Math 635 Lecture 13

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Review: Given (M, g) a Riemannian manifold, $\exists! G \in \mathfrak{X}(TM)$ s.t. the integral curves of G are the lifts of geodesics.

Notation: $\forall (q, v) \in TM$, $t \mapsto \gamma(t, q, m)$ is the geodesic with initial condition (q, v). It is the projection onto M of the integral curve of G, starting at (q, v). For given (q, v), $\gamma(t, q, v)$ has a maximal domain of definition, an interval in t, that depends on (q, v).

Lemma: (Lemma 1 from Last Time) $\forall p \in M$, $\exists V \subseteq M$, a neighborhood of p, and $\exists \varepsilon, \delta > 0$ s.t. γ is defined on the set $(-\delta, \delta) \times \{(q, v) \in TV : ||v|| < \varepsilon\}$. That is, $\gamma(t, q, v)$ is defined $\forall q \in V$, $v \in T_q V$, $||v|| < \varepsilon$, $|t| < \delta$, and γ is smooth as a map.

Defn: Given $V \subseteq M$, $\varepsilon > 0$, we define the <u>ε-ball tangent bundle</u>, by $B_\varepsilon(TV) = \{(q, v) \in TV : ||v|| < \varepsilon\}$. This is a fiber bundle $B_\varepsilon(TV) \to V$, whose fibers are open balls of radius ε, centered at 0.

Defn: We also define the <u>unit tangent bundle</u> of M, $S_1(TM)$, by $S_\varepsilon(TV) = \{(q, v) \in TV : ||v|| = \varepsilon\}$. This is a fiber bundle $S_1(TM) \to M$, whose fibers are S^{n-1}.

Ex: The unit tangent bundle of S^2 is isomorphic to

$$\left\{(\vec{q}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 : ||\vec{q}|| = 1, ||\vec{v}|| = 1, \vec{q} \cdot \vec{v} = 0\right\}$$

(The final condition is a tangency condition). In turn, this is diffeomorphic to SO(3) as manifolds, by $(\vec{q}, \vec{v}) \mapsto (\vec{q}, \vec{v}, \vec{q} \times \vec{v})$. Treating the three output vectors as columns of a matrix yields an orthogonal matrix with determinant 1.

Observe: G is tangent to $S_\varepsilon(TM)$, $\forall \varepsilon > 0$. This is a fancy way to say that, along a geodesic, speed is constant. Because $\frac{d}{dt}\langle \dot\gamma, \dot\gamma \rangle = 0$, we know $||\dot\gamma||$ is constant, so the integral curves of G are fully contained in $S_\varepsilon(TM)$.

Cor: If M is compact, every geodesic is defined $\forall t \in \mathbb{R}$, i.e., G is complete.

Proof: M is compact, so $\forall \varepsilon > 0$, $S_\varepsilon(TM)$ is compact, and any field on a compact manifold is complete. $\square$

Lemma: (Lemma 2 from Last Time) (Homogeneity of Geodesic Flow) Let $(q, v) \in TM$, $a > 0$. If $\gamma(t, q, v)$ is defined for $|t| < \delta$, then $\gamma(t, q, av)$ is defined for $|t| < \frac{\delta}{a}$, and $\gamma(t, q, av) = \gamma(at, q, v)$.

Proof: Check that both sides satisfy the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$, and have the same initial conditions (namely, (q, av)). $\square$

Prop: (Do Carmo 2.7) $\forall p \in M$, $\exists V \subseteq M$ a neighborhood of p, and $\varepsilon > 0$ s.t. $\forall (q, v) \in B_\varepsilon(TV)$, $\gamma(t, q, v)$ is defined for $|t| < 43$. (Note: we really just need it to be defined for $t = 1$, so we can get our exponential map. But 43 is such a nice number.)

Proof: Let V, δ, ε_1 be as in Lemma 1, so that $\forall (q, v) \in B_{\varepsilon_1}(TV)$, $\gamma(t, q, v)$ is defined for $|t| < \delta$. Choose $a > 0$ s.t. $|t| < \delta \Leftarrow a|t| = |at| < 43$ – specifically, choose $a = \frac{\delta}{43}$. Now, by Lemma 2, $\gamma(t, q, \frac{\delta}{43}v)$ is defined for $|t| < 43$ if $||v|| < \varepsilon_1$. Now define $\varepsilon = \varepsilon_1 \cdot \frac{\delta}{43}$. Thus, $\frac{\delta}{43}||v|| < \varepsilon_1 \Leftrightarrow ||v|| < \varepsilon$. $\square$

Defn: Let $p \in M$, $V \subset M$ a neighborhood of p, and ε as in the previous proposition. The we define

1. $\exp : B_\varepsilon(TV) \to M$
   $(q, v) \mapsto \gamma(1, q, v)$
2. $\exp_p : B_\varepsilon(0) \to M$
   $v \mapsto \gamma(1, p, v)$

Observe: Both exp and $\exp_p$ are differentiable.

Lemma: $\forall p \in M$, $d(\exp_p)_{v=0}$ is the identity.

$$\begin{tikzcd} T_0(T_pM) \arrow[r] & T_pM \\ T_pM \arrow[u, phantom, "\cong" rotate=90] \arrow[ur, "\text{Id (claimed)}"'] & \end{tikzcd}$$

Proof: Use curves to compute $d(\exp_p)_0$. Take a curve in T_pM, starting at $0 \in T_pM$, e.g., $t \mapsto tw$ for some $w \in T_pM$. Then

$$d(\exp_p)_0(w) = \frac{d}{dt} \exp_p(tw)\Big|_{t=0} = \frac{d}{dt}\gamma(1, p, tw)\Big|_{t=0} = \frac{d}{dt}\gamma(t, p, w)\Big|_{t=0} = w$$

$\square$

Cor: $\exp_p$ is a local diffeomorphism near 0, i.e., $\exists \mathscr{N} \subset T_pM$, a neighborhood of 0 such that $\exp_p\big|_{\mathscr{N}} : \mathscr{N} \xrightarrow{\sim} U \overset{\text{open}}{\subset} M$, for some U.