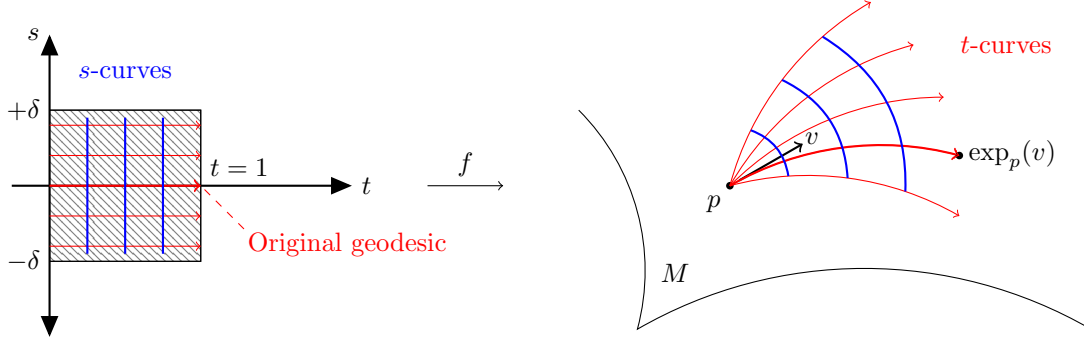


Math 635 Lecture 17

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3/1/21

Recall the setup from last time:



Defn: A vector field along f is a lift \tilde{f} of f to TM . I.e., \tilde{f} is defined such that the following diagram commutes:

$$\begin{array}{ccc} & & TM \\ & \nearrow \tilde{f} & \downarrow \pi \\ D & \xrightarrow{f} & M \end{array}$$

Note that such a lift isn't unique!

Ex: One such lift is $\tilde{f} = \begin{pmatrix} f_t \\ f_s \end{pmatrix}$. For such a \tilde{f} , we can define $\frac{D}{dt}\tilde{f}$ and $\frac{D}{ds}\tilde{f}$ by restricting \tilde{f} to t and s curves, respectively.

Prop: $\frac{D}{dt}f_s = \frac{D}{ds}f_t$ at each (t, s) .

Proof: We will compute in local coordinates (x^1, \dots, x^n) . Let $X_i = \frac{\partial}{\partial x^i}$, $\forall i$. We write $f(t, s) = (x^1(t, s), \dots, x^n(t, s))$, where $x^i(t, s) : \text{dom}(f) \rightarrow \mathbb{R}$. Note that we can write $f_s = \frac{\partial x^i}{\partial s} X_i(f(t, s))$, and likewise for f_t . We now compute

$$\frac{D}{dt}f_s = \frac{\partial^2 x^i}{\partial t \partial s} X_i + \frac{\partial x^i}{\partial s} \frac{D}{dt} X_i$$

We know $f_t = \frac{\partial x^j}{\partial t} X_j$, so because $\frac{D}{dt}$ is the covariant derivative with respect to f_t ,

$$\frac{D}{dt} X_i = \frac{\partial x^j}{\partial t} \nabla_{X_j} X_i \quad \frac{D}{dt} f_s = \frac{\partial^2 x^i}{\partial t \partial s} X_i + \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial t} \nabla_{X_j} X_i$$

Computing similarly, we also get

$$\frac{D}{ds} f_t = \frac{\partial^2 x^i}{\partial s \partial t} X_i + \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial s} \nabla_{X_j} X_i$$

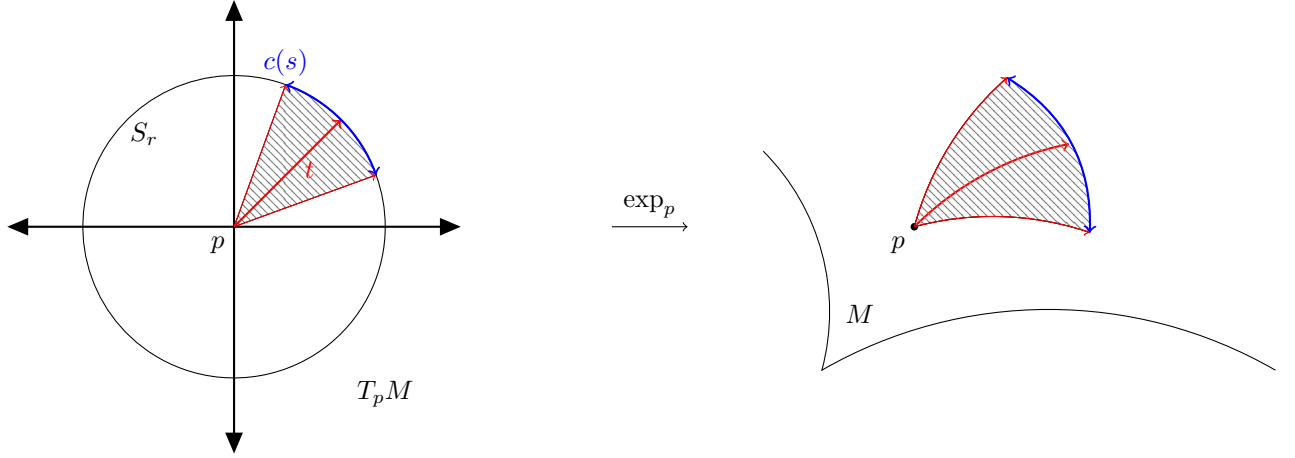
By Clairaut's theorem, $\frac{\partial^2 x^i}{\partial t \partial s} = \frac{\partial^2 x^i}{\partial s \partial t}$, so the first term of $\frac{D}{dt}f_s$ and $\frac{D}{ds}f_t$ are equal. Furthermore, because the Levi-Civita connection is torsion-free, $[X_i, X_j] = 0$, so $\nabla_{X_j} X_i = \nabla_{X_i} X_j$. This means we can swap the coefficients in the second term to show equality. We conclude that $\frac{D}{dt}f_s = \frac{D}{ds}f_t$. \square

Observe: We can ask if $\frac{D}{ds}$ and $\frac{D}{dt}$ commute. We'll see on Friday that the answer is no, because curvature comes into play.

We're now ready to prove Gauss' lemma...

Lemma: (Gauss' Lemma) In a normal neighborhood of p , radial geodesics are orthogonal to geodesic spheres.

Proof: Let $p \in M$ and $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \xrightarrow{\sim} \exp_p(B_\varepsilon(0))$ (with $B_\varepsilon(0) \subseteq T_p M$ and $\exp_p(B_\varepsilon(0)) \subseteq M$) is a diffeomorphism onto its image. Take $r \in (0, \varepsilon)$, so $S_r \subseteq T_p M$ is the sphere of radius r . Then choose any curve $s \mapsto c(s) \in S_r$, for an arbitrarily small domain $s \in (-\delta, \delta)$. Define $f(t, s) = \exp_p(tc(s))$. Illustration:



The key calculation we'll perform is $\frac{d}{dt} \langle f_t, f_s \rangle$; we want to show it's equal to 0. Well,

$$\begin{aligned}
 \frac{d}{dt} \langle f_t, f_s \rangle &= \left\langle \frac{D}{dt} f_t, f_s \right\rangle + \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
 &\stackrel{(1)}{=} \langle 0, f_s \rangle + \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
 &= \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
 &\stackrel{(2)}{=} \left\langle f_t, \frac{D}{ds} f_t \right\rangle \\
 &= \frac{1}{2} \left\langle \frac{D}{ds} f_t, f_t \right\rangle + \frac{1}{2} \left\langle f_t, \frac{D}{ds} f_t \right\rangle \\
 &= \frac{1}{2} \frac{d}{ds} \langle f_t, f_t \rangle \\
 &\stackrel{(3)}{=} \frac{1}{2} \frac{d}{ds} \|c(s)\|^2 \\
 &= \frac{1}{2} \frac{d}{ds} r^2 \\
 &= 0
 \end{aligned}$$

with (1) because $t \mapsto \exp_p(tv) = G(1, p, tv) = G(t, p, v)$ is a geodesic, (2) because of the proposition from earlier, and (3) because $\langle f_t, f_t \rangle$ is constant with respect to t , so we can choose to evaluate it at $t = 0$. Now, we can evaluate $\langle f_t, f_s \rangle|_{t=0} = \langle c(s), 0 \rangle = 0$, so we get that, for all t, s , $\langle f_t, f_s \rangle = 0$. \square

Why are we done? Well, we can find $f_t(t)$ by $f(t, s = 0)$ WOLOG, so $f(t, 0)$ is the velocity of the radial geodesic $t \mapsto \exp_p(tv(0))$, and $f(t, 0)$ is an arbitrary tangent vector to the geodesic sphere $\exp_p(S_r)$. So we conclude that the tangent space of a point q on the geodesic sphere is perpendicular to the geodesic $\exp_p(tv)$, where $\exp_p(v) = q$.

Cor: If $U = \exp_p(B_\varepsilon(0))$ is a normal neighborhood of p , and $q \in U$, then the shortest path from p to q is $t \mapsto \exp_p(tv)$ ($0 \leq t \leq 1$), where $\exp_p(v) = q$. (By path, we mean a continuous, piecewise smooth function, although we only need it to be C^1 for this proof.)

Proof: Assume $c : [0, 1] \rightarrow U$, with $c(0) = p$ and $c(1) = q$, is a smooth path, and its image is contained in U . Write $(\exp_p)^{-1}(c(t)) = r(t)w(t)$, where $r(t) \geq 0$ and $\|w(t)\| \equiv 1$. Consider the family $f(r, t) = \exp_p(rw(t))$, so that $c(t) = f(r(t), t)$. Then $\frac{dc}{dt} = \frac{dr}{dt}f_r + f_t$, and f_r and f_t are perpendicular for all t , so we can use the Pythagorean theorem to find

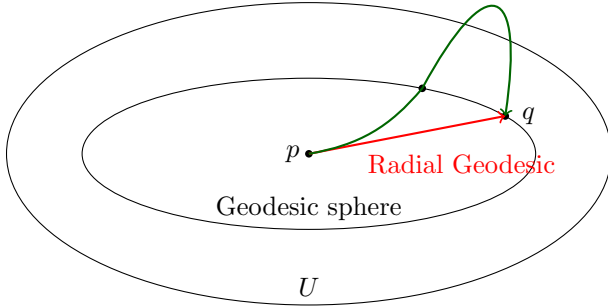
$$\left\| \frac{dc}{dt} \right\|^2 = \left\| \frac{dr}{dt} f_r \right\|^2 + \|f_t\|^2 = \left| \frac{dr}{dt} \right|^2 \|f_r\|^2 + \|f_t\|^2 = \left| \frac{dr}{dt} \right|^2 + \|f_t\|^2 \geq \left| \frac{dr}{dt} \right|^2$$

Using this inequality, we can bound the length of c :

$$\ell(c) = \int_0^1 \left\| \frac{dc}{dt} \right\| dt \geq \int_0^1 \left| \frac{dr}{dt} \right| dt \geq \int_0^1 \frac{dr}{dt} dt = r(1) - r(0) = r(1)$$

But $r(1)$ is the length of the radial geodesic $r \mapsto \exp_p(rw(1))$, joining p to q . (Note that equality holds iff $\|f_t\|^2 \equiv 0$, which is true iff c is the radial geodesic.)

Now, we must consider the case where the image of c is not contained in U . Well, there must be some $t_0 \in (0, 1)$ s.t. $c(t_0)$ is on the geodesic sphere passing through q . We know the length of $c : [0, t_0]$ is no smaller than the length of a radial geodesic directly to q , so the inequality still holds. See the illustration below:



Cor: $d(p, q) = \inf(\ell(c))$, over the set of all c joining p and q , is actually a distance function.

We showed all the other parts earlier – the only thing left to check is that $d(p, q) = 0 \Rightarrow p = q$. We'll prove this by contraposition next time, but the idea is to assume that p and q are distinct, and then construct a normal neighborhood of p that doesn't contain q . Then we know that $d(p, q)$ must be larger than the radius of the geodesic sphere, which is nonzero.