Math 635 Lecture 30

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Last time, we proved Cartan-Hadamard. To review:

Thm: (Cartan-Hadamard) If M is a complete, connected Riemannian manifold with $K \leq 0$, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Proof (sketch): From homework, we know $\exp_p$ is a local diffeomorphism. Then we show the following proposition: If $F : \tilde{M} \to M$ is a local isometry, with $\tilde{M}$ and M complete, then F is a covering map. Finally, we take $\tilde{M} = T_pM$, with the "pull-back" metric such that $\exp_p$ is a local isometry. Now, we apply the proposition. We just need the metric to be complete, and it is, because rays $t \mapsto tv$ (for $v \in T_pM \setminus \{0\}$) are geodesics with respect to the pull-back metric, and because $\exp_p$ is a local isometry, they map to geodesics on M. So T_pM with this metric has the property that all geodesics through $0 \in T_pM$ can be continued, $\forall t \in \mathbb{R}$, so T_pM is complete. $\square$

Why is this such a big deal?

Cor: If M is as in Cartan-Hadamard, and simply connected, then M is diffeomorphic to $\mathbb{R}^n$.

In general, for M complete with $K \leq 0$, M is said to be aspherical: $\forall n > 1$, $\pi_n(M) = 0$.

Observe: If M is complete, then M cannot be isometrically embedded as a proper open set of some Riemannian manifold W. This is true because covering maps are surjective.

Prop: If $M \subsetneq W$, and M is open, then M is not complete.

Proof: Let $p \in \partial M$, and U a normal neighborhood of p in W. Let $q \in M \cap U$. Let γ be a radial geodesic from p to q. Reverse t, so we have $\gamma^- : q \rightsquigarrow p$. For small t, γ^- is a geodesic in M. But for larger t, $\gamma^-(t) \notin M$, so M is not complete. $\square$

The Second Fundamental Form

(Refer to Chapter 6 of Do Carmo.)

Note: The first fundamental form is just the metric itself.

The setting and notation we'll consider is $\bar{M}$ a Riemannian manifold (known as the "ambient space"), $M \subseteq \bar{M}$ a submanifold with the induced metric, $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$, and ∇ the Levi-Civita connection on M. Further, if $p \in M$, $v \in T_p\bar{M}$, we'll write $v = v^T + v^\perp = v^T + v^N$, where v^T is the orthogonal projection of v onto T_pM (known as the "tangential component") and $v^\perp$ (or v^N) is its orthogonal complement in $(T_pM)^\perp$ (known as the "normal component").

Recall (from homework): $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$, where $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$ are (arbitrary) extensions of X and Y.

Defn: $\forall X, Y \in \mathfrak{X}(M)$, $B(X, Y) = (\bar{\nabla}_{\bar{X}} \bar{Y})^\perp$. So $\forall p \in M$, $B(X, Y)(p) = (\bar{\nabla}_{\bar{X}} \bar{Y})(p) - (\nabla_X Y)(p)$.

Note: $B(X, Y)(p) \in (T_pM)^\perp$.

Prop: $B(X,Y)(p)$ depends only on X_p and Y_p, and is therefore equivalent to tensors (maps) $\forall p \in M$:

$$B_p : T_pM \times T_pM \to (T_pM)^{\perp}$$

Moreover, $\forall p \in M$, B_p is bilinear and symmetric: $\forall x, y \in T_pM$, $B_p(x,y) = B_p(y,x)$.

Proof: We'll first show symmetry. Let $X, Y \in \mathfrak{X}(M)$. Then

$$B(X,Y) - B(Y,X) = \left(\bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}\right)\big|_M - (\nabla_X Y - \nabla_Y X) = \left[\bar{X}, \bar{Y}\right]\big|_M - [X,Y]$$

But we proved in homework (in Math 591) that on $\left[\bar{X}, \bar{Y}\right]\big|_M = [X,Y]$, so we have $B(X,Y) - B(Y,X) = 0$, and thus, B is symmetric.

Now, at p, we know that any connection depends pointwise in the "lower" entry. So $B(X,Y)(p)$ depends (with respect to X) only on X_p. From symmetry, the same is true for Y, so B is bilinear. $\square$

To get information from B, one chooses $\nu_p \in (T_pM)^{\perp}$ with $||\nu_p|| = 1$.

Defn:

- $H_\nu : T_pM \times T_pM \to \mathbb{R}$ by taking $H_\nu(x,y) = \langle B_p(x,y), \nu \rangle$.
- $I\!I_p(x) = H_\nu(x,x)$.
- $S_\nu : T_pM \to T_pM$, the shape operator, also known as the Weingarten map, is defined by $\forall x, y \in T_pM$,

$$\langle S_\nu(x), y \rangle = H_\nu(x,y) = \langle B(x,y), \nu \rangle$$

  S_ν exists and is well-defined because $\langle \ , \ \rangle$ is non-degenerate.

Observe: Because B is symmetric, H_ν is symmetric, so S_ν is self-adjoint. Thus, S_ν can be diagonalized, i.e., three's a basis $(e_1, \ldots, e_n)$ of T_pM such that $S_\nu(e_i) = \kappa_i e_i$, for $\kappa_i \in \mathbb{R}$.

Defn: The κ_i are the <u>principal curvatures</u> of M at p, and the e_i are the <u>principal directions</u>.

Looking ahead, we will prove if $\bar{M}$ is flat (e.g. $\bar{M} = \mathbb{R}^n$), then the intrinsic sectional curvature of M satisfies $K(e_i, e_j) = \kappa_i \kappa_j$, $\forall i \neq j$. When $\dim M = 2$, this is Gauss' "Theorem Egregium".