Math 635 Lecture 35

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Degree Theory

Given $F : M_1 \to M_2$, where M_1 and M_2 are compact, connected, oriented manifolds of dimension $n = \dim M_1 = \dim M_2$. Then $\exists \deg(F) \in \mathbb{Z}$ such that

a) $F^* : H^n(M_2) \to H^n(M_1)$ is "multiplication by $\deg F$".
b) $\forall a \in \Omega^n(M_2)$, $\int_{M_1} F^*\alpha = \deg(F) \int_{M_2} \alpha$.
c) If $q \in M_2$ is a regular value of F, so $F^{-1}(q) = \bigcup_{i=1}^N \{p_i\}$, then $\deg(F) = \sum_{i=1}^N (-1)^{p_i}$, where

$$(-1)^{p_i} = \begin{cases} 1 & dF_p \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

Application:

Thm: If $M_1 = \partial W$, and $f : M_1 \to M_2$ extends to a smooth function $\tilde{F} : W \to M_2$, then $\deg(F) = 0$.

$$\begin{tikzcd} M_1 \arrow[r, "F"] \arrow[d, hook, "\iota"'] & M_2 \\ W \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

Proof: Let $\alpha \in \Omega^p(M_2)$. Then

$$\int_{M_1} F^*\alpha = \int_{M_1} \iota^*\tilde{F}^*\alpha \overset{(1)}{=} \int_W d\tilde{F}^*\alpha = \int_W \tilde{F}^* d\alpha = 0$$

because α is a top-degree form, and (1) because of Stokes' theorem. $\square$

Cor: There is no C^∞ map $\overline{B}^n \to S^n$ (where $\overline{B}^n$ is the closed unit ball in $\mathbb{R}^n$) which is the identity on $S^n = \partial\overline{B}^n$.

Proof: $\deg(I_{S^n}) = 1 \neq 0$. $\square$

(We presented some examples of Gauss-Bonnet and degree theory, but I'm not yet talented enough with TikZ to reproduce them.)

The Laplacian and Hodge Theory

Let (M, g) be a Riemannian manifold.

Defn: The gradient is

$$\begin{aligned} \nabla : C^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto \nabla f \end{aligned}$$

where ∇f is metric-dual to df. That is, $\forall v \in T_pM$, $\langle \nabla f(p), v\rangle = df_p(v)$.

∇ has a product rule: $\nabla(fg) = f\nabla g + g\nabla f$.

Defn: The divergence $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ is defined by $\forall X \in \mathfrak{X}(M)$, $\mathcal{L}_X \operatorname{Vol} = (\operatorname{div} X)\operatorname{Vol}$, where $\operatorname{Vol} \in \Omega^n(M)$ is a volume form.

Observe: This is really a local definition, and div is independent of orientation.

Properties:

(a) In coordinate, $X = f^i \frac{\partial}{\partial x^i}$, then $\operatorname{div} X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} f^i)$. (Note: we're using the shorthand $\sqrt{g} = \sqrt{\det(g_{ij})}$).
(b) $\operatorname{div}(fX) = f \operatorname{div} X + X(f)$.
(c) $(\operatorname{div} X)(p) = \operatorname{tr}(T_pM \ni v \mapsto (\nabla_v X)(p) \in T_pM)$.

Thm: (Divergence Theorem) Let M be a manifold-with-boundary, with the boundary ∂M oriented by ν, an outward point unit normal vector field so that $\operatorname{Vol}_{\partial M} = \iota_\nu \operatorname{Vol}_M$. Then $\forall X \in \mathfrak{X}(M)$,

$$\int_M (\operatorname{div} X) \operatorname{Vol}_M = \int_{\partial M} \langle X, \nu \rangle \operatorname{Vol}_{\partial M}$$

The right-hand side is the flux of X out of M.

Cor: $(\operatorname{div} X)(p) = \lim_{\varepsilon \searrow 0} \frac{1}{\operatorname{Vol} B_\varepsilon(p)} \int_{B_\varepsilon(p)} (\operatorname{div} X) d\operatorname{Vol}_M \overset{\text{Thm}}{=} \lim_{\varepsilon \searrow 0} \frac{1}{\operatorname{Vol} B_\varepsilon(p)} \int_{S_\varepsilon(p)} \langle X, \nu \rangle \operatorname{Vol}_{S_\varepsilon(p)}$
"$(\operatorname{div} X)(p)$ is the infinitesimal flux per unit volume at p."

Observe: If $\partial M = \emptyset$, then we get $\int_M (\operatorname{div} X) \operatorname{Vol}_M = 0$. Proof: The left-hand side is simply

$$\int_M \mathcal{L}_X \operatorname{Vol}_M = \int_M d(\iota_X \operatorname{Vol}_M) = \int_{\partial M} j^*(\iota_X \operatorname{Vol}_M)$$

where $j : \partial M \hookrightarrow M$. Then, we check that $j^*(\iota_X \operatorname{Vol}_M) = \langle X, \nu \rangle \operatorname{Vol}_{\partial M}$. Use that $X = \langle X, \nu \rangle + \eta$, where η is tangent to ∂M. $\square$

Defn: The <u>laplacian</u> on functions is

$$\begin{aligned} \Delta : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto \Delta f = -\operatorname{div}(\nabla f) \end{aligned}$$

Note that some people don't use the negative sign.

Combining the previous formulas, in coordinates, we get

$$\begin{aligned} \Delta f &= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) \\ &= -\frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \text{lower order terms} \right) \\ &= -g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \text{lower order terms} \end{aligned}$$

So the highest order term only depend son the metric, not derivatives of the metric.

Next time, we'll consider the following commutative diagram:

$$\begin{tikzcd} C^\infty(M) \arrow[r, shift left, "\nabla"] \arrow[d, equal] & \mathfrak{X}(M) \arrow[l, shift left, "-\operatorname{div}"] \arrow[d, "\wr", "\text{metric dual}"'] \\ C^\infty(M) \arrow[r, shift left, "d"] & \Omega^1(M) \arrow[l, shift left, "\delta"] \end{tikzcd}$$

We'll see that $\delta = d^*$ (the adjoint) if we use ℓ^2 inner products on $C^\infty(M)$ and $\Omega^1(M)$. On functions, $\Delta = \delta d = d^* d$.