

Math 635 Lecture 28

Thomas Cohn

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We're currently trying to prove:

Thm: Let M be connected, $p \in M$ such that \exp_p is defined on all of $T_p M$. Let $q \in M$. Then there's a geodesic γ from p to q such that $d(p, q) = \ell(\gamma)$.

Continuing from last time, we want to show that $\sup \mathcal{T} = d(p, q)$, i.e., $d(p, q) \in \mathcal{T}$, i.e., $d(p, q) = d(p, q) + d(\gamma(d(p, q)), q)$, i.e., $d(\gamma(d(p, q)), q) = 0$. Because d is a distance function, it's enough to show $\gamma(d(p, q)) = q$.

Let $t_1 = \sup \mathcal{T}$. Because \mathcal{T} is closed, we know $t_1 \in \mathcal{T}$. Assume that $t_1 < d(p, q)$ (we will show a contradiction). Then $\exists \delta > 0$ s.t. $t_1 + \delta < d(p, q)$, and there exists a geodesic sphere S_δ centered at $\gamma(t_1)$ with radius δ . Let $y \in S_\delta$ minimizing the map

$$\begin{aligned} S_\delta &\rightarrow \mathbb{R} \\ x &\mapsto d(x, q) \end{aligned}$$

We want to show $t_1 + \delta \in \mathcal{T}$.

Claim 1: $d(y, q) = d(p, q) - (t_1 + \delta)$.

Proof: The lemma from last time implies that $d(p, q) - t_1 = d(\gamma(t_1), q) = \delta + d(y, q)$. \square

Claim 2: $d(p, y) = t_1 + \delta$.

Proof: \leq follows directly from the triangle inequality. \geq is true because $d(p, q) \leq d(p, y) + d(y, q)$, so

$$d(p, y) \geq d(p, q) - (d(p, q) - (t_1 + \delta)) \geq t_1 + \delta$$

by claim 1. \square

Claim 3: $y = \gamma(t_1 + \delta)$.

Proof: Consider the path $\gamma(t)$ for $0 \leq t \leq t_1$, followed by a radial geodesic from $\gamma(t_1)$ to y . So the path is overall from p to y . By claim 2, this path is length minimizing, so it's a geodesic with the same initial conditions as γ . This implies it must be γ , with the domain $0 \leq t \leq t_1 + \delta$.

Claims 1 and 3 together imply that $t_1 + \delta \in \mathcal{T}$, a contradiction. This completes the proof. \square

Thm: (Hopf-Rinow) Let M be connected. Assume $\exists p \in M$ s.t. \exp_p is defined on all of $T_p M$. Then

- (a) Every closed and bounded set is compact.
- (b) M is complete as a metric space.
- (c) M is geodesically complete, i.e., $\forall q \in M$, \exp_q is defined on all of $T_q M$.

In fact, these are all equivalent. We will show this by: assumption \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow assumption.

Proof: assumption \Rightarrow (a): Let $S \subset M$ be closed and bounded. Then $\forall y \in S$, $\exists \gamma_y$ a minimizing geodesic from p to y . S is bounded, so $\exists R > 0$ s.t. $\forall y \in S$, $\ell(\gamma_y) < R$. So $S \subset \exp_p(\overline{B_R(0)})$, where $B_R(0) \subset T_p M$. Because $\overline{B_R(0)}$ is compact and \exp_p is continuous, $\exp_p(\overline{B_R(0)})$ is also compact, so S is contained in a compact set, and is thus compact.

(a) \Rightarrow (b): This only requires facts of point-set topology. If (x_n) is Cauchy, then it is bounded. So its image is compact. Thus, there's a convergent subsequence, so (x_n) converges to its limit point.

(b) \Rightarrow (c): We prove this by contradiction. Assume $\exists q \in M$, $T > 0$, and $v \in T_q M$ with $\|v\| = 1$, such that $\gamma(t) = \exp_q(tv)$ exists $\forall t \in [0, T)$, but not beyond T . Take $t_1 < t_2 < T$ so that (t_n) converges to T , and let $x_n = \gamma(t_n)$. Note that $d(x_n, x_m) \leq |t_n - t_m|$, $\forall n, m$. Since (t_n) converges, (x_n) is Cauchy, so $\exists w \in M$ such that $\lim_{n \rightarrow \infty} x_n = w$. Let W be a totally normal neighborhood of w , and $\delta > 0$ such that $\forall x \in W$, the geodesic ball

$B(x, \delta) \supseteq W$. Then $\exists N \in \mathbb{N}$, such that $\forall n, m > N$, $|t_n - t_m| < \delta/2$, and $\gamma(t_m) \in W$. Pick $n > N$. Use \exp_{x_n} to “relaunch” the geodesic. $s \mapsto \exp_{x_n}(s\dot{\gamma}(t_n))$ exists for $|s| < \delta$. This extends γ past T , since $t_n + \delta > T$. Oops!

(c) \Rightarrow assumption: Trivial. \square

Defn: If M satisfies these properties, we call M a complete Riemannian manifold.

We obtain yet another version of Bonnet-Myer as a corollary:

Thm: (Bonnet-Myer III) Let M be a compact, connected Riemannian manifold, and assume $\text{Ric} > \left(\frac{\pi}{\ell}\right)^2$. Then M is compact and the diameter of M is no more than ℓ .

Proof: We already know that any geodesic with a length of at least ℓ is not minimizing. Also, any pair of points can be joined by a minimizing geodesic. Thus, the diameter of M is at most L , so M is bounded and compact. \square