Math 635 Lecture 16

Thomas Cohn

2/26/21

Notation: For M, a Riemannian manifold, the map $t \mapsto G(t, p, v)$ for $p \in M$, $v \in T_pM$, denotes the geodesic with initial conditions (p, v). We saw that, $\forall c \in \mathbb{R}$, if defined, $G(t, p, cv) = G(ct, p, v)$.

We also had the theorem that $\forall p \in M$, $\exists \varepsilon > 0$ s.t. $\forall v \in B_0(\varepsilon) \subseteq T_pM$ (recall that $B_0(\varepsilon) = \{v \in T_pM : ||v|| < \varepsilon\}$), $G(t, p, v)$ is defined for $t \in [0, 1]$. Based on that fact, we define $\exp_p(v) = G(1, p, v)$.

Lemma: $d(\exp_p)_{v=0} = \mathrm{Id}_{T_pM}$.

Cor: $\forall p \in M$, $\exists \varepsilon > 0$ s.t. $\exp_p : B_0(\varepsilon) \to M$ is a diffeomorphism onto its (open) image $U = \exp_p(B_0(\varepsilon))$.

Defn: Such a neighborhood U of p is called a <u>normal neighborhood</u> of p.

Warning: "Normal neighborhood" sometimes means any neighborhood that is the diffeomorphic image by $\exp_p$ of a neighborhood of $0 \in T_pM$.

Defn: <u>Normal coordinates</u> centered at $p \in M$ are any coordinates $(x^1, \ldots, x^n)$ of the form

$$\begin{tikzcd} U \arrow[r, "(\exp_p)^{-1}"] \arrow[rr, bend right, "(x^1,\ldots,x^n)"'] & T_pM \arrow[r, "\sim"] & \mathbb{R}^n \end{tikzcd}$$

where $n = \dim M$, U is a normal neighborhood, $(\exp_p)^{-1}$ is restricted to the image of $\exp_p$, and the mapping between T_pM and $\mathbb{R}^n$ is any orthogonal linear isomorphism.

Observe: The only choice needed to get normal coordinates on U is the identification $T_pM \cong \mathbb{R}^n$ that we select. Two different choices of identification will be related by an orthogonal matrix (that is, $y^i = a^i_j x^j$)

$$\begin{tikzcd} & & \mathbb{R}^n \arrow[dd, "\wr"', "(a^i_j)\in O(n)"] \\ U \arrow[urr, bend left, "(x^1,\ldots,x^n)"] \arrow[drr, bend right, "(y^1,\ldots,y^n)"'] \arrow[r] & T_pM \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & \\ & & \mathbb{R}^n \end{tikzcd}$$

Prop: In any normal coordinate system $(x^1, \ldots, x^n)$ centered at $p \in M$,

(a) $g_{ij}(0) = \delta_{ij}$
(b) $\forall \vec{v} \in \mathbb{R}^n$, $t \mapsto (tv^1, \ldots, tv^n)$, i.e., $x^i = tv^i$, is a geodesic.
We call these "radial geodesics", and they're precisely $G(t, p, v^i \frac{\partial}{\partial x^i})$.
(c) $\forall i, j, k$, $\Gamma^k_{ij}(0) = 0$
(d) $\forall i, j, k$, $\frac{\partial g_{ij}}{\partial x^k}(0) = 0$

Proof:

(a) Use the fact that $d(\exp_p)_0 = \mathrm{Id}$, and the isometry is orthogonal.
(b) By the definition of exp, the normal coordinates of $G(t, p, v^i \frac{\partial}{\partial x^i}) = (tv^1, \ldots, tv^n)$. (This is kind of tautological.)
(c) Use (b) and the geodesic equations: $\ddot{x}^k = -\dot{x}^i \dot{x}^j \Gamma^k_{ij}(x(t))$. Look at radial geodesics: $\ddot{x}^k = \ddot{(tv^k)} = 0$. So $\forall \vec{v} \in \mathbb{R}^n$, $v^i v^j \Gamma^k_{ij}(0) = 0$. This is a quadratic form in V; because $\forall k$, $\Gamma^k_{ij} = \Gamma^k_{ji}$, by the polarization identity for quadratic forms, $\Gamma^k_{ij} = 0$, $\forall i, j, k$.
(d) This is just an algebraic exercise. (Left for HW.)

$\square$

Lemma: (Polarization Identity) Let $\Gamma = (\Gamma_{ij})$ be a symmetric matrix, and let $Q(\vec{v}) = \vec{v}\Gamma\vec{v}^T$ be the quadratic form, for all column vectors $\vec{v}$. Then we can find Γ, and the quadratic form is 0 iff the matrix is 0.

The proof follows directly from the fact that

$$\vec{v}\Gamma\vec{w}^T = \frac{1}{4}(Q(\vec{v}+\vec{w}) - Q(\vec{v}-\vec{w}))$$

Observe that it's necessary to assume that the matrix is symmetric. If it's anti-symmetric, then Q is 0.

Observe that, in normal coordinates, $d(\exp_p)_{tv}(tv) = \sum_i v_i \frac{\partial}{\partial x^i}$.

Defn: Assume $U = \exp_p(B_\varepsilon(0))$ is a normal neighborhood for some $\varepsilon > 0$. $\forall r \in (0, \varepsilon)$, the image under $\exp_p$ of the ball $S_r(0) = \{v \in T_pM : ||v|| = r\}$ is a geodesic sphere: $\exp_p(S_r(0)) \subset U \subset M$.

Lemma: (Gauss' Lemma) In a normal neighborhood of p, radial geodesics are orthogonal to geodesic spheres. That is, $\langle d(\exp_p)_v(v), d(\exp_p)_v(w)\rangle = 0$ if $v \cdot w = 0$, because $d(\exp_p)_v(v)$ is tangent to the radial geodesic, and $v \cdot w = 0$ iff w is tangent at v to the sphere of radius $||v||$.

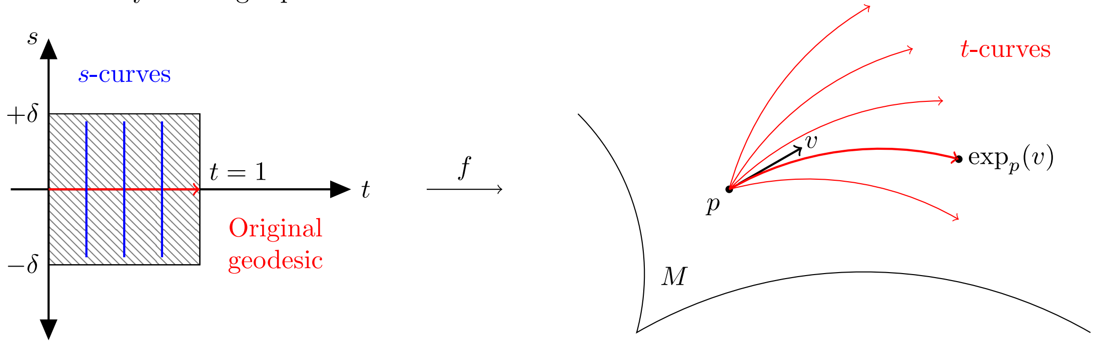
We need a new tool to deal with this!

Families of Curves

The idea is to extend a single radial geodesic to a family of radial geodesics, according to a parameter s. Consider a C^∞ map $f : [0,1] \times (-\delta, \delta) \to M$.

- For fixed s, $t \mapsto f(t,s)$ is a t-curve
- For fixed t, $s \mapsto f(t,s)$ is a s-curve

We're effectively creating a parametric surface:



If we let $f_t = \frac{\partial f}{\partial t}$, the velocity of a t-curve, and $f_s = \frac{\partial f}{\partial s}$, the velocity of an s-curve, then these define vector fields along each s and t curve (respectively).

Prop: For any family of curves as above, $\frac{D}{dt}f_s = \frac{D}{ds}f_t$. This follows from the Levi-Civita connection ∇ being torsion-free.

$\frac{D}{dt}f_s$ is the covariant derivative of the f_s vectors along t.