

# Math 635 Lecture 37

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4/16/21

Recall from last time: Let  $M$  be a compact, oriented, Riemannian manifold. Then the Laplacian on forms  $\Delta : \Omega^k \rightarrow \Omega^k$  is  $\Delta = \delta d + d\delta$ . (Note that these are technically different  $d$ 's and different  $\delta$ 's, because  $d$  increases the degree and  $\delta$  reduces it.) And we have the deRham complex

$$\dots \xrightarrow[\delta]{d} \Omega^{k-1} \xrightarrow[\delta]{d} \Omega^k \xrightarrow[\delta]{d} \Omega^{k+1} \xrightarrow[\delta]{d} \dots$$

where  $d^* = \delta$ , i.e.,  $\forall \alpha \in \Omega^{k-1}, \beta \in \Omega^k, \langle d\alpha, \beta \rangle_{\ell^2} = \langle \alpha, \delta\beta \rangle_{\ell^2}$ .

**Lemma:**

- (1)  $\delta^2 = 0$  (because  $\delta^2 = (d^2)^*$ )
- (2)  $\Delta^* = \Delta$ :  $\forall \alpha, \beta \in \Omega^k, \langle \Delta \alpha, \beta \rangle_{\ell^2} = \langle \alpha, \Delta \beta \rangle_{\ell^2}$
- (3)  $[\Delta, d] = 0, [\Delta, \delta] = 0$
- (4)  $\Delta \alpha = 0$  iff  $d\alpha = 0$  and  $\delta\alpha = 0$

Proof (3):  $(\delta d + d\delta)d = \delta d\delta + d\delta d$  and  $d(\delta d + d\delta) = d\delta d + d\delta d = d\delta d$ . So  $[\Delta, d] = 0$ . (Identical proof for  $[\Delta, \delta] = 0$ .)  $\square$

(4):  $\Leftarrow$  is obvious. For  $\Rightarrow$ , note that  $\Delta \alpha = 0$  implies

$$0 = \langle \Delta \alpha, \alpha \rangle = \langle \delta d\alpha, \alpha \rangle + \langle d\delta\alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle = \|d\alpha\|_{\ell^2}^2 + \|\delta\alpha\|_{\ell^2}^2$$

So  $d\alpha = 0$  and  $\delta\alpha = 0$ .  $\square$

## General Things about Linear Differential Operators

Let  $U \subseteq \mathbb{R}^n$ , consider  $C^\infty(U, \mathbb{C})$ .

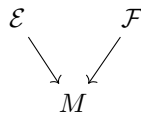
**Defn:** A differential operator  $P$  on  $C^\infty(U, \mathbb{C})$  is of the form  $\forall f \in C^\infty(U, \mathbb{C})$ ,

$$P(f) = \sum_{\alpha} c_{\alpha}(x)(D^{\alpha}f)(x) \quad \alpha \text{ multi-index with } |\alpha| \leq n \quad D^{\alpha} = \frac{1}{i} \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \frac{1}{i} \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}$$

(here, we're using  $i = \sqrt{-1}$ ). Note that  $P$  is local –  $P(f)(x)$  only depends on  $f$  in a neighborhood of  $x$ .

On manifolds, on  $C^\infty(M, \mathbb{C})$ , linear differential operators are generated as a ring by multiplying by functions. For vector bundles/systems, we have  $P : C^\infty(U, \mathbb{C}^r) \rightarrow C^\infty(U, \mathbb{C}^s)$ , where  $P = (P_{ij})_{s \times r}$ , and  $P_{ij}$  is a scalar differential operator.

In the manifold setting, suppose we have two bundles  $\mathcal{E}$  and  $\mathcal{F}$ :



Consider  $P : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$ . Under local trivializations on the same  $U \subseteq M$ , we have  $\Gamma(\mathcal{E}|_U) \cong C^\infty(U, \mathbb{C}^r) \cong \Gamma(\mathcal{F}|_U)$ . So  $P$  is locally the Euclidean case.

We already know a bunch of examples!

**Ex:**  $d : \Omega^k \rightarrow \Omega^{k+1}$  and  $\nabla : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E} \otimes T^*M)$  are differential operators of order 1.

The main invariant associated to a differential operator  $P$  is called its symbol. The symbol captures the top degree part of the operator. For an operator on  $C^\infty(U, \mathbb{C})$ , think of computing: take  $x_0 \in U, \xi_0 \in T_{x_0}^*U$  (i.e.  $(x_0, \xi_0) \in T^*U$ ). Pick  $\chi, \rho$  functions, both  $C_0^\infty(U, \mathbb{R})$  such that  $\chi \equiv 1$  near  $x_0$  and  $d\rho_{x_0} = \xi_0$ . Consider  $P(\chi e^{i\tau\rho})(x_0)$  for  $\tau \gg 1$ . We get

$$P(\chi e^{i\tau\rho})(x_0) = \sum_{|\alpha|=\overset{\text{def}}{m}=\deg P} c_\alpha(x_0) D^\alpha(\chi e^{i\tau\rho}) + \text{lower degree terms}$$

Look for the highest power of  $\tau$  on the right-hand side.

$$\tau^m e^{i\tau\rho(x_0)} \sum_{|\alpha|=m} c_\alpha(x_0) \underbrace{(\nabla\rho(x_0))^\alpha}_{\xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}} + \text{lower order terms in } \tau$$

Now, forget the  $\tau$  and forget the exponential.

**Defn:** The symbol of a differential operator  $P$  is

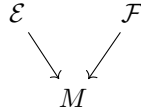
$$\sigma_P(x_0, \xi_0) = \sum_{|\alpha|=m} c_\alpha(x_0) \xi^\alpha = \lim_{\tau \rightarrow \infty} e^{-i\tau\rho(x_0)} \frac{1}{\tau^m} P(\chi e^{i\tau\rho})(x_0)$$

Conclusion: On manifolds,  $P : C^\infty(M, \mathbb{C}) \ni \cdot \mapsto \sigma_P : T^*M \rightarrow \mathbb{C}$  is well-defined. Specifically, it's a homogeneous polynomial in  $\chi$  of degree  $M$  on each fiber.

**Ex:** Let  $X \in \mathfrak{X}(M)$ . Then  $\sigma_{\mathcal{L}_X} : T^*M \rightarrow \mathbb{C}$  is  $\sigma_{\mathcal{L}_X}(p, \xi) = i \langle \xi, X_p \rangle$ .

**Ex:**  $\Delta : C^\infty(M) \ni \cdot \mapsto \sigma_\Delta : T^*M \rightarrow \mathbb{C}$  is  $\sigma_\Delta(p, \xi) = g^{ij}(p) \xi_i \xi_j = \|\xi\|^2 \text{Id}$ .  
(Recall: this is from the definition of  $\Delta$  in coordinates.)

If  $P : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$ ,



Then for  $(p, \xi) \in T^*M$ ,  $\sigma_P(p, \xi) : \mathcal{E}_p \rightarrow \mathcal{F}_p$  is a linear map between the fibers.

**Prop:** If  $P$  and  $Q$  are differential operators such that  $P \circ Q$  makes sense, then  $\sigma_{P \circ Q} = \sigma_P \circ \sigma_Q$ .

**Defn:**  $P$  is an elliptic operator iff  $\forall (p, \xi)$  with  $\xi \neq 0$ ,  $\sigma_P(p, \xi)$  is invertible.

**Ex:**  $P = \Delta$  is elliptic.  $\sigma_\Delta = \|\xi\|^2 \text{Id}$ , so it's invertible everywhere.  $\Delta$  has an approximate inverse  $G$  – “ $\sigma_G(p, \xi) = \frac{\text{Id}}{\|\xi\|^2}$ ”.  
 $\Delta \circ G - I$  and  $G \circ \Delta - I$  are smoothing operators – they're very small.

We're skipping a lot of stuff here, but...

**Thm:** (Spectral Theorem of the Laplacian) Consider  $\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$ , where  $M$  is a compact, oriented, Riemannian manifold. Then

- (1)  $\mathcal{U}^k = \ker \Delta$  has finite dimension.
- (2) There's an orthonormal basis (in the  $\ell^2$  sense) of  $\Omega^k$ ,  $\{\alpha_j\}$ , and  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots \rightarrow +\infty$  s.t.  $\Delta \alpha_j = \lambda_j \alpha_j$ .  
In other words, we can think of  $\Delta$  as an infinite matrix, that can be diagonalized by  $\alpha_j$  and  $\lambda_j$ . We'll write  $\forall \alpha \in \Omega^k, \alpha = \alpha^H + \sum_{\lambda > 0} \text{distinct } \alpha_\lambda$ .  $\alpha^H$  is the harmonic piece –  $\Delta \alpha^H = 0$ ,  $\Delta \alpha_\lambda = \lambda \alpha_\lambda$ . We define Green's operator  $G(\alpha) = \sum_{\lambda > 0} \frac{1}{\lambda} \alpha_\lambda$ . So  $(I - \Delta \circ G)(\alpha) = \alpha^H$ .

**Ex:** For  $\Delta$  on  $C^\infty(S^2)$ , we have  $\lambda = k(k+1)$  with multiplicity  $2k+1$  (for  $k \in \mathbb{Z}_{\geq 0}$ ).