Math 635 Lecture 38

Thomas Cohn

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Recall: The Laplace-Beltrami operator on forms: Δ on Ω^k, where $\Delta = d\delta + \delta d$.

Ex: On $\mathbb{R}^n$, Δ on functions (i.e. $\Omega^0 = C^\infty(\mathbb{R}^n, \mathbb{R})$). Then $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. (This is using our sign convention.)

For $\omega \in \mathbb{R}^n$, $\Delta(e^{i\omega\cdot x}) = ||\omega||^2 e^{i\omega\cdot x}$.

For forms $f_I dx^I$, where $I = \{i_1 < \cdots < i_k\}$ is a multi-index, the Laplacian is $\Delta(f_I dx^I) = (\Delta f_I) dx^I$.

Throughout our discussion today, we'll assume M is compact and oriented.

Ex: $\mathbb{T} = \mathbb{R}^n/\Lambda$, where $\Lambda = \{\sum k_i e_i, k_i \in \mathbb{Z}, e_1, \ldots, e_n \text{ linearly independent}\}$. $\Delta_{\mathbb{R}^n}$ makes sense on $C^\infty(\mathbb{T})$. If $\omega \in \mathbb{R}^n$ is such that $\forall k \in \Lambda$, $\omega \cdot k \in 2\pi\mathbb{Z}$, then $x \mapsto e^{i\omega\cdot x}$ is periodic w.r.t. Λ, so it's $C^\infty(\mathbb{T})$.

Thm: The eigenvalues of $\Lambda_{\mathbb{T}}$ on functions are $||\omega||^2$ for ω satisfying the above condition. Such ω constitute the *dual lattice* Λ^*.

Thm: For S^2, distinct eigenvalues are $k(k+1)$ for $k \in \{0, 1, 2, \ldots\}$, with corresponding muliplicities $2k+1$. That is, listed, with multiplicities, we have eigenvalues

$$\underbrace{0}_{k=0}, \underbrace{2, 2, 2}_{k=1}, \underbrace{6, 6, 6, 6, 6}_{k=2}, \ldots$$

In the general case, we have

Thm: (Spectral Theorem for the Laplacian on Forms) Let M be a compact, oriented manifold. There exists an orthonormal basis (in the ℓ^2 sense) of Ω^k, $\{\alpha_j : j = 0, 1, 2, \ldots\}$ an real eigenvalues $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to +\infty$ such that $\Delta\,\alpha_j = \lambda_j \alpha_j$. Moreover, $\lim_{j\to\infty} \lambda_j = +\infty$, i.e., the multiplicities are all finite.

This yields the spectral decomposition of $\alpha \in \Omega^k$:

$$\alpha = \alpha_H + \sum_{\lambda>0} \alpha_\lambda \qquad \Delta\,\alpha_\lambda = \lambda\alpha_\lambda, \Delta\,\alpha_H = 0$$

α_H is the "harmonic part", and the λ's are all distinct eigenvalues. This decomposition is unique and orthogonal.

Note: Because this is an infinite-dimensional space, there's a lot of analysis happening behind the scenes.

Observe: α is smooth iff $||\alpha_\lambda||_{\ell^2} = 0$ (really $\lambda^{-\infty}$). In other words, it decays rapidly as λ grows.

Defn: $G : \Omega^k \to \Omega^k$ is defined by $G(\alpha) = \sum_{\lambda>0} \frac{1}{\lambda}\alpha_\lambda$.

Observe: G is smooth.

By definition,

$$\Delta\, G(\alpha) = \sum_{\lambda>0} \frac{1}{\lambda}\,\Delta\,\alpha_\lambda = \sum_{\lambda>0} \frac{1}{\lambda}\lambda\alpha_\lambda = \sum_{\lambda>0} \alpha_\lambda = \alpha - \alpha_H$$

Lemma: $[G, d] = 0$ and $[G, \delta] = 0$.

Why should this be true? We've seen that $[\Delta, d] = 0$ and $[\Delta, \delta] = 0$. So $\forall \alpha_\lambda$, $\Delta\,\alpha_\lambda = \lambda\alpha_\lambda$. $d\alpha_\lambda$ is either 0 or an eigenform of Δ. So we get the commutative diagram

$$\begin{array}{ccc} \Omega^k & \xrightarrow{d} & \Omega^{k+1} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ \Omega^k & \xrightarrow{d} & \Omega^{k+1} \end{array}$$

Cor: (Hodge Decomposition Theorem) $\Omega^k = \mathcal{H}^k \oplus d(\Omega^{k-1}) \oplus \delta(\Omega^{k+1})$, where $\mathcal{H}^k = \ker(\Delta : \Omega^k \to \Omega^k)$ is the set of harmonic forms.

This decomposition is an orthogonal direct sum with respect to the ℓ^2 inner prouct. In fact, if $\alpha \in \Omega^k$,

$$\alpha = \alpha_H + \Delta\, G(\alpha) = \alpha_H + \underbrace{d(\delta G(\alpha))}_{\in d(\Omega^{k-1})} + \underbrace{\delta(\overbrace{d(G(\alpha))}^{G(d\alpha)})}_{\in \delta(\Omega^{k+1})}$$

Cor: If $\alpha \in \Omega^k$ is such that $d\alpha = 0$, then $[\alpha] = [\alpha_H]$ (where $[\,\cdot\,]$ is the equivalence class in deRham cohomology).

Proof:

$$\alpha = \alpha_H + d(\delta G(\alpha)) + \delta(\underbrace{dG(\alpha)}_{G(d\alpha)=0}) = \alpha_H + d(\delta G(\alpha))$$

□

Recall: $\Delta\,\alpha = 0$ iff $d\alpha = 0$ and $\delta\alpha = 0$. In fact, $\forall c \in H^k_{\mathrm{dR}}(M)$, $\exists! \alpha_H \in [c]$ such that $\Delta\,\alpha_H = 0$.

Cor: The Betti numbers of a manifold are finite.

Proof: $b_k = \dim H^k$. This is the multiplicity of k, which we know to be finite. □

Cor: (Poincaré Duality) The $\star$ operator induces $H^k(M) \cong H^{n-k}(M)$, where $n = \dim M$.

Proof: Follows from the fact that $[\Delta, \star] = 0$. □

Applications of Hodge Theory

Recall that $\forall p \in M$, $\forall u, v \in T_pM$, $\mathrm{Ric}^B_p(u, v) = \mathrm{tr}(T_pM \ni x \mapsto R_p(x, u)v \in T_pM)$. We saw that Ric^B is symmetric, and $\mathrm{Ric}(u) = \mathrm{Ric}^B(u, u)$.

Thm: (Bochner's Theorem) Assume M is connected, compact, and oriented.

(a) If $\mathrm{Ric} \geq 0$, $b_1 \leq \dim M$.
(b) If $\mathrm{Ric} \geq 0$ and $\exists p_0 \in M$ such that $\mathrm{Ric}^B_{p_0}$ is positive definite, then $b_1 = 0$.

Proof: Introduce a "rough Laplacian" associated to ∇. (Recall that $T^*M \cong TM$, and think of the connection as $\nabla : \gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$.) The rough Laplacian is $\nabla^*\nabla : \Omega^1 \to \Omega^1$. This works because we have $\nabla^* : \Gamma(T^*M \otimes T^*M) \to \Gamma(T^*M)$, using ℓ^2 inner products associated with the Euclidean structures on the bundles.

It turns out this rough Laplacian has the same ymbol as the ordinary Laplacian. In fact, hey only differ by a term of order 0! A calculation show that $\forall \alpha \in \Omega^1$, $\Delta\,\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}^B(\alpha^\sharp, \cdot)$, where $\alpha^\sharp$ is the metric dual to α. Well,

$$\begin{aligned} \Delta\,\alpha = 0 \Rightarrow 0 &= \int_M \langle \Delta\,\alpha, \alpha\rangle \, d\,\mathrm{Vol} \\ \Rightarrow 0 &= \underbrace{||\Delta\,\alpha||^2}_{=\langle \nabla^*\nabla\alpha, \alpha\rangle_{\ell^2}} + \int_M \mathrm{Ric}^B(\alpha^\sharp, \alpha^\sharp) d\,\mathrm{Vol} \\ &\Rightarrow \nabla\alpha = 0 \text{ and } \mathrm{Ric}(\alpha^\sharp) = 0 \\ &\Rightarrow \alpha \text{ is parallel-determined by any value } \alpha_p \in T^*_pM \end{aligned}$$