Math 635 Lecture 29

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Today's goal is to prove Cartan-Hadamard: If M is a complete, connected Riemannian manifold with $K \leq 0$, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Defn: A C^∞ map $F : \tilde{M} \to M$ is a smooth covering map iff $\forall p \in M$, there's a neighborhood V of p such that $F^{-1}(V) = \bigcup_\alpha U_\alpha$, where $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset$, and $\forall \alpha,\ F|_{U_\alpha}^V$ is a diffeomorphism. We say that V is evenly covered.

Observe: A smooth covering map $F : \tilde{M} \to M$ is always a local diffeomorphism, as the definition of local diffeomorphism is $\forall \tilde{p} \in \tilde{M}$, there's a neighborhood U of $\tilde{p}$ and V of $F(\tilde{p})$ such that $F|_U^V$ is a diffeomorphism.

Defn: A smooth map $F : \tilde{M} \to M$ between Riemannian manifolds is a local isometry iff $\forall \tilde{p} \in M$, there's a neighborhood U of $\tilde{p}$ and V of $F(\tilde{p})$, $F|_U^V$ is an isometry.

Some properties of a local isometry $F : \tilde{M} \to M$:

- F is a local diffeomorphism.
- If $\tilde{\gamma} : I \to \tilde{M}$ is a geodesic on $\tilde{M}$, then $\gamma = F \circ \tilde{\gamma}$ is a geodesic on M.
- If $c : [a, b] \to \tilde{M}$ is any path, $\ell(F \circ c) = \ell(c)$.

Lemma: Let $F : \tilde{M} \to M$ be a local isometry, where $\tilde{M}$ and M are connected, complete Riemannian manifolds. Then F is a surjective covering map.

Proof: The main property of F is $\forall p \in M$, $\tilde{p} \in F^{-1}(p)$, $\forall v \in T_pM$, $\exists! \tilde{v} \in T_{\tilde{p}}\tilde{M}$ such that $F_{*,\tilde{p}}(\tilde{v}) = v$, and also $F \circ (\exp_{\tilde{p}}(t\tilde{v})) = \exp_p(tv)$. We obtain the existence and uniqueness of $\tilde{v}$ because $dF_{\tilde{p}}$ is a bijection. And the equality with the exponential map is true because both sides are geodesics on M with the same initial conditions. We'll say that we can "lift" geodesics from M to $\tilde{M}$: choose $\tilde{p} \in F^{-1}(p)$. Then $\exists! \tilde{\gamma}$ geodesic on $\tilde{M}$ such that $(F \circ \gamma)(t) = \exp_p(tv)$ and $\tilde{\gamma}(0) = \tilde{p}$.

To show the map is surjective, let $\tilde{p} \in \tilde{M}$, and define $p = F(\tilde{p})$. Let $q \in M$. Then by completeness, there is a geodesic γ on M joining p to q – $\gamma(0) = p$ and $\gamma(T) = q$. Let $\tilde{\gamma}$ be the lift of γ to $\tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}$. $\forall t$, $(F \circ \tilde{\gamma})(t) = \gamma(t)$. So $F(\tilde{\gamma}(T)) = \gamma(T) = q$, so $q \in \operatorname{im} F$.

Next, we show the map is a covering map. Let $p \in M$. Since F is a local isometry, F_* is always bijective, so p is a regular value. Thus, $F^{-1}(p) = \bigsqcup_\alpha \{\tilde{p}_\alpha\}$ is the disjoint union of (at most) countably many points. Let $\varepsilon > 0$ such that there's an open geodesic all $B_\varepsilon(p) \subset M$, centered at p with radius ε. $\forall \alpha$, define the open metric ball $U_\alpha = \left\{\tilde{q} \in \tilde{M} \mid \tilde{d}(\tilde{p}_\alpha, \tilde{q}) < \varepsilon\right\}$. We claim that $F^{-1}(B_\varepsilon(p)) = \bigcup_\alpha U_\alpha$, and the conditions of being a covering map are satisfied by the U_α.

Claim 1: $\forall \alpha$, F maps U_α into $B_\varepsilon(p)$, and $F|_{U_\alpha}$ is a bijection (so as a result, the restriction of F is a diffeomorphism). Proof: Pick $\tilde{q} \in U_\alpha$. Let $\tilde{\gamma}$ be a geodesic segment in $\tilde{M}$ joining $\tilde{p}_\alpha$ to $\tilde{q}$. Then $\ell(\tilde{\gamma}) < \varepsilon$. Consider $\gamma = F \circ \tilde{\gamma}$, a geodesic of the same length, $\ell(\gamma) = \ell(\tilde{\gamma}) < \varepsilon$. Then $\operatorname{im}\gamma \subseteq B_\varepsilon(p)$, so $F(\tilde{q}) \in B_\varepsilon(p)$. Now, we construct the inverse of $F|_{U_\alpha}^{B_\varepsilon(p)}$. Start with some $q \in B_\varepsilon(p)$. Lift the radial geodesic from p to q u to $\tilde{}$, starting at $\tilde{p}_\alpha$. Then its endpoint is the inverse of $q \in U_\alpha$.

Claim 2: $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset$. Proof: We will show $\tilde{d}(\tilde{p}_\alpha, \tilde{b}_\beta) > 2\varepsilon$. By the triangle inequality, this suffices. Let $\tilde{\gamma}$ be he minimizing geodesic from $\tilde{p}_\alpha$ to $\tilde{p}_\beta$. Consider $\gamma = F \circ \tilde{\gamma}$. We claim that γ must exit $B_\varepsilon(p)$, because any geodesic contained in $B_\varepsilon(p)$ and passing through p is a radial geodesic, so it must be minimizing. It's not, so thus, $\ell(\gamma) > 2\varepsilon$.

Claim 3: $F^{-1}(B_\varepsilon) = \bigcup_\alpha U_\alpha$. Proof: $\supseteq$ is part of claim 1. For $\subseteq$, let $\tilde{q} \in F^{-1}(B_\varepsilon(p))$, so $F(q) \in B_\varepsilon(p)$. Then let γ be the radial geodesic from $F(\tilde{q})$ back to p. Let $\tilde{\gamma}$ be the lift of γ, starting at $\tilde{q}$. $\tilde{\gamma}$ ends at $\tilde{}$ such that $F(\tilde{p}) =$, so $\tilde{p} \in F^{-1}(p)$. Thus, $\exists \alpha$ s.t. $\tilde{p} = \tilde{p}_\alpha$, and $\ell(\tilde{\gamma}) = \ell(\gamma) < \varepsilon$. $\square$

Lemma: If M is such that $K \leq 0$ everywhere, then there are no conjugate points.

Proof: HW

Thm: (Cartan-Hadamard) Let M be a complete Riemannian manifold, with $K \leq 0$ everywhere, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Note that the second lemma implies $\exp_p$ has no critical points. The idea of the proof is we put a metric on T_pM that makes $\exp_p$ a local isometry. Then we have to check that this metric is complete. It is, because rays $t \mapsto tv$ are geodesics in this (crazy) metric, and they exist $\forall t$.