Math 635 Lecture 18

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Beginning where we left off last time...

Cor: Let M be a connected Riemannian manifold. Then $\forall p, q \in M$,

$$d(p,q) \stackrel{\text{def}}{=} \inf \left\{ \ell(c) \mid c \text{ continuous and piecewise } C^1 \text{ curve from } p \text{ to } q \right\}$$

is a distance function on M.

Proof: All that remains to be proved is $d(p,q) = 0 \Rightarrow p = q$. So by contraposition, it's enough to show $p \neq q \Rightarrow d(p,q) > 0$. Well, if $p \neq q$, then $\exists \varepsilon > 0$ s.t. $\exp_p\big|_{B_\varepsilon(0)}$ is a diffeomorphism onto its image, and $q \notin \exp_p(B_\varepsilon(0))$. Let c be a path from p to q. It's enough to show $\ell(c) > \varepsilon > 2$.

Well, by continuity, $\exists t_1 > 0$ s.t. $c(t_1) \in S_{\varepsilon/2}$, the geodesic sphere of radius $\varepsilon/2$ centered at p. But $d(p, c(t_1))$ is the length of any corresponding radial geodesic, which is $\varepsilon/2$. So $\ell(c) \geq \ell(c|_{[0,t_1]}) \geq \varepsilon/2 > 0$. $\square$

Thm: $\forall p \in M$, $\exists W$ a neighborhood of p, and $\exists \delta > 0$ s.t. $\forall q \in W$,

$$\exp_q\big|_{B_\delta(0)} : B_\delta(0) \xrightarrow{\sim} \exp_q(B_\delta(0))$$

is a diffeomorphism onto its image, and $W \subseteq \exp_q(B_\delta(0))$.

Proof: Recall that given $p \in M$, there's a neighborhood V of p, and $\varepsilon > 0$, such that $\exp : B_\varepsilon(TV) \to M$ is defined, where $B_\varepsilon(TV) = \{(q,v) \in TV \mid q \in V, ||v|| < \varepsilon\}$. In other words, $G(t,q,v)$ is defined past $t = 1$. Now, define

$$\begin{aligned} F : B_\varepsilon(TV) &\to M \times M \\ (q,v) &\mapsto (q, \exp_q(v)) \end{aligned}$$

In particular, $F(p,0) = (p,p)$. We claim that F is a local diffeomorphism near 0. To check this fact, introduce coordinates near p, and then "double" them to get coordinates on $M \times M$ near (p,p). Then the Jacobian is the block matrix

$$dF_{(p,0)} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

with the bottom-left entry being I because $\exp_q 0 = q$, and the bottom right entry being I because $d(\exp_p)_0 = \mathrm{Id}$. This matrix is invertible, so F is a local diffeomorphism. Thus, there's a neighborhood V' of p, with $V' \subset V$, and a $\delta > 0$, such that

$$F|_{B_\delta(TV')} \xrightarrow{\sim} F(B_\delta(TV'))$$

is a diffeomorphism onto its image, which is a neighborhood of $(p,p) \in M \times M$. Thus, $\exists W$, a neighborhood of p, such that $W \times W \subset F(B_\delta(TV'))$. In other words,

$$W \times W \subset \left\{ (q, \exp_q(v)) \mid q \in V', ||v|| < \delta \right\} \stackrel{F}{\cong} \{(q,v) \mid q \in V', ||v|| < \delta\}$$

So $\forall q \in W$, $\{q\} \times B_\delta(0) \xrightarrow{\sim} \{q\} \times \exp_q(B_\delta(0))$ via F under an appropriate restriction. $\square$

Defn: Such a neighborhood W is called a <u>totally normal neighborhood</u>.

Lemma: Let W be a totally normal neighborhood, and $p, q \in W$. Then there is a unique geodesic (up to reparameterization) joining p and q, and entirely contained in W. Moreover, this geodesic is the shortest path (i.e. continuous and piecewise C^1) joining p to q.

Proof: Let $\gamma : [0,1] \to W$ be a geodesic, with $\gamma(0) = p$ and $\gamma(1) = q$. Lift γ to T_pM by $\exp_p^{-1}\big|_W$. Then $\gamma(t) = \exp_p(c(t))$, where $t \mapsto c(t) \in T_pM$ and $c(0) = 0$. Note that $\dot\gamma(0) = \dot c(0)$, and $t \mapsto \exp_p(t\dot c(0))$ is a geodesic with the same initial conditions as γ. By the uniqueness (up to reparameterization) of geodesics with initial conditions, we must have $\gamma(t) = \exp_p(t\dot c(0))$, $\forall t$. This also implies that γ is the shortest path from p to q, since it's a radial geodesic. $\square$

Cor: All geodesics are locally length-minimizing.

Proof: Let $\gamma : I \to M$ be a geodesic. Take $t_0 \in \operatorname{Int} I$, i.e., $t_0 \in (a,b) \subseteq [a,b] \subseteq I$. By the existence of totally normal neighborhoods of $\gamma(t_0)$, if $b - a$ is small enough, then $\gamma([a,b])$ is contained in a totally normal neighborhood of $\gamma(t_0)$. $\square$

Cor: Suppose $\gamma : [0,1] \to M$ is a path (continuous, and piecewise C^1), and $d(\gamma(0), \gamma(1)) = \ell(\gamma)$. Then γ is a geodesic, and in paricular, it's C^∞.

Proof: The idea is that global length minimization leads to local. Let $t_0 \in [0,1]$. Again find a neighborhood $[a,b]$ of t_0 such that $\gamma([a,b])$ is contained in a totally normal neighborhood of $\gamma(t_0)$. Then γ must be the shortest path from $\gamma(a)$ to $\gamma(b)$. This means $\gamma|_{[a,b]}$ must be a geodesic, and γ is smooth in a neighborhood of t_0. $\square$