Math 635 Lecture 11

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2/12/21

Recall the theorem and definition stated at the end of the previous lecture...

Thm: Let M be a Riemannian manifold. Then $\exists! \nabla$ on $\mathcal{E} = TM \to M$ such that

(a) ∇ preserves the Riemannian metric. *(This depends on the choice of Riemannian metric.)*
(b) $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$. *(This does not depend on the choice of Riemannian metric.)*

Defn: This ∇ is called the Levi-Civita connection on M.

This theorem is sometimes known as the "Fundamental Theorem of Riemannian Geometry". The second condition – that $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$ – is sometimes called a "symmetry condition".

Proof of the theorem: We'll use properties (a) and (b) to find an expression for ∇. Let $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned} X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \underbrace{\langle \nabla_X Y, Z\rangle}_{\text{(III)}} + \underbrace{\langle Y, \nabla_X Z\rangle}_{\text{(I)}} + \underbrace{\langle \nabla_Y Z, X\rangle}_{\text{(II)}} + \underbrace{\langle Z, \nabla_Y X\rangle}_{\text{(IV)}} - \underbrace{\langle Y, \nabla_Z X\rangle}_{\text{(I)}} - \underbrace{\langle \nabla_Z Y, X\rangle}_{\text{(II)}} \\ &= \underbrace{\langle Y \nabla_X Z - \nabla_Z X\rangle}_{\text{(I)}} + \underbrace{\langle \nabla_Y Z - \nabla_Z Y, X\rangle}_{\text{(II)}} + \underbrace{\langle \nabla_X Y, Z\rangle}_{\text{(III)}} + \underbrace{\langle [Y, X] + \nabla_X Y, Z\rangle}_{\text{(IV)}} \\ &= \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle + \langle Z, [Y, X]\rangle + 2\langle Z, \nabla_X Y\rangle \end{aligned}$$

Now, we solve for $2\langle \nabla_X Y, Z\rangle$:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \Big(\langle X, [Y, Z]\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle\Big)$$

This is our defining expression for ∇, since choosing Z in all possible ways defines $\nabla_X Y$. We claim that defining ∇ in this way gives us the desired connection. (This part of the proof is tedious, so check it out in the textbook if you're interested.) $\square$

Computation of the Christoffel Symbols in Coordinates

Let $(x^1, \dots, x^n)$ be coordinates on $U \subseteq M$, and define $X_i = \frac{\partial}{\partial x^i}$, $g_{ij} = \langle X_i, X_j\rangle$. Note that $[X_i, X_j] = 0$.

Recall: The Christoffel symbols $\Gamma^\ell_{ij} \in C^\infty(U)$ are defined by $\nabla_{X_i} X_j = \Gamma^\ell_{ij} X_\ell$. We want to compute Γ^ℓ_{ij} using the defining expression above. Well,

$$2\langle \nabla_{X_i} X_j, X_k\rangle = 2\left\langle \Gamma^\ell_{ij} X_\ell, X_k\right\rangle = 2\Gamma^\ell_{ij} g_{\ell k} = X_i(g_{jk}) + X_j(g_{ki}) - X_k(g_{ij})$$

Now, we introduce the matrix g^{-1}, the inverse of $(g_{\ell k})$, with the notation $g^{-1} = (g^{km})$, so that $g_{\ell k} g^{km} = \delta^m_k$. If we multiply both sides by g^{-1} / g^{km}, and sum over k, we get

$$\underbrace{2\Gamma^\ell_{ij} \underbrace{g_{\ell k} g^{km}}_{=\delta^m_\ell}}_{=2\Gamma^m_{ij}} = \sum_k g^{km} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Thus,

$$\Gamma^m_{ij} = \frac{1}{2} \sum_k g^{km} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Observe: $[X_i, X_j] = 0 \Leftrightarrow \Gamma_{ij}^m = \Gamma_{ji}^m$. So the number of independent indices of the Christoffel symbols is $\frac{n(n+1)}{2} \cdot n = \frac{n^2(n+1)}{2}$. For example, when $n = 2$ (a surface), there are 6 Christoffel symbols.

Exer: (Do Carmo) For the upper half plane $\mathcal{H}$ (with the metric from HW1), show $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = \frac{1}{y}$, and $\Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$.

Exer: (HW3) If $M \subset \mathbb{R}^N$, with the induced Riemannian metric from the Euclidean metric on $\mathbb{R}^N$, and if $\gamma : [a, b] \to M$ and $V \in \Gamma_\gamma(TM)$, we can define

$$\frac{\bar{D}V}{dt} = \frac{d}{dt}(V : [a, b] \to \mathbb{R}^n)$$

Claim: If $\frac{D}{dt}$ is the operator associated with the Levi-Civita connection on M, then we have $\frac{DV}{dt}(t) = \pi_{\gamma(t)}[\frac{\bar{D}V}{dt}(t)]$, where $\pi_{\gamma(t)} : \mathbb{R}^N \to T_{\gamma(t)}M$ is the orthogonal projection.

Geodesics

Observe: TM is such that every curve γ into M has a natural lift to TM.

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "{(\gamma,\dot\gamma)}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

In an abuse of notation, we sometimes write "$\dot{\gamma}(t) = \frac{d\gamma}{dt} = (\gamma(t), \dot{\gamma}(t))$".
We can then consider $\frac{D}{dt}\frac{d\gamma}{dt}$, the acceleration of $\gamma \in \Gamma_\gamma(TM)$.

Defn: γ is a <u>geodesic</u> iff $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$.

Ex: Let $M = S^2 \hookrightarrow \mathbb{R}^3$. Then $\gamma(t) = (\cos(t), \sin(t), 0)$ is a geodesic, as $(\frac{\bar{D}}{dt}\dot{\gamma})(t) = \ddot{\gamma}(t) = -\gamma(t)$.
$\gamma(t) \perp T_{\gamma(t)}S^2 \Rightarrow \pi_{\gamma(t)}\frac{\bar{D}}{dt}\dot{\gamma}(t) = 0$.

Observe:

1. If γ is a geodesic, then $\frac{d}{dt}\left\langle \frac{d\gamma}{dt}\frac{d\gamma}{dt}\right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt}\right\rangle = 0$, so $\left|\left|\frac{d\gamma}{dt}\right|\right|$ is constant with respect to t. In other words, the "speed" of a geodesic is constant.
2. If γ is a geodesic, $c \in \mathbb{R}$, then $\gamma_c(t) \stackrel{\text{def}}{=} \gamma(ct)$ is also a geodesic. But other reparameterizations are generally not geodesics. The speed of γ_c is $|c|$ times the speed of γ.