

Math 635 Lecture 13

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Review: Given (M, g) a Riemannian manifold, $\exists! G \in \mathfrak{X}(TM)$ s.t. the integral curves of G are the lifts of geodesics.

Notation: $\forall (q, v) \in TM, t \mapsto \gamma(t, q, v)$ is the geodesic with initial condition (q, v) . It is the projection onto M of the integral curve of G , starting at (q, v) . For given (q, v) , $\gamma(t, q, v)$ has a maximal domain of definition, an interval in t , that depends on (q, v) .

Lemma: (Lemma 1 from Last Time) $\forall p \in M, \exists V \subseteq M$, a neighborhood of p , and $\exists \varepsilon, \delta > 0$ s.t. γ is defined on the set $(-\delta, \delta) \times \{(q, v) \in TV : \|v\| < \varepsilon\}$. That is, $\gamma(t, q, v)$ is defined $\forall q \in V, v \in T_q V, \|v\| < \varepsilon, |t| < \delta$, and γ is smooth as a map.

Defn: Given $V \subseteq M, \varepsilon > 0$, we define the ε -ball tangent bundle, by $B_\varepsilon(TV) = \{(q, v) \in TV : \|v\| < \varepsilon\}$. This is a fiber bundle $B_\varepsilon(TV) \rightarrow V$, whose fibers are open balls of radius ε , centered at 0.

Defn: We also define the unit tangent bundle of M , $S_1(TM)$, by $S_\varepsilon(TV) = \{(q, v) \in TV : \|v\| = \varepsilon\}$. This is a fiber bundle $S_1(TM) \rightarrow M$, whose fibers are S^{n-1} .

Ex: The unit tangent bundle of S^2 is isomorphic to

$$\{(\vec{q}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 : \|\vec{q}\| = 1, \|\vec{v}\| = 1, \vec{q} \cdot \vec{v} = 0\}$$

(The final condition is a tangency condition). In turn, this is diffeomorphic to $SO(3)$ as manifolds, by $(\vec{q}, \vec{v}) \mapsto (\vec{q}, \vec{v}, \vec{q} \times \vec{v})$. Treating the three output vectors as columns of a matrix yields an orthogonal matrix with determinant 1.

Observe: G is tangent to $S_\varepsilon(TM)$, $\forall \varepsilon > 0$. This is a fancy way to say that, along a geodesic, speed is constant. Because $\frac{d}{dt} \langle \dot{\gamma}, \dot{\gamma} \rangle = 0$, we know $\|\dot{\gamma}\|$ is constant, so the integral curves of G are fully contained in $S_\varepsilon(TM)$.

Cor: If M is compact, every geodesic is defined $\forall t \in \mathbb{R}$, i.e., G is complete.

Proof: M is compact, so $\forall \varepsilon > 0, S_\varepsilon(TM)$ is compact, and any field on a compact manifold is complete. \square

Lemma: (Lemma 2 from Last Time) (Homogeneity of Geodesic Flow) Let $(q, v) \in TM, a > 0$. If $\gamma(t, q, v)$ is defined for $|t| < \delta$, then $\gamma(t, q, av)$ is defined for $|t| < \frac{\delta}{a}$, and $\gamma(t, q, av) = \gamma(at, q, v)$.

Proof: Check that both sides satisfy the geodesic equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$, and have the same initial conditions (namely, (q, av)). \square

Prop: (Do Carmo 2.7) $\forall p \in M, \exists V \subseteq M$ a neighborhood of p , and $\varepsilon > 0$ s.t. $\forall (q, v) \in B_\varepsilon(TV)$, $\gamma(t, q, v)$ is defined for $|t| < 43$. (Note: we really just need it to be defined for $t = 1$, so we can get our exponential map. But 43 is such a nice number.)

Proof: Let V, δ, ε_1 be as in Lemma 1, so that $\forall (q, v) \in B_{\varepsilon_1}(TV)$, $\gamma(t, q, v)$ is defined for $|t| < \delta$. Choose $a > 0$ s.t. $|t| < \delta \iff a|t| = |at| < 43$ – specifically, choose $a = \frac{\delta}{43}$. Now, by Lemma 2, $\gamma(t, q, \frac{\delta}{43}v)$ is defined for $|t| < 43$ if $\|v\| < \varepsilon_1$. Now define $\varepsilon = \varepsilon_1 \cdot \frac{\delta}{43}$. Thus, $\frac{\delta}{43} \|v\| < \varepsilon_1 \iff \|v\| < \varepsilon$. \square

Defn: Let $p \in M, V \subset M$ a neighborhood of p , and ε as in the previous proposition. Then we define

1. $\exp : B_\varepsilon(TV) \rightarrow M$
 $(q, v) \mapsto \gamma(1, q, v)$
2. $\exp_p : B_\varepsilon(0) \rightarrow M$
 $v \mapsto \gamma(1, p, v)$

Observe: Both \exp and \exp_p are differentiable.

Lemma: $\forall p \in M$, $d(\exp_p)_{v=0}$ is the identity.

$$\begin{array}{ccc} T_0(T_p M) & \longrightarrow & T_p M \\ \parallel & \nearrow & \\ T_p M & \xrightarrow{\text{Id (claimed)}} & \end{array}$$

Proof: Use curves to compute $d(\exp_p)_0$. Take a curve in $T_p M$, starting at $0 \in T_p M$, e.g., $t \mapsto tw$ for some $w \in T_p M$. Then

$$d(\exp_p)_0(w) = \left. \frac{d}{dt} \exp_p(tw) \right|_{t=0} = \left. \frac{d}{dt} \gamma(1, p, tw) \right|_{t=0} = \left. \frac{d}{dt} \gamma(t, p, w) \right|_{t=0} = w$$

□

Cor: \exp_p is a local diffeomorphism near 0, i.e., $\exists \mathcal{N} \subset T_p M$, a neighborhood of 0 such that $\exp_p|_{\mathcal{N}} : \mathcal{N} \xrightarrow{\sim} U \subset M$, for some U .