Math 635 Lecture 5

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Some stuff we can do with Riemannian metrics:

- A Riemannian metric on M allows us to define the length of a tangent vector, and the angle between two tangent vectors. For $v, w \in T_pM$, $||v|| = \sqrt{\langle v, v\rangle_p}$, and $\langle v, w\rangle_p = ||v||\,||w|| \cos\theta$ defines θ, the angle between v and w.
- A Riemannian metric on M allows us to define the length of a curve in M. Say $\gamma : [a, b] \to M$. We define its length

$$L(\gamma) = \int_a^b ||\dot\gamma(t)||\; dt = \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}}\, dt$$

If $\mathrm{im}\gamma \subseteq U$, a coordinate patch, with coordinates $(x^1(t), \dots, x^k(t))$, then

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(\gamma(t))\dot x^i(t)\dot x^j(t)}\, dt \qquad g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle \in C^\infty(U)$$

Lemma: $L(\gamma)$ is invariant under re-parameterizations of γ. Consider $[\alpha, \beta] \underset{t}{\xrightarrow{\sim}} [a, b] \xrightarrow{\gamma} M$. Then

$$\int_\alpha^\beta \left|\left|\frac{d\gamma}{ds}(t(s))\right|\right|\; ds = \int_a^b \left|\left|\frac{d\gamma}{dt}(t)\right|\right|\; dt$$

Proof: The chain rule says $\frac{d\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt}$. So $\left|\left|\frac{d\gamma}{ds}\right|\right| = ||\frac{dt}{ds}||\left|\left|\frac{d\gamma}{dt}\right|\right|$. Now, use the change of variables formula for integrals. $\square$

Defn: Let M be a connected Riemannian manifold. We define the <u>Riemannian distance function</u> by, $\forall p, q \in M$, $d(p, q) = \inf\{L(\gamma) \mid \gamma$ is a continuous curve, or "path", joining p and q that is piecewise smooth$\}$.

Note: We can replace this definition with just "smooth" – the definitions are equivalent. But that's harder to prove, and this definition will be useful later on.

Note that because M is connected, it's path connected. This means the set of lengths of curves connecting pairs of points is nonempty, so the infimum exists. And because $L(\gamma) \geq 0$, $d(p, q) \geq 0$.

Thm: d is a metric, or distance function, i.e., $\forall p, q, r \in M$,
(i) $d(p, q) \geq 0$, with $d(p, q) = 0 \Leftrightarrow p = q$
(ii) $d(p, q) = d(q, p)$
(iii) $d(p, q) + d(q, r) \geq d(p, r)$

Proof: (This is only a partial proof)
(i) If $p = q$, take a trivial/constant path. $\dot\gamma \equiv 0$, so $L(\gamma) = 0$. The converse remains to be shown: that $d(p, q) = 0$ implies $p = q$. This will be a corollary of the "Gauss lemma", which we'll do later on.
(ii) We never assumed reparameterizations couldn't reverse the direction of the curve. They can, which directly proves $d(p, q) = d(q, p)$.
(iii) Among the paths joining p and r are paths that travel through q. Specifically, given any path from p to q and any path from q to r, we can concatenate them to get a path from p to r.

$\square$

Observe: The topology defined by d is the same as the given topology on M.

Sometimes, but not always, the infimum is attained, i.e., there exists a minimizing path. In fact, if such a path exists, it's always smooth.

Ex: Minimizing paths on the sphere are arcs of great circles – intersections of the sphere with hyperplanes through the origin.

Ex: Minimizing paths don't always exist! Consider $M = \mathbb{R}^2 \setminus \{0\}$. For $p = (43, 0)$ and $q = (-43, 0)$, $d(p, q) = 86$, but there's no path of that length between them (since you can't go through the origin).

Refer to Do Carmo, Chapter 1, §2 for more details.

Volume Element of an Oriented Riemannian Manifold

Reminder: An orientation on an orientable manifold M is determined by a class of top-degree differential forms, ν, with the defining property that $\forall p \in M$ and any $(v_1, \dots, v_n)$, a positive basis of T_pM, $\nu_p(v_1, \dots, v_n) > 0$.

In particular, ν is nowhere-vanishing. Conversely, a nowhere-vanishing top-degree form can be used to define positive bases, and in turn, an orientation.

Defn: If M is a orientable Riemannian manifold, its <u>volume form</u> ν is defined by the property that $\forall p \in M$, $\forall (v_1, \dots, v_n)$, a positive, orthonormal basis of T_pM, one has $\nu_p(v_1, \dots, v_n) = 1$.

Observe: If this condition holds for some positive orthonormal basis, it holds for all positive orthonormal bases. The important calculation is as follows: Fix $p \in M$, $(v_1, \dots, v_n)$ a positive orthonormal basis of T_pM, and $(e_1, \dots, e_n)$ any other ordered basis of T_pM. Then we can write each $e_i = \sum_{\ell=1}^n a_i^\ell v_\ell$. For any top-degree form ν, $\nu_p(e_1, \dots, e_n) = \det(a_i^\ell)\nu(v_1, \dots, v_n)$, so if $(e_1, \dots, e_n)$ is also positive and orthonormal, then $\det(a_i^\ell) = 1$. $\square$

Computation of the Volume Form in Coordinates

Start with $(x_1, \dots, x_n)$, a positive coordinate system with domain U. (Recall that this means $\forall p \in U$, $\left(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\right)$ is a positive basis of T_pM.) Apply Gram-Schmidt to each basis (pointwise). We obtain vector fields $v_1, \dots, v_n$ on U which are orthonormal at each point. And Gram-Schmidt shows the v's are related to the partial derivatives by a smooth matrix, so $\forall j$, $v_j \in \mathfrak{X}(U)$. And, by possibly permuting the v_i's, we can ensure it's a positive basis at every point.

In fact, let's write $\frac{\partial}{\partial x^i} = \sum_\ell a_i^\ell v_\ell$. Then

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \right\rangle = \sum_{k,\ell} a_i^\ell a_j^k \underbrace{\langle v_k, v_\ell \rangle}_{=\delta_{k,\ell}} = \sum_k a_i^k a_j^k = AA^T \quad \text{for} \quad A = (a_i^k)$$

So $\det(g_{ij}) = \det(A)^2 > 0$. On the other hand, with our Riemannian volume form ν,

$$\nu\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \underbrace{\det(A)}_{=\sqrt{\det(g_{ij})}} \underbrace{\nu(v_1, \dots, v_n)}_{=1} = \sqrt{\det(g_{ij})}$$

So in coordinates, $\nu = \sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$.

Defn: The volume of a subset U of a Riemannian manifold is $\mathrm{Vol}(U) = \int_U \nu$, where ν is the Riemannian volume form, if this integral is finite.

Lemma: For any coordinate system $(y^1, \dots, y^n)$ on U, positive or not, the Riemannian integral

$$\int_U \sqrt{\det\left(\left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle\right)} \underbrace{dy^1 \cdots dy^n}_{\text{Riemann integral}}$$

is equal to $\mathrm{Vol}(U)$.

This is true because the change of variables formula for a Riemann integral involves the absolute value of the Jacobian. So in the end, orientation is *not* needed to compute volumes of manifolds. In fact, we can even compute volumes of non-orientable manifolds! We generalize by using partitions of unity.