Math 635 Lecture 28

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We're currently trying to prove:

Thm: Let M be connected, $p \in M$ such that $\exp_p$ is defined on all of T_pM. Let $q \in M$. Then there's a geodesic γ from p to q such that $d(p,q) = \ell(\gamma)$.

Continuing from last time, we want to show that $\sup \mathscr{T} = d(p,q)$, i.e., $d(p,q) \in \mathscr{T}$, i.e., $d(p,q) = d(p,q) + d(\gamma(d(p,q)), q)$, i.e., $d(\gamma(d(p,q)), q) = 0$. Because d is a distance function, it's enough to show $\gamma(d(p,q)) = q$.

Let $t_1 = \sup \mathscr{T}$. Because $\mathscr{T}$ is closed, we know $t_1 \in \mathscr{T}$. Assume that $t_1 < d(p,q)$ (we will show a contradiction). Then $\exists \delta > 0$ s.t. $t_1 + \delta < d(p,q)$, and there exists a geodesic sphere S_δ centered at $\gamma(t_1)$ with radius δ. Let $y \in S_\delta$ minimizing the map

$$\begin{aligned} S_\delta &\to \mathbb{R} \\ x &\mapsto d(x,q) \end{aligned}$$

We want to show $t_1 + \delta \in \mathscr{T}$.

Claim 1: $d(y,q) = d(p,q) - (t_1 + \delta)$.
Proof: The lemma from last time implies that $d(p,q) - t_1 = d(\gamma(t_1), q) = \delta + d(y,q)$. $\square$

Claim 2: $d(p,y) = t_1 + \delta$.
Proof: $\leq$ follows directly from the triangle inequality. $\geq$ is true because $d(p,q) \leq d(p,y) + d(y,q)$, so

$$d(p,y) \geq d(p,q) - (d(p,q) - (t_1 + \delta)) \geq t_1 + \delta$$

by claim 1. $\square$

Claim 3: $y = \gamma(t_1 + \delta)$.
Proof: Consider the path $\gamma(t)$ for $0 \leq t \leq t_1$, followed by a radial geodesic from $\gamma(t_1)$ to y. So the path is overall from p to y. By claim 2, this path is length minimizing, so it's a geodesic with the same initial conditions as γ. This implies it must be γ, with the domain $0 \leq t \leq t_1 + \delta$.

Claims 1 and 3 together imply that $t_1 + \delta \in \mathscr{T}$, a contradiction. This completes the proof. $\square$

Thm: (Hopf-Rinow) Let M be connected. Assume $\exists p \in M$ s.t. $\exp_p$ is defined on all of T_pM. Then

(a) Every closed and bounded set is compact.
(b) M is complete as a metric space.
(c) M is geodesically complete, i.e., $\forall q \in M$, $\exp_q$ is defined on all of T_qM.

In fact, these are all equivalent. We will show this by: assumption $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ assumption.

Proof: assumption $\Rightarrow$ (a): Let $S \subset M$ be closed and bounded. Then $\forall y \in S$, $\exists \gamma_y$ a minimizing geodesic from p to y. S is bounded, so $\exists R > 0$ s.t. $\forall y \in S$, $\ell(\gamma_y) < R$. So $S \subset \exp_p\left(\overline{B_R(0)}\right)$, where $B_R(0) \subset T_pM$. Because $\overline{B_R(0)}$ is compact and $\exp_p$ is continuous, $\exp_p\left(\overline{B_R(0)}\right)$ is also compact, so S is contained in a compact set, and is thus compact.

(a) $\Rightarrow$ (b): This only requires facts of point-set topology. If (x_n) is Cauchy, then it is bounded. So its image is compact. Thus, there's a convergent subsequence, so (x_n) converges to its limit point.

(b) $\Rightarrow$ (c): We prove this by contradiction. Assume $\exists q \in M$, $T > 0$, and $v \in T_qM$ with $||v|| = 1$, such that $\gamma(t) = \exp_q(tv)$ exists $\forall t \in [0,)$, but not beyond T. Take $t_1 < t_2 <$ in $[0,T)$ so that (t_n) converges to T, and let $x_n = \gamma(t_n)$. Note that $d(x_n, x_m) \leq |t_n - t_m|$, $\forall n, m$. Since (t_n) converges, (x_n) is Cauchy, so $\exists w \in M$ such that

$\lim_{n\to\infty} x_n = w$. Let W be a totally normal neighborhood of w, and $\delta > 0$ such that $\forall x \in W$, the geodesic ball $B(x, \delta) \supseteq W$. Then $\exists N \in \mathbb{N}$, such that $\forall n, m > N$, $|t_n - t_m| < \delta/2$, and $\gamma(t_m) \in W$. Pick $n > N$. Use $\exp_{x_n}$ to "relaunch" the geodesic. $s \mapsto \exp_{x_n}(s\dot{\gamma}(t_n))$ exists for $|s| < \delta$. This extends γ past T, since $t_n + \delta > T$. Oops!

(c) $\Rightarrow$ assumption: Trivial. $\square$

Defn: If M satisfies these properties, we call M a <u>complete Riemannian manifold</u>.

We obtain yet another version of Bonnet-Myer as a corollary:

Thm: (Bonnet-Myer III) Let M be a compact, connected Riemannian manifold, and assume Ric $> \left(\frac{\pi}{\ell}\right)^2$. Then M is compact and the diameter of M is no more than ℓ.

Proof: We already know that any geodesic with a length of at least ℓ is not minimizing. Also, any pair of points can be joined by a minimizing geodesic. Thus, the diameter of M is at most L, so M is bounded and compact. $\square$