Math 635 Lecture 4

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Recall: A Riemannian metric g on M is a covariant 2-tensor (i.e. a tensor of type $(0,2)$), such that, at each point, g_p is symmetric and positive definite.

Prop: Any C^∞ manifold M has infinitely many Riemannian metrics.

Proof 1: We can embed $M \hookrightarrow \mathbb{R}^N$ (for some N), and use the induced metric. □

Proof 2: We can use a partition of unity. Let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M, and $\{\chi_\alpha\}$ a subordinate partition of unity. (Recall: A subordinate partition of unity means $\forall \alpha$, $\chi_\alpha \in C_0^\infty(M)$, $\{\operatorname{supp}\chi_\alpha\}$ is a locally finite cover of M, and $\forall p \in M$, $\sum_\alpha \chi_\alpha(p) = 1$.) $\forall \alpha$, let g^α be the Riemannian metric on U_α s.t. ϕ_α is an isometry (to the standard flat metric on $\mathbb{R}^n$). That is, $(g^\alpha)_{ij} = \delta_{ij}$. Define, $\forall p \in M$, $\forall u, v \in T_pM$,

$$g_p(u,v) = \sum_\alpha \underbrace{\chi_\alpha(p) g_p^\alpha(u,v)}_{\text{Interpret this as 0 if } p \notin U_\alpha.}$$

Basically, we're extending $\chi_\alpha g^\alpha$ to 0 outside U_α. Now that we've define g, we need to check that it satisfies the appropriate properties. Symmetric matrices form a closed group under addition, so clearly, every g_p is symmetric. And because $\chi_\alpha \geq 0$, we can also assert that g_p is positive semidefinite. Finally we have to check that $g_p(u,u) = 0 \Rightarrow u = 0$. Well, $\exists \alpha$ s.t. $\chi_\alpha(p) > 0$, so $p \in U_\alpha$. Thus, we must have $g_p^\alpha(u,u) = 0$, so we must have $u = 0$ by the positive definiteness of g_p^α. □

Observe: An analogous proof shows that any vector bundle has infinitely many C^∞ sections, where we use local trivializations instead of coordinates.

Note that positive-definiteness is important! It's not true that M has a non-degenerate symmetric covariant 2-tensor of any arbitrary signature. For example, general relativity cares about the signature $(-,+,+,+)$, and such a tensor (with all relevant properties) doesn't always exist.

Defn: Let G be a Lie group. (Recall that this means G is a manifold, and a group, with the group operation and inversion C^∞ functions with respect to the smooth structure.) $\forall k \in G$, we have the left translation map $L_k : G \to G$, where $L_k(h) = kh$. A Riemannian metric on G is said to be left-invariant iff $\forall k \in G$, L_k is an isometry.

Prop: Let $\mathfrak{g} = T_eG$, the Lie algebra (where $e \in G$ is the identity element of the group). Then given any positive definite inner product $\langle\ ,\ \rangle_e$ on $\mathfrak{g}$, there exists a unique left-invariant Riemannian metric g on G s.t. $g_e = \langle\ ,\ \rangle_e$.

Proof: $\forall k \in G$, we require (by left-invariantness) that $d(L_k)_e : (\mathfrak{g}, \langle\ ,\ \rangle_e) \to (T_kG, g_k)$ is an isometry. We define g this way – g_k is the "pushforward" of $\langle\ ,\ \rangle_e$ by $d(L_k)_e$. Specifically, $\forall u, v \in T_kG$,

$$g_k(u,v) = \left\langle d(L_k^{-1})(u), d(L_k^{-1})(v)\right\rangle_e = \langle d(L_{-k})(u), d(L_{-k})(v)\rangle_e$$

We claim that this is smooth, using the smoothness of the map $\begin{matrix} G \times G \to G \\ (k,h) \mapsto kh \end{matrix}$.

Question: We can also define everything in terms of right-invariant metrics. Do bi-invariant metrics exist? Answer (which we will show in HW): It depends on G!

If M and N are Riemannian manifolds, then $M \times N$ has a natural Riemannian metric. $\forall (p,q) \in M \times N$, $T_{(p,q)}(M \times N) = (T_pM) \oplus (T_qN)$. The product metric is defined as follows: Declare T_pM and T_qN to be orthogonal. With g^M, g^N the Riemannian metrics for M, N (respectively), we have the product metric on $M \times N$ (as a block matrix):

$$g^{M\times N} = \left(\begin{array}{c|c} g^M & 0 \\ \hline 0 & g^N \end{array}\right)$$

Defn: Let M, N be Riemannian manifolds. A smooth map $F : M \to N$ is a local isometry iff $\forall p \in M$,

$$dF_p : (T_pM, g_p^M) \to (T_{F(p)}N, g_{F(p)}^N)$$

is a linear isometry.

Note that local isometries are local diffeomorphisms, by the inverse function theorem.

Defn: A surjective C^∞ map $\pi : M \to N$ is a smooth covering map iff $\forall q \in N$, there's a neighborhood V of q s.t. π maps the connected components of $\pi^{-1}(U)$ diffeomorphically onto V. That is,

$$\pi^{-1}(U) = \bigsqcup_j U_j, \qquad \forall j,\, \pi|_{U_j} : U_j \xrightarrow{\sim} V$$

Defn: Let π be a smooth covering map. The automorphism group is $\mathrm{Aut}(\pi) = \{F : M \to N \mid \pi = F \circ \pi\}$. We say that such an F "shuffles the fibers". $\mathrm{Aut}(\pi)$ is a group under composition.

Defn: π is normal iff $\mathrm{Aut}(\pi)$ acts transitively on the fibers.

Ex: $S^n \xrightarrow{\pi} \mathbb{RP}^n$. $\mathrm{Aut}(\pi) = \{\text{identity map, antipodal map}\} \cong \mathbb{Z}_2$.
$\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/\mathbb{Z}^n$.

Ex: (HW) $\mathcal{H} = \big\{(x, y) \in \mathbb{R}^2 \mid y > 0\big\}$. For any genus g surface G, with $g > 1$, $\exists \pi : \mathcal{H} \to G$.

Thm: Let $\pi : M \to N$ be a smooth normal covering map. Let g^M be a Riemannian metric on M s.t. $\forall F \in \mathrm{Aut}(\pi)$ is an isometry. Then there's a unique Riemannian metric g_N on N s.t. π is a local isometry.

Proof: Let $q \in N$, and V a neighborhood of q. Then $\pi^{-1}(V) = \bigsqcup_j U_j$ (where the U_j are the connected components) such that $\pi|_{U_j} : U_j \xrightarrow{\sim} V$. We want to choose a metric on V s.t. $\forall j$, $\pi|_{U_j}$ is an isometry. Well, choose a j_0, and put such a metric on V. $\forall j \neq j_0$, $\exists F \in \mathrm{Aut}(\pi)$ s.t. $F(U_j) = F(U_{j_0})$. F is an isometry, by assumption, so π is an isometry on U_j. $\square$