned} \mathcal{R}_{ij} &\overset{\text{def}}{=} \mathrm{Ric}(e_i, e_j) \\ &= \mathrm{tr}(w \mapsto \mathcal{R}(w, e_i)e_j) \\ &= \sum_{k=1}^{n} (e_k\text{-component of } \mathcal{R}(e_k, e_i)e_j) \\ &= \mathcal{R}_{kij}^{k} \text{ (with implicit summation)} \end{aligned}$$

So $\mathcal{R}_{ij} = \mathcal{R}_{kij}^{k}$.

(As we will see later, we can interpret this as a sort of averaging.)

Side note: Why do we choose to fix the middle two elements? Well, if you fix the first two or last two, swapping the two unfixed elements flips the sign, so the trace would be identically zero.

Recall that we also have the shorthand notation $\mathcal{R}_{ijk\ell} = \langle \mathcal{R}(e_i, e_j)e_k, e_\ell \rangle$.

Exer: Check that $\mathcal{R}_{ijk\ell} = \mathcal{R}_{ijk}^{a} g_{a\ell}$, i.e., $\mathcal{R}_{ijk}^{b} = \mathcal{R}_{ijk\ell} g^{\ell b}$ (where $g^{\ell b}$ is the element of the matrix inverse of $g_{a\ell}$). This implies that $\mathcal{R}_{ij} = \mathcal{R}_{kij\ell} g^{\ell k}$.

Now, let's explore some properties of the Ricci Tensor.

Prop: Ric is symmetric.

Proof: Let $e_1, \ldots, e_n$ be an orthonormal frame. Then

$$\begin{aligned}
\mathcal{R}_{ij} &= \sum_k \langle \mathcal{R}(e_k, e_j)e_j, e_k \rangle \\
&= \sum_k \mathcal{R}_{kijk} \\
&= \sum_k \mathcal{R}_{jkki} \\
&= -\left(-\sum_k \mathcal{R}_{kjik} \right) \\
&= \sum_k \mathcal{R}_{kjik} \\
&= \mathcal{R}_{ji}
\end{aligned}$$

$\square$

Defn: The <u>Ricci curvature</u> is

$$\begin{aligned}
\mathrm{Ric} : \{(p, v) \mid p \in M, v \in T_pM, ||v|| = 1\} &\to \mathbb{R} \\
(p, v) &\mapsto \mathrm{Ric}_p(v, v) \underbrace{\frac{1}{n-1}}_{\text{Do Carmo only}}
\end{aligned}$$

Geometric Interpretation

We'll be using the Do Carmo convention. Let v be a unit tangent vector, and let $(e_1, \ldots, e_n)$ be an orthonormal basis of T_pM, with $e_n = v$. Then

$$\mathrm{Ric}(v) = \frac{1}{n-1} \sum_{i=1}^{n-1} K(e_i, v)$$

So we're effectively averaging the sectional curvature of the planes produced by v with each of the basis vectors $e_1, \ldots, e_{n-1}$. This is true because

$$\mathrm{Ric}(v, v) = \sum_{i=1}^{n} \underbrace{\langle \mathcal{R}(e_i, v)v, e_i \rangle}_{= \begin{cases} K(e_i, v) & i < n \\ 0 \end{cases}}$$

Defn: The <u>scalar curvature</u> $S : M \to \mathbb{R}$ is defined by $\forall p \in M$, $S(p) = \mathrm{tr}(\mathrm{Ric}_p)$.

Thm: (Bonet-Myers V2) Suppose (M, g) is a Riemannian manifold. If $\mathrm{Ric} > \left(\frac{\pi}{\ell}\right)^2$ (using the function form of Ric, not the tensor), then no geodesic of length ℓ is minimizing.

Proof: Pick $(E_1(0), \ldots, E_{n-1}(0))$, an orthonormal basis of $\mathrm{span}(\dot\gamma(0))^\perp$. Then parallel transport the E_j's along γ to get $E_j(t)$, a parallel frame (i.e. $\frac{D}{dt}E_j \equiv 0$) that is orthonormal at every t, and orthogonal to $\dot\gamma(t)$. Now, $\forall j$, let $V_j(t) = \sin(\pi t)E_j(t)$. We consider $n-1$ proper variations of γ, and $\forall j = 1, \ldots, n-1$, we have, as before,

$$E_j''(0) \underbrace{= \cdots =}_{\text{Same computation as Bonet-Myers V1}} \int_0^\ell \sin^2\left(t\frac{\pi}{\ell}\right) \left(\left(\frac{\pi}{\ell}\right)^2 - K(E_j, \dot\gamma) \right) dt$$

Now, we average:

$$\frac{1}{n-1} \sum_{j=1}^{n-1} E_j''(0) = \int_0^\ell \sin^2(t\frac{\pi}{\ell}) \left(\left(\frac{\pi}{\ell}\right)^2 - \mathrm{Ric}_{\gamma(t)}(\dot\gamma) \right) dt$$

This quantity is less than 0 by our assumption, so $\exists j$ such that $E_j''(0) < 0$, so the geodesic isn't minimizing. $\square$

Jacobi Fields

Our motivation comes from the second variation formula: it involves the operator $\frac{D^2}{dt^2}V + \mathcal{R}(V, \dot{\gamma})\dot{\gamma}$. We're interested in studying fields where this is equal to 0.

Defn: Let γ be a geodesic. Then $V \in \Gamma_\gamma(TM)$ is a Jacobi field iff

$$\frac{D^2}{dt^2}V + \mathcal{R}(V, \dot{\gamma})\dot{\gamma} = 0$$

Note: We don't require V to vanish at the endpoints.

Prop: Let $f : (-\varepsilon, \varepsilon) \times [0, a] \to M$ be a smooth variation of γ by geodesics, i.e., $\forall s$, $t \mapsto f(s, t)$ is a geodesic. Then the variation field $V = \partial_s f|_{s=0}$ is a Jacobi field.

Proof: Our assumption of f is true iff $\frac{D}{dt}\partial_t f = 0$, so

$$0 = \frac{D}{ds}\frac{D}{dt}\partial_t f = \frac{D}{dt}\frac{D}{ds}\partial_t f + \mathcal{R}(\partial_s f, \partial_t f)\partial_t f = \frac{D^2}{dt^2}\partial_s f + \mathcal{R}(\partial_s f, \partial_t f)\partial_t f$$

If we restrict to $s = 0$, we get the Jacobi operator. $\square$

Exer: (HW) Prove that the converse is true as well.

Claim: Let $E_i \in \Gamma_\gamma(TM)$, for $i = 1, \ldots, n$, be a parallel frame, and let $V(t) = f^i(t)E_i(t)$. Then the Jacobi equation on V is equivalent to a second order system of ordinary differential equations, that looks like $\ddot{f}^i + a_{ij}f^j = 0$.