Math 635 Lecture 15

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Some review:

- For (X, ω) a symplectic manifold, given $H \in C^\infty(X)$, $\exists! \Xi_H \in \mathfrak{X}(X)$ s.t. $\iota_{\xi_H}\omega = -dH$. Refer to Lee, Smooth Manifolds, Chapter 22. Warning: His ω is different from ours by a sign. But the ξ_H is the same.
- The flow of ξ_H preserves H, ω, $\frac{\omega^n}{n!}$, and the Liouville volume on regular level sets of H.
- Symmetries: We proved that $\forall G, H \in C^\infty(X)$, $[\Xi_G, \Xi_H] = 0$ iff $\omega(\Xi_G, \Xi_H) = \Xi_G(H) = -\Xi_H(G)$ is constant. We call this the Poisson bracket, and denote it by $\omega(\Xi_G, \Xi_H) = \{G, H\}$.

Defn: We say that G is a conserved quantity under the flow of Ξ_H if and only if G is constant along the trajectories of Ξ_H if and only if $\Xi_H(G) = 0$.

Observe: The above implies that G is a conserved quantity under the flow of Ξ_H if and only if $\{G, H\} = 0$, so Ξ_G preserves H if and only if Ξ_H preserves G.

We'll apply this as follows:

- Geodesic flow is a Hamilton flow for some $H : T^*M \to \mathbb{R}$.
- If we have a field on M generating isometries, we'll get a G on T^*M that preserves H. This implies that G is constant along geodesics.

Recall: If M is any C^∞ manifold, then $X = T^*M$ has a natural symplectic form ω, which, in standard local coordinates on T^*M, takes the form $\omega = dp_i \wedge dx^i = d\alpha$, where $\alpha = \sum p_i dx^i$ is the tautological 1-form.

A class of examples of Hamiltonians on $T*M$: Start with $X \in \mathfrak{X}(M)$. Then define

$$\begin{aligned} \ell_X : T^*M &\to \mathbb{R} \\ (x, \xi) &\mapsto \xi(X_x) \end{aligned}$$

Observe that $\ell_X(x, \xi)$ is linear in ξ, i.e., linear on the fibers.

In coordinates $(x^1, \dots, x^n)$, $X = f^i \frac{\partial}{\partial x^i}$, we get $(x^1, \dots, x^n, p_1, \dots, p_n)$ coordinates on T^*U. Then

$$\ell_X(x^1, \dots, x^n, p_1, \dots, p_n) = p_i f^i(X)$$

Recall Hamilton's equations for the flow of $\Xi_{\ell_X} \in \mathfrak{X}(T^*M)$:

$$\begin{cases} \dot{x}^i = \frac{\partial \ell_X}{\partial p_i} = f^i & \leftrightarrow \text{exactly the flow of } X \text{ itself} \\ \dot{p}^i = -\frac{\partial \ell_X}{\partial x^i} = -p_j \frac{\partial f^j}{\partial x^i} \end{cases}$$

Prop: Let $\phi_t : M \to M$ be the flow of X. Then the flow of Ξ_{ℓ_X} is $\tilde{\phi}_t : T^*M \to T^*M$, given by

$$\tilde{\phi}_t(x, \xi) = \big(\phi_t(x), ((d\phi_{-t})_{\phi_t(x)})^*\xi\big)$$

Let's unpack this. We know

$$d(\phi_{-t})_{\phi_t(x)} : T_{\phi_t(x)}M \to T_xM$$

Thus, its pullback is a map $T_x^* \to T_{\phi_t(x)}^*M$. So we have the following commutative diagram:

$$\begin{array}{ccc} T^*M & \xrightarrow{\tilde{\phi}_t} & T^*M \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi_t} & M \end{array}$$

Our main example is when (M, g) is a Riemannian manifold. Using g, $\forall x \in M$, we get $T_xM \cong T_x^*M$ by $\mathbb{F} : v \mapsto \langle \cdot, v\rangle$. We can assemble this into a "big map" between the total spaces of the bundle:

$$\begin{tikzcd} TM \arrow[rr, "\sim", "\mathbb{F}"'] \arrow[dr] & & T^*M \arrow[dl] \\ & M & \end{tikzcd}$$

by computing $\mathbb{F}$ fiberwise. In coordinates, let $(x^1, \dots, x^n)$ be coordinates on $U \overset{\text{open}}{\subset} M$. Then we get coordinates on TU and T^*U, and $\mathbb{F}$ is

$$\mathbb{F}(x^1, \dots, x^n, v^1, \dots, v^n) = (x^1, \dots, x^n, p_i = g_{ij}v^j)$$

Thm: Let $\begin{aligned} L : TM &\to \mathbb{R} \\ (x, v) &\mapsto \tfrac{1}{2} \|v\|^2 \end{aligned}$. Define $H : T^*M \to \mathbb{R}$ by $H = L \circ \mathbb{F}^{-1}$.

$$\begin{tikzcd} & TM \arrow[r, "\sim", "\mathbb{F}"'] \arrow[dl, "L"'] \arrow[dr] & T^*M \arrow[l, bend right, "\mathbb{F}^{-1}"'] \arrow[d] \arrow[dll, bend right=60, "H=L\circ\mathbb{F}^{-1}"'] \\ \mathbb{R} & & M \end{tikzcd}$$

Then $\mathbb{F}$ intertwines the geodesic flow on TM with the Hamilton flow of H.

Cor: Geodesic flow is volume preserving, and we can use the Hamiltonian to study symmetries and other things.

We could prove this now, but it would require a terribly long and boring computation. There's an elegant proof of this using a different point of view, which we will do later on.

Observe: $H(x, p) = \frac{1}{2} g^{ij}(x) p_i p_j$. $\dot{x}^i = \frac{\partial H}{\partial p_i} = g^{ij}(x) p_j$.

Application: Surfaces of revolution. Let $S = \partial_\Theta \in \mathfrak{X}(M)$ generate rotations: $\phi_t : M \to M$ where $\phi_{t+2\pi} = \phi_t$, and $\forall t$, ϕ_t is an isometry. (Compare with problem 1 on page 77 of Do Carmo.)

On the cotangent bundle, $\tilde{\phi}_k : T^*M \to T^*M$ preserves H; therefore, $\ell_{\partial_\Theta} : T^*M \to T^*M$ is a conserved quantity. But what is it geometrically? We can pass it to TM:

$$\begin{aligned} \ell_{\partial_\Theta} \circ \mathbb{F} : TM &\to \mathbb{R} \\ (x, v) &\mapsto \langle v, \partial_\Theta \rangle \end{aligned}$$

where $\langle\ ,\ \rangle$ is the Euclidean dot product (from the subspace-induced Riemannian metric). Again, let $\gamma(t)$ be a geodesic with speed 1. We know that

$$\langle \dot{\gamma}(t), \partial_\Theta \rangle = ||\partial_\Theta|| \cos \angle(\dot{\gamma}(t), \partial_\Theta)$$

is independent of t. It turns out $||\partial_\Theta|| = r$, the distance to the axis of symmetry, since the line of latitude has perimeter $2\pi \, ||\partial_\Theta||$. So we conclude that along a speed-1 geodesic, $\gamma(t) \cos \angle(\dot{\gamma}(t), \partial_\Theta)$ is independent of t.