Math 635 Lecture 8

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/5/21

Start with a vector bundle $\mathcal{E} \to M$, with connection ∇. Let $U \overset{\text{open}}{\subseteq} M$, and $(E_1, \dots, E_r)$ a moving frame on U. Note that we have the following commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, phantom, "\overset{\phi}{\cong}"] \arrow[dr, "\pi"] & U \times \mathbb{R}^r \arrow[d, "\pi_U"] \\
& U \arrow[ul, bend left, "s"]
\end{tikzcd}$$

$\forall s \in \Gamma(U)$, $\exists \vec{f} : U \to \mathbb{R}^r$, $\vec{f} = (f^1, \dots, f^r)$ s.t. $s = \sum_{i=1}^r f^i E_i$. Thus, we have an isomorphism

$$\begin{aligned} \Gamma(U) &\overset{\sim}{\to} C^\infty(U, \mathbb{R}^r) \\ s &\mapsto \vec{f} \end{aligned}$$

Under this isomorphism, $\nabla_X s$ corresponds with $\nabla_X \vec{f}$. We saw:

$$\nabla_X \vec{f} = d\vec{f}(X) + \vartheta(X)\vec{f}$$

treating $\vec{f}$ as a column vector, $d\vec{f}(X) = \begin{pmatrix} df^1(X) \\ \vdots \\ df^r(X) \end{pmatrix} = X\vec{f}$, and $\vartheta = (\theta_i^j)$ s.t. $\nabla_X E_i = \sum_j \theta_i^j(X) E_j$. So we can rewrite this as $\nabla_X = X + \vartheta(X)$ on $C^\infty(U, \mathbb{R}^r)$.

Parallelism

Let $\mathcal{E} \to M$ be a vector bundle, with connection ∇.

Defn: Given $\gamma : [a,b] \to M$ and $s \in \Gamma(\mathcal{E})$, we say s is <u>covariant constant</u>, or <u>parallel</u>, along γ if and only if $\forall t \in [a,b]$, $\nabla_{\dot{\gamma}(t)} s = 0 \in \mathcal{E}_{\gamma(t)} = \pi^{-1}(\gamma(t))$. Locally, this is true if and only if $\frac{d\vec{f}(t)}{dt} = -\vartheta(\dot{\gamma}(t))\vec{f}(t)$, where $\vec{f}(t) = \vec{f}(\gamma(t))$.

Today, we'll work with parallel transport. The idea is given γ, we can construct sections along γ that are covariant constant. Given a vector bundle $\mathcal{E} \to M$, with connection ∇, a curve $\gamma : [a,b] \to M$, and a section $s_a \in \mathcal{E}_{\gamma(a)}$, we want to "parallel transport" s_a along γ.

Defn: Given $\gamma : [a,b] \to M$ on a vector bundle $\pi : \mathcal{E} \to M$, a <u>section of $\mathcal{E}$ along γ</u> is a function $V : [a,b] \to \mathcal{E}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{E} \arrow[d, "\pi"] \\
{[a,b]} \arrow[ur, "V"] \arrow[r, "\gamma"] & M
\end{tikzcd}$$

That is, $\forall t \in [a,b]$, $\pi(V(t)) = \gamma(t)$, i.e., $V(t) \in \mathcal{E}_{\gamma(t)}$.

Our (nonstandard) notation is, for a curve γ, $\Gamma_\gamma(\mathcal{E})$ is the set of all such smooth V.

Ex: We can always just use a global section. If we have $s \in \Gamma(\mathcal{E})$, then $V = s \circ \gamma : [a,b] \to \mathcal{E}$ is a section along γ.

We can extend covariant differentiation (with respect to ∇) to $\Gamma_\gamma(\mathcal{E})$ for a given γ!

Prop: (Do Carmo, Chapter 2, Prop 2.2) Given $\mathcal{E}$, ∇, and γ, there is a unique operator

$$\frac{D}{dt} : \Gamma_\gamma(\mathcal{E}) \to \Gamma_\gamma(\mathcal{E})$$

such that
(a) $\frac{D}{dt}$ is $\mathbb{R}$-linear.
(b) $\frac{D}{dt}$ is a derivation: $\forall f \in C^\infty([a,b])$, $\forall V \in \Gamma_\gamma(\mathcal{E})$, $\frac{D}{dt}(fV) = f\frac{DV}{dt} + \dot{f}V$, where $\dot{f}(t) = \frac{df}{dt}$.
(c) $\forall s \in \Gamma(\mathcal{E})$, $\frac{D}{dt}(s \circ \gamma) = \nabla_{\dot{\gamma}(t)} s$.

Proof: Start with local uniqueness. Let $(E_1, \ldots, E_r)$ be a moving frame on $U \subseteq M$, with $U \cap \mathrm{Im}(\gamma) \neq \emptyset$. Let ϑ be the connection matrix associated with the E_i's. Assume that $\frac{D}{dt}$ exists. Then we have a commutative diagram:

$$\begin{tikzcd}
 & \pi^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "\pi"] & U \times \mathbb{R}^r \arrow[d, "\pi_U"] \\
\gamma^{-1}(U) \arrow[ur, "V|_{\gamma^{-1}(U)}"] \arrow[rr, "\gamma|_{\gamma^{-1}(U)}"] \arrow[urr, bend left, "\phi\circ(V|_{\gamma^{-1}(U)})"] & & U
\end{tikzcd}$$

$\forall t \in \gamma^{-1}(U)$, $\phi(V(t)) = (\gamma(t), (f^1(t), \ldots, f^r(t)))$. This defines $f^i(t)$. Now, we claim that (a), (b), and (c) together of $\frac{D}{dt}$ imply that, $\forall t \in \gamma^{-1}(U)$,

$$\frac{DV}{dt}(t) = \frac{D}{dt}\left(\sum_{i=1}^r f^i(t) E_i(\gamma(t))\right) = \sum_{i=1}^r f^i(t)(\nabla_{\dot{\gamma}(t)} E_i)(t) + \dot{f}^i(t) E_i(\gamma(t))$$

Existence: Define $\frac{D}{dt}$ locally, using the above formula, and then use trivializations $\{U_\alpha\}$ that cover, so that

$$\gamma^{-1}\left(\bigcup_\alpha U_\alpha\right) = [a,b]$$

Uniqueness: On the overlap of differing U_α's, the definitions must agree. $\square$

In fact, you can check: If we define in terms of $F_j = \sum_i a^i_j E_i$, then the definitions agree.

Defn: (Parallel Transport) For $\gamma : [a,b] \to M$, we define $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ by $\forall s_a \in \mathcal{E}_{\gamma(a)}$, $\mathcal{P}_\gamma(s_a) = V(b)$, where $V(b)$ is a solution at $t = b$ of $\frac{DV}{dt}(t) = 0$, $V(a) = s_a$, where $V \in \Gamma_\gamma(\mathcal{E})$. Locally: $\dot{\vec{f}} = -\vartheta(\dot{\gamma})\vec{f}$.

Observe: $\mathcal{P}_\gamma$ is $\mathbb{R}$-linear.

We can extend this to continuous, piecewise smooth γ by using composition: Say $\gamma_1 : [a,b] \to M$ and $\gamma_2 : [a', b'] \to M$ are two smooth curves, with $\gamma_1(b) = \gamma_2(a')$. Then for their concatenation $\gamma_2 \# \gamma_1$ (we won't use this notation often), we have $\mathcal{P}_{\gamma_2 \# \gamma_1} = \mathcal{P}_{\gamma_2} \circ \mathcal{P}_{\gamma_1}$.

In particular, reversing the direction of γ shows that $\mathcal{P}_\gamma$ is a bijection. If we have a loop, so $\mathcal{E}_{\gamma(a)} = \mathcal{E}_{\gamma(b)}$, then $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \circlearrowleft$ is called the holonomy of γ.