

Math 635 Lecture 21

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Review: second variation formula. If γ is a geodesic, and f a proper variation of γ , $V = \partial_s|_{s=0}$, then

$$E''(0) = - \int_0^a \left\langle V, \frac{D^2 V}{dt^2} + \mathcal{R}(V, \dot{\gamma})(\dot{\gamma}) \right\rangle dt$$

Next, we're going to take a closer look at the curvature, \mathcal{R} . Recall its definition: If $X, Y \in \mathfrak{X}(M)$, then the curvature is $\mathcal{R}(X, Y) : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, defined by $\mathcal{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$. It turns out that $\mathcal{R}(X, Y)$ is given by the action of a tensor \mathcal{R} of the form $\forall p \in M; u, v \in T_p M, \mathcal{R}_p(u, v) : T_p M \rightarrow T_p M$, a linear map.

As an operator, $\mathcal{R}(X, Y)(Z)_p = \mathcal{R}_p(X_p, Y_p)(Z_p) \in T_p M$. Also, \mathcal{R} shows up as an obstruction to finding a covariant-constant frame $\nabla_{E_i} E_j \equiv 0$.

Curvature Identities

(Covered in Do Carmo, Chapter 4, §2)

The first identity we consider is the Bianchi identity: $\forall X, Y, Z \in \mathfrak{X}(M), \mathcal{R}(X, Y)Z + \mathcal{R}(Z, X)Y + \mathcal{R}(Y, Z)X = 0$, due to ∇ being torsion-free. The proof is simply a computation, and can be found on page 91 of Do Carmo.

Prop: Introduce $X, Y, Z, T \in \mathfrak{X}(M)$, and define $(X, Y, Z, T) \stackrel{\text{def}}{=} \langle \mathcal{R}(X, Y)Z, T \rangle$.

- (a) $(X, Y, Z, T) + (Y, Z, X, T) + (Z, X, Y, T) = 0$ (Proved via the Bianchi identity)
- (b) $(X, Y, Z, T) = -(Y, X, Z, T)$ (Because $\mathcal{R}(X, Y) = -\mathcal{R}(Y, X)$)
- (c) $(X, Y, Z, T) = -(X, Y, T, Z)$ (Because ∇ preserves $\langle \cdot, \cdot \rangle$)
- (d) $(X, Y, Z, T) = (Z, T, X, Y)$ (Follow from the Bianchi identity and some algebra)

In coordinates (x^1, \dots, x^n) , $X_i = \frac{\partial}{\partial x^i}$, then \mathcal{R} has components in the coordinate system, $\mathcal{R}_{ijk}^\ell \in C^\infty(U)$ (where U is the domain of the coordinate chart) such that

$$\underbrace{\mathcal{R}(X_i, X_j)X_k}_{\in \mathfrak{X}(U)} = \mathcal{R}_{ijk}^\ell X_\ell$$

But how do we compute \mathcal{R}_{ijk}^ℓ ?

Lemma: $\mathcal{R}_{ijk}^\ell = \Gamma_{jk}^\ell \Gamma_{il}^s - \Gamma_{ik}^\ell \Gamma_{jl}^s + \partial_i \Gamma_{jk}^s - \partial_j \Gamma_{ik}^s$. (Recall that Do Carmo uses the opposite sign for \mathcal{R} .)

Proof: This is a messy computation. The complete details can be found on pages 92-93 of Do Carmo.

Observe that Γ depends on the first derivatives of g_{ij} , so \mathcal{R} depends on the second derivatives of g_{ij} .

Defn: $(X_i, X_j, X_k, X_s) \stackrel{\text{def}}{=} \mathcal{R}_{ijks}$. $X_{ijks} \stackrel{\text{def}}{=} \mathcal{R}_{ijk}^\ell g_{\ell s}$.

This allows us to rephrase the identities much more concisely:

- (a) $\mathcal{R}_{ijks} + \mathcal{R}_{jkis} + \mathcal{R}_{kij s} = 0$
- (b) $\mathcal{R}_{ijks} = -\mathcal{R}_{jik s}$
- (c) $\mathcal{R}_{ijks} = -\mathcal{R}_{ijs k}$
- (d) $\mathcal{R}_{ijks} = \mathcal{R}_{ksij}$

Nobody *really* understands this whole tensor. The whole thing is a monster. But we can understand parts of it.

One part which we can understand is the sectional curvature, which shows up in the second variation formula.

Defn: $\forall u, v \in T_p M$,

$$|u \wedge v| = \sqrt{||u||^2 ||v||^2 - \langle u, v \rangle^2}$$

is the area of the parallelogram spanned by u and v .

Lemma: If u and v are linearly independent, then

$$K(u, v) \stackrel{\text{def}}{=} \frac{\mathcal{R}(u, v, v, u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u, v)v, u \rangle}{|u \wedge v|^2}$$

K depends only on the plane $\pi(u, v) = \text{span}(u, v)$.

Proof: Check that the RHS is invariant under each of the following “moves”:

- $(u, v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u, v) \rightsquigarrow (v, u)$
- $(u, v) \rightsquigarrow (u + \lambda v, v)$

We will finish proving this next time.

Defn: $K(\pi) = K(u, v)$ is called the sectional curvature of p at π .