Math 635 Lecture 6

Thomas Cohn

2/1/21

A brief roadmap for the next few weeks:

$$\begin{tikzcd}
\text{Riemannian Metrics} \arrow[dr] & & \text{Connections} \arrow[d] \\
& & \begin{array}{c}\text{Parallel Transport}\\ \text{and Curvature}\end{array} \arrow[dl] \\
& \begin{array}{c}\text{If } M \text{ is Riemannian, then}\\ TM \text{ has a special connection}\end{array} \arrow[d] & \\
& \text{Etc.} &
\end{tikzcd}$$

First, some motivation... (See also Lee Riemannian Geometry, Chapter 4)

Start with $\gamma : (a, b) \to \mathbb{R}^n$. Let $Y = \sum_i f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$, a smooth vector field in $\mathbb{R}^n$. We can compute $\frac{d}{dt} Y(\gamma(t))\big|_{t=t_0}$ in $\mathbb{R}^n$. But what's really happening? Well, we're computing

$$\frac{d}{dt} Y(\gamma(t))\bigg|_{t=t_0} = \lim_{h\to 0} \frac{Y(\gamma(t_0+h)) - Y(\gamma(t_0))}{h}$$

But we can only do this in $\mathbb{R}^n$, not on manifolds in general! Strictly speaking, $Y(\gamma(t_0 + h)) \in T_{\gamma(t_0+h)}\mathbb{R}^n$ and $Y_{\gamma(t_0)} \in T_{\gamma(t_0)}\mathbb{R}^n$. We can take their difference because we're identifying all tangent spaces of $\mathbb{R}^n$ with each other, using translations of $\mathbb{R}^n$. And $\{\text{translations of } \mathbb{R}^n\} \cong \mathbb{R}^n$ as a vector space. In other words, we can translate vectors in $\mathbb{R}^n$ "parallel to themselves".

For $Y = \sum_i f^i \frac{\partial}{\partial x^i}$, we get a formula:

$$\frac{d}{dt} Y(\gamma(t))\bigg|_{t=t_0} = \sum_{i=1}^n df^i_{\gamma(t_0)}(\dot\gamma(t_0)) \frac{\partial}{\partial x^i} \overset{\text{def}}{=} \left(\bar\nabla_{\dot\gamma(t_0)} Y\right)(\gamma(t_0))$$

(Note that $\frac{\partial}{\partial x^i}$ is a constant frame on $\mathbb{R}^n$, so we can use $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^i}\big|_p$ interchangeably).

Defn: If $Y = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$, $p \in \mathbb{R}^n$, and $v \in T_pM$, we define

$$\left(\bar\nabla_v Y\right)(p) \overset{\text{def}}{=} \sum_{i=1}^n df^i_p(v) \frac{\partial}{\partial x^i}$$

We can think of $\left(\bar\nabla_v Y\right)(p)$ as a vector, which only depends on the values of Y along a curve γ as above.

The question remains: Is there something analogous to this on manifolds? It may look a bit like a Lie derivative, but note that $\bar\nabla$ is **not** a Lie derivative!

Recall: Given $X, Y \in \mathfrak{X}(M)$, we can define $\mathcal{L}_X Y$ using the flow φ of X:

$$(\mathcal{L}_X Y)(p) = \lim_{t\to 0} \frac{(\varphi_{-t})_{*,\varphi_t(p)}(X_{\varphi_t(p)}) - X_p}{t}$$

In this case, we need X as a vector field, whereas above, we just need a vector. And $\mathcal{L}_X Y$ is dependent on X, but there are infinitely many vector fields X s.t. $X_p = v$ (with v as above).

In fact, we need some additional structure on the manifold, because we cannot natural identify T_pM with T_qN, when $p \neq q$. This additional structure is called a connection.

Defn: Let $\mathcal{E} \to M$ be a vector bundle. A connection on $\mathcal{E}$ is an operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

that satisfies:

1) $\forall X, Y \in \mathfrak{X}(M)$, $\forall s \in \Gamma(\mathcal{E})$, $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
2) $\forall f \in C^\infty(M)$, $\nabla_{fX} s = f \nabla_X s$
3) $\nabla_X(fs) = f\nabla_X s + X(f)s = f\nabla_X s + df(X)s$

Because of properties 1 and 2, we say that a connection is "linear in X over $C^\infty(M)$".

Note that although our definition above uses vector fields, we will show that this dependence is pointwise.

Ex: $\nabla = \bar{\nabla}$ on $\mathcal{E} = T\mathbb{R}^n$.

Prop: If ∇ is a connection on $\mathcal{E} \to M$, then $\forall X \in \mathfrak{X}(M), s \in \Gamma(\mathcal{E}), p \in M$, $(\nabla_X s)(p) \in \mathcal{E}_p = \pi^{-1}(p)$ only depends on X_p and the values of s in an arbitrarily small open neighborhood of p.

Proof: Let U be a neighborhood of p; χ a bump function supported on U, with $\chi \equiv 1$ on $V \overset{\text{open}}{\subseteq} U$, a smaller open neighborhood of p. Consider $\nabla_X(\chi s) = X(\chi)s + \chi\nabla_X s$. Evaluate at p: the right hand side is just $(\nabla_X s)(p)$ because $\chi \equiv 1$ on V and $X(\chi) \equiv 0$ on V (because χ is constant on V, and X is a derivation). The computation of $(\nabla_X s)(p)$ can be localized to, say, a coordinate neighborhoood of p.

Let $X = \sum_i a^i \frac{\partial}{\partial x^i}$ in local coordinates. Then, by C^∞ linearity of ∇,

$$(\nabla_X s)(p) = \sum_i a^i(p)(\nabla_{\frac{\partial}{\partial x^i}} s)(p)$$

If $X(p) = 0$ (which is true iff $\forall j$, $a^j(p) = 0$), then $(\nabla_X s)(p) = 0$. So if $X(p) = \tilde{X}(p)$, then $(\nabla_X s)(p) = (\nabla_{\tilde{X}} s)(p)$. $\square$

Observe: Given ∇ and s, $(\nabla_X s)(p) \in \mathcal{E}_p$ depends only on $X(p)$, and does so linearly! So ∇ and s define a map

$$\begin{aligned} T_pM &\to \mathcal{E}_p \\ v &\mapsto (\nabla_v s)(p) \end{aligned}$$

which is itself an element of $T_p^*M \otimes \mathcal{E}_p$. Therefore, ∇ can be thought of as an operator $\nabla : \Gamma(\mathcal{E}) \to \mathcal{E}(T^*M \otimes \mathcal{E})$, whose image is "$\mathcal{E}$-valued differential forms".

Local Expression of a ∇

Let $\mathcal{E} \to M$ be a vector bundle, with connection ∇, $U \overset{\text{open}}{\subseteq} M$, and $(E_1, \dots, E_r)$ a moving frame of $\mathcal{E}$ over U. That is, $\forall j$, $E_j \in \Gamma(\mathcal{E}|_U)$, and at each $p \in U$, $(E_1(p), \dots, E_r(p))$ is a basis of $\mathcal{E}_p$. So if $s \in \Gamma(\mathcal{E}|_U)$, then $\exists f^i \in C^\infty(U)$ s.t. $s = \sum_j f^j E_j$. So $\forall X \in \mathfrak{X}(U)$, we get

$$\nabla_X s = \sum_{j=1}^r f^j \nabla_X E_j + X(f^j)E_j$$

But

$$\nabla_X E_j = \sum_{i=1}^r \theta_j^i(X) E_i$$

By the discussion above, $\forall i, j$, $\theta_j^i \in \Omega^1(U)$, a one-form. So we can define $\vartheta = (\theta_j^i)$, an $r \times r$ matrix of one-forms on U, depending on the moving frame $(E_1, \dots, E_r)$. In fact, this ϑ determines ∇ on U!