Math 635 Lecture 21

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Review: second variation formula. If γ is a geodesic, and f a proper variation of γ, $V = \partial_s|_{s=0}$, then

$$E''(0) = -\int_0^a \left\langle V, \frac{D^2V}{dt^2} + \mathcal{R}(V,\dot\gamma)(\dot\gamma) \right\rangle dt$$

Next, we're going to take a closer look at the curvature, $\mathcal{R}$. Recall its definition: If $X, Y \in \mathfrak{X}(M)$, then the curavture is $\mathcal{R}(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by $\mathcal{R}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. It turns out that $\mathcal{R}(X,Y)$ is given by the action of a tensor $\mathcal{R}$ of the form $\forall p \in M; u,v \in T_pM$, $\mathcal{R}_p(u,v) : T_pM \to T_pM$, a linear map.

As an operator, $\mathcal{R}(X,Y)(Z)_p = \mathcal{R}_p(X_p,Y_p)(Z_p) \in T_pM$. Also, $\mathcal{R}$ shows up as an obstruction to finding a covariant-constant frame $\nabla_{E_i}E_j \equiv 0$.

Curvature Identities

(Covered in Do Carmo, Chapter 4, §2)

The first identity we consider is the Bianchi identity: $\forall X,Y,Z \in \mathfrak{X}(M)$, $\mathcal{R}(X,Y)Z + \mathbb{R}(Z,X)Y + \mathbb{R}(Y,Z)X = 0$, due to ∇ being torsion-free. The proof is simply a computation, and can be found on page 91 of Do Carmo.

Prop: Introduce $X,Y,Z,T \in \mathfrak{X}(M)$, and define $(X,Y,Z,T) \stackrel{\text{def}}{=} \langle \mathcal{R}(X,Y)Z, T\rangle$.

(a) $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$ (Proved via the Bianchi identity)
(b) $(X,Y,Z,T) = -(Y,X,Z,T)$ (Because $\mathcal{R}(X,Y) = -\mathcal{R}(Y,X)$)
(c) $(X,Y,Z,T) = -(X,Y,T,Z)$ (Because ∇ perserves $\langle\ ,\ \rangle$)
(d) $(X,Y,Z,T) = (Z,T,X,Y)$ (Follow from the Bianchi identity and some algebra)

In coordinates $(x^1,\ldots,x^n)$, $X_i = \frac{\partial}{\partial x^i}$, then $\mathcal{R}$ has components in the coordinate system, $\mathcal{R}^\ell_{ijk} \in C^\infty(U)$ (where U is the domain of the coordinate chart) such that

$$\underbrace{\mathcal{R}(X_i,X_j)X_k}_{\in \mathfrak{X}(U)} = \mathcal{R}^\ell_{ijk}X_\ell$$

But how do we compute $\mathcal{R}^\ell_{ijk}$?

Lemma: $\mathcal{R}^\ell_{ijk} = \Gamma^\ell_{jk}\Gamma^s_{i\ell} - \Gamma^\ell_{ik}\Gamma^s_{j\ell} + \partial_i\Gamma^s_{jk} - \partial_j\Gamma^s_{ik}$. (Recall that Do Carmo uses the opposite sign for $\mathcal{R}$.)

Proof: This is a messy computation. The complete details can be found on pages 92-93 of Do Carmo.

Observe that Γ depends on the first derivatives of g_{ij}, so $\mathcal{R}$ depends on the second derivatives of g_{ij}.

Defn: $(X_i,X_j,X_k,X_s) \stackrel{\text{def}}{=} \mathcal{R}_{ijks}$. $X_{ijks} \stackrel{\text{def}}{=} \mathcal{R}^\ell_{ijk}g_{\ell s}$.

This allows us to rephrase the identities much more concisely:

(a) $\mathcal{R}_{ijks} + \mathcal{R}_{jkis} + \mathcal{R}_{kijs} = 0$
(b) $\mathcal{R}_{ijks} = -\mathcal{R}_{jiks}$
(c) $\mathcal{R}_{ijks} = -\mathcal{R}_{ijsk}$
(d) $\mathcal{R}_{ijks} = \mathcal{R}_{ksij}$

Nobody *really* understands this whole tensor. The whole thing is a monster. But we can understand parts of it.

One part which we can understand is the sectional curvature, which shows up in the second variation formula.

Defn: $\forall u, v \in T_pM$,

$$|u \wedge v| = \sqrt{||u||^2\, ||v||^2 - \langle u, v\rangle^2}$$

is the area of the parallelogram spanned by u and v.

Lemma: If u and v are linearly independent, then

$$K(u, v) \stackrel{\text{def}}{=} \frac{\mathcal{R}(u, v, v, u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u, v)v, u\rangle}{|u \wedge v|^2}$$

K depends only on the plane $\pi(u, v) = \text{span}(u, v)$.

Proof: Check that the RHS is invariant under each of the following "moves":

- $(u, v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u, v) \rightsquigarrow (v, u)$
- $(u, v) \rightsquigarrow (u + \lambda v, v)$

We will finish proving this next time.

Defn: $K(\pi) = K(u, v)$ is called the <u>sectional curvature</u> of p at π.