Math 635 Lecture 24

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Continuing with Jacobi fields from last time. Recall: For γ a geodesic, $J \in \Gamma_\gamma(TM)$ is a Jacobi field iff $\frac{D^2}{dt^2}J + \mathcal{R}(J,\dot\gamma)\dot\gamma = 0$. It's a fact that J is a Jacobi field iff $V = \partial_s f|_{s=0}$ for some variation f of γ by geodesics. We proved part of this in class, and will prove the rest in a homework problem.

Let $E_1, \dots, E_n \in \Gamma_\gamma(TM)$ be an orthonormal parallel frame. We can write any $J \in \Gamma_\gamma(TM)$ as $J(t) = f^i(t)E_i(t)$ for some smooth functions f^i. Using the fact that $\frac{D}{dt}E_i \equiv 0$, we get $\frac{D^2}{dt^2}J = \ddot{f}^i E_i$, and by linearity, the Jacobi equation becomes a system of ODEs. $\forall i$,

$$\ddot{f}^i + \underbrace{(E_i,\dot\gamma,\dot\gamma,E_j)}_{a_{ij}=a_{ji}} f^j \equiv 0 \qquad \text{and} \qquad (E_i,\dot\gamma,\dot\gamma,E_j)f^j = \mathcal{R}(J,\dot\gamma)\dot\gamma = \mathcal{R}(f^jE_j,\dot\gamma)\dot\gamma = f^j\mathcal{R}(E_j,\dot\gamma)\dot\gamma$$

The ith component of $\mathcal{R}(J,\dot\gamma)\dot\gamma$ is

$$a_{ij} = f^i(E_j,\dot\gamma,\dot\gamma,E_i) = f^i(\dot\gamma,E_i,E_j,\dot\gamma) = a_{ji}$$

So our system of equations is

$$\ddot{f}^i(t) + a_{ij}f^j(t) = 0 \quad 1 \le i \le n$$

Cor: A Jacobi field is uniquely determined by $J(0)$ and $\frac{DJ}{dt}(0)$. In fact, we have an isomorphism

$$\begin{aligned}\{\text{Jacobi fields along } \gamma\} &\cong T_{\gamma(0)}M \oplus T_{\gamma(0)}M \\ J &\mapsto (J(0), \tfrac{DJ}{dt}(0))\end{aligned}$$

This tells us that the space of Jacobi fields along γ has dimension $2n$.

Lemma: Let J be a Jacobi field. Then $\exists a, b \in \mathbb{R}$ such that $\langle J(t), \dot\gamma(t)\rangle = a + bt$.

Proof: It's enough to show $\frac{d}{dt}\langle J, \dot\gamma\rangle$ is constant. Well,

$$\frac{d}{dt}\left(\frac{d}{dt}\langle J(t),\dot\gamma(t)\rangle\right) = \frac{d}{dt}\left\langle \frac{DJ}{dt}(t),\dot\gamma(t)\right\rangle = \left\langle \frac{D^2J}{dt^2}(t),\dot\gamma(t)\right\rangle = -\langle \mathcal{R}(J,\dot\gamma)\dot\gamma(t),\dot\gamma(t)\rangle = 0$$

with the last equality because curvature is skew-symmetric. □

Cor: If $J(0)$ and $\frac{DJ}{dt}(0)$ are orthogonal to $\dot\gamma(0)$, then they remain orthogonal for all t.

Defn: A Jacobi field satisfying the above condition is called a normal Jacobi field. The set of normal Jacobi fields forms a dimension $2(n-1)$ subspace.

Lemma: $(\dot\gamma, t\dot\gamma)$ span the space of tangential Jacobi fields.

Lemma: The space of Jacobi fields has a natural symplectic structure. $\forall J_1, J_2$ Jacobi fields, the quantity

$$\Omega(J_1,J_2) = \left\langle J_1, \frac{DJ_2}{dt}\right\rangle - \left\langle \frac{DJ_1}{dt}, J_2\right\rangle$$

is constant w.r.t. t. We take Ω to be the symplectic form.

Observe: The space of normal Jacobi fields is a symplectic subspace (i.e. the restriction of Ω is still non-degenerate). It corresponds to a certain subspace of $T_{(\dot\gamma(0),\dot\gamma(0))}(T^*M)$.

We now check the lemma above. All we need to do is show $\Omega(J_1, J_2)$ is constant w.r.t. t. So we differentiate:

$$\frac{d}{dt}\Omega(J_1,J_2) = \frac{d}{dt}\left(\left\langle J_1, \frac{DJ_2}{dt}\right\rangle - \left\langle \frac{DJ_1}{dt}, J_2\right\rangle\right) = \cancel{\left\langle \frac{DJ_1}{dt}, \frac{DJ_2}{dt}\right\rangle} + \left\langle J_1, \frac{D^2 J_2}{dt^2}\right\rangle - \cancel{\left\langle \frac{DJ_1}{dt}, \frac{DJ_2}{dt}\right\rangle} - \left\langle \frac{D^2 J_1}{dt^2}, J_2\right\rangle$$

We can then use the Jacobi equation to cancel out the remaining terms. □

Ex: Let M be an oriented surface, and take $||\dot{\gamma}|| \equiv 1$. Then $(E_1, E_2) = (\dot{\gamma}, \dot{\gamma}^\perp)$ is an orthonormal frame. We write down the Jacobi equations:

$$(E_i, \dot{\gamma}, \dot{\gamma}, E_j) = \begin{cases} 0 & i = 1 \text{ or } j = 1 \\ k & i = j = 2 \end{cases}$$

where k is the Gaussian curvature. Write $J = f^1\dot{\gamma} + f^2\dot{\gamma}^\perp$. Then $\ddot{f}^1 = 0$ iff $f^1 = a + bt$, for $a, b \in \mathbb{R}$. And $\ddot{f}^2 = kf^2 = 0$ (assume k is constant for this problem). Then $f^2(t) = Ae^{i\sqrt{k}} + Be^{-i\sqrt{k}}$. f^2 must be real, so we solve this well-known type of differential equation, and if $k > 0$, we get the following:

$$J(t) = (A\cos\sqrt{k}t + B\sin\sqrt{k}t)\dot{\gamma}^\perp$$

If $J(0) = 0$, then $J(t) = B\sin(\sqrt{k}t)\dot{\gamma}^\perp$ Observe that $J(\frac{\pi}{\sqrt{k}}) = 0$. We say that "$\gamma(0)$ and $\gamma(\frac{\pi}{\sqrt{k}})$ are conjugate".
If $k < 0$, then we replace sin and cos with sinh and cosh. Then $J(0) = 0$ implies $J(t) = B\sinh(\sqrt{k}t)\dot{\gamma}^\perp$. In this case, $J(t) \neq 0$, $\forall t \neq 0$.

One application of this is computing $d(\exp_p)_v$ for $v \neq 0$.

Prop: Given $v, w \in T_pM$, $d(\exp_p)_v(w) = J(1)$, where J is the Jacobi field such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$.

Proof: $d(\exp_p)_v(w) = \frac{d}{dt}\exp_p(v + sw)\big|_{s=0}$. Define $f(s,t) = \exp_p(t(v + sw))$. $\forall s$, $t \mapsto f(s,t)$ is a geodesic. Define $J(t) = \partial_s f|_{s=0}$. We know this is a Jacobi field, and claim that $\frac{DJ}{dt}(0) = w$.