Math 635 Lecture 26

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3/22/21

Continuing from last time, we need to compute $\left\langle J^{(3)}, J' \right\rangle$.

Lemma: $J^{(3)} = -\frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma$. (Prove using properties of Jacobi fields.)

Claim: $\frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma\big|_{t=0} = \mathcal{R}(J',\dot\gamma)\dot\gamma|_{t=0}$.
Proof: Let $W \in \Gamma_\gamma(TM)$. Compute

$$\frac{d}{dt}\left\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle = \left\langle \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle + \left\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, \overset{0\text{ at }t=0}{\cancel{\frac{Dw}{dt}}}\right\rangle$$

and

$$\frac{d}{dt}\left\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle = \frac{d}{dt}(J,\dot\gamma,\dot\gamma,W) = \frac{d}{dt}(W,\dot\gamma,\dot\gamma,J) = \left\langle \overset{0}{\cancel{\frac{D}{dt}\mathcal{R}(W,\dot\gamma)\dot\gamma}}, J\right\rangle + \underbrace{\left\langle \mathcal{R}(W,\dot\gamma)\dot\gamma\right\rangle, \frac{DJ}{dt}}_{=(W,\dot\gamma,\dot\gamma,J')=(J',\dot\gamma,\dot\gamma,W)}$$

Thus, at $t=0$, we have $(J',\dot\gamma,\dot\gamma,W)|_{t=0} = \left\langle \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle\big|_{t=0} = \langle \mathcal{R}(J',\dot\gamma)\dot\gamma, W\rangle$.
We conclude that $\mathcal{R}(J',\dot\gamma)\dot\gamma = \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma$.

Thus, $\left\langle J^{(3)}, J'\right\rangle = \langle \mathcal{R}(J',\dot\gamma)\dot\gamma, J'\rangle = K_0(J',\dot\gamma)$. This proves the lemma, and completes the proof we began last time. □

Conjugate Points

Defn: Let γ be a geodesic, $p,q \in \mathrm{im}\gamma$ distinct. Then p,q are conjugate along γ iff there exists a nonzero Jacobi field of γ that vanishes at p and q.

Ex: With constant negative curvature $K<0$, suppose $\gamma(0)=p$, $\gamma(t_1)=q$. Then $J(t) = A\sinh(\sqrt{-K}t)W(t)$. So there are no conjugate points.

If $K>0$, then $J(t) = A\sin(\sqrt{K}t)$, so $t = \frac{\pi}{\sqrt{k}}$ yields a conjugate point.

Defn: The multiplicity of q as a conjugate point of p is $\dim\{J \mid J(q)=0, J(p)=0\}$.

Claim: Multiplicity is at most $n-1$, and $n-1$ is achieved by the sphere S^n.

Note: "Being conjugate" is a symmetric relation, but it's not transitive!

Prop: Let $q = \exp_p(tv_q)$, $v_q \in T_pM$. Then q is conjugate to p iff v_q is a critical point of $\exp_p$, iff $d(\exp_p)_{v_q}$ has a nontrivial kernel.

Proof: Simply recall how to compute $d(\exp_p)_{v_q}$: $J(1) = d(\exp_p)_{v_q}(w)$ when $J(0)=0$ and $J'(0)=w$ (by parameterizing γ by arc length, and rescaling so that $t_1=1$). □

Moreover, $\dim\ker d(\exp_p)_{v_q}$ is the multiplicity of q, and it is at most $n-1$, because $v_q \notin \ker d(\exp_p)_{v_q}$, because $d(\exp_p)_{v_q}(v_q) = \dot\gamma(t_1) \neq 0$, because γ is nontrivial, because $p \neq q$.

Thm: (I) Let $q = \exp_p(t_1 v)$, $||v||=1$ be a conjugate point of p. Then $\forall t_2 > t_1$, $t \mapsto \exp_p(tv)$ is *not* minimizing on $[0,t_2]$.

Proof: This is simply a nice application of the second variation form. □

Think of the sphere – if we go from the north pole to past the south pole, the curve isn't minimizing.

Note: There are no conjugate points on a cylinder, but not all geodesics are minimizing.

Note: The converse is not globally true.

Thm: (II) Let $p, q \in \mathrm{im}\gamma$, for γ a geodesic, and assume p, q are not conjguate, and there are no conjugate points between p and q. Then any proper variation of the arc $\widehat{pq}$ is such that $\forall s$ sufficiently small, the t-curves are longer than $\widehat{pq}$.

Thm: *Second Variation Formula, with one jump in V'.* Let γ be a geodesic, defined for $t \in [0, t_2]$. Suppose V is a proper, continuous, infinitesimal variation (with respect to s), and is smooth on $[0, t_1]$ and $[t_1, t_2]$ (assume 1-sided derivatives exist at t_1). Then

$$E''(0) = -\int_0^{t_2} \langle V, V'' + \mathcal{R}(V, \dot{\gamma})\dot{\gamma}\rangle \, dt - \langle V(t_1), \Delta V'(t_1)\rangle$$

where $\Delta V'(t_1) = V'(t_1^+) - V'(t_1^-)$ (using the one-sided derivatives).

Proof: See Do Carmo, page 197. (They prove this result for any number of jumps.)

Now, back to the philosophy that this is a sort of Hessian of the energy functional $\mathcal{E} : \{\text{paths } p \rightsquigarrow q\} \to \mathbb{R}$. In the above formula, V is a tangent vector, so the right hand side should be a quadratic form on V. But what is the associated bilinear form?

Prop: Given the same assumptions as above, with any number of jumps, we have

$$E''(0) = \int_0^{t_2} \langle V', V'\rangle - \langle \mathcal{R}(V, \dot{\gamma})\dot{\gamma}, V\rangle \, dt$$

and moreover, the symmetric bilinear form is

$$I(V, W) = \int_0^{t} \langle V', W'\rangle - \underbrace{\langle \mathcal{R}(V, \dot{\gamma})\dot{\gamma}, W\rangle}_{\text{pairwise symmetric in } V, W} \, dt$$

so $E''(0) = I(V, V)$ (and clearly, I is symmetric).