

# Math 635 Lecture 18

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Beginning where we left off last time...

**Cor:** Let  $M$  be a connected Riemannian manifold. Then  $\forall p, q \in M$ ,

$$d(p, q) \stackrel{\text{def}}{=} \inf \{ \ell(c) \mid c \text{ continuous and piecewise } C^1 \text{ curve from } p \text{ to } q \}$$

is a distance function on  $M$ .

Proof: All that remains to be proved is  $d(p, q) = 0 \Rightarrow p = q$ . So by contraposition, it's enough to show  $p \neq q \Rightarrow d(p, q) > 0$ . Well, if  $p \neq q$ , then  $\exists \varepsilon > 0$  s.t.  $\exp_p|_{B_\varepsilon(0)}$  is a diffeomorphism onto its image, and  $q \notin \exp_p(B_\varepsilon(0))$ . Let  $c$  be a path from  $p$  to  $q$ . It's enough to show  $\ell(c) > \varepsilon > 0$ .

Well, by continuity,  $\exists t_1 > 0$  s.t.  $c(t_1) \in S_{\varepsilon/2}$ , the geodesic sphere of radius  $\varepsilon/2$  centered at  $p$ . But  $d(p, c(t_1))$  is the length of any corresponding radial geodesic, which is  $\varepsilon/2$ . So  $\ell(c) \geq \ell(c|_{[0, t_1]}) \geq \varepsilon/2 > 0$ .  $\square$

**Thm:**  $\forall p \in M$ ,  $\exists W$  a neighborhood of  $p$ , and  $\exists \delta > 0$  s.t.  $\forall q \in W$ ,

$$\exp_q|_{B_\delta(0)} : B_\delta(0) \xrightarrow{\sim} \exp_q(B_\delta(0))$$

is a diffeomorphism onto its image, and  $W \subseteq \exp_q(B_\delta(0))$ .

Proof: Recall that given  $p \in M$ , there's a neighborhood  $V$  of  $p$ , and  $\varepsilon > 0$ , such that  $\exp : B_\varepsilon(TV) \rightarrow M$  is defined, where  $B_\varepsilon(TV) = \{(q, v) \in TV \mid q \in V, \|v\| < \varepsilon\}$ . In other words,  $G(t, q, v)$  is defined past  $t = 1$ . Now, define

$$\begin{aligned} F : B_\varepsilon(TV) &\rightarrow M \times M \\ (q, v) &\mapsto (q, \exp_q(v)) \end{aligned}$$

In particular,  $F(p, 0) = (p, p)$ . We claim that  $F$  is a local diffeomorphism near 0. To check this fact, introduce coordinates near  $p$ , and then “double” them to get coordinates on  $M \times M$  near  $(p, p)$ . Then the Jacobian is the block matrix

$$dF_{(p, 0)} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

with the bottom-left entry being  $I$  because  $\exp_q 0 = q$ , and the bottom right entry being  $I$  because  $d(\exp_p)_0 = \text{Id}$ . This matrix is invertible, so  $F$  is a local diffeomorphism. Thus, there's a neighborhood  $V'$  of  $p$ , with  $V' \subset V$ , and a  $\delta > 0$ , such that

$$F|_{B_\delta(TV')} \xrightarrow{\sim} F(B_\delta(TV'))$$

is a diffeomorphism onto its image, which is a neighborhood of  $(p, p) \in M \times M$ . Thus,  $\exists W$ , a neighborhood of  $p$ , such that  $W \times W \subset F(B_\delta(TV'))$ . In other words,

$$W \times W \subset \{(q, \exp_q(v)) \mid q \in V', \|v\| < \delta\} \stackrel{F}{\cong} \{(q, v) \mid q \in V', \|v\| < \delta\}$$

So  $\forall q \in W$ ,  $\{q\} \times B_\delta(0) \xrightarrow{\sim} \{q\} \times \exp_q(B_\delta(0))$  via  $F$  under an appropriate restriction.  $\square$

**Defn:** Such a neighborhood  $W$  is called a totally normal neighborhood.

**Lemma:** Let  $W$  be a totally normal neighborhood, and  $p, q \in W$ . Then there is a unique geodesic (up to reparameterization) joining  $p$  and  $q$ , and entirely contained in  $W$ . Moreover, this geodesic is the shortest path (i.e. continuous and piecewise  $C^1$ ) joining  $p$  to  $q$ .

Proof: Let  $\gamma : [0, 1] \rightarrow W$  be a geodesic, with  $\gamma(0) = p$  and  $\gamma(1) = q$ . Lift  $\gamma$  to  $T_p M$  by  $\exp_p^{-1}|_W$ . Then  $\gamma(t) = \exp_p(c(t))$ , where  $t \mapsto c(t) \in T_p M$  and  $c(0) = 0$ . Note that  $\dot{\gamma}(0) = \dot{c}(0)$ , and  $t \mapsto \exp_p(t\dot{c}(0))$  is a geodesic with the same initial conditions as  $\gamma$ . By the uniqueness (up to reparameterization) of geodesics with initial conditions, we must have  $\gamma(t) = \exp_p(t\dot{c}(0))$ ,  $\forall t$ . This also implies that  $\gamma$  is the shortest path from  $p$  to  $q$ , since it's a radial geodesic.  $\square$

**Cor:** All geodesics are locally length-minimizing.

Proof: Let  $\gamma : I \rightarrow M$  be a geodesic. Take  $t_0 \in \text{Int } I$ , i.e.,  $t_0 \in (a, b) \subseteq [a, b] \subseteq I$ . By the existence of totally normal neighborhoods of  $\gamma(t_0)$ , if  $b - a$  is small enough, then  $\gamma([a, b])$  is contained in a totally normal neighborhood of  $\gamma(t_0)$ .  $\square$

**Cor:** Suppose  $\gamma : [0, 1] \rightarrow M$  is a path (continuous, and piecewise  $C^1$ ), and  $d(\gamma(0), \gamma(1)) = \ell(\gamma)$ . Then  $\gamma$  is a geodesic, and in particular, it's  $C^\infty$ .

Proof: The idea is that global length minimization leads to local. Let  $t_0 \in [0, 1]$ . Again find a neighborhood  $[a, b]$  of  $t_0$  such that  $\gamma([a, b])$  is contained in a totally normal neighborhood of  $\gamma(t_0)$ . Then  $\gamma$  must be the shortest path from  $\gamma(a)$  to  $\gamma(b)$ . This means  $\gamma|_{[a, b]}$  must be a geodesic, and  $\gamma$  is smooth in a neighborhood of  $t_0$ .  $\square$