Math 635 Lecture 31

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Recall our definitions: $\bar{M}$ is a Riemannian manifold, $M \subseteq \bar{M}$ is a submanifold with the induced metric. $\forall X, Y \in \mathfrak{X}(M)$, $B(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y \in (T_pM)^\perp$, where $\bar{X}$ and $\bar{Y}$ are extensions of X and Y (respectively) to $\bar{M}$. (This is normal to M.) We can also think of it as $\bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_X Y + B(X,Y)$.

- B is a symmetric tensor – $\forall p \in M$, $B_p : T_pM \times T_pM \to (T_pM)^\perp \subseteq T_p\bar{M}$.
- If we define, $\forall p \in M$, $\nu_p \in (T_pM)^\perp$ a unit normal vector, then $S_\nu : T_pM \to T_pM$ is the shape operator, with the defining property $\forall x, y \in T_pM$, $\langle S_\nu(x), y\rangle = \langle B(x,y), \nu\rangle$.

Because B is symmetric, S_ν is self-adjoint, so there's an orthonormal eigenbasis $(e_1, \ldots, e_n)$ of T_pM and $\kappa_1, \ldots, \kappa_n$ such that $S_\nu(e_i) = \kappa_i e_i$. The e_i are the principal directions, and the κ_i are the principal curvatures.

Observe: If $\operatorname{codim} M = 1$, then $\delta_{ij}\kappa_i = \langle S_\nu(e_i), e_j\rangle = \langle B(e_i, e_j), \nu\rangle$ by orthonormality. This is just the ν-component of $B(e_i, e_j)$, so $B(e_i, e_j) = \delta_{ij}\kappa_i\nu$.

Thm: (Weingarten Formula) Pick $p \in M$ and $\nu \in (T_pM)^\perp$. Extend ν to a unit normal vector field N on M. Then $\forall x \in T_pM$, the Weingarten Formula says

$$S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

(This is computed via a curve in M with velocity x.)

Proof: Let $x, y \in T_pM$, and extend these to vector fields $\bar{X}$ and $\bar{Y}$ tangent to M. Then

$$\langle S_\nu(x), y\rangle = \langle B(x,y), N_p\rangle(p) = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y, N\right\rangle(p) = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y}, N\right\rangle(p)$$

because $\langle \nabla_X Y, N\rangle(p) = 0$, as $(\nabla_X Y)_p \in T_pM$ and $N \in (T_pM)^\perp$ are orthogonal. On the other hand, $0 = \bar{Y}, N\big|_M$, so

$$0 = \bar{X}\left\langle \bar{Y}, N\right\rangle = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y}, N\right\rangle + \left\langle \bar{Y}, \bar{\nabla}_{\bar{X}} N\right\rangle$$

The first term in the last expression is $\langle S_\nu(x), y\rangle$ by the above. So we have

$$\langle S_\nu(x), y\rangle = -\left\langle \bar{Y}, \bar{\nabla}_{\bar{X}} N\right\rangle = -\left\langle y, \bar{\nabla}_x N\right\rangle = \left\langle -\bar{\nabla}_x N, y\right\rangle \qquad \Rightarrow \qquad S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

$\square$

Consider the case where $\bar{M} = \mathbb{R}^{n+1}$, and $\dim M = n$. We identify tangent spaces with subspaces. Let U be the domain of N above. Then N is a map $N : U \to \mathbb{S}^n$, the unit sphere. We have

$$\begin{aligned} T_pM &\cong T_{N(p)}\mathbb{S}^n \\ S_{N_p} : T_pM \to T_pM &\cong T_{N(p)}\mathbb{S}^n \end{aligned}$$

Claim: $S_{N_p} : T_pM \to T_{N(p)}\mathbb{S}^n$ is equal to $-dN_p$.
Check this: Let $c(t)$ be a curve on M with $p = c(0)$, $x = \dot{c}(0)$. Then

$$S_{N_p}(x) = -\bar{\nabla}_x N = -\frac{d}{dt}N_{x(t)}\bigg|_{t=0} = dN_p(x)$$

Ex:

a) Let $M = \mathbb{S}^n_R$, the sphere of radius R in $\mathbb{R}^{n+1}$. Let N be the outward pointing unit normal.

$$\begin{aligned} N : \mathbb{S}^n_R &\to \mathbb{S}^n \\ p &\mapsto \tfrac{1}{R}p \end{aligned} \qquad \Rightarrow \qquad S_N = \frac{1}{R}\mathrm{Id}$$

In this case, every direction is principal, with corresponding principal curvature $\kappa = -\frac{1}{R}$.

b) Let M be a cylinder of radius R in $\mathbb{R}^3$. Specifically, with parameterization

$$\vec{r}(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

Let $N(\theta, z) = (\cos\theta, \sin\theta, 0)$. We have the basis of T_pM:

$$\begin{cases} \frac{\partial}{\partial\theta} = (-R\sin\theta, R\cos\theta, 0) \\ \frac{\partial}{\partial z} = (0, 0, 1) \end{cases}$$

Then we have $T_pM \cong T_{N_p}\mathbb{S}^2$. The matrix of dN with respect to the above basis is

$$\begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

So we conclude that $\frac{\partial}{\partial\theta}$ is a principal direction, with corresponding principal curvature $\kappa_1 = -\frac{1}{R}$. $\frac{\partial}{\partial z}$ is the other principal direction, with corresponding principal curvature $\kappa_2 = 0$.

Now, back to the general $M \subseteq \bar{M}$.

Thm: Let W, X, Y, Z be vector fields on $\bar{M}$ which are tangent to M. Then $\forall p \in M$,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z)\rangle - \langle B(W, Z), B(X, Y)\rangle$$

Proof: Next time...

Cor: If $x, y \in T_pM$ are orthonormal, then

$$\bar{K}(x, y) = K(x, y) + ||B(x, y)||^2 - \langle B(x, x), B(y, y)\rangle$$

Proof: Simply use $\bar{K}(x, y) = R(x, y, y, x)$. $\square$

Cor: If $\bar{M}$ is flat, and M has codimension 1, i.e., it's a hypersurface, then $\forall i \neq j$, $K(e_i, e_j) = \kappa_i\kappa_j$.

Ex: $\mathbb{S}^n_R$ has constant sectional curvature $\frac{1}{R^2}$.