Math 635 Lecture 22

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Sectional Curvature

Defn: $\forall p \in M$, $\forall u, v \in T_pM$ linearly independent, the sectional curvature is defined to be

$$K_p(u,v) \overset{\text{def}}{=} \frac{\mathcal{R}(u,v,v,u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u,v)v, u\rangle}{||u||^2\,||v||^2 - \langle u, v\rangle^2}$$

Last time, we claimed that $K_p(u,v)$ only depends on the plane $\pi = \operatorname{span}(u,v) \in \operatorname{Gr}(2, T_pM)$. So really, K_p is a function $K_p : \operatorname{Gr}(2, T_pM) \to \mathbb{R}$.

Proof of this claim: We want to show that the right-hand side is invariant under the following "moves":
- $(u,v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u,v) \rightsquigarrow (v,u)$
- $(u,v) \rightsquigarrow (u + \lambda v, v)$

This is easy to do using known properties of $\mathcal{R}$.

Observe: If $\dim M = 2$, then think about $\mathcal{R}$ and its symmetries. How many independent components does it have? Well, for $\mathcal{R}(u,v,u',v')$ (with u and v linearly independent), we can write (u',v') in terms of (u,v) via a change of basis:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

So $\mathcal{R}(u,v,u',v') = \det(A)\mathcal{R}(u,v,u,v) = -\det(A)\mathcal{R}(u,v,v,u)$. So we conclude that in dimension 2, there's only one degree of freedom! All the information that's contained in $\mathcal{R}$ reduces to knowing the function $K(p) = \mathcal{R}_p(e,f,f,e)$, where (e,f) is an orthonormal basis of T_pM $(K : M \to \mathbb{R})$.

Prop: K from above is the Gaussian curvature.

In dimension n, there are $\frac{n^2(n^2-1)}{12}$ degrees of freedom.

Prop: The function $K : \bigcup_{p \in M} \operatorname{Gr}(2, T_pM) \to \mathbb{R}$ completely determines $\mathcal{R}$.

> Proof: This is just an algebraic exercise. As a preliminary, start with a bilinear map $b : V \times V \to \mathbb{R}$. Note that b is the sum of a symmetric and antisymmetric bilinear form; this can be seen by writing $b(x,y) = x^T M y$ for a unique matrix M, and then noting that we can write $M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$. $M + M^T$ is symmetric, and $M - M^T$ is antisymmetric.
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> Now, if b is symmetric, then b is uniquely determined by the mapping $x \mapsto q(x) = b(x,x)$, because $q(x+y) = q(x) + q(y) + 2b(x,y)$. We can then solve for $b(x,y)$. (This is known as the "polarization identity".)
>
> **Cor:** If $b(x,x) = 0$, $\forall x \in V$, then b must be skew-symmetric.
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> We now continue the proof of the proposition. Let V be a vector space (e.g. T_pM). Let $\mathcal{R}, \mathcal{R}' : V \times V \times V \times V \to \mathbb{R}$ be multilinear maps, with the symmetries of the Riemannian curvature. Then define $D = \mathcal{R} - \mathcal{R}'$, and it has the same symmetries. Assume $\forall v, w, x$, $D(v,w,w,v) = 0$ (x will be used later). We want to show $D \equiv 0$.

Why is this true? Well,

$$\begin{aligned}
? &= D(v+w,x,x,v+w)\\
&= \underbrace{D(v,x,x,v)}_{=0} + D(v,x,x,w) + D(w,x,x,v) + \underbrace{D(w,x,x,w)}_{=0}\\
&= D(v,x,x,w) - D(v,x,x,w)\\
&= 0
\end{aligned}$$

Finally, we use the Bianchi identity, to show that

$$0 = D(u,v,w,t) + D(w,u,v,t) + D(v,w,u,t) = 3D(u,v,w,t)$$

□

Now, we return to studying the second variation formula. Recall: For a geodesic γ, and V the variation field of a proper variation of γ, we have

$$E''(0) = -\int_0^a \left\langle V, \frac{D^2}{dt^2}V + \mathcal{R}(V,\dot\gamma)\dot\gamma \right\rangle dt$$

Thm: (Bonnet-Myers V1) If sectional curvature K satisfies $K > \left(\frac{\pi}{\ell}\right)^2 > 0$ for some $\ell > 0$, then no geodesic of length ℓ is minimizing.

Proof: Let γ be a geodesic of length ℓ. We need to show the energy of γ is not a minimum. Let $\gamma : [0,\ell] \to M$, with $\gamma(0) = p$, $\gamma(\ell) = q$, and $||\dot\gamma|| = 1$. Pick $N \in T_pM$ a unit vector such that $\langle N, \dot\gamma(0)\rangle = 0$. Let $E(t)$ be the parallel transport of N along γ, so then $\frac{D}{dt}E \equiv 0$. Observe that $\forall t$, $\langle E(t), \dot\gamma(t)\rangle = 0$. So if we define

$$V(t) \overset{\text{def}}{=} \sin\left(t\frac{\pi}{\ell}\right) E(t)$$

then $V(0) = 0$ and $V(\ell) = 0$, so V is the variation field of a pinned variation. So now, we can substitute into the second variation formula. We compute

$$\frac{D}{dt}V(t) = \frac{\pi}{\ell}\cos\left(t\frac{\pi}{\ell}\right)E(t) + 0 \quad \Rightarrow \quad \frac{D^2}{dt^2}V(t) = -\left(\frac{\pi}{\ell}\right)^2 \sin\left(t\frac{\pi}{\ell}\right)E(t)$$

Thus,

$$\left\langle V, \frac{D^2}{dt^2}V \right\rangle = -\left(\frac{\pi}{\ell}\right)^2 \sin^2\left(t\frac{\pi}{\ell}\right)\underbrace{||v||^2}_{=1}$$

Also,

$$\langle V, \mathcal{R}(V,\dot\gamma)\dot\gamma\rangle = \sin^2\left(t\frac{\pi}{\ell}\right)\underbrace{\langle E, \mathcal{R}(E,\dot\gamma)\dot\gamma\rangle}_{=K(E,\dot\gamma)}$$

So for a proper variation of γ with variation field $V = f_s|_{s=0}$, we have

$$E''(0) = \int_0^\ell \underbrace{\sin^2\left(t\frac{\pi}{\ell}\right)}_{\geq 0}\underbrace{\left(\left(\frac{\pi}{\ell}\right)^2 - K(E,\dot\gamma)\right)}_{<0} dt$$

So $E''(0) < 0$, so for $|s| \ll 1$, $E(s) < E(0)$, so γ is not minimizing. □