Math 635 Lecture 14

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Hamiltonian Formulation of Geodesic Flow

Defn: A symplectic manifold is a pair (X,ω) where ω is a 2-form on X s.t. $d\omega = 0$ and ω is pointwise non-denerate: $\forall m \in X$, the map

$$\begin{aligned} \omega_m^\sharp : T_mX &\to T_m^*X \\ v &\mapsto -\omega_m(\cdot, v) \end{aligned}$$

is an isomorphism.

Note: This implies the dimension of X is even, since skew symmetric forms on odd dimensional spaces are singular.

Ex: Let $X = \mathbb{R}^{2n}$, with coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$. Then $\omega = \sum_i dp_i \wedge dx^i$, as a matrix, is $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

We will see, $\forall M$ smooth manifolds, that $X = T^*M$ is naturally a symplectic manifold. And on symplectic manifolds, we can define Hamiltonian dynamics.

Defn: Given $H \in C^\infty(X)$, where (X,ω) is a symplectic manifold, the Hamilton field of H, $\Xi_H \in \mathfrak{X}(X)$, is defined by the condition that $-\iota_{\Xi_H} = \omega(\cdot, \Xi_H) = dH$. Existence is guaranteed by the non-degeneracy of ω.

We want to compute a local formula: In $\mathbb{R}^{2n}$, $H : \mathbb{R}^{2n} \to \mathbb{R}$. Then $\Xi_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$. The flow/trajectory of Ξ_H are

$$\begin{cases} \dot{x}^i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t)) \\ \dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t)) \end{cases}$$

These are Hamilton's equations.

Exer: $H = \frac{1}{2m}\left|\left|p\right|\right|^2 + V(x)$ is Newton's second law, $\ddot{x} = -\nabla V$.

Properties of Hamiltonian flows (i.e. flows of Ξ_H):

(a) $\Xi_H(H) = 0$, i.e., H is constant along trajectories of Ξ_H.
Proof: $\Xi_H(H) = dH(\Xi_H) = \omega(\Xi_H, \Xi_H) = 0$ by antisymmetry. $\square$

(b) $\mathcal{L}_{\Xi_H}\omega = 0$.
Proof: Use Cartan's formula. $\mathcal{L}_{\Xi_H}\omega = \iota_{\Xi_H}\underbrace{d\omega}_{=0} + d(\underbrace{\iota_{\Xi_H}\omega}_{=-dH}) = -d^2H = 0$. $\square$

Volume elements (Liouville)

On any symplectic manifold (X,ω), the form $\frac{\omega^n}{n!}$ is a volume form.

Ex: In $\mathbb{R}^{2n}$, $\frac{\omega^n}{n!} = dp_1 \wedge dx^1 \wedge dp_2 \wedge dx^2 \wedge \cdots \wedge dp_n \wedge dx^n$.

If we're given a Hamiltonian $H \in C^\infty(X)$, and $c \in \mathbb{R}$ is a regular value of H, then let $\Sigma = H^{-1}(c) \hookrightarrow X$, a codim-1 submanifold. We claim that $\exists! \lambda \in \Omega^{2n-1}(X)$ s.t. in a neighborhood of Σ, $\frac{\omega^n}{n!} = \lambda \wedge dH$, and $\iota^*(\omega)$ is unique. This is a volume form on Σ.

Cor: The Hamilton flow of H preserves $\frac{\omega^n}{n!}$, and its restriction to any regular level set $H^{-1}(c)$ preserves the Liouville measure on that level set.

$$\phi_t^*\left(\frac{\omega^n}{n!}\right) = \frac{\omega^n}{n!}; \qquad \left(\phi_t|_{H^{-1}(c)}\right)^*(\iota^*\lambda) = (\iota^*\lambda)$$

Symmetries

Question: Given $H, G \in C^\infty(X)$, the Hamiltonian flows of H and G commute iff $[\Xi_G, \Xi_H] = 0$, which is true iff Ξ_H is ϕ_t^G-related to itself.

Lemma: For X connected, this is equivalent to $(\phi_t^G)^* dH = dH$.

Proof:

$$\begin{aligned}(\phi_t^G)^* dH = dH &\Leftrightarrow \mathcal{L}_{\Xi_G}(dH) = 0 \\ &\Leftrightarrow \cancel{\iota_{\Xi_G} d^2 H} + d(\iota_{\Xi_G} dH) = dH(\Xi_G) = 0 \\ &\Leftrightarrow d(dH(\Xi_G)) = 0 \\ &\Leftrightarrow dH(\Xi_G) = \Xi_G(H) \text{ is locally constant} \\ &\Leftrightarrow \Xi_G(H) \text{ is constant (because } X \text{ is connected)}\end{aligned}$$

This is a symmetric condition:

$$dH(\Xi_G) = \omega(\Xi_G, \Xi_H) = -\omega(\Xi_H, \Xi_G) = -dG(\Xi_H)$$

$\square$

The main example of (X, ω) is T^*M, for some arbitrary smooth manifold M.

Prop: For an arbitrary smooth manifold M, T^*M has a natural symplectic structure.

Proof: We'll show that T^*M has a natural, "tautological" 1-form, α, which is sometimes called a Liouville form, defined by:
For $(x, \xi) \in T^*M$, $x \in M$, $\xi \in T_x^*M$, let $v \in T_{(x,\xi)}(T^*M)$. Then $\alpha_{(x,\xi)}(v) = \xi(\underbrace{\pi_*(v)}_{\in T_x M})$. In coordinates, say we have $(x^1, \ldots, x^n)$ on $U \subset M$, and $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ on $T{*}U$. Then $\xi = p_i(\xi) dx^i$, so $v = a^i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial p_i}$. $\pi_*(v) = a^i \frac{\partial}{\partial x^i}$. $\xi(\pi_*(v)) = p_i(\xi) a^i$. Altogether, $\alpha = p_i dx^i$, and $\omega + d\alpha = dp_i \wedge dx^i$, just as in $\mathbb{R}^{2n}$.