Math 635 Lecture 3

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Review from Friday: Given a finite-dimensional vector space V (i.e. T_pM for some $p \in M$), a tensor of type (k, ℓ) is an element of

$$T^{(k,\ell)}(V) \stackrel{\text{def}}{=} \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \cong \{\text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}\}$$

Ex:

- Type $(0,1)$ is V^*, the set of linear maps $V \to \mathbb{R}$.
- Type $(1,0)$ is $(V^*)^* \cong V$, by $\begin{array}{l} V \xrightarrow{\sim} (V^*) \\ v \mapsto (\alpha \mapsto \alpha(v)) \end{array}$ (for $\alpha \in V^*$).
- Type $(0,2)$ is $V^* \otimes V^* \cong \{\text{bilinear } V \times V \to \mathbb{R}\}$.

Defn: We can take the <u>tensor product</u> (or <u>outer product</u>) of two tensors on the same vector space: given $\tau \in T^{(k,\ell)}(V)$ and $\tau' \in T^{(k',\ell')}(V)$, we define $\tau \otimes \tau' \in T^{(k+k',\ell+\ell')}(V)$ by

$$(\tau \otimes \tau')(\underbrace{v_1, \ldots, v_{k+k'}}_{\in V}, \underbrace{\alpha_1, \cdots, \alpha_{\ell+\ell'}}_{\in V^*}) \stackrel{\text{def}}{=} \tau(v_1, \ldots, v_k, \alpha_1, \ldots, \alpha_k) \cdot \tau'(v_{k+1}, \ldots, v_{k+k'}, \alpha_{\ell+1}, \ldots, \alpha_{\ell+\ell'})$$

Note: We have to check that this map is indeed multilinear.

Ex: If $\alpha, \beta \in T^{(0,1)}(V) = V^*$, then $\alpha \otimes \beta$ is the map $\begin{array}{l} V \times V \to \mathbb{R} \\ (v_1, v_2) \mapsto \alpha(v_1)\beta(v_2) \end{array}$.

Now, we want to consider tensors on manifolds. By the lemma from last time, given k, ℓ, there's a bundle $\pi : T^{(k,\ell)}(TM) \to M$ with fibers $T^{(k,\ell)}(T_pM) = T^{(k,\ell)}(\pi^{-1}(p))$ for $p \in M$. Given coordinates $(x^1, \ldots, x^n)$ on $U \subseteq M$, we have a smooth moving frame:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \;\middle|\; i_a, j_b \in \{1, \ldots, n\} \right\}$$

Last time, we stated that this is a basis, and any smooth section is a linear combination of these components, with C^∞ functions as coefficients:

$$\sum A^{i_1 \cdots i_k}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

Why is this true?

Exer: Check that $A^{i_1 \cdots i_k}_{j_1 \cdots j_\ell} = \tau(dx^{i_1}, \ldots, dx^{i_k}, \frac{\partial}{\partial x^{j_1}}, \ldots, pdx^{j_\ell})$.

Ex: Fix a smooth 1-form $\alpha \in \Omega^1(U)$. Then $\alpha = \sum_{j=1}^n a_j dx^j$, and $a_j = \alpha(\frac{\partial}{\partial x^j})$.

Defn: Some terminology: We say $\tau \in \Gamma(T^{(k,\ell)}(TM))$ is a <u>tensor</u>, or <u>tensor field</u>, on M of type (k, ℓ). k is the <u>contravariant degree</u>, and ℓ is the <u>covariant degree</u>.

Ex: Vector fields are contravariant (type $(1,0)$).
1-forms are covariant (type $(0,1)$).

Einstein Summation Notation

In coordinates, the covariant indices are subindices, and the contravariant indices are superindices. The convention of Einstein summation notation is if the same index appears exactly twice in a monomial – one upper and one lower – then summation over that index is implied.

Ex: (Linear case) $V = \mathbb{R}^n$, $\alpha \in V^* \otimes V \cong \mathrm{Hom}(V, V)$. Then we have

$$a^i_j v^j = \sum_{j=1}^n a^i_j v^j = \left[\begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \right]_i$$

Note that (a^i_j) is taken to be a matrix, where j is the column index and i the row index.

Looking ahead, we'll also have a contraction operation: $a \in V^* \otimes V \rightsquigarrow \alpha \in \mathrm{Hom}(V, V) \rightsquigarrow \mathrm{tr}(a) = a^i_i \in \mathbb{R}$. *This doesn't require coordinates, as trace is basis-invariant.*

Defn: A <u>Riemannian metric</u> g on a manifold M is a smooth $(0, 2)$ tensor, which, at every point, as a matrix, is symmetric ($g_p = g_p^T$) and positive definite ($\forall v \in T_pM$, $v^T g_p v \geq 0$, with equality iff $v = 0$).

Observe: $\forall p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear map. In coordinates $(x^1, \ldots, x^n)$ on $U \overset{\text{open}}{\subseteq} M$, $g = g_{ij}\, dx^i \otimes dx^j$, where $g_{ij} \in C^\infty(U)$, and $g_{ij} = g_{ji}$. In fact, because of symmetry, we can write (using Einstein summation notation)

$$\begin{aligned} g &= \frac{1}{2}\left[g_{ij}\, dx^i \otimes dx^j + g_{ji}\, dx^i \otimes dx^j\right] \\ &= \frac{1}{2}\left[g_{ij}\, dx^i \otimes dx^j + g_{ij}\, dx^j \otimes dx^i\right] \\ &= g_{ij} \frac{1}{2} \underbrace{\left[dx^i \otimes dx^j + dx^j \otimes dx^i\right]}_{\overset{\text{def}}{=} dx^i dx^j = dx^j dx^i, \text{ the symmetric product}} \end{aligned}$$

It's also standard notation to write $ds^2 \overset{\text{def}}{=} dx^i dx^j$, so we can write $g = g_{ij}\, ds^2$.

Defn: A <u>Riemannian manifold</u> is a pair (M, g), where M is a smooth manifold, and g is a Riemannian metric on M.

Defn: If (M, g^M) and (N, g^N) are Riemannian manifolds, an <u>isometry</u> $F : M \to N$ is a diffeomorphism s.t. $\forall p \in M$, $(dF)_p : T_pM \to T_{F(p)}N$ preserves the metric, i.e., s.t. $\forall u, v \in T_pM$, $g^N_{F(p)}(dF_p(u), dF_p(v)) = g^M_p(u, v)$.

Notation: Sometimes, we'll write $g_p(u, v) = \langle v, w \rangle_p = \langle v, w \rangle$. (We omit the subscript when the choice of metric is obvious.)

Ex: Let $M \subset \mathbb{R}^N$, a regular submanifold. $\forall p \in M$, $T_pM \subset \mathbb{R}^N$ can be identified with a subspace of $T_pR^N \cong \mathbb{R}^N$. Define $g_p = \langle\ ,\ \rangle$ by the usual inner product in $\mathbb{R}^N$.

Ex: Let $M = S^2 \subseteq \mathbb{R}^3$. We'll compute in coordinates in the upper hemisphere. Say our coordinates are (s, t) (with projection to the flat unit disk), then we have our inverse parameterization

$$(s, t) \mapsto (s, t, \sqrt{1 - s^2 - t^2}) \overset{\text{def}}{=} \vec{r}(s, t)$$

(with notation as in Calc III). Then under $T_pS^2 \hookrightarrow \mathbb{R}^3$, we have

$$\begin{aligned} \frac{\partial}{\partial s} &\mapsto \frac{\partial \vec{r}}{\partial s} = \left(1, 0, \frac{-s}{\sqrt{1 - s^2 - t^2}}\right) \\ \frac{\partial}{\partial t} &\mapsto \frac{\partial \vec{r}}{\partial t} = \left(0, 1, \frac{-t}{\sqrt{1 - s^2 - t^2}}\right) \end{aligned}$$

So our metric is

$$(g_{ij}) = \begin{pmatrix} \frac{\partial \vec{r}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial s} & \frac{\partial \vec{r}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial t} \\ \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial s} & \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 - t^2 & \frac{st}{1 - s^2 - t^2} \\ \frac{st}{1 - s^2 - t^2} & 1 - s^2 \end{pmatrix}$$

Observe: Any submanifold $S \subset M$ of a Riemannian manifold (M, g) inherits a Riemannian metric by $\forall p \in S$, $T_pS \hookrightarrow T_pM$.