Math 635 Lecture 27

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Continuing from last time...

Thm: Let $q = \exp_p(t, v)$, $||v|| = 1$ be a conjugate point of p. Then $\forall t_2 > t_1$, $t \mapsto \exp_p(tv)$ is not minimizing on $[0, t_2]$.

Proof: By the hypothesis, there's a Jacobi field J f γ such that $J \neq 0$, $J(0) = 0$, and $J(t_1) = 0$. We will construct a variation of γ on $[0, t_2]$ with $E'' < 0$. Define

$$\tilde{J}(t) = \begin{cases} J(t) & 0 \leq t \leq t_1 \\ 0 & t_1 \leq t \leq t_2 \end{cases}$$

Because $J(t_1) = 0$, this variation is continuous at t_1, so it's clearly continuous on $[0, t_2]$. Let $W \in \Gamma_\gamma(TM)$ be smooth, supported near t_1, and defined such that $W(t_1) = \Delta\tilde{J}'(t_1) \neq 0$. It's nonzero because $\Delta\tilde{J}'(t_1) = 0$ would imply that $J'(t_1) = 0$, which would mean $J = 0$, a contradiction with our original assumption. Now, we define the actual variation we're going to use. Let

$$V_\varepsilon = \tilde{J} + \varepsilon W$$

for some small $0 < \varepsilon \ll 1$. This is a proper variation of γ on $[0, t_2]$. Now compute $E''(0)$ (associated with V_ε).

$$E''(0) = I(V_\varepsilon, V_\varepsilon) = I(\tilde{J} + \varepsilon W, \tilde{J} + \varepsilon W) = I(\tilde{J}, \tilde{J}) + 2\varepsilon I(\tilde{J}, W) + \varepsilon^2 I(W, W)$$

where I is the bilinear form defined previously. Well,

$$I(\tilde{J}, \tilde{J}) = -\int_0^{t_2} \left\langle \tilde{J}, \cancelto{0}{\text{Jacobi operator on } \tilde{J}} \right\rangle - \left\langle \underbrace{\tilde{J}(t_1)}_{=0}, \Delta\tilde{J}'(t_1) \right\rangle dt = 0$$

$$I(\tilde{J}, W) = \int_0^{t_2} \left\langle \tilde{J}', W \right\rangle - \left\langle \mathcal{R}(\tilde{J}, \dot{\gamma})\dot{\gamma}, W \right\rangle dt$$

We use integration by parts, with $\frac{d}{dt}\left\langle W, \tilde{J}' \right\rangle = \left\langle W', \tilde{J}' \right\rangle + \left\langle W, \tilde{J}'' \right\rangle$, to compute

$$\int_0^{t_2} \left\langle \tilde{J}', W \right\rangle dt = -\int_0^{t_2} \left\langle W, \tilde{J}'' \right\rangle dt - \left\langle W(t_1), \Delta\tilde{J}'(t_1) \right\rangle$$

Now, we combine with the $\left\langle \mathcal{R}(\tilde{J}, \dot{\gamma})\dot{\gamma}, W \right\rangle$ term. Using the fact that $\tilde{J}$ satisfies the Jacobi equation, they cancel, and we're left with

$$I(\tilde{J}, W) = -\left\langle W(t_1), \Delta\tilde{J}'(t_1) \right\rangle = -\left|\left|\Delta\tilde{J}'(t_1)\right|\right|^2 < 0$$

Thus, $E''(0) = \varepsilon^2 I(W, W) - 2\varepsilon\left|\left|\Delta\tilde{J}'(t_1)\right|\right|^2$. So for $\varepsilon \ll 1$, $E''(0) <$, so for s small enough, t-curves in a variation of γ with $\tilde{V}_\varepsilon$ are shorter than γ. $\square$

Completeness

(Chpater 7 in Do Carmo)

Defn: M is geodesically complete iff $\forall p \in M$, $\exp_p$ is defined on all of T_pM.

Ex: If M is compact, M is geodesically complete.

Why? Well, if M is compact, then the unit tangent bundle $TM_{\mathbb{1}} = \{(p, v) \in TM : ||v|| = 1\}$ is compact. So geodesic flow is given by the flow of a certain field on $TM_{\mathbb{1}}$ (up to scaling by time), and smooth vector fields on compact manifolds are complete. $\square$

Defn: M is complete iff (M, d) is a complete metric space.

The **big idea** we're working towards is
Thm: (Hopf-Rinow) M is geodesically complete iff M is a complete metric space.

Thm: Let M be connected. Let $p \in M$ such that $\exp_p$ is defined on all of T_pM. Fix $q \in M$. Then there's a geodesic γ from p to q, and $d(p, q) = \ell(\gamma)$.

Proof: Let $\varepsilon > 0$ be such that there's a geodesic sphere S_ε of radius ε centered at p. Let $p' \in S_\varepsilon$ be a point minimizing the map

$$\begin{aligned} S_\varepsilon &\to \mathbb{R} \\ x &\mapsto d(x, q) \end{aligned}$$

That is, p' is the point on S_ε which is closest to q. By compactness, p' exists, and $p' = \exp_p(\varepsilon v)$ for some $v \in T_pM$ with $||v|| = 1$. Now, we want to show $\exp_p(d(p, q)v) = q$...

Lemma: $d(p, q) = \underbrace{d(p, p')}_{=\varepsilon} + d(p', q)$.

Proof: $\leq$ is just a direct application of the triangle inequality. For $\geq$, let c be any path from p to q, and let w be the point where c intersects S_ε. Then $\ell(c) = \ell(\widehat{pw}) + \ell(\widehat{wq}) \geq \varepsilon + d(p', q)$. Now, take the infimum over all such paths c. We get

$$d(p, q) = \inf_c \ell(c) \geq \varepsilon + d(p', q) = d(p, p') + d(p', q)$$

$\square$

Returning to the proof of the theorem, introduce $\mathscr{T} \overset{\text{def}}{=} \{t \in [0, d(p, q)] \mid d(p, q) = t + d(\gamma(t), q)\}$. We observe the following facts about $\mathscr{T}$:

- $\mathscr{T} \neq \emptyset$ because $\varepsilon \in \mathscr{T}$ by the lemma.
- $\mathscr{T}$ is closed, because it's the preimage of a closed set under a continuous function.
- $\forall t \in \mathscr{T}$, $d(\gamma(t), p) = t$.

We want to show $d(p, q) = \sup \mathscr{T}$. We will argue this by contradiction: assume $t_1 \overset{\text{def}}{=} \sup \mathscr{T} < d(p, q)$. Then $t_1 + \delta < d(p, q)$. S_δ exists centered at $\gamma(t)$, so then we'll show $t + \delta \in \mathscr{T}$, thus contradicting the definition of t_1 as the supremum of $\mathscr{T}$. We will do this next time.