Math 635 Leture 34

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Pontryagin Classes

Observe: $\mathrm{Pf}|_{\mathrm{so}(n)} : \mathrm{so}(n) \to \mathbb{R}$ (for n even) is $\mathrm{Ad}_{\mathrm{SO}(n)}$-invariant. So $\mathrm{Pf}(\Omega)$ is independent of choice of frame, where Ω is the curvature matrix with respect to that frame. We also have the Chern-Weil morphism:

$$\underbrace{\left\{\mathrm{Ad}_{\mathrm{SO}(n)}\text{invariant polynomials on so}(n)\right\}}_{I(\mathrm{so}(n))} \to \Omega^*(M)$$

where $p \mapsto p(\Omega)$. An amazing fact is that $p(\Omega)$ is always closed! So we get $I(\mathrm{so}(n)) \to H^*_{dR}(M)$, the deRham cohomology.

Ex: Elements of $I(\mathrm{so}(r))$ are Pontryagin polynomials. Let $A \in \mathrm{so}(r)$ be skew-symmetric (as usual, with r even). We claim that $A^T = -A$ implies the characteristic polynomial is even.

$$\det(\lambda I - A) = \sum_{k=0}^{r/2} \lambda^{r-2k} P_k(A)$$

where $P_k(A)$ is homogeneous, and of degree $2r$.

We can apply this idea to a rank-r vector bundle $\mathcal{E} \to M$. The idea is to use a metric on each fiber of $\mathcal{E}$ to get an orthonormal frame, and then the connection to get curvature forms. We get $P_r(\Omega)$, a differential form of degree $4r$. (Ω is the curvature matrix w.r.t. the orthonormal moving frame.) $[P_r(\Omega)] \in H^{4r}(M)$.

Thm: The cohomology classes are independent of the connection chosen – they're purely topological, and associated to $\mathcal{E}$.

So Gauss-Bonnet implies that $[\mathscr{K}\, dV] \in H^n(M)$ is independent of the metric.

Now, back to Gauss-Bonnet. We want to show

$$\int\limits_M \mathscr{K}\, dV = \frac{\mathrm{Vol}(S^n)}{2}\chi(M) \qquad \text{using} \qquad \int\limits_M \mathscr{K}\, dV = \int\limits_M N^*(dV_{S^n})$$

where N is the Gauss spherical map.

Degree Theory: *What happens when you pullback a top-degree form.* (See Lee Differentiable Manifolds page 457.)

Preliminary (but still important) result:

Thm: Let M be a compact, connected, oriented manifold. (Note: It must have empty boundary.) Then the integration map

$$\begin{aligned} \textstyle\int_M : H^n(M) &\to \mathbb{R} \\ [\omega] &\mapsto \textstyle\int_M \omega \end{aligned}$$

is an isomorphism! (We know it's well-defined by Stokes' theorem.) As a result, $\dim H^n(M) = 1$.

Proof: We'll work with compactly-supported forms in open sets. Observe that $\int_M$ is nonzero – $\int_M d\,\mathrm{Vol} > 0$. We know $\int_M$ is a linear map. So we need to show $\forall \omega \in \Omega^n(M)$ such that $\int_M \omega = 0$, $\exists \eta \in \Omega^{n-1}(M)$ such that $d\eta = \omega$.

Step 1: Assume $\omega \in \Omega^n_0(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \omega = 0$. Then we claim $\exists \eta \in \Omega^{n-1}_0(\mathbb{R}^n)$ such that $d\eta = \omega$. Observe that the homotopy axiom implies $H^k(\mathbb{R}^n) = 0$ for $k > 0$, so such an η exists, and the claim is that η can be chosen to have compact support. For this, see Lemma 17.27 in Lee.

Now, back to the manifold case. Let $\{U_i\}$ be a finite cover M (possible by compactness) such that $\forall i$, $U_i \cong \mathbb{R}^n$ diffeomorphically. WOLOG if $M_k = U_1 \cup \cdots \cup U_k$, then $M_k \cap U_{k+1} = \emptyset$. (Use M's connectedness, and renumber the U_i if necessary. If no such U_i existed, then union all of them, and we would have two disjoint open sets that cover M, making it disconnected. Oops!)

Introduction: If $\omega \in \Omega_0^n(M_k)$ is such that $\int_{M_k} \omega = 0$, then $\exists \eta \in \Omega_0^{n-1}(M_k)$ such that $d\eta = \omega$. For $k = 1$, see the previous claim. Then use induction and a partition of unity to complete the proof. (See Lee for the full details.) $\square$

Defn: Let $F : M_1 \to M_2$ be smooth, where M_1 and M_2 are compact, connected, oriented manifolds of the same dimension, $\dim M_1 = \dim M_2 = n$. Consider $F^* : H^n(M_2) \to H^n(M_1)$. By the previous result, we know that $\dim H^n(M_2) = \dim H^n(M_1) = 1$. Thus, F^* is multiplication by a scalar, and that number is called the degree of F.

$\forall c \in H^n(M_2)$, $\int_{M_1} F^*(c) = \deg F \int_{M_2} c$. That is,

$$\begin{array}{ccc} H^n(M_2) & \xrightarrow{F^*} & H^n(M_1) \\ \Big\downarrow{\scriptstyle \int_{M_2}} & & \Big\downarrow{\scriptstyle \int_{M_1}} \\ \mathbb{R} & \xrightarrow[\text{mult. by } \deg F]{} & \mathbb{R} \end{array}$$

Thm: Let $q \in M_2$ be a regular value of F. Write $F^{-1}(q) = \bigcup_{i=1}^N \{p_i\}$. This is a zero-manifold and compact, so it's the finite disjoint union of points. Define

$$(-1)^{p_i} = \begin{cases} 1 & dF_p \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

Then

$$\deg(F) = \sum_{i=1}^n (-1)^{p_i} \in \mathbb{Z}$$

Proof: F is a local diffeomorphism at each p_i. So we can argue that $\exists V$ a neighborhood of q and U_i a neighborhood of each p_i such that $F|_{U_i}^V$ is a diffeomorphism. That is, F is evenly covered at q. Let $\omega \in \Omega_0^n(V)$ be a bump function such that $\int_V \omega = \int_{M_2} \omega = 1$ (by extending ω to 0 on M_2 outside of V). What is $\int_{M_1} F^*\omega$? Well, it's equal to $\deg(F) \cdot 1 = \deg(F)$. But $F^{-1}(V)$ is the union of the U_i's, so

$$\int_{M_1} F^*(\omega) = \sum_{i=1}^N \underbrace{\int_{U_i} (F|_{U_i})^*\omega}_{=\pm 1 \text{ by diffeo invariance of integrals}} = \sum_{i=1}^N (-1)^{p_i}$$

$\square$

Cor: Gauss-Bonnet reduces to the (purely topological) statement $\deg(N) = \frac{1}{2}\chi(M)$ ($N : M \to S^n$ is the Gauss map).

Observe:

1. If $F, F' : M_1 \to M_2$ are homotopic, then $\deg(F) = \deg(F')$, because $F^* = (F')^*$.
2. If $M_1 = \partial W$, and $F : M_1 \to M_2$ extends to $\tilde{F} : W \to M_2$, hen the degree of F is 0.

$$\begin{tikzcd} M_1 \arrow[r, "F"] \arrow[d, hook] & M_2 \\ W \arrow[ur, "\tilde{F}"] & \end{tikzcd}$$

Prove this by using Stokes theorem to show $F^* = 0$.