Math 635 Lecture 36

Thomas Cohn

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Recall from last time: Let M be a Riemannian manifold. We defined the differential operators ∇ (gradient) and div (divergence), and we have

$$\begin{array}{ccc} C^\infty(M) & \underset{\substack{\mathrm{div}\\ \text{or}\\ -\mathrm{div}}}{\overset{\nabla}{\rightleftarrows}} & \mathfrak{X}(M) \\ \| & & \|\wr \text{ (metric dual)} \\ C^\infty(M) & \underset{\substack{-\delta\\ \text{or}\\ \delta}}{\overset{d}{\rightleftarrows}} & \Omega^1(M) \end{array}$$

"$\delta = -\operatorname{div}$ on the differential form side"

We also defined the Laplacian on functions $\Delta : C^\infty(M) \to C^\infty(M)$ by $\Delta = \delta \circ d$ iff $\Delta f = -\operatorname{div}(\nabla f)$.

ℓ^2 Inner Products

Defn: Assume M is oriented. $\forall f, g \in C^\infty(M)$, we define the $\underline{\ell^2 \text{ inner product}}$ by

$$\langle f, g \rangle_{\ell^2} = \int_M fg \; d\,\mathrm{Vol}$$

We can extend this to sections of real vector bundles over M, $\mathcal{E} \xrightarrow{\pi} M$. Put a Euclidean structure on the fibers of $\mathcal{E}$: $\forall p \in M$, $\langle\ ,\ \rangle_p$ is a Euclidean inner product on $\mathcal{E}_p = \pi^{-1}(p)$, varying smoothly with p.

Defn: $\forall s, t \in \Gamma_0(\mathcal{E})$ (compactly supported sections). Then we define the $\underline{\ell^2 \text{ inner product}}$ by

$$\langle s, t \rangle_{\ell^2} = \int_M \underbrace{\langle s(p), t(p) \rangle_p}_{\text{function of } p} d\,\mathrm{Vol}$$

Consider the case $\mathcal{E} = \bigwedge^k(T^*M)$. Then the Euclidean structure on $\bigwedge^k(T^*M)$ is induced by the Riemannian metric. For $k = 1$, we simply have $T^*M \cong TM$ by the metric dual. For general k, $\forall p \in M$, let $V = T_p^*M$. We define an inner product on $\bigwedge^k V$:

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \overset{\text{def}}{=} \det\left(\langle v_i, w_j \rangle\right)_{ij}$$

Check: If $(e_1, \ldots, e_n)$ is an orthonormal basis of V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}$ is an orthonormal basis of $\bigwedge^k V$.

In this way, we get the notion of an ℓ^2 inner product of any two k-forms $\alpha, \beta \in \Omega^k(M)$ by

$$\langle \alpha, \beta \rangle_{\ell^2} = \int_M \langle \alpha_p, \beta_p \rangle_p \, d\,\mathrm{Vol}$$

Prop: $\forall f \in C^\infty(M), X \in \mathfrak{X}(M)$, one has

$$\langle \nabla f, X \rangle_{\ell^2} = -\langle f, \operatorname{div} X \rangle_{\ell^2}$$

That is, $\forall f \in \Omega^0(M), \alpha \in \Omega^1(M)$,

$$\langle df, \alpha \rangle_{\ell^2} = \langle f, \delta\alpha \rangle_{\ell^2}$$

That is, $\delta = d^*$, the adjoint of d, so $\Delta = d^*d$.

Proof: Start with $\mathcal{L}_{fX}(d\,\mathrm{Vol}) = f\mathcal{L}_X(d\,\mathrm{Vol}) + Xf$. Now integrate:

$$\int\limits_M \mathcal{L}_{fX}(d\,\mathrm{Vol}) = \int\limits_M \operatorname{div}(fX)d\,\mathrm{Vol} = 0$$

because $\partial M = \emptyset$. So we have

$$0 = \int\limits_M f\operatorname{div}(X)d\,\mathrm{Vol} + \int\limits_M \underbrace{\langle X, \nabla F\rangle}_{X(f)=df(X)=\langle \nabla f, X\rangle} d\,\mathrm{Vol}$$

So $0 = \langle f, \operatorname{div} X\rangle_{\ell^2} + \langle X, \nabla f\rangle_{\ell^2}$. $\square$

Cor: $\langle \Delta f, g\rangle_{\ell^2} = \langle f, \Delta g\rangle_{\ell^2}$.

Now, generalize to Ω^k. (The previous discussion was for $k = 0$.)

$$\Omega^k \underset{\delta = d^* = ?}{\overset{d}{\rightleftarrows}} \Omega^{k+1}$$

Is there a δ? What is it?

In local coordinates, δ is *also* a differential operator of degree 1. Try integrating by parts!

Preliminary linear algebra: the Hodge star operator. Let V be an n-dimensional vector space, oriented, with an inner product. We claim that $\forall k$, there exists $\star : \bigwedge^k V \to \bigwedge^{n-k} V$ linear such that for any positive oriented basis $(e_1, \dots, e_n)$ of V, $\star(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n$.

Ex: For $V = \mathbb{R}^3$ with the standard orientation,

$$\begin{aligned} \star : \textstyle\bigwedge^2 V &\to \textstyle\bigwedge^1 V \\ dx^1 \wedge dx^2 &\mapsto dx^3 \end{aligned}$$

(Now do it cyclically.)

Note: $\dim \bigwedge^k = \binom{n}{k} = \binom{n}{n-k} = \dim \bigwedge^{n-k}$.

Observe: On $\mathbb{R}^3$ in the calc 3 context, for $X \in \mathfrak{X}(\mathbb{R}^3)$, we define

$$\operatorname{curl} X = \nabla \times X \in \mathfrak{X}(M)$$

What is this object? Well,

$$\begin{tikzcd} \mathfrak{X}(\mathbb{R}^3) \arrow[r, phantom, "\cong"] \arrow[rrrr, bend right, "\mathrm{curl}"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "\star"] & \Omega^1(\mathbb{R}^3) \arrow[r, phantom, "\cong"] & \mathfrak{X}(\mathbb{R}^3) \end{tikzcd}$$

Note that this only works for dim $= 3$.

Some properties of $\star$:

1. We have

$$\begin{tikzcd} \bigwedge^k \arrow[r, "\star"] \arrow[rr, bend right, "(-1)^{k(n-k)}\mathrm{Id}"'] & \bigwedge^{n-k} \arrow[r, "\star"] & \bigwedge^k \end{tikzcd}$$

   because

$$e_1 \wedge \cdots \wedge e_k \overset{\star}{\mapsto} e_{k+1} \wedge \cdots \wedge e_n \overset{\star}{\mapsto} (-1)^\sigma e_1 \wedge \cdots \wedge e_k$$

   "$n-k$ signs, k times".
2. $\star : \bigwedge^n V \to \bigwedge^0 V = \mathbb{R}$ has $\star(\mathrm{Vol}) = 1$.
3. $\forall \alpha, \beta \in \bigwedge^k V$, $\langle \alpha, \beta\rangle = \star(\alpha \wedge (\star\beta)) \in \mathbb{R}$.

Cor: Apply/extend $\star$ to forms on a compact, oriented, Riemannian manifold M (with $\dim M = n$), $\Omega^k(M)$, by acting pointwise: $\star : \Omega^k(M) \to \Omega^{n-k}(M)$. Note: $\forall \alpha, \beta \in \Omega^k(M)$, $\langle \alpha, \beta\rangle_{\ell^2} = \int_M \alpha \wedge (\star\beta)$.

Back to our main question:

Prop: The adjoint δ of $d : \Omega^k \to \Omega^{k+1}$ is $\delta = (-1)^{nk+1} \star d\star$.

Note: If $\beta \in \Omega^{k+1}$, $\star\beta \in \Omega^{n-k-1}$, so $d \star \beta \in \Omega^{n-k}$, so $\star d \star \beta \in \Omega^k$. Superficially, $\delta = \star d\star : \Omega^{k+1} \to \Omega^k$. Now, we prove it:

Proof: Let $\alpha \in \Omega^k, \beta \in \Omega^{k+1}$. We want to show $\langle d\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \delta\beta\rangle_{\ell^2}$. We'll use integration by parts. Starting with the fact that $0 = \int_M d(\alpha \wedge \star\beta)$, because $\alpha \wedge \star\beta$ is a $n-1$ form, so $d(\alpha \wedge \star\beta)$ is a top-degree form. By Stokes' theorem, since we have an empty boundary, this integral is 0. Well,

$$0 = \int\limits_M d(\alpha \wedge \star\beta) = \underbrace{\int\limits_M d\alpha \wedge \star\beta}_{=\langle d\alpha, \beta\rangle_{\ell^2}} + (-1)^k \int\limits_M \alpha \wedge (d \star \beta)$$

So

$$\langle d\alpha, \beta\rangle_{\ell^2} = (-1)^? \int\limits_M \alpha \wedge (d \star \beta) = (-1)^? \int\limits_M \alpha \wedge (\star\star) d \star \beta = (-1)^? \langle \alpha, \star d \star \beta\rangle_{\ell^2} = (-1)^? \langle \alpha, \delta\beta\rangle$$

(We didn't do the sign computations, but they do work out.) $\square$

Defn: The Laplacian on forms $\Delta : \Omega^k(M) \to \Omega^k(M)$ is $\Delta = \delta d + d\delta$.