Math 635 Lecture 9

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An Example of Connections and Parallelism

Take $M = S^1$, with the coordinate $0 \leq x \leq 2\pi$. Let $\mathcal{E} = M \times \mathbb{R}^2 = S^1 \times \mathbb{R}^2$, with the global frame $(E_1 = (1,0), E_2 = (0,1))$, $\forall x \in S^1$. Then elements of $\Gamma(\mathcal{E})$ are functions $\vec{f}(x) = (f^1(x), f^2(x))$.

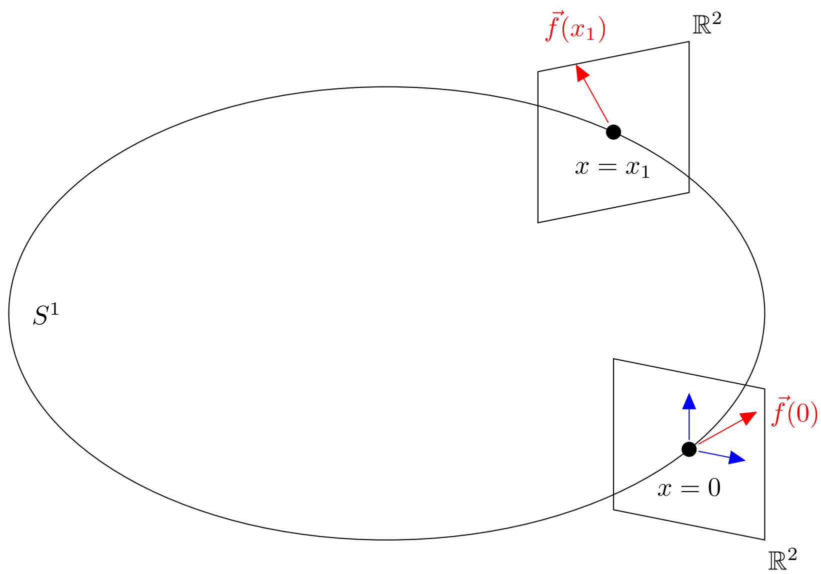
Suppose ∇ is given by ϑ, any 2×2 matrix of 1-forms on M. So $\theta_i^j = \alpha_i^j dx$, for $\alpha_i^j \in C^\infty(S^1)$. Thus,

$$\nabla_{\frac{\partial}{\partial x}} \vec{f} = \underbrace{\begin{pmatrix} \frac{df^1}{dx} \\ \frac{df^2}{dx} \end{pmatrix}}_{= \frac{d}{dx}\vec{f}} + \underbrace{\left(\alpha_i^j\right)}_{\text{because } dx\left(\frac{\partial}{\partial x}\right) = 1} \vec{f}$$

Consider the simple case $\vartheta = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, for some $c \in \mathbb{R} \setminus \{0\}$. Then

$$\vec{f} \text{ is parallel} \quad \Leftrightarrow \quad \nabla_{\frac{\partial}{\partial x}} \vec{f} = 0 \quad \Leftrightarrow \quad \begin{cases} \frac{df^1}{dx} = cf^2 \\ \frac{df^2}{dx} = -cf^1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} f^1(x) = f^1(0)\cos(cx) + f^2(0)\sin(cx) \\ f^2(x) = -f^1(0)\sin(cx) + f^2(0)\cos(cx) \end{cases}$$

This final object is the solution to the parallel transport problem, starting at $x = 0$ with $\left(\begin{smallmatrix} f^1(0) \\ f^2(0) \end{smallmatrix}\right) = \vec{f}(0)$.



Additionally, we can map from $\vec{f}(0)$ to $\vec{f}(2\pi)$ with the holonomy matrix:

$$\begin{pmatrix} \cos(2\pi c) & -\sin(2\pi c) \\ \sin(2\pi c) & \cos(2\pi c) \end{pmatrix}$$

Thus, the holonomy is trivial (i.e., the holonomy matrix is the identity matrix I_2) iff $c \in \mathbb{Z}$.

Exer: Compute $\nabla_{\frac{\partial}{\partial x}} E_1$ and $\nabla_{\frac{\partial}{\partial x}} E_2$. Note: It won't be 0!

Curvature of a Connection on a Bundle

Say ∇ is a connection on the vector bundle $\mathcal{E} \to M$. Curvature is a local object, and there are two approaches to describe it.

First Approach

Begin with a moving frame $(E_1, \ldots, E_r)$ (which leads to the connection matrix $\vartheta = (\theta_i^j)$).

Question: Can we change the moving frame to $(F_1, \ldots, F_r)$ so that the new connection matrix, $\tilde{\vartheta}$, is 0?

Well, if $A = (a_j^i)$ is defined so that $F_j = a_j^i E_i$ (using Einstein summation notation), then we saw (in homework) that

$$\tilde{\vartheta} = A^{-1} dA + A^{-1} \vartheta A$$

So

$$\tilde{\vartheta} = 0 \quad \Leftrightarrow \quad dA = -\vartheta A \quad \Leftrightarrow \quad \forall i, j, da_j^i = -\theta_k^i a_j^k$$

We suspect that we'll encounter problems from the fact that $d^2 = 0$. Assume such a solution A exists. Then, using the product rule, $0 = -(d\theta_k^i) a_j^k + \theta_k^i \wedge da_j^k$. But we know $da_j^k = -\theta_\ell^k a_j^\ell$, so for the connection matrix to be trivial, we must have

$$(d\theta_k^i) a_j^k + \theta_k^i \wedge \theta_\ell^k a_j^\ell = 0, \quad \forall i, j$$

Note: $\theta_k^j \wedge \theta_\ell^k = (\vartheta \wedge \vartheta)_\ell^j$ by definition. So using matrix notation, we can write this necessary condition as

$$(d\vartheta) A + (\vartheta \wedge \vartheta) A = 0$$

A is invertible, so this condition is true iff $d\vartheta + \vartheta \wedge \vartheta = 0$.

Defn: $\Omega \stackrel{\text{def}}{=} d\vartheta + \vartheta \wedge \vartheta$ is a matrix of 2-forms, called the <u>curvature matrix</u> of ∇ with respect to the moving frame $(E_1, \ldots, E_r)$.

The connection is trivializable only if $\Omega = 0$ (i.e. this is a necessary condition). There is no reason in principle this should happen, so it's very rare!

Second Approach

Fix $X, Y \in \mathfrak{X}(M)$. Consider $\nabla_X, \nabla_Y : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ as operators, acting linearly on smooth sections.

Question: What is their commutator (as operators)?

Well, if $\mathcal{E} = M \times \mathbb{R}^r$ and $\vartheta \equiv 0$, then $\nabla_X \vec{f} = X(f) + 0$. So $\nabla_X = X$. Therefore, $[\nabla_X, \nabla_Y] = [X, Y] = \nabla_{[X,Y]}$, by definition of the commutator of vector fields.

In general, let $E_1, \ldots, E_r$ be a moving frame, and $\vartheta = (\theta_i^j)$ the connection matrix as before. Then $\nabla_X = X + \vartheta(X)$, acting on functions $\vec{f} : U \to \mathbb{R}^r$ ($\vartheta(X)$ is a matrix of functions). Likewise, $\nabla_Y = Y + \vartheta(Y)$. So we can compute:

$$[\nabla_X, \nabla_Y] = [X + \vartheta(X), Y + \vartheta(Y)] = [X, Y] + [\vartheta(X), \vartheta(Y)] + \underbrace{[X, \vartheta(Y)]}_{\text{(I)}} + \underbrace{[\vartheta(X), Y]}_{\text{(II)}}$$

For (I), we have

$$\Big([X, \vartheta(Y)]\,(\vec{f})\Big)^i = X\,\underbrace{(\theta_j^i(Y) f^j)}_{\substack{i\text{th component} \\ \text{of } \vartheta(Y)(\vec{f})}} - \theta_j^i(Y) X(f^j) = (X\theta_j^i(Y)) f^j$$

by Leibniz's rule. So (I) is $X((\theta_j^i)(Y)) = X(\vartheta(Y))$ as a matrix. Similarly, (II) is $-Y((\theta_j^i)(X)) = -Y(\vartheta(X))$ as matrix.

Claim: This eventually leads to $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + d\vartheta(X, Y) + [\vartheta(X), \vartheta(Y)]$.
Proof: Use the fact that for any 1-form ω, and for any $X, Y \in \mathfrak{X}(M)$, $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

Observe: $[\vartheta(X), \vartheta(Y)] = (\vartheta \wedge \vartheta)(X, Y)$, by definition of $\wedge$. So we get the same object using both approaches!

Defn: $\mathcal{R}(X, Y) \stackrel{\text{def}}{=} d\vartheta(X, Y) + [\vartheta(X), \vartheta(Y)]$ is the <u>curvature operator</u>.