Math 635 Lecture 1

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

1/20/21

The overall goal of this course is to build a notion of distance on a manifold. But in order to have such a notion, we need additional structure. Specifically, we need a Riemannian metric: a C^∞ assignment of an inner product to each tangent space T_pM, $\forall p \in M$.

Ex: Say $S \subset \mathbb{R}^N$ is a submanifold. $\forall p \in S$, $T_pS \hookrightarrow \mathbb{R}^N$, so it inherits the Euclidean inner product of $\mathbb{R}^N$. We can use this to define the length of a curve $\gamma : (a, b) \to S$ – $\int_a^b ||\dot{\gamma}(s)||\ ds$, where $||\dot{\gamma}|| = \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}$.

For a systematic study of these ideas, we need to review:

- Vector bundles
- Connections on vector bundles

The basic idea of a connection is that a Riemannian metric allows us to talk about parallel transport of vectors along curves. Specifically, a connection allows us to differentiate sections of a vector bundle along curves in the base.

Vector Bundles (Review)

Defn: A <u>vector bundle</u> (of rank r) over M is a surjective submersion $\pi : \mathcal{E} \to M$ s.t.

a) For each $p \in M$, the fiber $\mathcal{E}_p \overset{\text{def}}{=} \pi^{-1}(p)$ is a vector space (of rank r).

b) There's an (open[1]) covering $\{U_\alpha\}$ of M, and diffeomorphisms $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ called <u>local trivializations</u>, such that

   i) $\forall \alpha$, the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

   ii) $\forall p \in U,\ \chi_\alpha|_{\mathcal{E}_p} : \mathcal{E}_p \to \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$ is a linear isomorphism.

Colloquially, we think of $\mathcal{E}$ as a family $\{\mathcal{E}_p\}$ of vector spaces, parameterized by M.

Ex: TM and T^*M are examples of vector bundles.

Defn: Given a vector bundle $\pi : \mathcal{E} \to M$, let $\Gamma(\mathcal{E}) \overset{\text{def}}{=} \{s : M \to \mathcal{E} \mid s \text{ smooth and } \pi \circ s \equiv I_M\}$, the set of mooth sections of $\mathcal{E}$.

Defn: The <u>zero section</u> is s s.t. $\forall p \in M$, $s(p) = 0 \in \mathcal{E}_p$.

Ex: If $\mathcal{E} = TM$, then $\mathcal{E}_p = T_pM$, so
- $\Gamma(TM) = \mathfrak{X}(M)$, the set of all smooth vector fields on M.
- $\Gamma(T^*M)$ is the set of all smooth 1-forms on M.

Observe: $\Gamma(\mathcal{E})$ is a module over $C^\infty(M)$ – $\forall s, t \in \Gamma(\mathcal{E}), f \in C^\infty(M)$, $(fs + t)(p) = f(p)s(p) + t(p)$.

Defn: A <u>moving frame</u> is a collection of sections $(E_1, \ldots, E_r)$ over U s.t. $\forall p \in U$, $(E_1(p), \ldots, E_r(p))$ is a basis of $\mathcal{E}_p$.

[1] In the future, assume that any covering is open, unless explicitly stated otherwise.

Observe: Using our notion of modules, a local trivialization $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ defines a moving frame on U. If $(x^1, \dots, x^n)$ are coordinates on $U \subseteq M$, then $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ is a moving frame of TM on U.

Given χ, define $E_j(p) \in \mathcal{E}_p$ such that $\chi(p, E_j(p)) = (p, \langle 0, \dots, 1, \dots, 0\rangle)$ (with a 1 in the jth entry). Then we have

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, shift left, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[ll, shift left, "\chi^{-1}"] \arrow[dl] \\
& U \arrow[ur, bend right, "p \mapsto (p, \langle 0, \dots, 1, \dots, 0\rangle)"'] &
\end{tikzcd}$$

Then $\forall s \in \Gamma(\mathcal{E}|_U)$, $\exists f_j : U \to \mathbb{R}$ such that $\forall p \in U$,

$$s(p) = \sum_{j=1}^{r} f_j(p) E_j(p)$$

This is true for any section $s : U \to \pi^{-1}(U) = \mathcal{E}|_U$, and, in fact, these f_j's are unique, simply because $s(p) \in \mathcal{E}_p$, and $\{E_j(p)\}$ is a basis of $\mathcal{E}_p$.

Lemma: s is smooth iff $\forall j$, f_j is smooth.

Defn: The trivial bundle of rank r is $\pi : M \times \mathbb{R}^r \to M$.

Operations with Vector Bundles

Principle: Any operation or construction that one can do with vector spaces, that is natural with respect to linear isomorphisms, can be extended to vector bundles, by doing such an operation or construction fiber-wise.

Ex:
- $V \rightsquigarrow V^*$
- $V, W \rightsquigarrow V \oplus W$
- $V, W \rightsquigarrow V \otimes W$
- $V, W \rightsquigarrow \mathrm{Hom}(V, W)$
- $V \subset w \rightsquigarrow W/V$

Idea: Given any two bundles $\mathcal{E}, \mathcal{F} \to M$ over M, we can form the "Whitney direct sum", $\mathcal{E} \oplus \mathcal{F} \to M$, such that $\forall p \in M$, $(\mathcal{E} \oplus \mathcal{F})_p = \mathcal{E}_p \oplus \mathcal{F}_p$. This already defines what $\mathcal{E} \oplus \mathcal{F}$ has to be as a set:

$$\mathcal{E} \oplus \mathcal{F} = \bigsqcup_{p \in M} \mathcal{E}_p \oplus \mathcal{F}_p$$

The question is: how do we give such a set the topology and C^∞ structure of a vector bundle?

Math 635 Lecture 2

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

1/22/21

Lemma: Let M be a C^∞ manifold. Assume $\forall p \in M$, we have a vector space $\mathcal{E}_p$ (of dimension r). Let $\mathcal{E} = \bigsqcup_{p \in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \to M$ the natural projection, where $\mathcal{E}_p \mapsto p \in M$. Assume we're given $\{U_\alpha\}$, a cover of M, plus bijections χ_α such that $\forall \alpha$, the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

i.e. $\chi_\alpha(\mathcal{E}_p) = \{p\} \times \mathbb{R}^r$.

Observe that this gives us $\forall \alpha, \beta$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$, a map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$, by:

$$\begin{tikzcd} & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\chi_\alpha"'] \arrow[dr, "\chi_\beta"] & \\ (U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[rr, "\chi_\beta \circ \chi_\alpha^{-1}"] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\ (p, v) \arrow[rr, mapsto] & & (p, \tau_{\alpha,\beta}(p) v) \end{tikzcd}$$

$$p \in U_\alpha \cap U_\beta$$
$$v \in \mathbb{R}^r$$

This mapping is a "change of trivialization", like a transition map for vector bundles. The matrix of $\tau_{\alpha,\beta}$ depends on p, but it's always a linear map.

If, $\forall \alpha, \beta$, $\tau_{\alpha,\beta}$ is C^∞, then there is a unique topology on $\mathcal{E}$, and a unique differentiable structure, that makes $\pi : \mathcal{E} \to M$ a smooth vector bundle, and each χ_α is a diffeomorphism, i.e., a smooth local trivialization.

Observe: On triple intersections, $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall p$, one has $\tau_{\alpha,\beta}(p)\tau_{\beta,\gamma}(p) = \tau_{\alpha,\gamma}(p)$. This is called a "cocycle condition".

Now, imageine starting with a cover $\{U_\alpha\}$ and C^∞ maps $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ satisfying the cocycle condition. Then, if we choose, $\forall p$, $\mathcal{E}_p = \mathbb{R}^r$, and $\chi_\alpha = I_n$, then we get a vector bundle $\mathcal{E} \to M$. So $\{U_\alpha; \tau_{\alpha,\beta}\}$, called a Čech cocycle, is all we need to put together a vector bundle.

Now, we apply the lemma to construct new bundles from old ones.

Ex: Given $\mathcal{E}', \mathcal{E}'' \to M$ vector bundles, with ranks r' and r'', respectively, define, $\forall p \in M$, $(\mathcal{E}' \oplus \mathcal{E}'')_p = \mathcal{E}'_p \oplus \mathcal{E}''_p$. Consider a cover $\{U_\alpha\}$ of M, and local trivializations for both $\mathcal{E}'$ and $\mathcal{E}''$. We get, $\forall \alpha$,

$$\chi'_\alpha : (\pi')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r'}$$
$$\chi''_\alpha : (\pi'')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r''}$$

Now define $\pi : \mathcal{E}' \oplus \mathcal{E}'' \to M$, and

$$\begin{aligned} \chi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''}) \\ (p, (v', v'')) &\mapsto (p, (\chi'_\alpha(v'), \chi''_\alpha(v''))) \\ v' &\in \mathcal{E}'_p \\ v'' &\in \mathcal{E}''_p \end{aligned}$$

We have to check that the $\tau_{\alpha,\beta}$ are smooth. Note that $\tau_{\alpha,\beta}(p)$ will be a block diagonal matrix, and the two diagonal blocks vary smoothly, since χ_α and χ_β are diffeomorphisms. Thus, $\mathcal{E}' \oplus \mathcal{E}'' \to M$ is a natural bundle. This is called the <u>Whitney direct sum</u> or <u>Whitney sum</u>.

Observe:

- $\text{rank}(\mathcal{E}' \oplus \mathcal{E}'') = \text{rank}(\mathcal{E}') + \text{rank}(\mathcal{E}'')$
- $\dim(\mathcal{E}' \oplus \mathcal{E}'') = r' + r'' + n$, where n is the dimension of the total space.

Similarly, we can define $\mathcal{E}' \otimes \mathcal{E}'' \to M$ with fibers $(\mathcal{E}' \otimes \mathcal{E}'')_p = \mathcal{E}'_p \otimes \mathcal{E}''_p$. $\text{rank}(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r''$, and $\dim(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r'' + n$.

Review of Tensor Products

Defn: Let V, W be finite-dimensional vector spaces. Their tensor product $V \otimes W$ is the free vector space over $V \times W$, modulo an equivalence relation, $\sim$. The free vector space over $V \times W$ is the set of all formal finite linear combinations of pairs $(v, w) \in V \times W$, and $\sim$ is defined such that

$$\begin{cases} (v_1 + v_2, w) \sim (v_1, w) + (v_2, w) \\ (\lambda v, w) \sim \lambda(v, w) \end{cases}$$

Notation: $\forall (v, w) \in V \times W$, $v \otimes w = [(v, w)]$.

Claim: If $(e_1, \dots, e_k)$, $(f_1, \dots, f_\ell)$ are bases of V, W, then $\{e_i \otimes f_j \mid i \in \{1, \dots, k\}, j \in \{1, \dots, \ell\}\}$ is a basis of $V \otimes W$.

Cor: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

The universal property of $V \otimes W$: We have a bilinear map

$$\begin{aligned} V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

It's "universal" in that for any bilinear map, there's a unique linear map such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[rr, "(v,w)\mapsto v\otimes w"] \arrow[dr, "\text{bilinear map}"'] & & V \otimes W \arrow[dl, "\text{linear map}"] \\ & Z & \end{tikzcd}$$

There are other realizations of $V \otimes W$.

$$V \otimes W \cong \text{Hom}(V^*, W) \qquad \text{where} \qquad v \otimes w \mapsto \begin{pmatrix} V^* & \to & W \\ \alpha & \mapsto & \alpha(v)w \end{pmatrix}$$

This is completely natural – we don't need to choose a basis.

Note: This isomorphism, as it's currently written, actually doesn't work! In an exercise, we prove that most tensors in $V \otimes W$ cannot be written as $v \otimes w$ for any $v \in V, w \in W$ (i.e. most tensors are not "pure tensors"). Rather, they have to be written as linear combinations of pure tensors. So it would appear that this isomoprhism doesn't work. Fortunately, this map will work if we instead consider linear combinations of pure tensors.

Cor: $V^* \otimes V^* \cong \text{Hom}((V^*)^*, V^*) = \text{Hom}(V, V^*) \overset{\text{claim}}{\cong} \text{Bil}(V \times V, \mathbb{R})$, by

$$\begin{aligned} \text{Hom}(V, V^*) &\cong \text{Bil}(V \times V, \mathbb{R}) \\ (f : V \to V^*) &\mapsto \begin{pmatrix} V \times V \to \mathbb{R} \\ (v_1, v_2) \mapsto (f(v_1))(v_2) \end{pmatrix} \end{aligned}$$

(Note that $f(v_1) \in V^*$.)

Thus, $V \otimes V \cong \text{Bil}(V^* \times V^*, \mathbb{R})$. This is the realization we will use!

Note: When taking multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \cong \{\text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}\}$$

Now, we put everything together, and define tensor bundles.

Defn: Given a smooth manifold M, $\forall p \in M$, we define

$$T^{(k,\ell)}(T_pM) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{\ell}$$

k is called the contravariant degree, and ℓ is called the covariant degree. Using the lemma from before, we get tensor bundles

$$\bigsqcup_{p \in M} T^{(k,\ell)}(T_pM) = T^{(k,\ell)}(TM) \to M$$

Exer: Compute the rank of this bundle.

In coordinate $(x^1, \ldots, x^n)$ on $U \overset{\text{open}}{\subseteq} M$, we have a moving frame of $T^{(k,\ell)}(TM)$:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \;\middle|\; i_a, j_b \in \{1, \ldots, n\} \right\}$$

Defn: Any smooth section of a tensor bundle is called a tensor.

Note that any tensor is a combination of these basis elements with C^∞ functions as coefficients:

$$\sum A^{i_1 \cdots i_k}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

Defn: A Riemannian metric is a symmetric, positive-definite $(0,2)$ tensor $g_p : T_pM \times T_pM \to \mathbb{R}$.

Math 635 Lecture 3

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

1/25/21

Review from Friday: Given a finite-dimensional vector space V (i.e. T_pM for some $p \in M$), a tensor of type (k, ℓ) is an element of

$$T^{(k,\ell)}(V) \overset{\text{def}}{=} \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \cong \{\text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}\}$$

Ex:

- Type $(0,1)$ is V^*, the set of linear maps $V \to \mathbb{R}$.
- Type $(1,0)$ is $(V^*)^* \cong V$, by $\begin{aligned} V &\xrightarrow{\sim} (V^*) \\ v &\mapsto (\alpha \mapsto \alpha(v)) \end{aligned}$ (for $\alpha \in V^*$).
- Type $(0,2)$ is $V^* \otimes V^* \cong \{\text{bilinear } V \times V \to \mathbb{R}\}$.

Defn: We can take the <u>tensor product</u> (or <u>outer product</u>) of two tensors on the same vector space: given $\tau \in T^{(k,\ell)}(V)$ and $\tau' \in T^{(k',\ell')}(V)$, we define $\tau \otimes \tau' \in T^{(k+k',\ell+\ell')}(V)$ by

$$(\tau \otimes \tau')(\underbrace{v_1, \ldots, v_{k+k'}}_{\in V}, \underbrace{\alpha_1, \cdots, \alpha_{\ell+\ell'}}_{\in V^*}) \overset{\text{def}}{=} \tau(v_1, \ldots, v_k, \alpha_1, \ldots, \alpha_k) \cdot \tau'(v_{k+1}, \ldots, v_{k+k'}, \alpha_{\ell+1}, \ldots, \alpha_{\ell+\ell'})$$

Note: We have to check that this map is indeed multilinear.

Ex: If $\alpha, \beta \in T^{(0,1)}(V) = V^*$, then $\alpha \otimes \beta$ is the map $\begin{aligned} V \times V &\to \mathbb{R} \\ (v_1, v_2) &\mapsto \alpha(v_1)\beta(v_2) \end{aligned}$.

Now, we want to consider tensors on manifolds. By the lemma from last time, given k, ℓ, there's a bundle $\pi : T^{(k,\ell)}(TM) \to M$ with fibers $T^{(k,\ell)}(T_pM) = T^{(k,\ell)}(\pi^{-1}(p))$ for $p \in M$. Given coordinates $(x^1, \ldots, x^n)$ on $U \subseteq M$, we have a smooth moving frame:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \;\middle|\; i_a, j_b \in \{1, \ldots, n\} \right\}$$

Last time, we stated that this is a basis, and any smooth section is a linear combination of these components, with C^∞ functions as coefficients:

$$\sum A^{i_1 \cdots i_k}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

Why is this true?

Exer: Check that $A^{i_1 \cdots i_k}_{j_1 \cdots j_\ell} = \tau(dx^{i_1}, \ldots, dx^{i_k}, \frac{\partial}{\partial x^{j_1}}, \ldots, pdx^{j_\ell})$.

Ex: Fix a smooth 1-form $\alpha \in \Omega^1(U)$. Then $\alpha = \sum_{j=1}^n a_j dx^j$, and $a_j = \alpha(\frac{\partial}{\partial x^j})$.

Defn: Some terminology: We say $\tau \in \Gamma(T^{(k,\ell)}(TM))$ is a <u>tensor</u>, or <u>tensor field</u>, on M of type (k, ℓ). k is the <u>contravariant degree</u>, and ℓ is the <u>covariant degree</u>.

Ex: Vector fields are contravariant (type $(1,0)$).
1-forms are covariant (type $(0,1)$).

Einstein Summation Notation

In coordinates, the covariant indices are subindices, and the contravariant indices are superindices. The convention of Einstein summation notation is if the same index appears exactly twice in a monomial – one upper and one lower – then summation over that index is implied.

Ex: (Linear case) $V = \mathbb{R}^n$, $\alpha \in V^* \otimes V \cong \mathrm{Hom}(V, V)$. Then we have

$$a^i_j v^j = \sum_{j=1}^n a^i_j v^j = \left[\begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \right]_i$$

Note that (a^i_j) is taken to be a matrix, where j is the column index and i the row index.

Looking ahead, we'll also have a contraction operation: $a \in V^* \otimes V \rightsquigarrow \alpha \in \mathrm{Hom}(V, V) \rightsquigarrow \mathrm{tr}(a) = a^i_i \in \mathbb{R}$. *This doesn't require coordinates, as trace is basis-invariant.*

Defn: A <u>Riemannian metric</u> g on a manifold M is a smooth $(0, 2)$ tensor, which, at every point, as a matrix, is symmetric ($g_p = g_p^T$) and positive definite ($\forall v \in T_pM$, $v^T g_p v \geq 0$, with equality iff $v = 0$).

Observe: $\forall p \in M$, $g_p : T_pM \times T_pM \to \mathbb{R}$ is a bilinear map. In coordinates $(x^1, \ldots, x^n)$ on $U \overset{\text{open}}{\subseteq} M$, $g = g_{ij}\, dx^i \otimes dx^j$, where $g_{ij} \in C^\infty(U)$, and $g_{ij} = g_{ji}$. In fact, because of symmetry, we can write (using Einstein summation notation)

$$\begin{aligned} g &= \frac{1}{2} \left[g_{ij}\, dx^i \otimes dx^j + g_{ji}\, dx^i \otimes dx^j \right] \\ &= \frac{1}{2} \left[g_{ij}\, dx^i \otimes dx^j + g_{ij}\, dx^j \otimes dx^i \right] \\ &= g_{ij} \frac{1}{2} \underbrace{\left[dx^i \otimes dx^j + dx^j \otimes dx^i \right]}_{\overset{\text{def}}{=} dx^i dx^j = dx^j dx^i, \text{ the symmetric product}} \end{aligned}$$

It's also standard notation to write $ds^2 \overset{\text{def}}{=} dx^i dx^j$, so we can write $g = g_{ij}\, ds^2$.

Defn: A <u>Riemannian manifold</u> is a pair (M, g), where M is a smooth manifold, and g is a Riemannian metric on M.

Defn: If (M, g^M) and (N, g^N) are Riemannian manifolds, an <u>isometry</u> $F : M \to N$ is a diffeomorphism s.t. $\forall p \in M$, $(dF)_p : T_pM \to T_{F(p)}N$ preserves the metric, i.e., s.t. $\forall u, v \in T_pM$, $g^N_{F(p)}(dF_p(u), dF_p(v)) = g^M_p(u, v)$.

Notation: Sometimes, we'll write $g_p(u, v) = \langle v, w \rangle_p = \langle v, w \rangle$. (We omit the subscript when the choice of metric is obvious.)

Ex: Let $M \subset \mathbb{R}^N$, a regular submanifold. $\forall p \in M$, $T_pM \subset \mathbb{R}^N$ can be identified with a subspace of $T_pR^N \cong \mathbb{R}^N$. Define $g_p = \langle\ ,\ \rangle$ by the usual inner product in $\mathbb{R}^N$.

Ex: Let $M = S^2 \subseteq \mathbb{R}^3$. We'll compute in coordinates in the upper hemisphere. Say our coordinates are (s, t) (with projection to the flat unit disk), then we have our inverse parameterization

$$(s, t) \mapsto (s, t, \sqrt{1 - s^2 - t^2}) \overset{\text{def}}{=} \vec{r}(s, t)$$

(with notation as in Calc III). Then under $T_pS^2 \hookrightarrow \mathbb{R}^3$, we have

$$\begin{aligned} \frac{\partial}{\partial s} &\mapsto \frac{\partial \vec{r}}{\partial s} = \left(1, 0, \frac{-s}{\sqrt{1 - s^2 - t^2}} \right) \\ \frac{\partial}{\partial t} &\mapsto \frac{\partial \vec{r}}{\partial t} = \left(0, 1, \frac{-t}{\sqrt{1 - s^2 - t^2}} \right) \end{aligned}$$

So our metric is

$$(g_{ij}) = \begin{pmatrix} \frac{\partial \vec{r}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial s} & \frac{\partial \vec{r}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial t} \\ \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial s} & \frac{\partial \vec{r}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial t} \end{pmatrix} = \begin{pmatrix} 1 - t^2 & \frac{st}{1 - s^2 - t^2} \\ \frac{st}{1 - s^2 - t^2} & 1 - s^2 \end{pmatrix}$$

Observe: Any submanifold $S \subset M$ of a Riemannian manifold (M, g) inherits a Riemannian metric by $\forall p \in S$, $T_pS \hookrightarrow T_pM$.

Math 635 Lecture 4

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

1/27/21

Recall: A Riemannian metric g on M is a covariant 2-tensor (i.e. a tensor of type $(0, 2)$), such that, at each point, g_p is symmetric and positive definite.

Prop: Any C^∞ manifold M has infinitely many Riemannian metrics.

Proof 1: We can embed $M \hookrightarrow \mathbb{R}^N$ (for some N), and use the induced metric. $\square$

Proof 2: We can use a partition of unity. Let $\{U_\alpha, \phi_\alpha\}$ be an atlas of M, and $\{\chi_\alpha\}$ a subordinate partition of unity. (Recall: A subordinate partition of unity means $\forall \alpha$, $\chi_\alpha \in C_0^\infty(M)$, $\{\operatorname{supp} \chi_\alpha\}$ is a locally finite cover of M, and $\forall p \in M$, $\sum_\alpha \chi_\alpha(p) = 1$.) $\forall \alpha$, let g^α be the Riemannian metric on U_α s.t. ϕ_α is an isometry (to the standard flat metric on $\mathbb{R}^n$). That is, $(g^\alpha)_{ij} = \delta_{ij}$. Define, $\forall p \in M$, $\forall u, v \in T_pM$,

$$g_p(u, v) = \sum_\alpha \underbrace{\chi_\alpha(p) g_p^\alpha(u, v)}_{\text{Interpret this as 0 if } p \notin U_\alpha.}$$

Basically, we're extending $\chi_\alpha g^\alpha$ to 0 outside U_α. Now that we've define g, we need to check that it satisfies the appropriate properties. Symmetric matrices form a closed group under addition, so clearly, every g_p is symmetric. And because $\chi_\alpha \geq 0$, we can also assert that g_p is positive semidefinite. Finally we have to check that $g_p(u, u) = 0 \Rightarrow u = 0$. Well, $\exists \alpha$ s.t. $\chi_\alpha(p) > 0$, so $p \in U_\alpha$. Thus, we must have $g_p^\alpha(u, u) = 0$, so we must have $u = 0$ by the positive definiteness of g_p^α. $\square$

Observe: An analogous proof shows that any vector bundle has infinitely many C^∞ sections, where we use local trivializations instead of coordinates.

Note that positive-definiteness is important! It's not true that M has a non-degenerate symmetric covariant 2-tensor of any arbitrary signature. For example, general relativity cares about the signature $(-, +, +, +)$, and such a tensor (with all relevant properties) doesn't always exist.

Defn: Let G be a Lie group. (Recall that this means G is a manifold, and a group, with the group operation and inversion C^∞ functions with respect to the smooth structure.) $\forall k \in G$, we have the left translation map $L_k : G \to G$, where $L_k(h) = kh$. A Riemannian metric on G is said to be <u>left-invariant</u> iff $\forall k \in G$, L_k is an isometry.

Prop: Let $\mathfrak{g} = T_eG$, the Lie algebra (where $e \in G$ is the identity element of the group). Then given any positive definite inner product $\langle\ ,\ \rangle_e$ on $\mathfrak{g}$, there exists a unique left-invariant Riemannian metric g on G s.t. $g_e = \langle\ ,\ \rangle_e$.

Proof: $\forall k \in G$, we require (by left-invariantness) that $d(L_k)_e : (\mathfrak{g}, \langle\ ,\ \rangle_e) \to (T_kG, g_k)$ is an isometry. We define g this way – g_k is the "pushforward" of $\langle\ ,\ \rangle_e$ by $d(L_k)_e$. Specifically, $\forall u, v \in T_kG$,

$$g_k(u, v) = \left\langle d(L_k^{-1})(u), d(L_k^{-1})(v) \right\rangle_e = \langle d(L_{-k})(u), d(L_{-k})(v) \rangle_e$$

We claim that this is smooth, using the smoothness of the map $\begin{array}{c} G \times G \to G \\ (k, h) \mapsto kh \end{array}$.

Question: We can also define everything in terms of right-invariant metrics. Do bi-invariant metrics exist? Answer (which we will show in HW): It depends on G!

If M and N are Riemannian manifolds, then $M \times N$ has a natural Riemannian metric. $\forall (p, q) \in M \times N$, $T_{(p,q)}(M \times N) = (T_pM) \oplus (T_qN)$. The product metric is defined as follows: Declare T_pM and T_qN to be orthogonal. With g^M, g^N the Riemannian metrics for M, N (respectively), we have the product metric on $M \times N$ (as a block matrix):

$$g^{M \times N} = \left(\begin{array}{c|c} g^M & 0 \\ \hline 0 & g^N \end{array}\right)$$

Defn: Let M, N be Riemannian manifolds. A smooth map $F : M \to N$ is a local isometry iff $\forall p \in M$,

$$dF_p : (T_pM, g_p^M) \to (T_{F(p)}N, g_{F(p)}^N)$$

is a linear isometry.

Note that local isometries are local diffeomorphisms, by the inverse function theorem.

Defn: A surjective C^∞ map $\pi : M \to N$ is a smooth covering map iff $\forall q \in N$, there's a neighborhood V of q s.t. π maps the connected components of $\pi^{-1}(U)$ diffeomorphically onto V. That is,

$$\pi^{-1}(U) = \bigsqcup_j U_j, \qquad \forall j,\, \pi|_{U_j} : U_j \xrightarrow{\sim} V$$

Defn: Let π be a smooth covering map. The automorphism group is $\mathrm{Aut}(\pi) = \{F : M \to N \mid \pi = F \circ \pi\}$. We say that such an F "shuffles the fibers". $\mathrm{Aut}(\pi)$ is a group under composition.

Defn: π is normal iff $\mathrm{Aut}(\pi)$ acts transitively on the fibers.

Ex: $S^n \xrightarrow{\pi} \mathbb{RP}^n$. $\mathrm{Aut}(\pi) = \{\text{identity map, antipodal map}\} \cong \mathbb{Z}_2$.
$\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n/\mathbb{Z}^n$.

Ex: (HW) $\mathcal{H} = \big\{(x, y) \in \mathbb{R}^2 \mid y > 0\big\}$. For any genus g surface G, with $g > 1$, $\exists \pi : \mathcal{H} \to G$.

Thm: Let $\pi : M \to N$ be a smooth normal covering map. Let g^M be a Riemannian metric on M s.t. $\forall F \in \mathrm{Aut}(\pi)$ is an isometry. Then there's a unique Riemannian metric g_N on N s.t. π is a local isometry.

Proof: Let $q \in N$, and V a neighborhood of q. Then $\pi^{-1}(V) = \bigsqcup_j U_j$ (where the U_j are the connected components) such that $\pi|_{U_j} : U_j \xrightarrow{\sim} V$. We want to choose a metric on V s.t. $\forall j$, $\pi|_{U_j}$ is an isometry. Well, choose a j_0, and put such a metric on V. $\forall j \neq j_0$, $\exists F \in \mathrm{Aut}(\pi)$ s.t. $F(U_j) = F(U_{j_0})$. F is an isometry, by assumption, so π is an isometry on U_j. $\square$

Math 635 Lecture 5

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

1/29/21

Some stuff we can do with Riemannian metrics:

- A Riemannian metric on M allows us to define the length of a tangent vector, and the angle between two tangent vectors. For $v, w \in T_pM$, $||v|| = \sqrt{\langle v, v\rangle_p}$, and $\langle v, w\rangle_p = ||v||\,||w|| \cos\theta$ defines θ, the angle between v and w.
- A Riemannian metric on M allows us to define the length of a curve in M. Say $\gamma : [a, b] \to M$. We define its length

$$L(\gamma) = \int_a^b ||\dot\gamma(t)||\, dt = \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)}}\, dt$$

If $\mathrm{im}\gamma \subseteq U$, a coordinate patch, with coordinates $(x^1(t), \ldots, x^k(t))$, then

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(\gamma(t))\dot x^i(t)\dot x^j(t)}\, dt \qquad g_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle \in C^\infty(U)$$

Lemma: $L(\gamma)$ is invariant under re-parameterizations of γ. Consider $[\alpha, \beta] \underset{t}{\xrightarrow{\sim}} [a, b] \xrightarrow{\gamma} M$. Then

$$\int_\alpha^\beta \left|\left|\frac{d\gamma}{ds}(t(s))\right|\right|\, ds = \int_a^b \left|\left|\frac{d\gamma}{dt}(t)\right|\right|\, dt$$

Proof: The chain rule says $\frac{d\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt}$. So $\left|\left|\frac{d\gamma}{ds}\right|\right| = \left|\left|\frac{dt}{ds}\right|\right| \left|\left|\frac{d\gamma}{dt}\right|\right|$. Now, use the change of variables formula for integrals. $\square$

Defn: Let M be a connected Riemannian manifold. We define the <u>Riemannian distance function</u> by, $\forall p, q \in M$, $d(p, q) = \inf\{L(\gamma) \mid \gamma$ is a continuous curve, or "path", joining p and q that is piecewise smooth$\}$.

Note: We can replace this definition with just "smooth" – the definitions are equivalent. But that's harder to prove, and this definition will be useful later on.

Note that because M is connected, it's path connected. This means the set of lengths of curves connecting pairs of points is nonempty, so the infimum exists. And because $L(\gamma) \geq 0$, $d(p, q) \geq 0$.

Thm: d is a metric, or distance function, i.e., $\forall p, q, r \in M$,

(i) $d(p, q) \geq 0$, with $d(p, q) = 0 \Leftrightarrow p = q$
(ii) $d(p, q) = d(q, p)$
(iii) $d(p, q) + d(q, r) \geq d(p, r)$

Proof: (This is only a partial proof)

(i) If $p = q$, take a trivial/constant path. $\dot\gamma \equiv 0$, so $L(\gamma) = 0$. The converse remains to be shown: that $d(p, q) = 0$ implies $p = q$. This will be a corollary of the "Gauss lemma", which we'll do later on.
(ii) We never assumed reparameterizations couldn't reverse the direction of the curve. They can, which directly proves $d(p, q) = d(q, p)$.
(iii) Among the paths joining p and r are paths that travel through q. Specifically, given any path from p to q and any path from q to r, we can concatenate them to get a path from p to r.

$\square$

Observe: The topology defined by d is the same as the given topology on M.

Sometimes, but not always, the infimum is attained, i.e., there exists a minimizing path. In fact, if such a path exists, it's always smooth.

Ex: Minimizing paths on the sphere are arcs of great circles – intersections of the sphere with hyperplanes through the origin.

Ex: Minimizing paths don't always exist! Consider $M = \mathbb{R}^2 \setminus \{0\}$. For $p = (43, 0)$ and $q = (-43, 0)$, $d(p, q) = 86$, but there's no path of that length between them (since you can't go through the origin).

Refer to Do Carmo, Chapter 1, §2 for more details.

Volume Element of an Oriented Riemannian Manifold

Reminder: An orientation on an orientable manifold M is determined by a class of top-degree differential forms, ν, with the defining property that $\forall p \in M$ and any $(v_1, \dots, v_n)$, a positive basis of T_pM, $\nu_p(v_1, \dots, v_n) > 0$.

In particular, ν is nowhere-vanishing. Conversely, a nowhere-vanishing top-degree form can be used to define positive bases, and in turn, an orientation.

Defn: If M is a orientable Riemannian manifold, its <u>volume form</u> ν is defined by the property that $\forall p \in M$, $\forall (v_1, \dots, v_n)$, a positive, orthonormal basis of T_pM, one has $\nu_p(v_1, \dots, v_n) = 1$.

Observe: If this condition holds for some positive orthonormal basis, it holds for all positive orthonormal bases. The important calculation is as follows: Fix $p \in M$, $(v_1, \dots, v_n)$ a positive orthonormal basis of T_pM, and $(e_1, \dots, e_n)$ any other ordered basis of T_pM. Then we can write each $e_i = \sum_{\ell=1}^n a_i^\ell v_\ell$. For any top-degree form ν, $\nu_p(e_1, \dots, e_n) = \det(a_i^\ell)\nu(v_1, \dots, v_n)$, so if $(e_1, \dots, e_n)$ is also positive and orthonormal, then $\det(a_i^\ell) = 1$. $\square$

Computation of the Volume Form in Coordinates

Start with $(x_1, \dots, x_n)$, a positive coordinate system with domain U. (Recall that this means $\forall p \in U$, $(\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p)$ is a positive basis of T_pM.) Apply Gram-Schmidt to each basis (pointwise). We obtain vector fields $v_1, \dots, v_n$ on U which are orthonormal at each point. And Gram-Schmidt shows the v's are related to the partial derivatives by a smooth matrix, so $\forall j$, $v_j \in \mathfrak{X}(U)$. And, by possibly permuting the v_i's, we can ensure it's a positive basis at every point.

In fact, let's write $\frac{\partial}{\partial x^i} = \sum_\ell a_i^\ell v_\ell$. Then

$$g_{ij} = \left\langle \frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p \right\rangle = \sum_{k,\ell} a_i^\ell a_j^k \underbrace{\langle v_k, v_\ell \rangle}_{=\delta_{k,\ell}} = \sum_k a_i^k a_j^k = AA^T \quad \text{for} \quad A = (a_i^k)$$

So $\det(g_{ij}) = \det(A)^2 > 0$. On the other hand, with our Riemannian volume form ν,

$$\nu\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \underbrace{\det(A)}_{=\sqrt{\det(g_{ij})}} \underbrace{\nu(v_1, \dots, v_n)}_{=1} = \sqrt{\det(g_{ij})}$$

So in coordinates, $\nu = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n$.

Defn: The volume of a subset U of a Riemannian manifold is $\mathrm{Vol}(U) = \int_U \nu$, where ν is the Riemannian volume form, if this integral is finite.

Lemma: For any coordinate system $(y^1, \dots, y^n)$ on U, positive or not, the Riemannian integral

$$\int_U \sqrt{\det\left(\left\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right\rangle\right)} \underbrace{dy^1 \cdots dy^n}_{\text{Riemann integral}}$$

is equal to $\mathrm{Vol}(U)$.

This is true because the change of variables formula for a Riemann integral involves the absolute value of the Jacobian. So in the end, orientation is *not* needed to compute volumes of manifolds. In fact, we can even compute volumes of non-orientable manifolds! We generalize by using partitions of unity.

Math 635 Lecture 6

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/1/21

A brief roadmap for the next few weeks:

$$\begin{tikzcd}
\text{Riemannian Metrics} \arrow[dr] & & \text{Connections} \arrow[d] \\
& & \text{Parallel Transport and Curvature} \arrow[dl] \\
& \text{If } M \text{ is Riemannian, then } TM \text{ has a special connection} \arrow[d] & \\
& \text{Etc.} &
\end{tikzcd}$$

First, some motivation... (See also Lee Riemannian Geometry, Chapter 4)

Start with $\gamma : (a,b) \to \mathbb{R}^n$. Let $Y = \sum_i f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$, a smooth vector field in $\mathbb{R}^n$. We can compute $\frac{d}{dt} Y(\gamma(t))\big|_{t=t_0}$ in $\mathbb{R}^n$. But what's really happening? Well, we're computing

$$\frac{d}{dt} Y(\gamma(t))\bigg|_{t=t_0} = \lim_{h\to 0} \frac{Y(\gamma(t_0+h)) - Y(\gamma(t_0))}{h}$$

But we can only do this in $\mathbb{R}^n$, not on manifolds in general! Strictly speaking, $Y(\gamma(t_0+h)) \in T_{\gamma(t_0+h)}\mathbb{R}^n$ and $Y_{\gamma(t_0)} \in T_{\gamma(t_0)}\mathbb{R}^n$. We can take their difference because we're identifying all tangent spaces of $\mathbb{R}^n$ with each other, using translations of $\mathbb{R}^n$. And $\{\text{translations of } \mathbb{R}^n\} \cong \mathbb{R}^n$ as a vector space. In other words, we can translate vectors in $\mathbb{R}^n$ "parallel to themselves".

For $Y = \sum_i f^i \frac{\partial}{\partial x^i}$, we get a formula:

$$\frac{d}{dt} Y(\gamma(t))\bigg|_{t=t_0} = \sum_{i=1}^n df^i_{\gamma(t_0)}(\dot\gamma(t_0)) \frac{\partial}{\partial x^i} \stackrel{\text{def}}{=} \left(\bar\nabla_{\dot\gamma(t_0)} Y\right)(\gamma(t_0))$$

(Note that $\frac{\partial}{\partial x^i}$ is a constant frame on $\mathbb{R}^n$, so we can use $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^i}\big|_p$ interchangeably).

Defn: If $Y = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$, $p \in \mathbb{R}^n$, and $v \in T_pM$, we define

$$\left(\bar\nabla_v Y\right)(p) \stackrel{\text{def}}{=} \sum_{i=1}^n df^i_p(v) \frac{\partial}{\partial x^i}$$

We can think of $\left(\bar\nabla_v Y\right)(p)$ as a vector, which only depends on the values of Y along a curve γ as above.

The question remains: Is there something analogous to this on manifolds? It may look a bit like a Lie derivative, but note that $\bar\nabla$ is **not** a Lie derivative!

Recall: Given $X, Y \in \mathfrak{X}(M)$, we can define $\mathcal{L}_X Y$ using the flow φ of X:

$$(\mathcal{L}_X Y)(p) = \lim_{t\to 0} \frac{(\varphi_{-t})_{*,\varphi_t(p)}(X_{\varphi_t(p)}) - X_p}{t}$$

In this case, we need X as a vector field, whereas above, we just need a vector. And $\mathcal{L}_X Y$ is dependent on X, but there are infinitely many vector fields X s.t. $X_p = v$ (with v as above).

In fact, we need some additional structure on the manifold, because we cannot natural identify T_pM with T_qN, when $p \neq q$. This additional structure is called a connection.

Defn: Let $\mathcal{E} \to M$ be a vector bundle. A <u>connection</u> on $\mathcal{E}$ is an operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

that satisfies:

1) $\forall X, Y \in \mathfrak{X}(M)$, $\forall s \in \Gamma(\mathcal{E})$, $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
2) $\forall f \in C^\infty(M)$, $\nabla_{fX} s = f \nabla_X s$
3) $\nabla_X (fs) = f\nabla_X s + X(f)s = f\nabla_X s + df(X)s$

Because of properties 1 and 2, we say that a connection is "linear in X over $C^\infty(M)$".

Note that although our definition above uses vector fields, we will show that this dependence is pointwise.

Ex: $\nabla = \bar{\nabla}$ on $\mathcal{E} = T\mathbb{R}^n$.

Prop: If ∇ is a connection on $\mathcal{E} \to M$, then $\forall X \in \mathfrak{X}(M), s \in \Gamma(\mathcal{E}), p \in M$, $(\nabla_X s)(p) \in \mathcal{E}_p = \pi^{-1}(p)$ only depends on X_p and the values of s in an arbitrarily small open neighborhood of p.

Proof: Let U be a neighborhood of p; χ a bump function supported on U, with $\chi \equiv 1$ on $V \overset{\text{open}}{\subseteq} U$, a smaller open neighborhood of p. Consider $\nabla_X(\chi s) = X(\chi)s + \chi\nabla_X s$. Evaluate at p: the right hand side is just $(\nabla_X s)(p)$ because $\chi \equiv 1$ on V and $X(\chi) \equiv 0$ on V (because χ is constant on V, and X is a derivation). The computation of $(\nabla_X s)(p)$ can be localized to, say, a coordinate neighborhoood of p.

Let $X = \sum_i a^i \frac{\partial}{\partial x^i}$ in local coordinates. Then, by C^∞ linearity of ∇,

$$(\nabla_X s)(p) = \sum_i a^i(p)(\nabla_{\frac{\partial}{\partial x^i}} s)(p)$$

If $X(p) = 0$ (which is true iff $\forall j$, $a^j(p) = 0$), then $(\nabla_X s)(p) = 0$. So if $X(p) = \tilde{X}(p)$, then $(\nabla_X s)(p) = (\nabla_{\tilde{X}} s)(p)$. $\square$

Observe: Given ∇ and s, $(\nabla_X s)(p) \in \mathcal{E}_p$ depends only on $X(p)$, and does so linearly! So ∇ and s define a map

$$\begin{aligned} T_pM &\to \mathcal{E}_p \\ v &\mapsto (\nabla_v s)(p) \end{aligned}$$

which is itself an element of $T_p^*M \otimes \mathcal{E}_p$. Therefore, ∇ can be thought of as an operator $\nabla : \Gamma(\mathcal{E}) \to \mathcal{E}(T^*M \otimes \mathcal{E})$, whose image is "$\mathcal{E}$-valued differential forms".

Local Expression of a ∇

Let $\mathcal{E} \to M$ be a vector bundle, with connection ∇, $U \overset{\text{open}}{\subseteq} M$, and $(E_1, \ldots, E_r)$ a moving frame of $\mathcal{E}$ over U. That is, $\forall j$, $E_j \in \Gamma(\mathcal{E}|_U)$, and at each $p \in U$, $(E_1(p), \ldots, E_r(p))$ is a basis of $\mathcal{E}_p$. So if $s \in \Gamma(\mathcal{E}|_U)$, then $\exists f^i \in C^\infty(U)$ s.t. $s = \sum_j f^j E_j$. So $\forall X \in \mathfrak{X}(U)$, we get

$$\nabla_X s = \sum_{j=1}^r f^j \nabla_X E_j + X(f^j)E_j$$

But

$$\nabla_X E_j = \sum_{i=1}^r \theta_j^i(X) E_i$$

By the discussion above, $\forall i, j$, $\theta_j^i \in \Omega^1(U)$, a one-form. So we can define $\vartheta = (\theta_j^i)$, an $r \times r$ matrix of one-forms on U, depending on the moving frame $(E_1, \ldots, E_r)$. In fact, this ϑ determines ∇ on U!

Math 635 Lecture 7

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/3/21

Review: Connections on Vector Bundles

Given $\mathcal{E} \to M$ a vector bundle, a connection is an operator

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{E}) &\to \Gamma(\mathcal{E}) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

with universal quantifiers $\forall X, Y \in \mathfrak{X}(M)$, $s, t \in \Gamma(\mathcal{E})$, and $f \in C^\infty(M)$:

(1) $\nabla_{X+Y} s = \nabla_X s + \nabla_Y s$
(2) $\nabla_{fX} s = f \nabla_X s$
(3) $\nabla_X (s + t) = \nabla_X s + \nabla_X t$
(4) $\nabla_X (fs) = f \nabla_X s + X(f) s$

Properties (1) and (2) together are written as ∇ is linear in X over $C^\infty(M)$. Note that property (4) implies that $\forall c \in \mathbb{R}$, $\nabla_X (cs) = c \nabla_X s$.

Last time, we saw that $(\nabla_X s)(p)$ only depends on $X_p \in T_p M$ and $s|_U$ for any (arbitrarily small) neighborhood U of p. In other words, if we fix a section $s \in \Gamma(\mathcal{E})$, $\nabla_\cdot s$ is a 1-form that takes values in sections. And if we fix $X \in \mathfrak{X}(M)$, then $\nabla_X \cdot : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is a derivation.

Ex: Say $\mathcal{E} = M \times \mathbb{R}^r \to M$. (This is locally the general case.) Then $\Gamma(M \times \mathbb{R}^r) \ni s \leftrightarrow \vec{f} : M \to \mathbb{R}^r$ by $\forall p \in M$, $s(p) = (p, \vec{f}(p))$. This is the same as having a global moving frame $\Gamma(M \times \mathbb{R}^r) \ni E_i \leftrightarrow \vec{f}_i(p) = (0, \ldots, 1, \ldots, 0)$ (with a 1 in the ith entry), for $i = 1, \ldots, r$, because $s = \sum_{i=1}^r f^i E_i$ for $f^i \in C^\infty(M, \mathbb{R})$, where $\vec{f} = (f^1, \ldots, f^r)$.

Suppose we have a connection ∇. Last time, we defined $\vartheta = (\theta_i^j)$, the connection matrix associated with $(E_1, \ldots, E_r)$, by $\forall X$, $\forall j$,

$$\nabla_X E_j = \sum_{i=1}^r \theta_i^j (X) E_j$$

$\forall i, j$, $\theta_i^j \in \Omega^1(M)$ (a C^∞ one-form). Then:

$$\nabla_X s = \sum_{i=1}^r \nabla_X (f^i E_i) = \sum_{i=1}^r X(f^i) E_i + \sum_{i=1}^r f^i \nabla_X E_i = \sum_{i=1}^r X(f^i) E_i + \sum_{i,j=1}^r f^i \theta_i^j (X) E_j$$

The corresponding vector-valued function on M is

$$\nabla_X \vec{f} = \left(X(f^1) + \sum_{i=1}^r \theta_i^1 (X) f^i, \ldots, X(f^r) + \sum_{i=1}^r \theta_i^r (X) f^i \right) = \big(X(f^1), \ldots, X(f^r) \big) + \left(\sum_{i=1}^r \theta_i^1 (X) f^i, \ldots, \sum_{i=1}^r \theta_i^r (X) f^i \right)$$

So in vector/matrix notation, with $\vec{f}$ as a column vector, we write

$$\nabla_X \vec{f} = d\vec{f}(X) + \vartheta(X) \vec{f}$$

where $\vartheta(X) = (\theta_i^j (X))$ with lower index i being the columns, and upper index j being the rows.

Conversely, here, ϑ can be any $r \times r$ matrix of one-forms, and this can be used to define a connection on the trivial bundle!

Observe: The previous calculation is valid locally, given some moving frame $(E_1, \dots, E_r)$ of $\mathcal{E} \to M$ on $U \subseteq M$. Suppose $(F_1, \dots, F_r)$ is another moving frame on U. Then $\forall i$, $F_i = \sum_j a_i^j E_j$, where the matrix $A = (a_i^j)$ is invertible at each $p \in U$, and $\forall i, j$, $a_i^j \in C^\infty(U)$. Given ∇, we get ϑ, the connection matrix corresponding to the E_j's, and $\tilde{\vartheta}$, the connection matrix corresponding to the F_j's.

Exer: Check that $\tilde{\vartheta} = A^{-1}dA + A^{-1}\vartheta A$.

A special feature of the case where $\mathcal{E} = TM$ ($r = n = \dim M$). Consider again a moving frame $(E_1, \dots, E_n)$. In this case, we can write $X = \sum_k a^k E_k$. In the generic case,

$$\nabla_X E_i = \sum_j \theta_i^j(X) E_j = \sum_{j,k} a^k \underbrace{\theta_i^j(E_k)}_{\substack{\text{Christoffel}\\ \text{Symbols}}} E_j$$

Defn: $\forall i, j, k$, $\Gamma_{ki}^j \overset{\text{def}}{=} \theta_i^j(E_k) \in C^\infty(U)$ are the <u>Christoffel symbols</u>.

Note that $\forall i, k$, we get $\nabla_{E_k} E_i = \sum_j \Gamma_{ki}^j E_j$. So Γ_{ki}^j determines ϑ, and therefore ∇ on U.

Now, back to the general case: vector bundle $\mathcal{E} \to M$ with connection ∇. We want to look at "parallelism" and "parallel transport".

Defn: Let $\gamma : [a, b] \to M$, $s \in \Gamma(\mathcal{E})$. Then s is said to be <u>covariant constant</u>, or <u>parallel</u>, iff $\forall t \in [a, b]$, $(\nabla_{\dot{\gamma}(t)} s)(\gamma(t)) = 0$.

Let's analyze this equation... It will turn out to be a system of ordinary differential equations!

Let $(E_1, \dots, E_r)$ be a moving frame on U, a neighborhood of $\gamma(t)$. For the time being, just assume $\mathrm{Im}(\gamma) \subseteq U$. We can write $s = \sum_i f^i E_i$, for some $f^i \in C^\infty(U)$. Let ϑ be the connection matrix w.r.t. the E_i's.
Introduce $\vec{f} = (f^1, \dots, f^r) : U \to \mathbb{R}^r$. We saw that $\nabla_{\dot{\gamma}} \vec{f} = d\vec{f}(\dot{\gamma}) + \vartheta(\dot{\gamma})\vec{f}$. Define $f^i(t) = f^i(\gamma(t))$, so here, $\vec{f}(t) = \vec{f}(\gamma(t))$. Then

$$d\vec{f}(\dot{\gamma}) = \frac{d}{dt}\vec{f} \qquad \text{and} \qquad \frac{d}{dt}\vec{f} + \vartheta(\dot{\gamma})\vec{f} = \vec{0}$$

(taking $\vec{f}$ to be the column vector of the $f^i(t)$'s, and $\vartheta(\dot{\gamma})$ being a t-dependent matrix).

Math 635 Lecture 8

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/5/21

Start with a vector bundle $\mathcal{E} \to M$, with connection ∇. Let $U \overset{\text{open}}{\subseteq} M$, and $(E_1, \dots, E_r)$ a moving frame on U. Note that we have the following commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, phantom, "\overset{\phi}{\cong}"] \arrow[dr, "\pi"] & U \times \mathbb{R}^r \arrow[d, "\pi_U"] \\ & U \arrow[ul, bend left, "s"] \end{tikzcd}$$

$\forall s \in \Gamma(U)$, $\exists \vec{f} : U \to \mathbb{R}^r$, $\vec{f} = (f^1, \dots, f^r)$ s.t. $s = \sum_{i=1}^r f^i E_i$. Thus, we have an isomorphism

$$\begin{aligned} \Gamma(U) &\xrightarrow{\sim} C^\infty(U, \mathbb{R}^r) \\ s &\mapsto \vec{f} \end{aligned}$$

Under this isomorphism, $\nabla_X s$ corresponds with $\nabla_X \vec{f}$. We saw:

$$\nabla_X \vec{f} = d\vec{f}(X) + \vartheta(X)\vec{f}$$

treating $\vec{f}$ as a column vector, $d\vec{f}(X) = \begin{pmatrix} df^1(X) \\ \vdots \\ df^r(X) \end{pmatrix} = X\vec{f}$, and $\vartheta = (\theta_i^j)$ s.t. $\nabla_X E_i = \sum_j \theta_i^j(X) E_j$. So we can rewrite this as $\nabla_X = X + \vartheta(X)$ on $C^\infty(U, \mathbb{R}^r)$.

Parallelism

Let $\mathcal{E} \to M$ be a vector bundle, with connection ∇.

Defn: Given $\gamma : [a, b] \to M$ and $s \in \Gamma(\mathcal{E})$, we say s is <u>covariant constant</u>, or <u>parallel</u>, along γ if and only if $\forall t \in [a, b]$, $\nabla_{\dot{\gamma}(t)} s = 0 \in \mathcal{E}_{\gamma(t)} = \pi^{-1}(\gamma(t))$. Locally, this is true if and only if $\frac{d\vec{f}(t)}{dt} = -\vartheta(\dot{\gamma}(t))\vec{f}(t)$, where $\vec{f}(t) = \vec{f}(\gamma(t))$.

Today, we'll work with parallel transport. The idea is given γ, we can construct sections along γ that are covariant constant. Given a vector bundle $\mathcal{E} \to M$, with connection ∇, a curve $\gamma : [a, b] \to M$, and a section $s_a \in \mathcal{E}_{\gamma(a)}$, we want to "parallel transport" s_a along γ.

Defn: Given $\gamma : [a, b] \to M$ on a vector bundle $\pi : \mathcal{E} \to M$, a <u>section of $\mathcal{E}$ along γ</u> is a function $V : [a, b] \to \mathcal{E}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "V"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

That is, $\forall t \in [a, b]$, $\pi(V(t)) = \gamma(t)$, i.e., $V(t) \in \mathcal{E}_{\gamma(t)}$.

Our (nonstandard) notation is, for a curve γ, $\Gamma_\gamma(\mathcal{E})$ is the set of all such smooth V.

Ex: We can always just use a global section. If we have $s \in \Gamma(\mathcal{E})$, then $V = s \circ \gamma : [a, b] \to \mathcal{E}$ is a section along γ.

We can extend covariant differentiation (with respect to ∇) to $\Gamma_\gamma(\mathcal{E})$ for a given γ!

Prop: (Do Carmo, Chapter 2, Prop 2.2) Given $\mathcal{E}$, ∇, and γ, there is a unique operator

$$\frac{D}{dt} : \Gamma_\gamma(\mathcal{E}) \to \Gamma_\gamma(\mathcal{E})$$

such that

(a) $\frac{D}{dt}$ is $\mathbb{R}$-linear.
(b) $\frac{D}{dt}$ is a derivation: $\forall f \in C^\infty([a,b])$, $\forall V \in \Gamma_\gamma(\mathcal{E})$, $\frac{D}{dt}(fV) = f\frac{DV}{dt} + \dot{f}V$, where $\dot{f}(t) = \frac{df}{dt}$.
(c) $\forall s \in \Gamma(\mathcal{E})$, $\frac{D}{dt}(s \circ \gamma) = \nabla_{\dot{\gamma}(t)} s$.

Proof: Start with local uniqueness. Let $(E_1, \ldots, E_r)$ be a moving frame on $U \subseteq M$, with $U \cap \operatorname{Im}(\gamma) \neq \emptyset$. Let ϑ be the connection matrix associated with the E_i's. Assume that $\frac{D}{dt}$ exists. Then we have a commutative diagram:

$$\begin{tikzcd}
& \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"] & & U \times \mathbb{R}^r \arrow[dl, "\pi_U"] \\
\gamma^{-1}(U) \arrow[ur, "V|_{\gamma^{-1}(U)}"] \arrow[rr, "\gamma|_{\gamma^{-1}(U)}"] \arrow[urrr, bend left, "\phi\circ(V|_{\gamma^{-1}(U)})"] & & U &
\end{tikzcd}$$

$\forall t \in \gamma^{-1}(U)$, $\phi(V(t)) = (\gamma(t), (f^1(t), \ldots, f^r(t)))$. This defines $f^i(t)$. Now, we claim that (a), (b), and (c) together of $\frac{D}{dt}$ imply that, $\forall t \in \gamma^{-1}(U)$,

$$\frac{DV}{dt}(t) = \frac{D}{dt}\left(\sum_{i=1}^r f^i(t) E_i(\gamma(t))\right) = \sum_{i=1}^r f^i(t)(\nabla_{\dot{\gamma}(t)} E_i)(t) + \dot{f}^i(t) E_i(\gamma(t))$$

Existence: Define $\frac{D}{dt}$ locally, using the above formula, and then use trivializations $\{U_\alpha\}$ that cover, so that

$$\gamma^{-1}\left(\bigcup_\alpha U_\alpha\right) = [a,b]$$

Uniqueness: On the overlap of differing U_α's, the definitions must agree. $\square$

In fact, you can check: If we define in terms of $F_j = \sum_i a_j^i E_i$, then the definitions agree.

Defn: (Parallel Transport) For $\gamma : [a,b] \to M$, we define $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ by $\forall s_a \in \mathcal{E}_{\gamma(a)}$, $\mathcal{P}_\gamma(s_a) = V(b)$, where $V(b)$ is a solution at $t = b$ of $\frac{DV}{dt}(t) = 0$, $V(a) = s_a$, where $V \in \Gamma_\gamma(\mathcal{E})$. Locally: $\dot{\vec{f}} = -\vartheta(\dot{\gamma})\vec{f}$.

Observe: $\mathcal{P}_\gamma$ is $\mathbb{R}$-linear.

We can extend this to continuous, piecewise smooth γ by using composition: Say $\gamma_1 : [a,b] \to M$ and $\gamma_2 : [a', b'] \to M$ are two smooth curves, with $\gamma_1(b) = \gamma_2(a')$. Then for their concatenation $\gamma_2 \# \gamma_1$ (we won't use this notation often), we have $\mathcal{P}_{\gamma_2 \# \gamma_1} = \mathcal{P}_{\gamma_2} \circ \mathcal{P}_{\gamma_1}$.

In particular, reversing the direction of γ shows that $\mathcal{P}_\gamma$ is a bijection. If we have a loop, so $\mathcal{E}_{\gamma(a)} = \mathcal{E}_{\gamma(b)}$, then $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \circlearrowleft$ is called the holonomy of γ.

Math 635 Lecture 9

Professor Alejandro Uribe-Ahumada
Transcribed by Thomas Cohn

2/8/21

An Example of Connections and Parallelism

Take $M = S^1$, with the coordinate $0 \leq x \leq 2\pi$. Let $\mathcal{E} = M \times \mathbb{R}^2 = S^1 \times \mathbb{R}^2$, with the global frame $(E_1 = (1,0), E_2 = (0,1))$, $\forall x \in S^1$. Then elements of $\Gamma(\mathcal{E})$ are functions $\vec{f}(x) = (f^1(x), f^2(x))$.

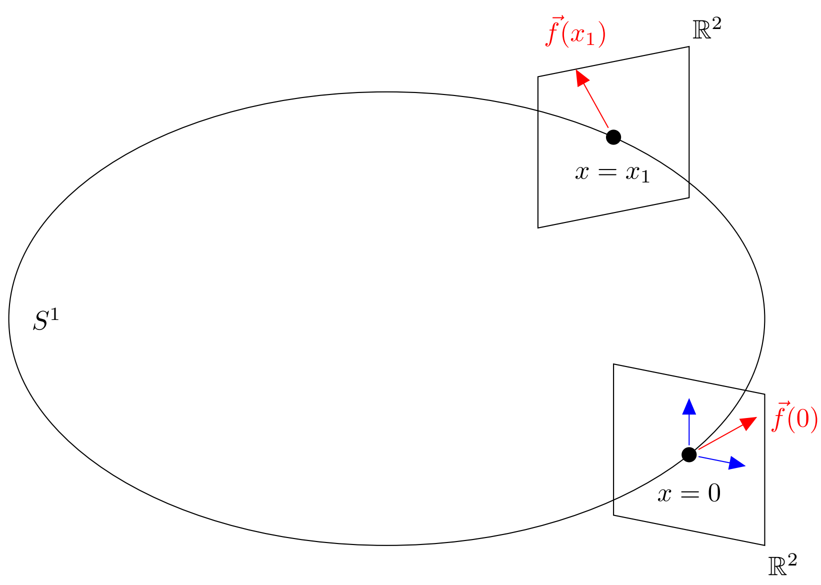
Suppose ∇ is given by ϑ, any 2×2 matrix of 1-forms on M. So $\theta_i^j = \alpha_i^j dx$, for $\alpha_i^j \in C^\infty(S^1)$. Thus,

$$\nabla_{\frac{\partial}{\partial x}} \vec{f} = \underbrace{\begin{pmatrix} \frac{df^1}{dx} \\ \frac{df^2}{dx} \end{pmatrix}}_{=\frac{d}{dx}\vec{f}} + \underbrace{\left(\alpha_i^j\right)}_{\text{because } dx\left(\frac{\partial}{\partial x}\right) = 1} \vec{f}$$

Consider the simple case $\vartheta = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, for some $c \in \mathbb{R} \setminus \{0\}$. Then

$$\vec{f} \text{ is parallel} \quad \Leftrightarrow \quad \nabla_{\frac{\partial}{\partial x}} \vec{f} = 0 \quad \Leftrightarrow \quad \begin{cases} \frac{df^1}{dx} = cf^2 \\ \frac{df^2}{dx} = -cf^1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} f^1(x) = f^1(0)\cos(cx) + f^2(0)\sin(cx) \\ f^2(x) = -f^1(0)\sin(cx) + f^2(0)\cos(cx) \end{cases}$$

This final object is the solution to the parallel transport problem, starting at $x = 0$ with $\left(\begin{smallmatrix} f^1(0) \\ f^2(0) \end{smallmatrix}\right) = \vec{f}(0)$.



Additionally, we can map from $\vec{f}(0)$ to $\vec{f}(2\pi)$ with the holonomy matrix:

$$\begin{pmatrix} \cos(2\pi c) & -\sin(2\pi c) \\ \sin(2\pi c) & \cos(2\pi c) \end{pmatrix}$$

Thus, the holonomy is trivial (i.e., the holonomy matrix is the identity matrix I_2) iff $c \in \mathbb{Z}$.

Exer: Compute $\nabla_{\frac{\partial}{\partial x}} E_1$ and $\nabla_{\frac{\partial}{\partial x}} E_2$. Note: It won't be 0!

Curvature of a Connection on a Bundle

Say ∇ is a connection on the vector bundle $\mathcal{E} \to M$. Curvature is a local object, and there are two approaches to describe it.

First Approach

Begin with a moving frame $(E_1, \dots, E_r)$ (which leads to the connection matrix $\vartheta = (\theta_i^j)$).

Question: Can we change the moving frame to $(F_1, \dots, F_r)$ so that the new connection matrix, $\tilde{\vartheta}$, is 0?

Well, if $A = (a_j^i)$ is defined so that $F_j = a_j^i E_i$ (using Einstein summation notation), then we saw (in homework) that

$$\tilde{\vartheta} = A^{-1} dA + A^{-1} \vartheta A$$

So

$$\tilde{\vartheta} = 0 \quad \Leftrightarrow \quad dA = -\vartheta A \quad \Leftrightarrow \quad \forall i, j, da_j^i = -\theta_k^i a_j^k$$

We suspect that we'll encounter problems from the fact that $d^2 = 0$. Assume such a solution A exists. Then, using the product rule, $0 = -(d\theta_k^i) a_j^k + \theta_k^i \wedge da_j^k$. But we know $da_j^k = -\theta_\ell^k a_j^\ell$, so for the connection matrix to be trivial, we must have

$$(d\theta_k^i) a_j^k + \theta_k^i \wedge \theta_\ell^k a_j^\ell = 0, \quad \forall i, j$$

Note: $\theta_k^j \wedge \theta_\ell^k = (\vartheta \wedge \vartheta)_\ell^j$ by definition. So using matrix notation, we can write this necessary condition as

$$(d\vartheta) A + (\vartheta \wedge \vartheta) A = 0$$

A is invertible, so this condition is true iff $d\vartheta + \vartheta \wedge \vartheta = 0$.

Defn: $\Omega \stackrel{\text{def}}{=} d\vartheta + \vartheta \wedge \vartheta$ is a matrix of 2-forms, called the curvature matrix of ∇ with respect to the moving frame $(E_1, \dots, E_r)$.

The connection is trivializable only if $\Omega = 0$ (i.e. this is a necessary condition). There is no reason in principle this should happen, so it's very rare!

Second Approach

Fix $X, Y \in \mathfrak{X}(M)$. Consider $\nabla_X, \nabla_Y : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ as operators, acting linearly on smooth sections.

Question: What is their commutator (as operators)?

Well, if $\mathcal{E} = M \times \mathbb{R}^r$ and $\vartheta \equiv 0$, then $\nabla_X \vec{f} = X(f) + 0$. So $\nabla_X = X$. Therefore, $[\nabla_X, \nabla_Y] = [X, Y] = \nabla_{[X,Y]}$, by definition of the commutator of vector fields.

In general, let $E_1, \dots, E_r$ be a moving frame, and $\vartheta = (\theta_i^j)$ the connection matrix as before. Then $\nabla_X = X + \vartheta(X)$, acting on functions $\vec{f} : U \to \mathbb{R}^r$ ($\vartheta(X)$ is a matrix of functions). Likewise, $\nabla_Y = Y + \vartheta(Y)$. So we can compute:

$$[\nabla_X, \nabla_Y] = [X + \vartheta(X), Y + \vartheta(Y)] = [X, Y] + [\vartheta(X), \vartheta(Y)] + \underbrace{[X, \vartheta(Y)]}_{\text{(I)}} + \underbrace{[\vartheta(X), Y]}_{\text{(II)}}$$

For (I), we have

$$\left([X, \vartheta(Y)] \, (\vec{f}) \right)^i = X \, \underbrace{(\theta_j^i(Y) f^j)}_{\substack{i\text{th component} \\ \text{of } \vartheta(Y)(\vec{f})}} - \theta_j^i(Y) X(f^j) = (X \theta_j^i(Y)) f^j$$

by Leibniz's rule. So (I) is $X((\theta_j^i)(Y)) = X(\vartheta(Y))$ as a matrix. Similarly, (II) is $-Y((\theta_j^i)(X)) = -Y(\vartheta(X))$ as matrix.

Claim: This eventually leads to $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + d\vartheta(X, Y) + [\vartheta(X), \vartheta(Y)]$.
Proof: Use the fact that for any 1-form ω, and for any $X, Y \in \mathfrak{X}(M)$, $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

Observe: $[\vartheta(X), \vartheta(Y)] = (\vartheta \wedge \vartheta)(X, Y)$, by definition of $\wedge$. So we get the same object using both approaches!

Defn: $\mathcal{R}(X, Y) \stackrel{\text{def}}{=} d\vartheta(X, Y) + [\vartheta(X), \vartheta(Y)]$ is the curvature operator.

Math 635 Lecture 10

Professor Alejandro Uribe-Ahumada
Transcribed by Thomas Cohn

2/10/21

Recall from last time:

Defn: Given a vector bundle $\mathcal{E} \to M$ with connection ∇, $X, Y \in \mathfrak{X}(M)$, the curvature operator $\mathcal{R}$ of ∇, evlaulated on (X, Y), is

$$\begin{aligned} \mathcal{R}(X,Y) &: \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}) \\ (X,Y) &\mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

Observe: When Do Carmo defines the curvature operator (Chapter 4, Definition 2.1, in the case where $\mathcal{E} = TM$), they use the opposite sign.

Observe: $\mathcal{R}$ is given by a tensor! What does that mean? Last time, using the second approach, we computed locally in a moving frame $(E_1, \ldots, E_r)$ (with associated connection matrix ϑ) that $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + d\vartheta(X,Y) + [\vartheta(X), \vartheta(Y)]$. So $\mathcal{R}$ has, for its components in the given frame, the components of the vector

$$(d\vartheta(X,Y) + [\vartheta(X), \vartheta(Y)])\vec{f} \qquad s = f^i E_i, \vec{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^r \end{pmatrix}$$

Defn: $\Omega \overset{\text{def}}{=} d\vartheta + \vartheta \wedge \vartheta$ is the curvature matrix of ∇ with respect to the moving frame $(E_1, \ldots, E_r)$.

(This is true because we observed $(\vartheta \wedge \vartheta)(X,Y) = [\vartheta(X), \vartheta(Y)]$.)

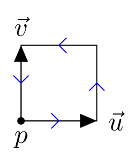
In fact, $\forall p \in U = \operatorname{dom}(E_i)$, $\mathcal{R}(X,Y)(s)(p) \in \mathcal{E}_p$ is the image of $s(p)$ by the linear transformation $\mathcal{E}_p \to \mathcal{E}_p$ whose matrix (in the basis $(E_1(p), \ldots, E_r(p))$) is $\Omega_p(X_p, Y_p)$.

The virtue of this definition is that it's a well-defined global object! But it turns out to be a differential operator of order 0, meaning there's no derivatives, so it's just multiplication. At each point it's given by a linear transformation of the fibers, with the matrix determined by X_p and Y_p.

Observe: The dependence on X and Y is punctual! $\forall p \in M$, $\mathcal{R}(X,Y)(s)(p)$ depends only on $X_p, Y_p \in T_pM$ and $s(p) \in \mathcal{E}_p$.

$\mathcal{R}$, as an object, is "an End-$\mathcal{E}$ valued 2-form on M". That is, $\forall p \in M$, $\forall u, v \in T_pM$, $\mathcal{R}_p(u,v) : \mathcal{E}_p \to \mathcal{E}_p$ is a linear map, and $\mathcal{R}(\cdot, \cdot)$ is bilinear and skew-symmetric.

Intuition: $\mathcal{R}$ is given by infinitesimal holonomy. Given a tiny loop at p below, the holonomy of the path is approximately $\exp(\mathcal{R}_p(u,v))$ (using the matrix exponential).



Even though we're taling about an operator, it's given by a tensor. $\mathcal{R}$ itself is a section of

$$\underbrace{T^*M \otimes T^*M}_{\text{2-form part}} \otimes \underbrace{\mathcal{E}_p \otimes \mathcal{E}_p}_{\text{Endomorphism part}}$$

We're well on our way to defining the Levi-Civita connection!

Consider a vector bundle $\mathcal{E} \to M$, now with a positive definite inner product on each fiber. (In the case where $\mathcal{E} = TM$, this exactly is a Riemannian metric.)

Defn: A connection ∇ on $\mathcal{E}$ (with $\langle\ ,\ \rangle$) is said to preserve $\langle\ ,\ \rangle$ iff $\forall \gamma : [a,b] \to M$, parallel transport $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ is an isometry, i.e., $\forall u, v \in \mathcal{E}_{\gamma(a)}$, $\langle \mathcal{P}_\gamma(u), \mathcal{P}_\gamma(v)\rangle_{\gamma(B)} = \langle u, v\rangle_{\gamma(a)}$.

Prop: Given a vector bundle $\mathcal{E} \to M$, inner product $\langle\ ,\ \rangle$ on each fiber, and a connection ∇, the following are equivalent:

(a) ∇ preserves $\langle\ ,\ \rangle$.
(b) $\forall s, t \in \Gamma(\mathcal{E})$, $\forall X \in \mathfrak{X}(M)$, $X(\langle s, t\rangle) = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$. Note that $\langle s, t\rangle$ is a function on M, which we can differentiate with respect to X. We can think of this as a sort of "product rule".
(c) $\forall (E_1, \ldots, E_r)$ local orthonormal frame (which exists by Gram-Schmidt), the connection matrix ϑ is skew symmetric, i.e., $\forall i, j$, $\theta^i_j = -\theta^j_i$.

Proof: First, we show that (b) $\Leftrightarrow$ (c). Let $(E_1, \ldots, E_r)$ be our local orthonormal frame. Then there are functions f^i, g^j such that $s = f^i E_i$ and $t = g^j E_j$. Thus, we can form $\vec{f}, \vec{g}$, and by orthonormality of the frame

$$\langle s, t\rangle = \sum_{i=1}^{r}\sum_{j=1}^{r} f^i g^j \underbrace{\langle E_i, E_j\rangle}_{=\delta_{ij}} = \sum_{i=1}^{r} f^i g^i = \vec{f} \cdot \vec{g}$$

Thus, with a slight abuse of notation,

$$\langle \nabla_X s, t\rangle \text{ ``}{=}\text{''} (\nabla_X \vec{f}) \cdot \vec{g} = (X(\vec{f}) + \vartheta(X)\vec{f}) \cdot \vec{g}$$

And

$$\langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle = \underbrace{X(\vec{f}) \cdot \vec{g} + \vec{f} \cdot X(\vec{g})}_{=X(\vec{f}\cdot\vec{g})=X(\langle s,t\rangle)} + (\vartheta(X)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(X)\vec{g})$$

So the product rule holds iff $\forall s, t$ / $\forall \vec{f}, \vec{g}$, $(\vartheta(X)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(X)\vec{g}) = 0$, which is true iff $\vartheta(X)$ is skew-symmetric.

In order to show (a), we just change the setting a bit. Let $\gamma : [a,b] \to M$ be a smooth curve. Take $V, W \in \Gamma_\gamma(\mathcal{E})$. We claim that, just as above, we get

$$\underbrace{\left\langle \frac{DV}{dt}, W \right\rangle}_{\text{a function of } t} + \left\langle V, \frac{DW}{dt} \right\rangle - \frac{d}{dt}\langle V, W\rangle = (\vartheta(\dot\gamma)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot\gamma)\vec{g})$$

Assume V and W are parallel along γ. By definition, this means $\frac{DV}{dt} = \frac{DW}{dt} = 0$. Then

$$-\frac{d}{dt}\langle V, W\rangle = (\vartheta(\dot\gamma)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot\gamma)\vec{g})$$

Well,

$$\begin{aligned} \nabla \text{ preserves } \langle\ ,\ \rangle &\Leftrightarrow \frac{d}{dt}\langle V, W\rangle = 0,\ \forall V, W \text{ parallel} \\ &\Leftrightarrow (\vartheta(\dot\gamma)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot\gamma)\vec{g}) = 0 \text{ in all instances} \\ &\Leftrightarrow \vartheta \text{ is skew symmetric} \end{aligned}$$

□

Thm: Let M be a Riemannian manifold. Then $\exists! \nabla$ on $\mathcal{E} = TM \to M$ such that

(a) ∇ preserves the Riemannian metric. *(This depends on the choice of Riemannian metric.)*
(b) $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$. *(This does not depend on the choice of Riemannian metric.)*

Defn: This ∇ is called the <u>Levi-Civita connection</u> on M.

Math 635 Lecture 11

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/12/21

Recall the theorem and definition stated at the end of the previous lecture...

Thm: Let M be a Riemannian manifold. Then $\exists! \nabla$ on $\mathcal{E} = TM \to M$ such that

(a) ∇ preserves the Riemannian metric. *(This depends on the choice of Riemannian metric.)*
(b) $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$. *(This does not depend on the choice of Riemannian metric.)*

Defn: This ∇ is called the <u>Levi-Civita connection</u> on M.

This theorem is sometimes known as the "Fundamental Theorem of Riemannian Geometry". The second condition – that $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$ – is sometimes called a "symmetry condition".

Proof of the theorem: We'll use properties (a) and (b) to find an expression for ∇. Let $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle &= \underbrace{\langle \nabla_X Y, Z\rangle}_{\text{(III)}} + \underbrace{\langle Y, \nabla_X Z\rangle}_{\text{(I)}} + \underbrace{\langle \nabla_Y Z, X\rangle}_{\text{(II)}} + \underbrace{\langle Z, \nabla_Y X\rangle}_{\text{(IV)}} - \underbrace{\langle Y, \nabla_Z X\rangle}_{\text{(I)}} - \underbrace{\langle \nabla_Z Y, X\rangle}_{\text{(II)}} \\
&= \underbrace{\langle Y \nabla_X Z - \nabla_Z X\rangle}_{\text{(I)}} + \underbrace{\langle \nabla_Y Z - \nabla_Z Y, X\rangle}_{\text{(II)}} + \underbrace{\langle \nabla_X Y, Z\rangle}_{\text{(III)}} + \underbrace{\langle [Y, X] + \nabla_X Y, Z\rangle}_{\text{(IV)}} \\
&= \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle + \langle Z, [Y, X]\rangle + 2\langle Z, \nabla_X Y\rangle
\end{aligned}$$

Now, we solve for $2\langle \nabla_X Y, Z\rangle$:

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \Big(\langle X, [Y, Z]\rangle + \langle Y, [X, Z]\rangle + \langle Z, [Y, X]\rangle\Big)$$

This is our defining expression for ∇, since choosing Z in all possible ways defines $\nabla_X Y$. We claim that defining ∇ in this way gives us the desired connection. (This part of the proof is tedious, so check it out in the textbook if you're interested.) $\square$

Computation of the Christoffel Symbols in Coordinates

Let $(x^1, \ldots, x^n)$ be coordinates on $U \subseteq M$, and define $X_i = \frac{\partial}{\partial x^i}$, $g_{ij} = \langle X_i, X_j\rangle$. Note that $[X_i, X_j] = 0$.

Recall: The Christoffel symbols $\Gamma^\ell_{ij} \in C^\infty(U)$ are defined by $\nabla_{X_i} X_j = \Gamma^\ell_{ij} X_\ell$. We want to compute Γ^ℓ_{ij} using the defining expression above. Well,

$$2\langle \nabla_{X_i} X_j, X_k\rangle = 2\left\langle \Gamma^\ell_{ij} X_\ell, X_k\right\rangle = 2\Gamma^\ell_{ij} g_{\ell k} = X_i(g_{jk}) + X_j(g_{ki}) - X_k(g_{ij})$$

Now, we introduce the matrix g^{-1}, the inverse of $(g_{\ell k})$, with the notation $g^{-1} = (g^{km})$, so that $g_{\ell k} g^{km} = \delta^m_k$. If we multiply both sides by g^{-1} / g^{km}, and sum over k, we get

$$\underbrace{2\Gamma^\ell_{ij} \underbrace{g_{\ell k} g^{km}}_{=\delta^m_\ell}}_{=2\Gamma^m_{ij}} = \sum_k g^{km}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Thus,

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k g^{km}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Observe: $[X_i, X_j] = 0 \Leftrightarrow \Gamma_{ij}^m = \Gamma_{ji}^m$. So the number of independent indices of the Christoffel symbols is $\frac{n(n+1)}{2} \cdot n = \frac{n^2(n+1)}{2}$. For example, when $n = 2$ (a surface), there are 6 Christoffel symbols.

Exer: (Do Carmo) For the upper half plane $\mathcal{H}$ (with the metric from HW1), show $\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$, $\Gamma_{11}^2 = \frac{1}{y}$, and $\Gamma_{12}^1 = \Gamma_{22}^2 = -\frac{1}{y}$.

Exer: (HW3) If $M \subset \mathbb{R}^N$, with the induced Riemannian metric from the Euclidean metric on $\mathbb{R}^N$, and if $\gamma : [a, b] \to M$ and $V \in \Gamma_\gamma(TM)$, we can define

$$\frac{\bar{D}V}{dt} = \frac{d}{dt}(V : [a, b] \to \mathbb{R}^n)$$

Claim: If $\frac{D}{dt}$ is the operator associated with the Levi-Civita connection on M, then we have $\frac{DV}{dt}(t) = \pi_{\gamma(t)}[\frac{\bar{D}V}{dt}(t)]$, where $\pi_{\gamma(t)} : \mathbb{R}^N \to T_{\gamma(t)}M$ is the orthogonal projection.

Geodesics

Observe: TM is such that every curve γ into M has a natural lift to TM.

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "(\gamma,\dot{\gamma})"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

In an abuse of notation, we sometimes write "$\dot{\gamma}(t) = \frac{d\gamma}{dt} = (\gamma(t), \dot{\gamma}(t))$".
We can then consider $\frac{D}{dt}\frac{d\gamma}{dt}$, the acceleration of $\gamma \in \Gamma_\gamma(TM)$.

Defn: γ is a <u>geodesic</u> iff $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$.

Ex: Let $M = S^2 \hookrightarrow \mathbb{R}^3$. Then $\gamma(t) = (\cos(t), \sin(t), 0)$ is a geodesic, as $(\frac{\bar{D}}{dt}\dot{\gamma})(t) = \ddot{\gamma}(t) = -\gamma(t)$.
$\gamma(t) \perp T_{\gamma(t)}S^2 \Rightarrow \pi_{\gamma(t)}\frac{\bar{D}}{dt}\dot{\gamma}(t) = 0$.

Observe:

1. If γ is a geodesic, then $\frac{d}{dt}\left\langle \frac{d\gamma}{dt} \frac{d\gamma}{dt} \right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{D}{dt}\frac{d\gamma}{dt} \right\rangle = 0$, so $\left|\left|\frac{d\gamma}{dt}\right|\right|$ is constant with respect to t. In other words, the "speed" of a geodesic is constant.
2. If γ is a geodesic, $c \in \mathbb{R}$, then $\gamma_c(t) \overset{\text{def}}{=} \gamma(ct)$ is also a geodesic. But other reparameterizations are generally not geodesics. The speed of γ_c is $|c|$ times the speed of γ.

Math 635 Lecture 12

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/15/21

Recall:
Defn: $\gamma : [a,b] \to M$ (for M a Riemannian manifold) is a <u>geodesic</u> iff $\frac{D}{dt}\dot{\gamma} = 0$.

Recall: $\dot{\gamma}$ is the natural lift of γ along γ. We say $\dot{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$, so there's some ambiguity in the notation.

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

Review: In coordinates on $U \subset M$, we write $\gamma(t) = (x^1(t), \ldots, x^n(t))$, with each $x^i \in C^\infty([a,b], M)$. Then γ is a geodesic iff $\ddot{x}^k(t) = -\dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(\gamma(t))$, where $\Gamma^k_{ij} : U \to \mathbb{R}$ are the Christoffel symbols.

Observe: If ∇ is trivial, i.e., the "flat case", then $\Gamma^k_{ij} = 0$. So $\ddot{x}^k = 0$, and $\forall k$, $x^k(t) = tv^k(0) + x^k(0)$. See Do Carmo, Chapter 3, §2 for more details.

We want to rewrite the geodesic equations, locally, as a first order system in twice as many unknowns. We introduce $v^1, \ldots, v^n$, which we call the "velocities", such that $v_k \overset{\text{def}}{=} \dot{x}^k$, and $\dot{v}^k = -\Gamma^k_{ij}(\gamma(t))v^iv^j$ are the "accelerations".

Note that time derivatives have been solved in all cases, so there is a unique solution (for a small time interval) given $x^k(0)$ and $v^k(0)$, for $k = 1, \ldots, n$.

Lemma: (Do Carmo 2.3) $\exists! G \in \mathfrak{X}(TM)$ s.t. the integral curves of G are precisely of the form $\dot{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$, where γ is a geodesic. In other words, the integral curves of G are precisely the lifts to TM of geodesics on M. (Integral curves of G are locally solutions to the above system of differential equations.)

Proof: First, we'll prove local existence and uniqueness of G in coordinates. Let $V \subset M$ be a coordinate neighborhood, with coordinates $(x^1, \ldots, x^n)$, inducing coordinates $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ on TV by $v = \sum_{i=1}^n v^i \, \partial_{x^i}|_p$ for $(p,v) \in TV$. Then $G = \sum_{i=1}^n a_i \partial_{x^i} + b_i \partial_{v^i}$, for some $a_i, b_i \in C^\infty(TV)$ (note that this is true for any vector field on TV).

Now, comparing with the system of differential equations, we can see that we must have $a_i = v^i = \dot{x}^i$, $\forall i$. So $G(x,v) = v^i\partial_{x^i} - \Gamma^k_{ij}(x)v^iv^j\partial_{v^k}$ iff the integral curves of V solve the system of equations.

Finally, local existence and uniqueness implies global existence and uniqueness by covering M with coordinate charts. $\square$

Observe: The vector field G can be described using T^*M and its symplectic form, and $TM \to T^*M$ by $T_pM \to T_p^*M$ using $\langle\ ,\ \rangle_p$. In the future, we'll also consider the Hamiltonian picture...

Now, we want to think about the flow of G on TM. Let $X \in \mathfrak{X}(M)$. Given any $m \in M$, there's a neighborhood $\mathcal{U} \subseteq M$ of m, $\delta > 0$, and $\varphi : (-\delta, \delta) \times \mathcal{U} \to M$ smooth such that $\forall \mu \in \mathcal{U}$, $t \mapsto \varphi(t, \mu)$ is the integral curve of X s.t. $\varphi(0, \mu) = X_\mu$, and $\forall t$, $\frac{d}{dt}\varphi(t,\mu) = X_{\varphi(t,\mu)}$.

Now, apply this to $\mathcal{M} = TM$, $X = G$, and $m = (p, 0)$ for $p \in M$. Then $\exists \mathcal{U} \subseteq \mathcal{M}$ and $\delta > 0$ as in the theorem. So we have $\{(q,0) \in TM : q \in M\} \cong M$.

Claim: $\exists V \subseteq M$, a neighborhood of p, and $\varepsilon > 0$ s.t. $\{(q,v) \in TM \mid q \in V, ||v|| < \varepsilon\} \subseteq \mathcal{U}$.

We get

$$\begin{tikzcd}(-\delta,\delta)\times\{(q,v)\mid q\in V, ||v||<\varepsilon\} \arrow[r, "\varphi"] \arrow[dr, "\gamma \overset{\text{def}}{=} \pi\circ\varphi"'] & TM \arrow[d, "\pi"] \\ & M\end{tikzcd}$$

An important property of γ is that $\forall(q,v)$, $t\mapsto\gamma(t,q,v)$ is *the unique* geodesic s.t. $\gamma(0,q,v)=q$ and $\left.\dfrac{d}{dt}\gamma(t,q,v)\right|_{t=0}=v$.

Lemma: By reparameterizing geodesics by a constant factor in time, one can show (keeping the notation from our previous discussion) that, for $a>0$, then $\gamma(t,q,av)=\gamma(at,q,v)$, provided that both sies are defined.

Proof: Check that both sides are geodesics, with the same initial conditions. Then by uniqueness of geodesics, they're equivalent. $\square$

Math 635 Lecture 13

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/17/21

Review: Given (M, g) a Riemannian manifold, $\exists! G \in \mathfrak{X}(TM)$ s.t. the integral curves of G are the lifts of geodesics.

Notation: $\forall (q, v) \in TM$, $t \mapsto \gamma(t, q, m)$ is the geodesic with initial condition (q, v). It is the projection onto M of the integral curve of G, starting at (q, v). For given (q, v), $\gamma(t, q, v)$ has a maximal domain of definition, an interval in t, that depends on (q, v).

Lemma: (Lemma 1 from Last Time) $\forall p \in M$, $\exists V \subseteq M$, a neighborhood of p, and $\exists \varepsilon, \delta > 0$ s.t. γ is defined on the set $(-\delta, \delta) \times \{(q, v) \in TV : ||v|| < \varepsilon\}$. That is, $\gamma(t, q, v)$ is defined $\forall q \in V$, $v \in T_q V$, $||v|| < \varepsilon$, $|t| < \delta$, and γ is smooth as a map.

Defn: Given $V \subseteq M$, $\varepsilon > 0$, we define the <u>ε-ball tangent bundle</u>, by $B_\varepsilon(TV) = \{(q, v) \in TV : ||v|| < \varepsilon\}$. This is a fiber bundle $B_\varepsilon(TV) \to V$, whose fibers are open balls of radius ε, centered at 0.

Defn: We also define the <u>unit tangent bundle</u> of M, $S_1(TM)$, by $S_\varepsilon(TV) = \{(q, v) \in TV : ||v|| = \varepsilon\}$. This is a fiber bundle $S_1(TM) \to M$, whose fibers are S^{n-1}.

Ex: The unit tangent bundle of S^2 is isomorphic to

$$\left\{(\vec{q}, \vec{v}) \in \mathbb{R}^3 \times \mathbb{R}^3 : ||\vec{q}|| = 1, ||\vec{v}|| = 1, \vec{q} \cdot \vec{v} = 0\right\}$$

(The final condition is a tangency condition). In turn, this is diffeomorphic to SO(3) as manifolds, by $(\vec{q}, \vec{v}) \mapsto (\vec{q}, \vec{v}, \vec{q} \times \vec{v})$. Treating the three output vectors as columns of a matrix yields an orthogonal matrix with determinant 1.

Observe: G is tangent to $S_\varepsilon(TM)$, $\forall \varepsilon > 0$. This is a fancy way to say that, along a geodesic, speed is constant. Because $\frac{d}{dt}\langle \dot{\gamma}, \dot{\gamma} \rangle = 0$, we know $||\dot{\gamma}||$ is constant, so the integral curves of G are fully contained in $S_\varepsilon(TM)$.

Cor: If M is compact, every geodesic is defined $\forall t \in \mathbb{R}$, i.e., G is complete.

Proof: M is compact, so $\forall \varepsilon > 0$, $S_\varepsilon(TM)$ is compact, and any field on a compact manifold is complete. $\square$

Lemma: (Lemma 2 from Last Time) (Homogeneity of Geodesic Flow) Let $(q, v) \in TM$, $a > 0$. If $\gamma(t, q, v)$ is defined for $|t| < \delta$, then $\gamma(t, q, av)$ is defined for $|t| < \frac{\delta}{a}$, and $\gamma(t, q, av) = \gamma(at, q, v)$.

Proof: Check that both sides satisfy the geodesic equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, and have the same initial conditions (namely, (q, av)). $\square$

Prop: (Do Carmo 2.7) $\forall p \in M$, $\exists V \subseteq M$ a neighborhood of p, and $\varepsilon > 0$ s.t. $\forall (q, v) \in B_\varepsilon(TV)$, $\gamma(t, q, v)$ is defined for $|t| < 43$. (Note: we really just need it to be defined for $t = 1$, so we can get our exponential map. But 43 is such a nice number.)

Proof: Let V, δ, ε_1 be as in Lemma 1, so that $\forall (q, v) \in B_{\varepsilon_1}(TV)$, $\gamma(t, q, v)$ is defined for $|t| < \delta$. Choose $a > 0$ s.t. $|t| < \delta \Leftarrow a|t| = |at| < 43$ – specifically, choose $a = \frac{\delta}{43}$. Now, by Lemma 2, $\gamma(t, q, \frac{\delta}{43}v)$ is defined for $|t| < 43$ if $||v|| < \varepsilon_1$. Now define $\varepsilon = \varepsilon_1 \cdot \frac{\delta}{43}$. Thus, $\frac{\delta}{43}||v|| < \varepsilon_1 \Leftrightarrow ||v|| < \varepsilon$. $\square$

Defn: Let $p \in M$, $V \subset M$ a neighborhood of p, and ε as in the previous proposition. The we define

1. $\exp : B_\varepsilon(TV) \to M$
   $(q, v) \mapsto \gamma(1, q, v)$
2. $\exp_p : B_\varepsilon(0) \to M$
   $v \mapsto \gamma(1, p, v)$

Observe: Both exp and $\exp_p$ are differentiable.

Lemma: $\forall p \in M$, $d(\exp_p)_{v=0}$ is the identity.

$$\begin{tikzcd} T_0(T_pM) \arrow[r] & T_pM \\ T_pM \arrow[u, phantom, "\cong" sloped] \arrow[ur, "\text{Id (claimed)}"'] & \end{tikzcd}$$

Proof: Use curves to compute $d(\exp_p)_0$. Take a curve in T_pM, starting at $0 \in T_pM$, e.g., $t \mapsto tw$ for some $w \in T_pM$. Then

$$d(\exp_p)_0(w) = \frac{d}{dt}\exp_p(tw)\Big|_{t=0} = \frac{d}{dt}\gamma(1,p,tw)\Big|_{t=0} = \frac{d}{dt}\gamma(t,p,w)\Big|_{t=0} = w$$

$\square$

Cor: $\exp_p$ is a local diffeomorphism near 0, i.e., $\exists \mathscr{N} \subset T_pM$, a neighborhood of 0 such that $\exp_p\big|_{\mathscr{N}} : \mathscr{N} \xrightarrow{\sim} U \overset{\text{open}}{\subset} M$, for some U.

Math 635 Lecture 14

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/19/21

Hamiltonian Formulation of Geodesic Flow

Defn: A symplectic manifold is a pair (X, ω) where ω is a 2-form on X s.t. $d\omega = 0$ and ω is pointwise non-denerate: $\forall m \in X$, the map

$$\begin{aligned} \omega_m^\sharp : T_m X &\to T_m^* X \\ v &\mapsto -\omega_m(\cdot, v) \end{aligned}$$

is an isomorphism.

Note: This implies the dimension of X is even, since skew symmetric forms on odd dimensional spaces are singular.

Ex: Let $X = \mathbb{R}^{2n}$, with coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$. Then $\omega = \sum_i dp_i \wedge dx^i$, as a matrix, is $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.

We will see, $\forall M$ smooth manifolds, that $X = T^*M$ is naturally a symplectic manifold. And on symplectic manifolds, we can define Hamiltonian dynamics.

Defn: Given $H \in C^\infty(X)$, where (X, ω) is a symplectic manifold, the Hamilton field of H, $\Xi_H \in \mathfrak{X}(X)$, is defined by the condition that $-\iota_{\Xi_H} = \omega(\cdot, \Xi_H) = dH$. Existence is guaranteed by the non-degeneracy of ω.

We want to compute a local formula: In $\mathbb{R}^{2n}$, $H : \mathbb{R}^{2n} \to \mathbb{R}$. Then $\Xi_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$. The flow/trajectory of Ξ_H are

$$\begin{cases} \dot{x}^i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t)) \\ \dot{p}_i(t) = -\frac{\partial H}{\partial x^i}(x(t), p(t)) \end{cases}$$

These are Hamilton's equations.

Exer: $H = \frac{1}{2m}\,||p||^2 + V(x)$ is Newton's second law, $\ddot{x} = -\nabla V$.

Properties of Hamiltonian flows (i.e. flows of Ξ_H):

(a) $\Xi_H(H) = 0$, i.e., H is constant along trajectories of Ξ_H.
Proof: $\Xi_H(H) = dH(\Xi_H) = \omega(\Xi_H, \Xi_H) = 0$ by antisymmetry. □

(b) $\mathcal{L}_{\Xi_H}\omega = 0$.
Proof: Use Cartan's formula. $\mathcal{L}_{\Xi_H}\omega = \iota_{\Xi_H}\underbrace{d\omega}_{=0} + d(\underbrace{\iota_{\Xi_H}\omega}_{=-dH}) = -d^2H = 0$. □

Volume elements (Liouville)

On any symplectic manifold (X, ω), the form $\frac{\omega^n}{n!}$ is a volume form.

Ex: In $\mathbb{R}^{2n}$, $\frac{\omega^n}{n!} = dp_1 \wedge dx^1 \wedge dp_2 \wedge dx^2 \wedge \cdots \wedge dp_n \wedge dx^n$.

If we're given a Hamiltonian $H \in C^\infty(X)$, and $c \in \mathbb{R}$ is a regular value of H, then let $\Sigma = H^{-1}(c) \hookrightarrow X$, a codim-1 submanifold. We claim that $\exists! \lambda \in \Omega^{2n-1}(X)$ s.t. in a neighborhood of Σ, $\frac{\omega^n}{n!} = \lambda \wedge dH$, and $\iota^*(\omega)$ is unique. This is a volume form on Σ.

Cor: The Hamilton flow of H preserves $\frac{\omega^n}{n!}$, and its restriction to any regular level set $H^{-1}(c)$ preserves the Liouville measure on that level set.

$$\phi_t^*\left(\frac{\omega^n}{n!}\right) = \frac{\omega^n}{n!}; \qquad \left(\phi_t|_{H^{-1}(c)}\right)^*(\iota^*\lambda) = (\iota^*\lambda)$$

Symmetries

Question: Given $H, G \in C^\infty(X)$, the Hamiltonian flows of H and G commute iff $[\Xi_G, \Xi_H] = 0$, which is true iff Ξ_H is ϕ_t^G-related to itself.

Lemma: For X connected, this is equivalent to $(\phi_t^G)^* dH = dH$.

Proof:

$$\begin{aligned}(\phi_t^G)^* dH = dH &\Leftrightarrow \mathcal{L}_{\Xi_G}(dH) = 0\\ &\Leftrightarrow \cancel{\iota_{\Xi_G} d^2 H} + d(\iota_{\Xi_G} dH) = dH(\Xi_G) = 0\\ &\Leftrightarrow d(dH(\Xi_G)) = 0\\ &\Leftrightarrow dH(\Xi_G) = \Xi_G(H) \text{ is locally constant}\\ &\Leftrightarrow \Xi_G(H) \text{ is constant (because } X \text{ is connected)}\end{aligned}$$

This is a symmetric condition:

$$dH(\Xi_G) = \omega(\Xi_G, \Xi_H) = -\omega(\Xi_H, \Xi_G) = -dG(\Xi_H)$$

□

The main example of (X, ω) is T^*M, for some arbitrary smooth manifold M.

Prop: For an arbitrary smooth manifold M, T^*M has a natural symplectic structure.

Proof: We'll show that T^*M has a natural, "tautological" 1-form, α, which is sometimes called a Liouville form, defined by:
For $(x, \xi) \in T^*M$, $x \in M$, $\xi \in T_x^*M$, let $v \in T_{(x,\xi)}(T^*M)$. Then $\alpha_{(x,\xi)}(v) = \xi(\underbrace{\pi_*(v)}_{\in T_x M})$. In coordinates, say we have $(x^1, \ldots, x^n)$ on $U \subset M$, and $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ on $T{*}U$. Then $\xi = p_i(\xi) dx^i$, so $v = a^i \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial p_i}$. $\pi_*(v) = a^i \frac{\partial}{\partial x^i}$. $\xi(\pi_*(v)) = p_i(\xi) a^i$. Altogether, $\alpha = p_i dx^i$, and $\omega + d\alpha = dp_i \wedge dx^i$, just as in $\mathbb{R}^{2n}$.

Math 635 Lecture 15

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

2/22/21

Some review:

- For (X, ω) a symplectic manifold, given $H \in C^\infty(X)$, $\exists! \Xi_H \in \mathfrak{X}(X)$ s.t. $\iota_{\xi_H}\omega = -dH$. Refer to Lee, Smooth Manifolds, Chapter 22. Warning: His ω is different from ours by a sign. But the ξ_H is the same.
- The flow of ξ_H preserves H, ω, $\frac{\omega^n}{n!}$, and the Liouville volume on regular level sets of H.
- Symmetries: We proved that $\forall G, H \in C^\infty(X)$, $[\Xi_G, \Xi_H] = 0$ iff $\omega(\Xi_G, \Xi_H) = \Xi_G(H) = -\Xi_H(G)$ is constant. We call this the Poisson bracket, and denote it by $\omega(\Xi_G, \Xi_H) = \{G, H\}$.

Defn: We say that G is a conserved quantity under the flow of Ξ_H if and only if G is constant along the trajectories of Ξ_H if and only if $\Xi_H(G) = 0$.

Observe: The above implies that G is a conserved quantity under the flow of Ξ_H if and only if $\{G, H\} = 0$, so Ξ_G preserves H if and only if Ξ_H preserves G.

We'll apply this as follows:

- Geodesic flow is a Hamilton flow for some $H : T^*M \to \mathbb{R}$.
- If we have a field on M generating isometries, we'll get a G on T^*M that preserves H. This implies that G is constant along geodesics.

Recall: If M is any C^∞ manifold, then $X = T^*M$ has a natural symplectic form ω, which, in standard local coordinates on T^*M, takes the form $\omega = dp_i \wedge dx^i = d\alpha$, where $\alpha = \sum p_i dx^i$ is the tautological 1-form.

A class of examples of Hamiltonians on $T{*}M$: Start with $X \in \mathfrak{X}(M)$. Then define

$$\begin{aligned} \ell_X : T^*M &\to \mathbb{R} \\ (x, \xi) &\mapsto \xi(X_x) \end{aligned}$$

Observe that $\ell_X(x, \xi)$ is linear in ξ, i.e., linear on the fibers.

In coordinates $(x^1, \dots, x^n)$, $X = f^i \frac{\partial}{\partial x^i}$, we get $(x^1, \dots, x^n, p_1, \dots, p_n)$ coordinates on T^*U. Then

$$\ell_X(x^1, \dots, x^n, p_1, \dots, p_n) = p_i f^i(X)$$

Recall Hamilton's equations for the flow of $\Xi_{\ell_X} \in \mathfrak{X}(T^*M)$:

$$\begin{cases} \dot{x}^i = \frac{\partial \ell_X}{\partial p_i} = f^i & \leftrightarrow \text{exactly the flow of } X \text{ itself} \\ \dot{p}^i = -\frac{\partial \ell_X}{\partial x^i} = -p_j \frac{\partial f^j}{\partial x^i} \end{cases}$$

Prop: Let $\phi_t : M \to M$ be the flow of X. Then the flow of Ξ_{ℓ_X} is $\tilde{\phi}_t : T^*M \to T^*M$, given by

$$\tilde{\phi}_t(x, \xi) = \big(\phi_t(x), ((d\phi_{-t})_{\phi_t(x)})^* \xi\big)$$

Let's unpack this. We know

$$d(\phi_{-t})_{\phi_t(x)} : T_{\phi_t(x)}M \to T_x M$$

Thus, its pullback is a map $T^*_x \to T^*_{\phi_t(x)}M$. So we have the following commutative diagram:

$$\begin{tikzcd} T^*M \arrow[r, "\tilde{\phi}_t"] \arrow[d] & T^*M \arrow[d] \\ M \arrow[r, "\phi_t"] & M \end{tikzcd}$$

Our main example is when (M,g) is a Riemannian manifold. Using g, $\forall x \in M$, we get $T_xM \cong T_x^*M$ by $\mathbb{F} : v \mapsto \langle \cdot, v\rangle$. We can assemble this into a "big map" between the total spaces of the bundle:

$$\begin{tikzcd} TM \arrow[rr, "\sim", "\mathbb{F}"'] \arrow[dr] & & T^*M \arrow[dl] \\ & M & \end{tikzcd}$$

by computing $\mathbb{F}$ fiberwise. In coordinates, let $(x^1,\ldots,x^n)$ be coordinates on $U \overset{\text{open}}{\subset} M$. Then we get coordinates on TU and T^*U, and $\mathbb{F}$ is

$$\mathbb{F}(x^1,\ldots,x^n,v^1,\ldots,v^n) = (x^1,\ldots,x^n,p_i = g_{ij}v^j)$$

Thm: Let $\begin{aligned} L : TM &\to \mathbb{R} \\ (x,v) &\mapsto \tfrac{1}{2}\,||v||^2 \end{aligned}$. Define $H : T^*M \to \mathbb{R}$ by $H = L \circ \mathbb{F}^{-1}$.

$$\begin{tikzcd} & TM \arrow[rr, "\sim", "\mathbb{F}"'] \arrow[dl, "L"'] \arrow[dr] & & T^*M \arrow[ll, bend right, "\mathbb{F}^{-1}"'] \arrow[dl] \arrow[lll, bend right=60, "H=L\circ\mathbb{F}^{-1}"'] \\ \mathbb{R} & & M & \end{tikzcd}$$

Then $\mathbb{F}$ intertwines the geodesic flow on TM with the Hamilton flow of H.

Cor: Geodesic flow is volume preserving, and we can use the Hamiltonian to study symmetries and other things.

We could prove this now, but it would require a terribly long and boring computation. There's an elegant proof of this using a different point of view, which we will do later on.

Observe: $H(x,p) = \frac{1}{2}g^{ij}(x)p_ip_j$. $\dot{x}^i = \frac{\partial H}{\partial p_i} = g^{ij}(x)p_j$.

Application: Surfaces of revolution. Let $S = \partial_\Theta \in \mathfrak{X}(M)$ generate rotations: $\phi_t : M \to M$ where $\phi_{t+2\pi} = \phi_t$, and $\forall t$, ϕ_t is an isometry. (Compare with problem 1 on page 77 of Do Carmo.)

On the cotangent bundle, $\tilde{\phi}_k : T^*M \to T^*M$ preserves H; therefore, $\ell_{\partial_\Theta} : T^*M \to T^*M$ is a conserved quantity. But what is it geometrically? We can pass it to TM:

$$\begin{aligned} \ell_{\partial_\Theta} \circ \mathbb{F} : TM &\to \mathbb{R} \\ (x,v) &\mapsto \langle v, \partial_\Theta\rangle \end{aligned}$$

where $\langle\ ,\ \rangle$ is the Euclidean dot product (from the subspace-induced Riemannian metric). Again, let $\gamma(t)$ be a geodesic with speed 1. We know that

$$\langle\dot\gamma(t), \partial_\Theta\rangle = ||\partial_\Theta||\cos\angle(\dot\gamma(t), \partial_\Theta)$$

is independent of t. It turns out $||\partial_\Theta|| = r$, the distance to the axis of symmetry, since the line of latitude has perimeter $2\pi\,||\partial_\Theta||$. So we conclude that along a speed-1 geodesic, $\gamma(t)\cos\angle(\dot\gamma(t), \partial_\Theta)$ is independent of t.

Math 635 Lecture 16

Professor Alejandro Uribe-Ahumada
Transcribed by Thomas Cohn

2/26/21

Notation: For M, a Riemannian manifold, the map $t \mapsto G(t,p,v)$ for $p \in M$, $v \in T_pM$, denotes the geodesic with initial conditions (p,v). We saw that, $\forall c \in \mathbb{R}$, if defined, $G(t,p,cv) = G(ct,p,v)$.

We also had the theorem that $\forall p \in M$, $\exists \varepsilon > 0$ s.t. $\forall v \in B_0(\varepsilon) \subseteq T_pM$ (recall that $B_0(\varepsilon) = \{v \in T_pM : ||v|| < \varepsilon\}$), $G(t,p,v)$ is defined for $t \in [0,1]$. Based on that fact, we define $\exp_p(v) = G(1,p,v)$.

Lemma: $d(\exp_p)_{v=0} = \mathrm{Id}_{T_pM}$.

Cor: $\forall p \in M$, $\exists \varepsilon > 0$ s.t. $\exp_p : B_0(\varepsilon) \to M$ is a diffeomorphism onto its (open) image $U = \exp_p(B_0(\varepsilon))$.

Defn: Such a neighborhood U of p is called a <u>normal neighborhood</u> of p.

Warning: "Normal neighborhood" sometimes means any neighborhood that is the diffeomorphic image by $\exp_p$ of a neighborhood of $0 \in T_pM$.

Defn: <u>Normal coordinates</u> centered at $p \in M$ are any coordinates $(x^1, \ldots, x^n)$ of the form

$$\begin{tikzcd} U \arrow[r, "(\exp_p)^{-1}"] \arrow[rr, bend right, "(x^1,\ldots,x^n)"'] & T_pM \arrow[r, "\sim"] & \mathbb{R}^n \end{tikzcd}$$

where $n = \dim M$, U is a normal neighborhood, $(\exp_p)^{-1}$ is restricted to the image of $\exp_p$, and the mapping between T_pM and $\mathbb{R}^n$ is any orthogonal linear isomorphism.

Observe: The only choice needed to get normal coordinates on U is the identification $T_pM \cong \mathbb{R}^n$ that we select. Two different choices of identification will be related by an orthogonal matrix (that is, $y^i = a^i_j x^j$)

$$\begin{tikzcd} & & \mathbb{R}^n \arrow[dd, "\wr", "(a^i_j) \in O(n)"'] \\ U \arrow[r] \arrow[urr, bend left, "(x^1,\ldots,x^n)"] \arrow[drr, bend right, "(y^1,\ldots,y^n)"'] & T_pM \arrow[ur, phantom, "\cong"] \arrow[dr, phantom, "\cong"] & \\ & & \mathbb{R}^n \end{tikzcd}$$

Prop: In any normal coordinate system $(x^1, \ldots, x^n)$ centered at $p \in M$,

(a) $g_{ij}(0) = \delta_{ij}$
(b) $\forall \vec{v} \in \mathbb{R}^n$, $t \mapsto (tv^1, \ldots, tv^n)$, i.e., $x^i = tv^i$, is a geodesic.
We call these "radial geodesics", and they're precisely $G(t,p,v^i \frac{\partial}{\partial x^i})$.
(c) $\forall i,j,k$, $\Gamma^k_{ij}(0) = 0$
(d) $\forall i,j,k$, $\frac{\partial g_{ij}}{\partial x^k}(0) = 0$

Proof:

(a) Use the fact that $d(\exp_p)_0 = \mathrm{Id}$, and the isometry is orthogonal.
(b) By the definition of exp, the normal coordinates of $G(t,p,v^i \frac{\partial}{\partial x^i}) = (tv^1, \ldots, tv^n)$. (This is kind of tautological.)
(c) Use (b) and the geodesic equations: $\ddot{x}^k = -\dot{x}^i \dot{x}^j \Gamma^k_{ij}(x(t))$. Look at radial geodesics: $\ddot{x}^k = \ddot{(tv^k)} = 0$. So $\forall \vec{v} \in \mathbb{R}^n$, $v^i v^j \Gamma^k_{ij}(0) = 0$. This is a quadratic form in V; because $\forall k$, $\Gamma^k_{ij} = \Gamma^k_{ji}$, by the polarization identity for quadratic forms, $\Gamma^k_{ij} = 0$, $\forall i,j,k$.

(d) This is just an algebraic exercise. (Left for HW.)

□

Lemma: (Polarization Identity) Let $\Gamma = (\Gamma_{ij})$ be a symmetric matrix, and let $Q(\vec{v}) = \vec{v}\Gamma\vec{v}^T$ be the quadratic form, for all column vectors $\vec{v}$. Then we can find Γ, and the quadratic form is 0 iff the matrix is 0.

The proof follows directly from the fact that

$$\vec{v}\Gamma\vec{w}^T = \frac{1}{4}(Q(\vec{v}+\vec{w}) - Q(\vec{v}-\vec{w}))$$

Observe that it's necessary to assume that the matrix is symmetric. If it's anti-symmetric, then Q is 0.

Observe that, in normal coordinates, $d(\exp_p)_{tv}(tv) = \sum_i v_i \frac{\partial}{\partial x^i}$.

Defn: Assume $U = \exp_p(B_\varepsilon(0))$ is a normal neighborhood for some $\varepsilon > 0$. $\forall r \in (0, \varepsilon)$, the image under $\exp_p$ of the ball $S_r(0) = \{v \in T_pM : ||v|| = r\}$ is a geodesic sphere: $\exp_p(S_r(0)) \subset U \subset M$.

Lemma: (Gauss' Lemma) In a normal neighborhood of p, radial geodesics are orthogonal to geodesic spheres. That is, $\langle d(\exp_p)_v(v), d(\exp_p)_v(w)\rangle = 0$ if $v \cdot w = 0$, because $d(\exp_p)_v(v)$ is tangent to the radial geodesic, and $v \cdot w = 0$ iff w is tangent at v to the sphere of radius $||v||$.

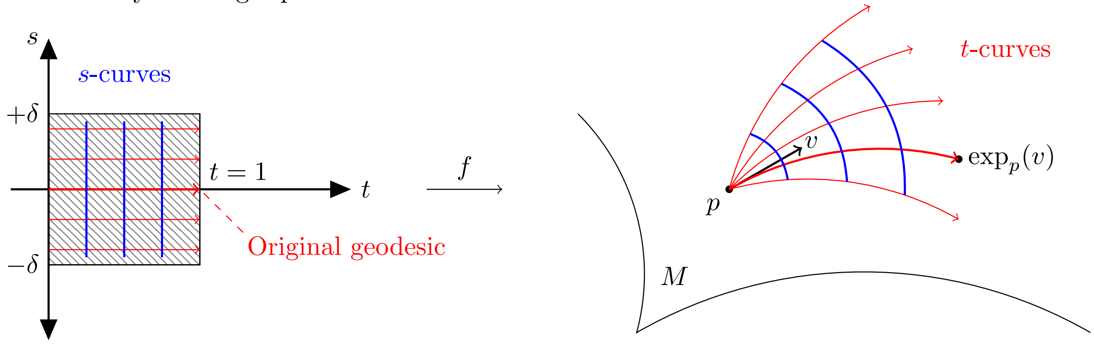
We need a new tool to deal with this!

Families of Curves

The idea is to extend a single radial geodesic to a family of radial geodesics, according to a parameter s. Consider a C^∞ map $f : [0,1] \times (-\delta, \delta) \to M$.

- For fixed s, $t \mapsto f(t,s)$ is a t-curve
- For fixed t, $s \mapsto f(t,s)$ is a s-curve

We're effectively creating a parametric surface:



If we let $f_t = \frac{\partial f}{\partial t}$, the velocity of a t-curve, and $f_s = \frac{\partial f}{\partial s}$, the velocity of an s-curve, then these define vector fields along each s and t curve (respectively).

Prop: For any family of curves as above, $\frac{D}{dt} f_s = \frac{D}{ds} f_t$. This follows from the Levi-Civita connection ∇ being torsion-free.

$\frac{D}{dt} f_s$ is the covariant derivative of the f_s vectors along t.

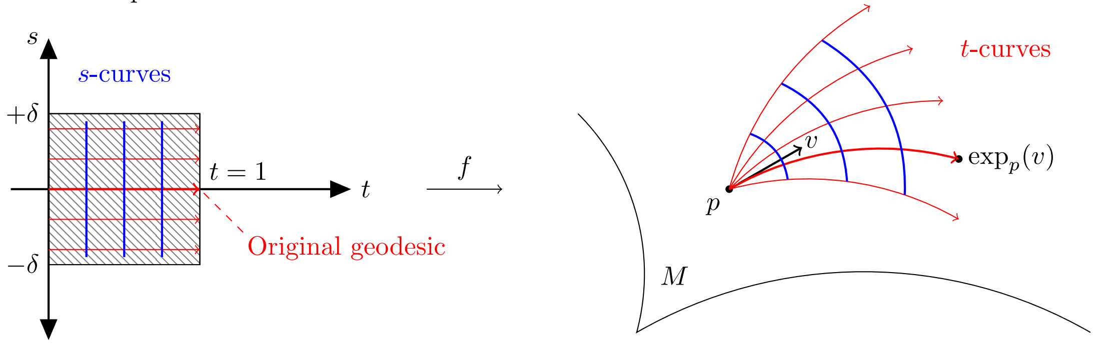
Math 635 Lecture 17

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/1/21

Recall the setup from last time:



Defn: A <u>vector field along f</u> is a lift $\tilde{f}$ of f to TM. I.e., $\tilde{f}$ is defined such that the following diagram commutes:

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ D \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & M \end{tikzcd}$$

Note that such a lift isn't unique!

Ex: One such lift is $\tilde{f} = \left\{ \begin{array}{l} f_t \\ f_s \end{array} \right.$. For such a $\tilde{f}$, we can define $\frac{D}{dt}\tilde{f}$ and $\frac{D}{ds}\tilde{f}$ by restricting $\tilde{f}$ to t and s curves, respectively.

Prop: $\frac{D}{dt}f_s = \frac{D}{ds}f_t$ at each (t, s).

Proof: We will compute in local coordinates $(x^1, \ldots, x^n)$. Let $X_i = \frac{\partial}{\partial x^i}$, $\forall i$. We write $f(t, s) = (x^1(t, s), \ldots, x^n(t, s))$, where $x^i(t, s) : \mathrm{dom}(f) \to \mathbb{R}$. Note that we can write $f_s = \frac{\partial x^i}{\partial s} X_i(f(t, s))$, and likewise for f_t. We now compute

$$\frac{D}{dt} f_s = \frac{\partial^2 x^i}{\partial t \partial s} X_i + \frac{\partial x^i}{\partial s} \frac{D}{dt} X_i$$

We know $f_t = \frac{\partial x^j}{\partial t} X_j$, so because $\frac{D}{dt}$ is the covariant derivative with respect to f_t,

$$\frac{D}{dt} X_i = \frac{\partial x^j}{\partial t} \nabla_{X_j} X_i \qquad \frac{D}{dt} f_s = \frac{\partial^2 x^i}{\partial t \partial s} X_i + \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial t} \nabla_{X_j} X_i$$

Computing similarly, we also get

$$\frac{D}{ds} f_t = \frac{\partial^2 x^i}{\partial s \partial t} X_i + \frac{\partial x^i}{\partial t} \frac{\partial x^j}{\partial s} \nabla_{X_j} X_i$$

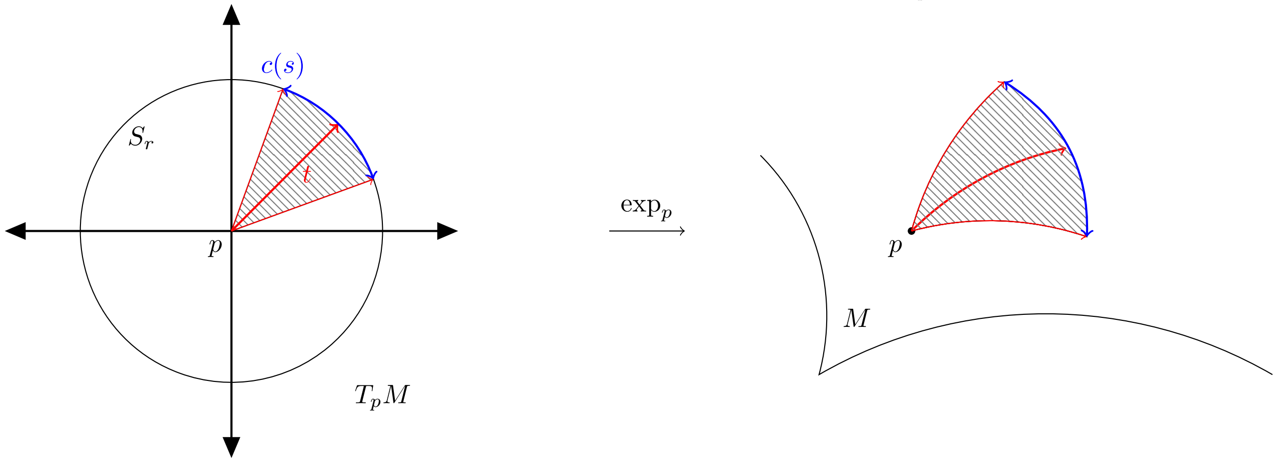
By Clairaut's theorem, $\frac{\partial^2 x^i}{\partial t \partial s} = \frac{\partial^2 x^i}{\partial s \partial t}$, so the first term of $\frac{D}{dt} f_s$ and $\frac{D}{ds} f_t$ are equal. Furthermore, because the Levi-Civita connection is torsion-free, $[X_i, X_j] = 0$, so $\nabla_{X_j} X_i = \nabla_{X_i} X_j$. This means we can swap the coefficients in the second term to show equality. We conclude that $\frac{D}{dt} f_s = \frac{D}{ds} f_t$. $\square$

Observe: We can ask if $\frac{D}{ds}$ and $\frac{D}{dt}$ commute. We'll see on Friday that the answer is no, because curvature comes into play.

We're now ready to prove Gauss' lemma...

Lemma: (Gauss' Lemma) In a normal neighborhood of p, radial geodesics are orthogonal to geodesic spheres.

Proof: Let $p \in M$ and $\varepsilon > 0$ such that $\exp_p : B_\varepsilon(0) \xrightarrow{\sim} \exp_p(B_\varepsilon(0))$ (with $B_\varepsilon(0) \subseteq T_pM$ and $\exp_p(B_\varepsilon(0)) \subseteq M$) is a diffeomorphism onto its image. Take $r \in (0, \varepsilon)$, so $S_r \subseteq T_pM$ is the sphere of radius r. Then choose any curve $s \mapsto c(s) \in S_r$, for an arbtirarily small domain $s \in (-\delta, \delta)$. Define $f(t,s) = \exp_p(tc(s))$. Illustration:



The key calculation we'll perform is $\frac{d}{dt}\langle f_t, f_s\rangle$; we want to show it's equal to 0. Well,

$$
\begin{aligned}
\frac{d}{dt}\langle f_t, f_s\rangle &= \left\langle \frac{D}{dt} f_t, f_s \right\rangle + \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
&\overset{(1)}{=} \langle 0, f_s\rangle + \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
&= \left\langle f_t, \frac{D}{dt} f_s \right\rangle \\
&\overset{(2)}{=} \left\langle f_t, \frac{D}{ds} f_t \right\rangle \\
&= \frac{1}{2}\left\langle \frac{D}{ds} f_t, f_t \right\rangle + \frac{1}{2}\left\langle f_t, \frac{D}{ds} f_t \right\rangle \\
&= \frac{1}{2}\frac{d}{ds}\langle f_t, f_t\rangle \\
&\overset{(3)}{=} \frac{1}{2}\frac{d}{ds}\left|\left|c(s)^2\right|\right| \\
&= \frac{1}{2}\frac{d}{ds} r^2 \\
&= 0
\end{aligned}
$$

with (1) because $t \mapsto \exp_p(tv) = G(1, p, tv) = G(t, p, v)$ is a geodesic, (2) because of the proposition from earlier, and (3) because $\langle f_t, f_t\rangle$ is constant with repsect to t, so we can choose to evaluate it at $t = 0$. Now, we can evaluate $\langle f_t, f_s\rangle|_{t=0} = \langle c(s), 0\rangle = 0$, so we get that, for all t, s, $\langle f_t, f_s\rangle = 0$. $\square$

Why are we done? Well, we can find $f_t(t)$ by $f(t, s = 0)$ WOLOG, so $f(t, 0)$ is the velocity of the radial geodeisc $t \mapsto \exp_p(tv(0))$, and $f(t, 0)$ is an arbitrary tangent vector to the geodesic sphere $\exp_p(S_r)$. So we conclude that the tangent space of a point q on the geodesic sphere is perpendicular to the geodesic $\exp_p(tv)$, where $\exp_p(v) = q$.

Cor: If $U = \exp_p(B_\varepsilon(0))$ is a normal neighborhood of p, and $q \in U$, then the shortest path from p to q is $t \mapsto \exp_p(tv)$ $(0 \le t \le 1)$, where $\exp_p(v) = q$. (By path, we mean a continuous, piecewise C^1 function.)

Proof: Assume $c : [0,1] \to U$, with $c(0) = p$ and $c(1) = q$, is a smooth path, and its image is contained in U. Write $(\exp_p)^{-1}(c(t)) = r(t)w(t)$, where $r(t) \ge 0$ and $||w(t)|| \equiv 1$. Consider the family $f(r,t) = \exp_p(rw(t))$, so that $c(t) = f(r(t), t)$. Then $\frac{dc}{dt} = \frac{dr}{dt} f_r + f_t$, and f_r and f_t are perpendicular for all t, so we can use the Pythagorean theorem to find

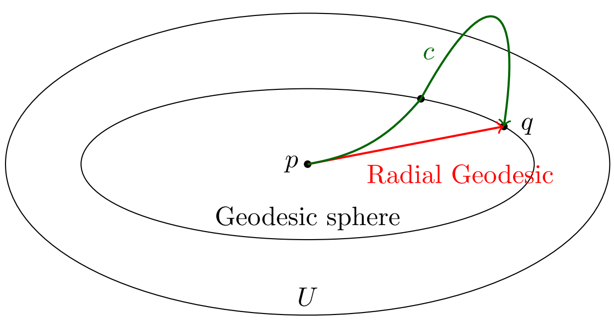
$$\left|\left|\frac{dc}{dt}\right|\right|^2 = \left|\left|\frac{dr}{dt} f_r\right|\right|^2 + ||f_t||^2 = \left|\frac{dr}{dt}\right|^2 ||f_r||^2 + ||f_t||^2 = \left|\frac{dr}{dt}\right|^2 + ||f_t||^2 \ge \left|\frac{dr}{dt}\right|^2$$

Using this inequality, we can bound the length of c:

$$\ell(c) = \int_0^1 \left|\left|\frac{dc}{dt}\right|\right| dt \ge \int_0^1 \left|\frac{dr}{dt}\right| dt \ge \int_0^1 \frac{dr}{dt} dt = r(1) - r(0) = r(1)$$

But $r(t)$ is the length of the radial geodesic $r \mapsto \exp_p(rw(1))$, joining p to q. (Note that equality holds iff $||f_t||^2 \equiv 0$, which is true iff c is the radial geodesic.)

Now, we must consider the case where the image of c is not contained in U. Well, there must be some $t_0 \in (0,1)$ s.t. $c(t_0)$ is on the geodesic sphere psasing through q. We know the length of $c : [0, t_0]$ is no smaller than the length of a radial geodesic directly to q, so the inequality still holds. See the illustration below:



Cor: $d(p,q) = \inf(\ell(c))$, over the set of all c joining p and q, is actually a distance function.

We showed all the other parts earlier – the only thing left to check is that $d(p,q) = 0 \Rightarrow p = q$. We'll prove this by contraposition next time, but the idea is to assume that p and q are distinct, and then construct a normal neighborhood of p that doesn't contain q. Then we know that $d(p,q)$ must be larger than the radius of the geodesic sphere, which is nonzero.

Math 635 Lecture 18

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/3/21

Beginning where we left off last time...

Cor: Let M be a connected Riemannian manifold. Then $\forall p, q \in M$,

$$d(p,q) \stackrel{\text{def}}{=} \inf \left\{ \ell(c) \mid c \text{ continuous and piecewise } C^1 \text{ curve from } p \text{ to } q \right\}$$

is a distance function on M.

Proof: All that remains to be proved is $d(p,q) = 0 \Rightarrow p = q$. So by contraposition, it's enough to show $p \neq q \Rightarrow d(p,q) > 0$. Well, if $p \neq q$, then $\exists \varepsilon > 0$ s.t. $\exp_p\big|_{B_\varepsilon(0)}$ is a diffeomorphism onto its image, and $q \notin \exp_p(B_\varepsilon(0))$. Let c be a path from p to q. It's enough to show $\ell(c) > \varepsilon > 2$.

Well, by continuity, $\exists t_1 > 0$ s.t. $c(t_1) \in S_{\varepsilon/2}$, the geodesic sphere of radius $\varepsilon/2$ centered at p. But $d(p, c(t_1))$ is the length of any corresponding radial geodesic, which is $\varepsilon/2$. So $\ell(c) \geq \ell(c|_{[0,t_1]}) \geq \varepsilon/2 > 0$. $\square$

Thm: $\forall p \in M$, $\exists W$ a neighborhood of p, and $\exists \delta > 0$ s.t. $\forall q \in W$,

$$\exp_q\big|_{B_\delta(0)} : B_\delta(0) \xrightarrow{\sim} \exp_q(B_\delta(0))$$

is a diffeomorphism onto its image, and $W \subseteq \exp_q(B_\delta(0))$.

Proof: Recall that given $p \in M$, there's a neighborhood V of p, and $\varepsilon > 0$, such that $\exp : B_\varepsilon(TV) \to M$ is defined, where $B_\varepsilon(TV) = \{(q,v) \in TV \mid q \in V, ||v|| < \varepsilon\}$. In other words, $G(t,q,v)$ is defined past $t = 1$. Now, define

$$\begin{aligned} F : B_\varepsilon(TV) &\to M \times M \\ (q,v) &\mapsto (q, \exp_q(v)) \end{aligned}$$

In particular, $F(p,0) = (p,p)$. We claim that F is a local diffeomorphism near 0. To check this fact, introduce coordinates near p, and then "double" them to get coordinates on $M \times M$ near (p,p). Then the Jacobian is the block matrix

$$dF_{(p,0)} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix}$$

with the bottom-left entry being I because $\exp_q 0 = q$, and the bottom right entry being I because $d(\exp_p)_0 = \text{Id}$. This matrix is invertible, so F is a local diffeomorphism. Thus, there's a neighborhood V' of p, with $V' \subset V$, and a $\delta > 0$, such that

$$F|_{B_\delta(TV')} \xrightarrow{\sim} F(B_\delta(TV'))$$

is a diffeomorphism onto its image, which is a neighborhood of $(p,p) \in M \times M$. Thus, $\exists W$, a neighborhood of p, such that $W \times W \subset F(B_\delta(TV'))$. In other words,

$$W \times W \subset \left\{ (q, \exp_q(v)) \mid q \in V', ||v|| < \delta \right\} \stackrel{F}{\cong} \{(q,v) \mid q \in V', ||v|| < \delta\}$$

So $\forall q \in W$, $\{q\} \times B_\delta(0) \xrightarrow{\sim} \{q\} \times \exp_q(B_\delta(0))$ via F under an appropriate restriction. $\square$

Defn: Such a neighborhood W is called a <u>totally normal neighborhood</u>.

Lemma: Let W be a totally normal neighborhood, and $p, q \in W$. Then there is a unique geodesic (up to reparameterization) joining p and q, and entirely contained in W. Moreover, this geodesic is the shortest path (i.e. continuous and piecewise C^1) joining p to q.

Proof: Let $\gamma : [0,1] \to W$ be a geodesic, with $\gamma(0) = p$ and $\gamma(1) = q$. Lift γ to T_pM by $\exp_p^{-1}\big|_W$. Then $\gamma(t) = \exp_p(c(t))$, where $t \mapsto c(t) \in T_pM$ and $c(0) = 0$. Note that $\dot\gamma(0) = \dot c(0)$, and $t \mapsto \exp_p(t\dot c(0))$ is a geodesic with the same initial conditions as γ. By the uniqueness (up to reparameterization) of geodesics with initial conditions, we must have $\gamma(t) = \exp_p(t\dot c(0))$, $\forall t$. This also implies that γ is the shortest path from p to q, since it's a radial geodesic. $\square$

Cor: All geodesics are locally length-minimizing.

Proof: Let $\gamma : I \to M$ be a geodesic. Take $t_0 \in \operatorname{Int} I$, i.e., $t_0 \in (a,b) \subseteq [a,b] \subseteq I$. By the existence of totally normal neighborhoods of $\gamma(t_0)$, if $b - a$ is small enough, then $\gamma([a,b])$ is contained in a totally normal neighborhood of $\gamma(t_0)$. $\square$

Cor: Suppose $\gamma : [0,1] \to M$ is a path (continuous, and piecewise C^1), and $d(\gamma(0), \gamma(1)) = \ell(\gamma)$. Then γ is a geodesic, and in paricular, it's C^∞.

Proof: The idea is that global length minimization leads to local. Let $t_0 \in [0,1]$. Again find a neighborhood $[a,b]$ of t_0 such that $\gamma([a,b])$ is contained in a totally normal neighborhood of $\gamma(t_0)$. Then γ must be the shortest path from $\gamma(a)$ to $\gamma(b)$. This means $\gamma|_{[a,b]}$ must be a geodesic, and γ is smooth in a neighborhood of t_0. $\square$

Math 635 Lecture 19

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/5/21

The Variantional Point of View of Geodesics

This material is covered in Do Carmo, chapter 9 §2, and in parts of Lee Riemannian Manifolds, chapters 6 and 10.

Throughout these notes, let M be a Riemannian manifold, $p, q \in M$, and $a > 0$.

Defn: $\Omega_{pq}^a = \{c : [0,a] \to M \mid c \text{ is continuous and piecewise } C^1, c(0) = p, c(a) = q\}$. In this case, piecewise C^1 means $\forall c$, there's a partition $0 = t_0 < t_1 < \cdots < t_N = a$ such that $\forall i,\ c|_{[t_i,t_{i+1}]}$ is C^1. In other words, c is C^1 on (t_i, t_{i+1}), and the one sided limits

$$\lim_{t\to t_i^-} \frac{dc}{dt} \qquad \lim_{t\to t_{i+1}^+} \frac{dc}{dt}$$

exist.

Defn: We define two functionals on this space.

1. From Lee: The length functional $L : \Omega_{pq}^a \to \mathbb{R}$
   $$c \mapsto L(c) = \int_0^a \left|\left|\tfrac{dc}{dt}\right|\right|\, dt$$
   Any minima of L, if one exists, corresponds to shortest paths between p and q.
2. From Do Carmo: The energy functional $E : \Omega_{pq}^a \to \mathbb{R}$
   $$c \mapsto E(c) = \tfrac{1}{2}\int_0^a \left|\left|\tfrac{dc}{dt}\right|\right|^2 dt$$

Lemma: $\forall c \in \Omega_{pq}^a$, $L(c)^2 \le 2aE(c)$, with equality iff $||\dot{c}||$ is constant.

Proof: Use the Cauchy-Shwarz inequality for functions on $[0,a]$: If $f, g : [0,1] \to \mathbb{R}$, then

$$\left(\int_0^a fg\, dt\right)^2 \le \left(\int_0^a f^2\, dt\right)\left(\int_0^a g^2\, dt\right)$$

Given a path c, apply Cauchy-Schwarz to $f = \left|\left|\frac{dc}{dt}\right|\right|$, with $g \equiv 1$. $\square$

Cor: Suppose $\gamma \in \Omega_{pq}^a$ is a minimizing geodesic. Then $\forall c \in \Omega_{pq}^a$, $E(\gamma) \le E(c)$, with equality iff c is a minimizing geodesic.

Proof: $||\dot{\gamma}||$ is constant, so $E(\gamma) = \frac{1}{2a}L(\gamma)^2 \le \frac{1}{2a}L(c)^2 \le E(c)$. Thus, if $E(\gamma) = E(c)$, then everything must be equal, so $L(\gamma) = L(c)$, so c is a minimizing path, so c is a geodesic. $\square$

We want to look for minimizers of E. This is a hard problem, and in fact, the may not exist.

The idea of the calculus of variations is to differentiate E, and then look for critical points. This is crazy, because Ω_{pq}^a is not a manifold. But one can still define variations of $\gamma \in \Omega_{pq}^a$. Colloquially, these are smooth paths in Ω_{pq}^a that pass through γ.

Defn: Let $\gamma \in \Omega_{pq}^a$. A <u>proper variation</u> (or <u>pinned variation</u>) of γ is

$$f : \underset{s}{(-\varepsilon,\varepsilon)} \times \underset{t}{[0,a]} \to M$$

such that

(i) $f(0,t) = \gamma(t)$
(ii) There is a partition $0 = t_0 < t_1 < \cdots < t_N = a$ such that $\forall i,\ f|_{(-\varepsilon,\varepsilon)\times[t_i,t_{i+1}]}$ is C^1.

(iii) $\forall s$, $f(s,0) = p$ and $f(s,a) = q$. That is, $\forall s \in (-\varepsilon, \varepsilon)$, $(t \mapsto f(s,t)) \in \Omega_{pq}^a$. In this sense, $s \mapsto (t \mapsto f(s,t))$ is a "smooth" curve in Ω_{pq}^a, passing through γ at $t = 0$.

Conditions (i) and (ii) are what make it a variation; condition (iii) makes it proper/pinned.

Defn: For f a proper variation of γ, $\forall t$, $V(t) = \partial_s f(0,t) \in T_{\gamma(t)}M$ is the variation field of the variation. So $V \in \Gamma_\gamma(TM)$.

Observe: Because the variation is proper, we must have $V(0) = 0$.

Lemma: Given any $V \in \Gamma_\gamma(TM)$ s.t. $V(0) = 0$, $V(a) = 0$, there exists a proper variation f whose variation field is V.

Proof: Let $f(s,t) = \exp_{\gamma(t)}(sV(t))$. We need $|s| < \varepsilon$ to be nonzero, but because of the compactness of the curve, we can construct a finite subcover, so that $|s| > 0$. $\square$

Idea: Differentiate the energy E w.r.t. a given varation of $\gamma \in \Omega^a_{pq}$.
Computation: Let f as above, a proper variation of $\gamma \in \Omega^a_{pq}$. Define

$$E(s_0) = \frac{1}{2}\int_0^a ||\partial_t f(s,t)||^2 \, dt = E(f|_{s=s_0})$$

Then compute

$$\begin{aligned}
\frac{dE}{ds} &= \frac{1}{2}\int_0^a \frac{d}{ds}\langle \partial_t f, \partial_t f\rangle \, dt \\
&= \frac{\cancel{1}}{\cancel{2}}\int_0^a \cancel{2}\left\langle \frac{D}{ds}\partial_t f, \partial_t f\right\rangle dt \\
&= \int_0^a \left\langle \frac{D}{dt}\partial_s f, \partial_t f\right\rangle dt \\
\text{(integration by parts)} \quad &= [\langle \partial_s f, \partial_t f\rangle]_{t=0}^{t=a} - \int_0^a \left\langle \partial_s f, \frac{D}{dt}, \partial_t f\right\rangle dt
\end{aligned}$$

Where the integration by parts succeeds because

$$\frac{d}{dt}\langle \partial_s f, \partial_t f\rangle = \left\langle \frac{D}{dt}\partial_s f, \partial_t f\right\rangle + \left\langle \partial_s f, \frac{D}{dt}\partial_t f\right\rangle$$

Note that we have to adjust the term outside of the integral due to corners, but the boundary terms will all appear in a similar form. We conclude with

$$\frac{d}{ds}E(0) = -\int_0^a \left\langle \underbrace{V(t)}_{\partial_s f|_{s=0}}, \frac{D}{dt}\frac{d\gamma}{dt}\right\rangle dt - \underbrace{\sum_i \langle V(t_i), \underbrace{\Delta\dot\gamma(t_i)}_{=\frac{d\gamma}{dt}(t_i^+) - \frac{d\gamma}{dt}(t_i^-)}\rangle}_{\text{corner terms}}$$

which is the first variation formula.

Now, choose

$$V(t) = \begin{cases} \frac{D}{dt}\frac{d\gamma}{dt} & t \in (t_i, t_{i+1}) \text{ for some } i \\ V(t_i) & t = t_i \text{ for some } i \end{cases}$$

We conclude that if $\gamma \in \Omega^a_{pq}$ is a critical point of E in the sense that for all variations f, $E'(0) = 0$, then γ is a geodesic!

Next time, we'll use the second derivative test. This is where curvature will appear!

Math 635 Lecture 20

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/8/21

Review: Derivation of the first variation formula. Suppose $\gamma \in \Omega_{pq}^a$, $\gamma : [0,a] \to M$, with $\gamma(0) = p$ and $\gamma(a) = q$. Let

$$f : \underset{s}{(-\varepsilon, \varepsilon)} \times \underset{t}{[0,a]} \to M$$

be a proper variation of γ, which is C^∞ on rectangles $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$, where $0 = t_0 < t_1 < \cdots < t_N = a$ is a partition. We have our energy formula

$$E(s) = \frac{1}{2} \int_0^a ||\partial_t f(s,t)||^2 \, dt$$

The key step is computing

$$\frac{dE}{ds} = \int_0^a \left\langle \frac{D}{ds} \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle dt = \int_0^a \left\langle \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle dt = -\int_0^a \left\langle \frac{\partial f}{\partial s}, \frac{D}{dt} \partial f t \right\rangle dt + \text{(boundary terms)}$$

So really, we're intergating on each segment $[t_i, t_{i+1}]$:

$$\int_{t_i}^{t_{i+1}} \frac{d}{dt} \langle \partial_s f, \partial_t f \rangle \, dt \bigg|_{s=0} = \left\langle V(t_{i+1}), \dot\gamma(t_{i+1}^-) - \dot\gamma(t_i^+) \right\rangle$$

where $V = \partial_s f|_{s=0} \in \Gamma_\gamma(TM)$ is the variation field. When we sum over i, we get

$$\left\langle V(t_i), \dot\gamma(t_i^-) - \dot\gamma(t_i^+) \right\rangle = \langle V(t_i), \Delta\dot\gamma(t_i) \rangle$$

Cor: If γ is such that, for all proper variations of γ, $\frac{dE}{ds}(s = 0) = 0$, then γ is a geodesic. (And the converse is true as well.)

Proof: Choose $V(t)$ as follows

$$V(t) = \begin{cases} \frac{D}{dt}\dot\gamma(t) & t \neq t_i, \forall i \\ \Delta\dot\gamma(t_i) & t = t_i \end{cases}$$

Then we get

$$0 = \frac{dE}{ds}(0) = \int_0^a \left|\left| \frac{D}{dt}\dot\gamma(t) \right|\right|^2 dt + \sum ||\Delta\dot\gamma(t)||^2$$

This is the case iff $\left|\left| \frac{D}{dt}\dot\gamma(t) \right|\right| \equiv 0$, and $\forall i$, $\Delta\dot\gamma(t_i) = 0$. Thus, γ is a geodesic. $\square$

Observe that one can replace the "energy" functional $E : \Omega_{pq}^a \to \mathbb{R}$ with other functionals.

Ex: $V \in C^\infty(M)$, $\mathscr{L} : \Omega_{pq}^a \to \mathbb{R}$

$\gamma \mapsto \int_0^a \frac{1}{2} ||\dot\gamma||^2 - V(\gamma(t)) \, dt$

We call this functional the Lagrangian.

Question: Which curves satisfy $\frac{d\mathscr{L}}{ds}(0) = 0$ for all variations? The answer is curves that follow Newton's second law, $\frac{D}{dt}\dot\gamma = -\nabla V(\gamma(t))$.

Ex: Given a particle rolling from a point p to a point q in a vertical plane under the influence of gravity, what curve will minimize the time it takes? The answer is the brachistochrone curve.

Now, we examine the second variation. Let $\gamma \in \Omega^a_{pq}$ be a geodesic, and $E : \Omega^a_{pq} \to \mathbb{R}$. Let f be a proper C^∞ variation of γ (i.e. no jumps). Then compute $\frac{d^2}{ds^2}E(s)\Big|_{s=0}$. Well,

$$\frac{d}{ds}E(f_s) = -\int_0^a \left\langle \partial_s f, \frac{D}{dt}\partial_t f \right\rangle dt$$

So

$$\frac{d^2}{ds^2}E(s) = -\cancel{\int_0^a \left\langle \frac{D}{ds}\partial_s f, \frac{D}{dt}\partial_t f \right\rangle} dt - \int_0^a \left\langle \partial_s f, \frac{D}{ds}\frac{D}{dt}\partial_t f \right\rangle dt$$

where the first term is eliminated because $\frac{D}{dt}\partial_t f$ vanishes at $s = 0$, because γ is a geodesic.

Lemma: $\left[\frac{D}{ds}, \frac{D}{dt}\right] = \mathcal{R}(\partial_s f, \partial_t f)$ as an operator acting on vector fields V along f. (Recall: $\mathcal{R}$ is the curvature of ∇.)

$$\begin{tikzcd} & TM \arrow[d] \\ f : (-\varepsilon, \varepsilon) \times [0, a] \arrow[ur, "V"] \arrow[r] & M \end{tikzcd}$$

Recall that $\mathcal{R}(\partial_s f, \partial_t f)_{f(s,t)} : T_{f(s,t)}M \to T_{f(s,t)}M$. So the lemma really says that

$$\frac{D}{ds}\frac{D}{dt}V - \frac{D}{dt}\frac{D}{ds}V = \mathcal{R}(\partial_s f, \partial_t f)(V)$$

So why is the lemma true? Well, assume for simplicity that f is an embedding away from p and q. We can extend $\partial_s f$ and $\partial_t f$ to fields X and Y (respectively) on M. Then $\frac{D}{ds}$ "$=$" ∇_X and $\frac{D}{dt}$ "$=$" ∇_Y, and by the definition of $\mathcal{R}$,

$$[\nabla_X, \nabla_Y] = \mathcal{R}(X, Y) + \nabla_{[X,Y]}$$

But $[X, Y]|_{\operatorname{Im} f} = 0$, because $X = \partial_s f$ and $Y = \partial_t f$ on $\operatorname{Im} f$ Thus,

$$\begin{aligned} \frac{d^2}{ds^2}E(s)\Big|_{s=0} &= -\int_0^a \left\langle V, \left(\frac{D}{dt}\frac{D}{ds} + \mathcal{R}(\partial_s f, \partial_t f)\right)\partial_t f\Big|_{s=0} \right\rangle dt \\ &= -\int_0^a \left\langle V, \frac{D}{dt}\frac{D}{ds}V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt \\ &= -\int_0^a \left\langle V, \frac{D^2}{dt^2}V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt \end{aligned}$$

Thus, we conclude that

$$\frac{d^2E}{ds^2}\Big|_{s=0} = -\int_0^a \left\langle V, \frac{D^2}{dt^2}V + \mathcal{R}(V, \dot\gamma)(\dot\gamma) \right\rangle dt$$

Observe: This is quadratic in V. But of course this is true, since it's a Hessian!

$$\langle V, \mathcal{R}(V, \dot\gamma)(\dot\gamma)\rangle \sim \underbrace{\langle \mathcal{R}(V, W)(W), V\rangle}_{\text{scalar, related to “sectional curvature”}}$$

We can think of $\frac{D^2}{dt^2}V + \mathcal{R}(V, \dot\gamma)(\dot\gamma)$ as an operator on $V \in \Gamma_\gamma(TM)$ called the "Jacobi operator". Elements of its kernel are called "Jacobi fields".

Math 635 Lecture 21

Professor Alejandro Uribe-Ahumada
Transcribed by Thomas Cohn

3/10/21

Review: second variation formula. If γ is a geodesic, and f a proper variation of γ, $V = \partial_s|_{s=0}$, then

$$E''(0) = -\int_0^a \left\langle V, \frac{D^2 V}{dt^2} + \mathcal{R}(V, \dot{\gamma})(\dot{\gamma}) \right\rangle dt$$

Next, we're going to take a closer look at the curvature, $\mathcal{R}$. Recall its definition: If $X, Y \in \mathfrak{X}(M)$, then the curavture is $\mathcal{R}(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by $\mathcal{R}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. It turns out that $\mathcal{R}(X,Y)$ is given by the action of a tensor $\mathcal{R}$ of the form $\forall p \in M; u, v \in T_pM$, $\mathcal{R}_p(u,v) : T_pM \to T_pM$, a linear map.

As an operator, $\mathcal{R}(X,Y)(Z)_p = \mathcal{R}_p(X_p, Y_p)(Z_p) \in T_pM$. Also, $\mathcal{R}$ shows up as an obstruction to finding a covariant-constant frame $\nabla_{E_i} E_j \equiv 0$.

Curvature Identities

(Covered in Do Carmo, Chapter 4, §2)

The first identity we consider is the Bianchi identity: $\forall X, Y, Z \in \mathfrak{X}(M)$, $\mathcal{R}(X,Y)Z + \mathbb{R}(Z,X)Y + \mathbb{R}(Y,Z)X = 0$, due to ∇ being torsion-free. The proof is simply a computation, and can be found on page 91 of Do Carmo.

Prop: Introduce $X, Y, Z, T \in \mathfrak{X}(M)$, and define $(X,Y,Z,T) \overset{\text{def}}{=} \langle \mathcal{R}(X,Y)Z, T\rangle$.

(a) $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$ (Proved via the Bianchi identity)
(b) $(X,Y,Z,T) = -(Y,X,Z,T)$ (Because $\mathcal{R}(X,Y) = -\mathcal{R}(Y,X)$)
(c) $(X,Y,Z,T) = -(X,Y,T,Z)$ (Because ∇ perserves $\langle\ ,\ \rangle$)
(d) $(X,Y,Z,T) = (Z,T,X,Y)$ (Follow from the Bianchi identity and some algebra)

In coordinates $(x^1, \dots, x^n)$, $X_i = \frac{\partial}{\partial x^i}$, then $\mathcal{R}$ has components in the coordinate system, $\mathcal{R}_{ijk}^\ell \in C^\infty(U)$ (where U is the domain of the coordinate chart) such that

$$\underbrace{\mathcal{R}(X_i, X_j)X_k}_{\in \mathfrak{X}(U)} = \mathcal{R}_{ijk}^\ell X_\ell$$

But how do we compute $\mathcal{R}_{ijk}^\ell$?

Lemma: $\mathcal{R}_{ijk}^\ell = \Gamma_{jk}^\ell \Gamma_{i\ell}^s - \Gamma_{ik}^\ell \Gamma_{j\ell}^s + \partial_i \Gamma_{jk}^s - \partial_j \Gamma_{ik}^s$. (Recall that Do Carmo uses the opposite sign for $\mathcal{R}$.)

Proof: This is a messy computation. The complete details can be found on pages 92-93 of Do Carmo.

Observe that Γ depends on the first derivatives of g_{ij}, so $\mathcal{R}$ depends on the second derivatives of g_{ij}.

Defn: $(X_i, X_j, X_k, X_s) \overset{\text{def}}{=} \mathcal{R}_{ijks}$. $X_{ijks} \overset{\text{def}}{=} \mathcal{R}_{ijk}^\ell g_{\ell s}$.

This allows us to rephrase the identities much more concisely:

(a) $\mathcal{R}_{ijks} + \mathcal{R}_{jkis} + \mathcal{R}_{kijs} = 0$
(b) $\mathcal{R}_{ijks} = -\mathcal{R}_{jiks}$
(c) $\mathcal{R}_{ijks} = -\mathcal{R}_{ijsk}$
(d) $\mathcal{R}_{ijks} = \mathcal{R}_{ksij}$

Nobody *really* understands this whole tensor. The whole thing is a monster. But we can understand parts of it.

One part which we can understand is the sectional curvature, which shows up in the second variation formula.

Defn: $\forall u, v \in T_pM$,

$$|u \wedge v| = \sqrt{||u||^2 \, ||v||^2 - \langle u, v \rangle^2}$$

is the area of the parallelogram spanned by u and v.

Lemma: If u and v are linearly independent, then

$$K(u,v) \overset{\text{def}}{=} \frac{\mathcal{R}(u,v,v,u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u,v)v, u \rangle}{|u \wedge v|^2}$$

K depends only on the plane $\pi(u,v) = \text{span}(u,v)$.

Proof: Check that the RHS is invariant under each of the following "moves":

- $(u,v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u,v) \rightsquigarrow (v,u)$
- $(u,v) \rightsquigarrow (u + \lambda v, v)$

We will finish proving this next time.

Defn: $K(\pi) = K(u,v)$ is called the <u>sectional curvature</u> of p at π.

Math 635 Lecture 22

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/12/21

Sectional Curvature

Defn: $\forall p \in M$, $\forall u, v \in T_pM$ linearly independent, the <u>sectional curvature</u> is defined to be

$$K_p(u,v) \stackrel{\text{def}}{=} \frac{\mathcal{R}(u,v,v,u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u,v)v, u\rangle}{||u||^2\,||v||^2 - \langle u,v\rangle^2}$$

Last time, we claimed that $K_p(u,v)$ only depends on the plane $\pi = \operatorname{span}(u,v) \in \operatorname{Gr}(2, T_pM)$. So really, K_p is a function $K_p : \operatorname{Gr}(2,T_pM) \to \mathbb{R}$.

Proof of this claim: We want to show that the right-hand side is invariant under the following "moves":
- $(u,v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u,v) \rightsquigarrow (v,u)$
- $(u,v) \rightsquigarrow (u + \lambda v, v)$

This is easy to do using known properties of $\mathcal{R}$.

Observe: If $\dim M = 2$, then think about $\mathcal{R}$ and its symmetries. How many independent components does it have? Well, for $\mathcal{R}(u,v,u',v')$ (with u and v linearly independent), we can write (u',v') in terms of (u,v) via a change of basis:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} u \\ v \end{bmatrix}$$

So $\mathcal{R}(u,v,u',v') = \det(A)\mathcal{R}(u,v,u,v) = -\det(A)\mathcal{R}(u,v,v,u)$. So we conclude that in dimension 2, there's only one degree of freedom! All the information that's contained in $\mathcal{R}$ reduces to knowing the function $K(p) = \mathcal{R}_p(e,f,f,e)$, where (e,f) is an orthonormal basis of T_pM ($K : M \to \mathbb{R}$).

Prop: K from above is the Gaussian curvature.

In dimension n, there are $\frac{n^2(n^2-1)}{12}$ degrees of freedom.

Prop: The function $K : \bigcup_{p \in M} \operatorname{Gr}(2, T_pM) \to \mathbb{R}$ completely determines $\mathcal{R}$.

Proof: This is just an algebraic exercise. As a preliminary, start with a bilinear map $b : V \times V \to \mathbb{R}$. Note that b is the sum of a symmetric and antisymmetric bilinear form; this can be seen by writing $b(x,y) = x^T M y$ for a unique matrix M, and then noting that we can write $M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$. $M + M^T$ is symmetric, and $M - M^T$ is antisymmetric.

Now, if b is symmetric, then b is uniquely determined by the mapping $x \mapsto q(x) = b(x,x)$, because $q(x+y) = q(x) + q(y) + 2b(x,y)$. We can then solve for $b(x,y)$. (This is known as the "polarization identity".)

Cor: If $b(x,x) = 0$, $\forall x \in V$, then b must be skew-symmetric.

We now continue the proof of the proposition. Let V be a vector space (e.g. T_pM). Let $\mathcal{R}, \mathcal{R}' : V \times V \times V \times V \to \mathbb{R}$ be multilinear maps, with the symmetries of the Riemannian curvature. Then define $D = \mathcal{R} - \mathcal{R}'$, and it has the same symmetries. Assume $\forall v, w, x$, $D(v,w,w,v) = 0$ (x will be used later). We want to show $D \equiv 0$.

Why is this true? Well,

$$\begin{aligned}
? &= D(v+w,x,x,v+w) \\
&= \underbrace{D(v,x,x,v)}_{=0} + D(v,x,x,w) + D(w,x,x,v) + \underbrace{D(w,x,x,w)}_{=0} \\
&= D(v,x,x,w) - D(v,x,x,w) \\
&= 0
\end{aligned}$$

Finally, we use the Bianchi identity, to show that

$$0 = D(u,v,w,t) + D(w,u,v,t) + D(v,w,u,t) = 3D(u,v,w,t)$$

□

Now, we return to studying the second variation formula. Recall: For a geodesic γ, and V the variation field of a proper variation of γ, we have

$$E''(0) = -\int_0^a \left\langle V, \frac{D^2}{dt^2}V + \mathcal{R}(V,\dot\gamma)\dot\gamma \right\rangle dt$$

Thm: (Bonnet-Myers V1) If sectional curvature K satisfies $K > \left(\frac{\pi}{\ell}\right)^2 > 0$ for some $\ell > 0$, then no geodesic of length ℓ is minimizing.

Proof: Let γ be a geodesic of length ℓ. We need to show the energy of γ is not a minimum. Let $\gamma : [0,\ell] \to M$, with $\gamma(0) = p$, $\gamma(\ell) = q$, and $||\dot\gamma|| = 1$. Pick $N \in T_pM$ a unit vector such that $\langle N, \dot\gamma(0)\rangle = 0$. Let $E(t)$ be the parallel transport of N along γ, so then $\frac{D}{dt}E \equiv 0$. Observe that $\forall t$, $\langle E(t), \dot\gamma(t)\rangle = 0$. So if we define

$$V(t) \overset{\text{def}}{=} \sin\left(t\frac{\pi}{\ell}\right) E(t)$$

then $V(0) = 0$ and $V(\ell) = 0$, so V is the variation field of a pinned variation. So now, we can substitute into the second variation formula. We compute

$$\frac{D}{dt}V(t) = \frac{\pi}{\ell}\cos\left(t\frac{\pi}{\ell}\right)E(t) + 0 \quad \Rightarrow \quad \frac{D^2}{dt^2}V(t) = -\left(\frac{\pi}{\ell}\right)^2 \sin\left(t\frac{\pi}{\ell}\right)E(t)$$

Thus,

$$\left\langle V, \frac{D^2}{dt^2}V \right\rangle = -\left(\frac{\pi}{\ell}\right)^2 \sin^2\left(t\frac{\pi}{\ell}\right)\underbrace{||v||^2}_{=1}$$

Also,

$$\langle V, \mathcal{R}(V,\dot\gamma)\dot\gamma\rangle = \sin^2\left(t\frac{\pi}{\ell}\right)\underbrace{\langle E, \mathcal{R}(E,\dot\gamma)\dot\gamma\rangle}_{=K(E,\dot\gamma)}$$

So for a proper variation of γ with variation field $V = f_s|_{s=0}$, we have

$$E''(0) = \int_0^\ell \underbrace{\sin^2\left(t\frac{\pi}{\ell}\right)}_{\geq 0}\underbrace{\left(\left(\frac{\pi}{\ell}\right)^2 - K(E,\dot\gamma)\right)}_{<0} dt$$

So $E''(0) < 0$, so for $|s| \ll 1$, $E(s) < E(0)$, so γ is not minimizing. □

Math 635 Lecture 23

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/15/21

Ricci Curvature

For a Riemannian manifold, the full Riemannian curvature at a point $p \in M$ is $\mathcal{R}_p \in V^* \otimes V^* \otimes V^* \otimes V$, where $V = T_pM$. The Ricci Tensor is a partial trace of $\mathcal{R}$.

Defn: $\forall p \in M$, $\forall u, v \in T_pM$, we define the Ricci tensor by

$$\mathrm{Ric}_p(u,v) = \mathrm{tr}\begin{bmatrix} T_pM & \to T_pM \\ w & \mapsto \mathcal{R}_p(u,v)w \end{bmatrix}$$

Note: The Do Carmo definition of the Ricci tensor has the same convention as ours. It deals with the minus sign by putting w in the second slot of $\mathcal{R}$.

Note that the trace of an arbitrary linear operator $F : V \to V$ can be defined without coordinates as follows. Think of $\mathrm{Hom}(V,V) \cong V^* \otimes V$. The evaluation map $(\alpha, v) \mapsto \alpha(v)$ for $\alpha \in V^*$, $v \in V$ is bilinear, so it corresponds to

$$\begin{aligned} \mathrm{tr} : V^* \otimes V &\to \mathbb{R} \\ F &\mapsto \mathrm{tr}(F) \end{aligned}$$

So Ric_p is bilinear.

We want to compute $\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R}$ in terms of components of $\mathcal{R}$ with respect to an arbitrary frame. Let $e_1, \dots, e_n$ be a local frame of TM. Then $\mathcal{R}(e_i, e_j)e_k = \mathcal{R}_{ijk}^{\ell} e_\ell$.

Defn:

$$\begin{aligned} \mathcal{R}_{ij} &\overset{\text{def}}{=} \mathrm{Ric}(e_i, e_j) \\ &= \mathrm{tr}(w \mapsto \mathcal{R}(w, e_i)e_j) \\ &= \sum_{k=1}^{n} (e_k\text{-component of } \mathcal{R}(e_k, e_i)e_j) \\ &= \mathcal{R}_{kij}^{k} \text{ (with implicit summation)} \end{aligned}$$

So $\mathcal{R}_{ij} = \mathcal{R}_{kij}^{k}$.

(As we will see later, we can interpret this as a sort of averaging.)

Side note: Why do we choose to fix the middle two elements? Well, if you fix the first two or last two, swapping the two unfixed elements flips the sign, so the trace would be identically zero.

Recall that we also have the shorthand notation $\mathcal{R}_{ijk\ell} = \langle \mathcal{R}(e_i, e_j)e_k, e_\ell \rangle$.

Exer: Check that $\mathcal{R}_{ijk\ell} = \mathcal{R}_{ijk}^{a} g_{a\ell}$, i.e., $\mathcal{R}_{ijk}^{b} = \mathcal{R}_{ijk\ell} g^{\ell b}$ (where $g^{\ell b}$ is the element of the matrix inverse of $g_{a\ell}$). This implies that $\mathcal{R}_{ij} = \mathcal{R}_{kij\ell} g^{\ell k}$.

Now, let's explore some properties of the Ricci Tensor.

Prop: Ric is symmetric.

Proof: Let $e_1, \ldots, e_n$ be an orthonormal frame. Then

$$\begin{aligned}
\mathcal{R}_{ij} &= \sum_k \langle \mathcal{R}(e_k, e_j)e_j, e_k \rangle \\
&= \sum_k \mathcal{R}_{kijk} \\
&= \sum_k \mathcal{R}_{jkki} \\
&= -\left(-\sum_k \mathcal{R}_{kjik} \right) \\
&= \sum_k \mathcal{R}_{kjik} \\
&= \mathcal{R}_{ji}
\end{aligned}$$

□

Defn: The <u>Ricci curvature</u> is

$$\begin{aligned}
\mathrm{Ric} : \{(p, v) \mid p \in M, v \in T_pM, ||v|| = 1\} &\to \mathbb{R} \\
(p, v) &\mapsto \mathrm{Ric}_p(v, v) \underbrace{\frac{1}{n-1}}_{\text{Do Carmo only}}
\end{aligned}$$

Geometric Interpretation

We'll be using the Do Carmo convention. Let v be a unit tangent vector, and let $(e_1, \ldots, e_n)$ be an orthonormal basis of T_pM, with $e_n = v$. Then

$$\mathrm{Ric}(v) = \frac{1}{n-1} \sum_{i=1}^{n-1} K(e_i, v)$$

So we're effectively averaging the sectional curvature of the planes produced by v with each of the basis vectors $e_1, \ldots, e_{n-1}$. This is true because

$$\mathrm{Ric}(v, v) = \sum_{i=1}^{n} \underbrace{\langle \mathcal{R}(e_i, v)v, e_i \rangle}_{= \begin{cases} K(e_i, v) & i < n \\ 0 \end{cases}}$$

Defn: The <u>scalar curvature</u> $S : M \to \mathbb{R}$ is defined by $\forall p \in M$, $S(p) = \mathrm{tr}(\mathrm{Ric}_p)$.

Thm: (Bonet-Myers V2) Suppose (M, g) is a Riemannian manifold. If $\mathrm{Ric} > \left(\frac{\pi}{\ell}\right)^2$ (using the function form of Ric, not the tensor), then no geodesic of length ℓ is minimizing.

Proof: Pick $(E_1(0), \ldots, E_{n-1}(0))$, an orthonormal basis of $\mathrm{span}(\dot{\gamma}(0))^\perp$. Then parallel transport the E_j's along γ to get $E_j(t)$, a parallel frame (i.e. $\frac{D}{dt}E_j \equiv 0$) that is orthonormal at every t, and orthogonal to $\dot{\gamma}(t)$. Now, $\forall j$, let $V_j(t) = \sin(\pi t)E_j(t)$. We consider $n-1$ proper variations of γ, and $\forall j = 1, \ldots, n-1$, we have, as before,

$$E_j''(0) \underbrace{= \cdots =}_{\text{Same computation as Bonet-Myers V1}} \int_0^\ell \sin^2\left(t\frac{\pi}{\ell}\right) \left(\left(\frac{\pi}{\ell}\right)^2 - K(E_j, \dot{\gamma}) \right) dt$$

Now, we average:

$$\frac{1}{n-1} \sum_{j=1}^{n-1} E_j''(0) = \int_0^\ell \sin^2(t\frac{\pi}{\ell}) \left(\left(\frac{\pi}{\ell}\right)^2 - \mathrm{Ric}_{\gamma(t)}(\dot{\gamma}) \right) dt$$

This quantity is less than 0 by our assumption, so $\exists j$ such that $E_j''(0) < 0$, so the geodesic isn't minimizing. □

Jacobi Fields

Our motivation comes from the second variation formula: it involves the operator $\frac{D^2}{dt^2}V + \mathcal{R}(V,\dot{\gamma})\dot{\gamma}$. We're interested in studying fields where this is equal to 0.

Defn: Let γ be a geodesic. Then $V \in \Gamma_\gamma(TM)$ is a Jacobi field iff

$$\frac{D^2}{dt^2}V + \mathcal{R}(V,\dot{\gamma})\dot{\gamma} = 0$$

Note: We don't require V to vanish at the endpoints.

Prop: Let $f : (-\varepsilon,\varepsilon) \times [0,a] \to M$ be a smooth variation of γ by geodesics, i.e., $\forall s$, $t \mapsto f(s,t)$ is a geodesic. Then the variation field $V = \partial_s f|_{s=0}$ is a Jacobi field.

Proof: Our assumption of f is true iff $\frac{D}{dt}\partial_t f = 0$, so

$$0 = \frac{D}{ds}\frac{D}{dt}\partial_t f = \frac{D}{dt}\frac{D}{ds}\partial_t f + \mathcal{R}(\partial_s f, \partial_t f)\partial_t f = \frac{D^2}{dt^2}\partial_s f + \mathcal{R}(\partial_s f, \partial_t f)\partial_t f$$

If we restrict to $s = 0$, we get the Jacobi operator. $\square$

Exer: (HW) Prove that the converse is true as well.

Claim: Let $E_i \in \Gamma_\gamma(TM)$, for $i = 1,\ldots,n$, be a parallel frame, and let $V(t) = f^i(t)E_i(t)$. Then the Jacobi equation on V is equivalent to a second order system of ordinary differential equations, that looks like $\ddot{f}^i + a_{ij}f^j = 0$.

Math 635 Lecture 24

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/17/21

Continuing with Jacobi fields from last time. Recall: For γ a geodesic, $J \in \Gamma_\gamma(TM)$ is a Jacobi field iff $\frac{D^2}{dt^2}J + \mathcal{R}(J,\dot\gamma)\dot\gamma = 0$. It's a fact that J is a Jacobi field iff $V = \partial_s f|_{s=0}$ for some variation f of γ by geodesics. We proved part of this in class, and will prove the rest in a homework problem.

Let $E_1, \dots, E_n \in \Gamma_\gamma(TM)$ be an orthonormal parallel frame. We can write any $J \in \Gamma_\gamma(TM)$ as $J(t) = f^i(t)E_i(t)$ for some smooth functions f^i. Using the fact that $\frac{D}{dt}E_i \equiv 0$, we get $\frac{D^2}{dt^2}J = \ddot f^i E_i$, and by linearity, the Jacobi equation becomes a system of ODEs. $\forall i$,

$$\ddot f^i + \underbrace{(E_i, \dot\gamma, \dot\gamma, E_j)}_{a_{ij}=a_{ji}} f^j \equiv 0 \qquad \text{and} \qquad (E_i,\dot\gamma,\dot\gamma,E_j)f^j = \mathcal{R}(J,\dot\gamma)\dot\gamma = \mathcal{R}(f^jE_j,\dot\gamma)\dot\gamma = f^j\mathcal{R}(E_j,\dot\gamma)\dot\gamma$$

The ith component of $\mathcal{R}(J,\dot\gamma)\dot\gamma$ is

$$a_{ij} = f^i(E_j,\dot\gamma,\dot\gamma,E_i) = f^i(\dot\gamma,E_i,E_j,\dot\gamma) = a_{ji}$$

So our system of equations is

$$\ddot f^i(t) + a_{ij}f^j(t) = 0 \quad 1 \le i \le n$$

Cor: A Jacobi field is uniquely determined by $J(0)$ and $\frac{DJ}{dt}(0)$. In fact, we have an isomorphism

$$\begin{aligned} \{\text{Jacobi fields along } \gamma\} &\cong T_{\gamma(0)}M \oplus T_{\gamma(0)}M \\ J &\mapsto (J(0), \tfrac{DJ}{dt}(0)) \end{aligned}$$

This tells us that the space of Jacobi fields along γ has dimension $2n$.

Lemma: Let J be a Jacobi field. Then $\exists a, b \in \mathbb{R}$ such that $\langle J(t), \dot\gamma(t)\rangle = a + bt$.

Proof: It's enough to show $\frac{d}{dt}\langle J, \dot\gamma\rangle$ is constant. Well,

$$\frac{d}{dt}\left(\frac{d}{dt}\langle J(t),\dot\gamma(t)\rangle\right) = \frac{d}{dt}\left\langle \frac{DJ}{dt}(t), \dot\gamma(t)\right\rangle = \left\langle \frac{D^2J}{dt^2}(t),\dot\gamma(t)\right\rangle = -\langle \mathcal{R}(J,\dot\gamma)\dot\gamma(t),\dot\gamma(t)\rangle = 0$$

with the last equality because curvature is skew-symmetric. □

Cor: If $J(0)$ and $\frac{DJ}{dt}(0)$ are orthogonal to $\dot\gamma(0)$, then they remain orthogonal for all t.

Defn: A Jacobi field satisfying the above condition is called a <u>normal Jacobi field</u>. The set of normal Jacobi fields forms a dimension $2(n-1)$ subspace.

Lemma: $(\dot\gamma, t\dot\gamma)$ span the space of tangential Jacobi fields.

Lemma: The space of Jacobi fields has a natural symplectic structure. $\forall J_1, J_2$ Jacobi fields, the quantity

$$\Omega(J_1,J_2) = \left\langle J_1, \frac{DJ_2}{dt}\right\rangle - \left\langle \frac{DJ_1}{dt}, J_2\right\rangle$$

is constant w.r.t. t. We take Ω to be the symplectic form.

Observe: The space of normal Jacobi fields is a symplectic subspace (i.e. the restriction of Ω is still non-degenerate). It corresponds to a certain subspace of $T_{(\dot\gamma(0),\dot\gamma(0))}(T^*M)$.

We now check the lemma above. All we need to do is show $\Omega(J_1, J_2)$ is constant w.r.t. t. So we differentiate:

$$\frac{d}{dt}\Omega(J_1, J_2) = \frac{d}{dt}\left(\left\langle J_1, \frac{DJ_2}{dt}\right\rangle - \left\langle \frac{DJ_1}{dt}, J_2\right\rangle\right) = \cancel{\left\langle \frac{DJ_1}{dt}, \frac{DJ_2}{dt}\right\rangle} + \left\langle J_1, \frac{D^2J_2}{dt^2}\right\rangle - \cancel{\left\langle \frac{DJ_1}{dt}, \frac{DJ_2}{dt}\right\rangle} - \left\langle \frac{D^2J_1}{dt^2}, J_2\right\rangle$$

We can then use the Jacobi equation to cancel out the remaining terms. □

Ex: Let M be an oriented surface, and take $||\dot{\gamma}|| \equiv 1$. Then $(E_1, E_2) = (\dot{\gamma}, \dot{\gamma}^{\perp})$ is an orthonormal frame. We write down the Jacobi equations:

$$(E_i, \dot{\gamma}, \dot{\gamma}, E_j) = \begin{cases} 0 & i = 1 \text{ or } j = 1 \\ k & i = j = 2 \end{cases}$$

where k is the Gaussian curvature. Write $J = f^1\dot{\gamma} + f^2\dot{\gamma}^{\perp}$. Then $\ddot{f}^1 = 0$ iff $f^1 = a + bt$, for $a, b \in \mathbb{R}$. And $\ddot{f}^2 = kf^2 = 0$ (assume k is constant for this problem). Then $f^2(t) = Ae^{i\sqrt{k}} + Be^{-i\sqrt{k}}$. f^2 must be real, so we solve this well-known type of differential equation, and if $k > 0$, we get the following:

$$J(t) = (A\cos\sqrt{k}t + B\sin\sqrt{k}t)\dot{\gamma}^{\perp}$$

If $J(0) = 0$, then $J(t) = B\sin(\sqrt{k}t)\dot{\gamma}^{\perp}$ Observe that $J(\frac{\pi}{\sqrt{k}}) = 0$. We say that "$\gamma(0)$ and $\gamma(\frac{\pi}{\sqrt{k}})$ are conjugate".
If $k < 0$, then we replace sin and cos with sinh and cosh. Then $J(0) = 0$ implies $J(t) = B\sinh(\sqrt{k}t)\dot{\gamma}^{\perp}$. In this case, $J(t) \neq 0$, $\forall t \neq 0$.

One application of this is computing $d(\exp_p)_v$ for $v \neq 0$.

Prop: Given $v, w \in T_pM$, $d(\exp_p)_v(w) = J(1)$, where J is the Jacobi field such that $J(0) = 0$ and $\frac{DJ}{dt}(0) = w$.

Proof: $d(\exp_p)_v(w) = \frac{d}{dt}\exp_p(v + sw)\big|_{s=0}$. Define $f(s,t) = \exp_p(t(v + sw))$. $\forall s$, $t \mapsto f(s,t)$ is a geodesic. Define $J(t) = \partial_s f|_{s=0}$. We know this is a Jacobi field, and claim that $\frac{DJ}{dt}(0) = w$.

Math 635 Lecture 25

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/19/21

Space of Constant Curvature

Such spaces are locally isometric to a sphere (if the curvature is positive), a Euclidean space (if the curvature is zero), or a hyperbolic space (if the curvature is negative).

Lemma: M has constant sectional curvature $K_0 \in \mathbb{R}$ iff $\forall X, Y, W, Z \in \mathfrak{X}(M)$,

$$(X, Y, W, Z) = K_0(\langle Y, W\rangle \langle X, Z\rangle - \langle X, W\rangle \langle Y, Z\rangle)$$

Proof: Note that by definition of K,

$$(X, Y, Y, X) = K(X, Y) \underbrace{(\langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2)}_{=|X\wedge Y|}$$

This is a special case of the identity we want! Proving $\Leftarrow$ is trivial. As for proving $\Rightarrow$, we show that the right hand side has the same symmetry properties as the left hand side. (This is just a messy computation.) Then we use the fact that we know K can determine $\mathcal{R}$ if it's applied everywhere. $\square$

Cor: Let M have constant curvature, γ a geodesic of speed 1, and J a normal Jacobi field. Then the Jacobi equation becomes $J'' + KJ = 0$. (Recall the notation: $J' = \frac{DJ}{dt}$, $J'' = \frac{D^2J}{dt^2}$.)

Proof: By the lemma, we have, $\forall X \in \mathfrak{X}(M)$,

$$\langle \mathcal{R}(J, \dot\gamma)\dot\gamma, X\rangle = (J, \dot\gamma, \dot\gamma, X) = K(||\dot\gamma||^2 \langle J, X\rangle - \langle J, \dot\gamma\rangle \langle \dot\gamma, X\rangle)$$

This is equal to $K\langle J, X\rangle$ for all X, $|mcR(J, \dot\gamma)\gamma = KJ$. $\square$

Now, we consider a generalization of the surface case. Let γ be a geodesic on M, with constant curvature.
Let $N(t) \in \Gamma_\gamma(TM)$ be a unit normal and parallel field, determined by $N(0)$. In the surface case, we call this $\dot\gamma^\perp$. Define $J(t) = \varphi(t)N(t)$. Then J is Jacobi iff $\ddot\varphi + K\varphi = 0$.

Cor: If $J(0) = 0$, then $J(t) = \begin{cases} A\sin(\sqrt{K}t)N & K > 0 \\ tN & K = 0 \\ A\sinh(\sqrt{-K}t)N & K < 0 \end{cases}$

Recall the differential of the exponenial map:

Thm: $d(\exp_p)_v(w) = J(1)$, where J is the Jacobi field along $\gamma : t \mapsto \exp_p(tv)$ s.t. $J(0) = 0$, $J'(0) = w$.

Proof: Define $f(s, t) = \exp_p(t(v + sw))$. Observe: $d(\exp_p)_v(w) = \partial_s f|_{s=0,t=1}$. Let $J(t) = \partial_s f(s = 0, t)$. This is a Jacobi field, since the t curves are geodesics. Now, we just need to check that J satisfies the initial conditions.

$$J(0) = 0, \qquad J' = \frac{D}{dt}\partial_s f = \frac{D}{ds}\partial_t f, \qquad \partial_t f(s, t = 0) = d(\exp)_0(v + sw) = v + sw \in T_pM$$

$s \mapsto \partial_t f(s, 0)$ is entirely contained in T_pM. Still, it's a field along $s \mapsto f(s, 0) = p$, so it's just a constant "curve". Finally, $\frac{D}{ds}\partial_t f = \frac{d}{ds}(v + sw) = w$. $\square$

Next, we examine a result of the "rate of spreading of geodesics". As before, define $f(s, t) = \exp_p(t(v + sw))$.

Prop: Take $v, w \in T_pM$ orthonormal, and let $\pi = \operatorname{span}(v, w)$, $\gamma(t) = \exp_p(tv)$. Let $J(t)$ be the Jacobi field s.t. $J(0) = 0$, $J'(0) = w$. (Again, exactly as before.) Then

$$||J(t)|| = t - \frac{t^3}{6} K_p(\pi) + O(t^3) \qquad \text{as} \qquad t \to 0$$

as $t \to \infty$.

Interpretation: Rate of spreading of the ray $t(v + sw)$ with respect to $t \mapsto tv$. $\frac{\partial}{\partial s} |t(v + sw)| = tw$. We want to measure the same object on the manifold after exponentiation. We do so with respect to infinitesimal change in s.

$$||J(t)|| \sim \begin{cases} < t & K_p(\pi) > 0 \\ > t & K_p(\pi) < 0 \end{cases}$$

Proof of the proposition: It's true iff

$$||J(t)||^2 = t^2 - \frac{t^4}{3} K_p(\pi) + O(t^4)$$

So we need to compute 4 derivatives of $\langle J, J \rangle$ at zero. Define $a_k = \langle J, J \rangle^{(k)} (t = 0)$. Then clearly $a_0 = 0$. $\langle J, J \rangle' = 2 \langle J', J \rangle$, so $a_1 = 0$.

$$\frac{1}{2} \langle J, J \rangle'' = \langle J'', J \rangle + \langle J', J' \rangle \quad \Rightarrow \quad a_2 = ||w||^2 = 1$$

$$\frac{1}{2} \langle J, J \rangle^{(3)} = \left\langle J^{(3)}, J \right\rangle + \langle J'', J' \rangle + 2 \langle J'', J' \rangle = \underbrace{\left\langle J^{(3)}, J \right\rangle}_{=0} + 3 \underbrace{\langle J'', J' \rangle}_{= -\langle \mathcal{R}(J, \dot\gamma)\dot\gamma, J' \rangle} = 0$$

because $J(0) = 0$ by the Jacobi equations. Finally, we compute the fourth derivative:

$$\frac{1}{2} \langle J, J \rangle^{(4)} = \cancelto{0}{\left\langle J^{(4)}, J \right\rangle} + 3 \left(\underbrace{\left\langle J^{(3)}, J' \right\rangle}_{?} + \underbrace{\left\langle J'', J'' \right\rangle}_{=0} \right)$$

We will figure out $\left\langle J^{(3)}, J' \right\rangle$ in the next lecture.

Math 635 Lecture 26

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/22/21

Continuing from last time, we need to compute $\left\langle J^{(3)}, J' \right\rangle$.

Lemma: $J^{(3)} = -\frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma$. (Prove using properties of Jacobi fields.)

Claim: $\frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma\big|_{t=0} = \mathcal{R}(J',\dot\gamma)\dot\gamma|_{t=0}$.
Proof: Let $W \in \Gamma_\gamma(TM)$. Compute

$$\frac{d}{dt}\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, W\rangle = \left\langle \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle + \left\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, \cancelto{0 \text{ at } t=0}{\frac{Dw}{dt}}\right\rangle$$

and

$$\frac{d}{dt}\langle \mathcal{R}(J,\dot\gamma)\dot\gamma, W\rangle = \frac{d}{dt}(J,\dot\gamma,\dot\gamma,W) = \frac{d}{dt}(W,\dot\gamma,\dot\gamma,J) = \left\langle \frac{D}{dt}\cancelto{0}{\mathcal{R}(W,\dot\gamma)\dot\gamma}, J\right\rangle + \underbrace{\langle \mathcal{R}(W,\dot\gamma)\dot\gamma\rangle, \frac{DJ}{dt}}_{=(W,\dot\gamma,\dot\gamma,J')=(J',\dot\gamma,\dot\gamma,W)}$$

Thus, at $t=0$, we have $(J',\dot\gamma,\dot\gamma,W)|_{t=0} = \left\langle \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma, W\right\rangle\big|_{t=0} = \langle \mathcal{R}(J',\dot\gamma)\dot\gamma, W\rangle$.
We conclude that $\mathcal{R}(J',\dot\gamma)\dot\gamma = \frac{D}{dt}\mathcal{R}(J,\dot\gamma)\dot\gamma$.

Thus, $\left\langle J^{(3)}, J'\right\rangle = \langle \mathcal{R}(J',\dot\gamma)\dot\gamma, J'\rangle = K_0(J',\dot\gamma)$. This proves the lemma, and completes the proof we began last time. $\square$

Conjugate Points

Defn: Let γ be a geodesic, $p, q \in \operatorname{im}\gamma$ distinct. Then p, q are conjugate along γ iff there exists a nonzero Jacobi field of γ that vanishes at p and q.

Ex: With constant negative curvature $K < 0$, suppose $\gamma(0) = p$, $\gamma(t_1) = q$. Then $J(t) = A\sinh(\sqrt{-K}t)W(t)$. So there are no conjugate points.

If $K > 0$, then $J(t) = A\sin(\sqrt{K}t)$, so $t = \frac{\pi}{\sqrt{k}}$ yields a conjugate point.

Defn: The multiplicity of q as a conjugate point of p is $\dim\{J \mid J(q) = 0, J(p) = 0\}$.

Claim: Multiplicity is at most $n-1$, and $n-1$ is achieved by the sphere S^n.

Note: "Being conjugate" is a symmetric relation, but it's not transitive!

Prop: Let $q = \exp_p(tv_q)$, $v_q \in T_pM$. Then q is conjugate to p iff v_q is a critical point of $\exp_p$, iff $d(\exp_p)_{v_q}$ has a nontrivial kernel.

Proof: Simply recall how to compute $d(\exp_p)_{v_q}$: $J(1) = d(\exp_p)_{v_q}(w)$ when $J(0) = 0$ and $J'(0) = w$ (by parameterizing γ by arc length, and rescaling so that $t_1 = 1$). $\square$

Moreover, $\dim\ker d(\exp_p)_{v_q}$ is the multiplicity of q, and it is at most $n-1$, because $v_q \notin \ker d(\exp_p)_{v_q}$, because $d(\exp_p)_{v_q}(v_q) = \dot\gamma(t_1) \neq 0$, because γ is nontrivial, because $p \neq q$.

Thm: (I) Let $q = \exp_p(t_1 v)$, $||v|| = 1$ be a conjugate point of p. Then $\forall t_2 > t_1$, $t \mapsto \exp_p(tv)$ is *not* minimizing on $[0, t_2]$.

Proof: This is simply a nice application of the second variation form. □

Think of the sphere – if we go from the north pole to past the south pole, the curve isn't minimizing.

Note: There are no conjugate points on a cylinder, but not all geodesics are minimizing.

Note: The converse is not globally true.

Thm: (II) Let $p, q \in \mathrm{im}\gamma$, for γ a geodesic, and assume p, q are not conjguate, and there are no conjugate points between p and q. Then any proper variation of the arc $\widehat{pq}$ is such that $\forall s$ sufficiently small, the t-curves are longer than $\widehat{pq}$.

Thm: *Second Variation Formula, with one jump in V'.* Let γ be a geodesic, defined for $t \in [0, t_2]$. Suppose V is a proper, continuous, infinitesimal variation (with respect to s), and is smooth on $[0, t_1]$ and $[t_1, t_2]$ (assume 1-sided derivatives exist at t_1). Then

$$E''(0) = -\int_0^{t_2} \langle V, V'' + \mathcal{R}(V, \dot{\gamma})\dot{\gamma}\rangle \; dt - \langle V(t_1), \Delta V'(t_1)\rangle$$

where $\Delta V'(t_1) = V'(t_1^+) - V'(t_1^-)$ (using the one-sided derivatives).

Proof: See Do Carmo, page 197. (They prove this result for any number of jumps.)

Now, back to the philosophy that this is a sort of Hessian of the energy functional $\mathcal{E} : \{\text{paths } p \rightsquigarrow q\} \to \mathbb{R}$. In the above formula, V is a tangent vector, so the right hand side should be a quadratic form on V. But what is the associated bilinear form?

Prop: Given the same assumptions as above, with any number of jumps, we have

$$E''(0) = \int_0^{t_2} \langle V', V'\rangle - \langle \mathcal{R}(V, \dot{\gamma})\dot{\gamma}, V\rangle \; dt$$

and moreover, the symmetric bilinear form is

$$I(V, W) = \int_0^{t} \langle V', W'\rangle - \underbrace{\langle \mathcal{R}(V, \dot{\gamma})\dot{\gamma}, W\rangle}_{\text{pairwise symmetric in } V, W} \; dt$$

so $E''(0) = I(V, V)$ (and clearly, I is symmetric).

Math 635 Lecture 27

Professor Alejandro Uribe-Ahumada
Transcribed by Thomas Cohn

3/24/21

Continuing from last time...

Thm: Let $q = \exp_p(t, v)$, $||v|| = 1$ be a conjugate point of p. Then $\forall t_2 > t_1$, $t \mapsto \exp_p(tv)$ is not minimizing on $[0, t_2]$.

Proof: By the hypothesis, there's a Jacobi field J f γ such that $J \neq 0$, $J(0) = 0$, and $J(t_1) = 0$. We will construct a variation of γ on $[0, t_2]$ with $E'' < 0$. Define

$$\tilde{J}(t) = \begin{cases} J(t) & 0 \leq t \leq t_1 \\ 0 & t_1 \leq t \leq t_2 \end{cases}$$

Because $J(t_1) = 0$, this variation is continuous at t_1, so it's clearly continuous on $[0, t_2]$. Let $W \in \Gamma_\gamma(TM)$ be smooth, supported near t_1, and defined such that $W(t_1) = \Delta\tilde{J}'(t_1) \neq 0$. It's nonzero because $\Delta\tilde{J}'(t_1) = 0$ would imply that $J'(t_1) = 0$, which would mean $J = 0$, a contradiction with our original assumption. Now, we define the actual variation we're going to use. Let

$$V_\varepsilon = \tilde{J} + \varepsilon W$$

for some small $0 < \varepsilon \ll 1$. This is a proper variation of γ on $[0, t_2]$. Now compute $E''(0)$ (associated with V_ε).

$$E''(0) = I(V_\varepsilon, V_\varepsilon) = I(\tilde{J} + \varepsilon W, \tilde{J} + \varepsilon W) = I(\tilde{J}, \tilde{J}) + 2\varepsilon I(\tilde{J}, W) + \varepsilon^2 I(W, W)$$

where I is the bilinear form defined previously. Well,

$$I(\tilde{J}, \tilde{J}) = -\int_0^{t_2} \left\langle \cancelto{0}{\tilde{J}, \text{Jacobi operator on } \tilde{J}} \right\rangle - \left\langle \underbrace{\tilde{J}(t_1)}_{=0}, \Delta\tilde{J}'(t_1) \right\rangle dt = 0$$

$$I(\tilde{J}, W) = \int_0^{t_2} \left\langle \tilde{J}', W \right\rangle - \left\langle \mathcal{R}(\tilde{J}, \dot{\gamma})\dot{\gamma}, W \right\rangle dt$$

We use integration by parts, with $\frac{d}{dt}\left\langle W, \tilde{J}' \right\rangle = \left\langle W', \tilde{J}' \right\rangle + \left\langle W, \tilde{J}'' \right\rangle$, to compute

$$\int_0^{t_2} \left\langle \tilde{J}', W \right\rangle dt = -\int_0^{t_2} \left\langle W, \tilde{J}'' \right\rangle dt - \left\langle W(t_1), \Delta\tilde{J}'(t_1) \right\rangle$$

Now, we combine with the $\left\langle \mathcal{R}(\tilde{J}, \dot{\gamma})\dot{\gamma}, W \right\rangle$ term. Using the fact that $\tilde{J}$ satisfies the Jacobi equation, they cancel, and we're left with

$$I(\tilde{J}, W) = -\left\langle W(t_1), \Delta\tilde{J}'(t_1) \right\rangle = -\left|\left|\Delta\tilde{J}'(t_1)\right|\right|^2 < 0$$

Thus, $E''(0) = \varepsilon^2 I(W, W) - 2\varepsilon \left|\left|\Delta\tilde{J}'(t_1)\right|\right|^2$. So for $\varepsilon \ll 1$, $E''(0) <$, so for s small enough, t-curves in a variation of γ with $\tilde{V}_\varepsilon$ are shorter than γ. $\square$

Completeness

(Chpater 7 in Do Carmo)

Defn: M is geodesically complete iff $\forall p \in M$, $\exp_p$ is defined on all of T_pM.

Ex: If M is compact, M is geodesically complete.

Why? Well, if M is compact, then the unit tangent bundle $TM_{\mathbb{1}} = \{(p, v) \in TM : ||v|| = 1\}$ is compact. So geodesic flow is given by the flow of a certain field on $TM_{\mathbb{1}}$ (up to scaling by time), and smooth vector fields on compact manifolds are complete. $\square$

Defn: M is complete iff (M, d) is a complete metric space.

The **big idea** we're working towards is
Thm: (Hopf-Rinow) M is geodesically complete iff M is a complete metric space.

Thm: Let M be connected. Let $p \in M$ such that $\exp_p$ is defined on all of T_pM. Fix $q \in M$. Then there's a geodesic γ from p to q, and $d(p, q) = \ell(\gamma)$.

Proof: Let $\varepsilon > 0$ be such that there's a geodesic sphere S_ε of radius ε centered at p. Let $p' \in S_\varepsilon$ be a point minimizing the map

$$\begin{aligned} S_\varepsilon &\to \mathbb{R} \\ x &\mapsto d(x, q) \end{aligned}$$

That is, p' is the point on S_ε which is closest to q. By compactness, p' exists, and $p' = \exp_p(\varepsilon v)$ for some $v \in T_pM$ with $||v|| = 1$. Now, we want to show $\exp_p(d(p, q)v) = q$...

Lemma: $d(p, q) = \underbrace{d(p, p')}_{=\varepsilon} + d(p', q)$.

Proof: $\leq$ is just a direct application of the triangle inequality. For $\geq$, let c be any path from p to q, and let w be the point where c intersects S_ε. Then $\ell(c) = \ell(\widehat{pw}) + \ell(\widehat{wq}) \geq \varepsilon + d(p', q)$. Now, take the infimum over all such paths c. We get

$$d(p, q) = \inf_c \ell(c) \geq \varepsilon + d(p', q) = d(p, p') + d(p', q)$$

$\square$

Returning to the proof of the theorem, introduce $\mathscr{T} \stackrel{\text{def}}{=} \{t \in [0, d(p, q)] \mid d(p, q) = t + d(\gamma(t), q)\}$. We observe the following facts about $\mathscr{T}$:

- $\mathscr{T} \neq \emptyset$ because $\varepsilon \in \mathscr{T}$ by the lemma.
- $\mathscr{T}$ is closed, because it's the preimage of a closed set under a continuous function.
- $\forall t \in \mathscr{T}$, $d(\gamma(t), p) = t$.

We want to show $d(p, q) = \sup \mathscr{T}$. We will argue this by contradiction: assume $t_1 \stackrel{\text{def}}{=} \sup \mathscr{T} < d(p, q)$. Then $t_1 + \delta < d(p, q)$. S_δ exists centered at $\gamma(t)$, so then we'll show $t + \delta \in \mathscr{T}$, thus contradicting the definition of t_1 as the supremum of $\mathscr{T}$. We will do this next time.

Math 635 Lecture 28

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/26/21

We're currently trying to prove:

Thm: Let M be connected, $p \in M$ such that $\exp_p$ is defined on all of T_pM. Let $q \in M$. Then there's a geodesic γ from p to q such that $d(p,q) = \ell(\gamma)$.

Continuing from last time, we want to show that $\sup \mathscr{T} = d(p,q)$, i.e., $d(p,q) \in \mathscr{T}$, i.e., $d(p,q) = d(p,q) + d(\gamma(d(p,q)),q)$, i.e., $d(\gamma(d(p,q)),q) = 0$. Because d is a distance function, it's enough to show $\gamma(d(p,q)) = q$.

Let $t_1 = \sup \mathscr{T}$. Because $\mathscr{T}$ is closed, we know $t_1 \in \mathscr{T}$. Assume that $t_1 < d(p,q)$ (we will show a contradiction). Then $\exists \delta > 0$ s.t. $t_1 + \delta < d(p,q)$, and there exists a geodesic sphere S_δ centered at $\gamma(t_1)$ with radius δ. Let $y \in S_\delta$ minimizing the map

$$\begin{aligned} S_\delta &\to \mathbb{R} \\ x &\mapsto d(x,q) \end{aligned}$$

We want to show $t_1 + \delta \in \mathscr{T}$.

Claim 1: $d(y,q) = d(p,q) - (t_1 + \delta)$.
Proof: The lemma from last time implies that $d(p,q) - t_1 = d(\gamma(t_1),q) = \delta + d(y,q)$. $\square$

Claim 2: $d(p,y) = t_1 + \delta$.
Proof: $\leq$ follows directly from the triangle inequality. $\geq$ is true because $d(p,q) \leq d(p,y) + d(y,q)$, so

$$d(p,y) \geq d(p,q) - (d(p,q) - (t_1 + \delta)) \geq t_1 + \delta$$

by claim 1. $\square$

Claim 3: $y = \gamma(t_1 + \delta)$.
Proof: Consider the path $\gamma(t)$ for $0 \leq t \leq t_1$, followed by a radial geodesic from $\gamma(t_1)$ to y. So the path is overall from p to y. By claim 2, this path is length minimizing, so it's a geodesic with the same initial conditions as γ. This implies it must be γ, with the domain $0 \leq t \leq t_1 + \delta$.

Claims 1 and 3 together imply that $t_1 + \delta \in \mathscr{T}$, a contradiction. This completes the proof. $\square$

Thm: (Hopf-Rinow) Let M be connected. Assume $\exists p \in M$ s.t. $\exp_p$ is defined on all of T_pM. Then

(a) Every closed and bounded set is compact.
(b) M is complete as a metric space.
(c) M is geodesically complete, i.e., $\forall q \in M$, $\exp_q$ is defined on all of T_qM.

In fact, these are all equivalent. We will show this by: assumption $\Rightarrow$ (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ assumption.

Proof: assumption $\Rightarrow$ (a): Let $S \subset M$ be closed and bounded. Then $\forall y \in S$, $\exists \gamma_y$ a minimizing geodesic from p to y. S is bounded, so $\exists R > 0$ s.t. $\forall y \in S$, $\ell(\gamma_y) < R$. So $S \subset \exp_p\left(\overline{B_R(0)}\right)$, where $B_R(0) \subset T_pM$. Because $\overline{B_R(0)}$ is compact and $\exp_p$ is continuous, $\exp_p\left(\overline{B_R(0)}\right)$ is also compact, so S is contained in a compact set, and is thus compact.

(a) $\Rightarrow$ (b): This only requires facts of point-set topology. If (x_n) is Cauchy, then it is bounded. So its image is compact. Thus, there's a convergent subsequence, so (x_n) converges to its limit point.

(b) $\Rightarrow$ (c): We prove this by contradiction. Assume $\exists q \in M, T > 0$, and $v \in T_qM$ with $||v|| = 1$, such that $\gamma(t) = \exp_q(tv)$ exists $\forall t \in [0,)$, but not beyond T. Take $t_1 < t_2 <$ in $[0,T)$ so that (t_n) converges to T, and let $x_n = \gamma(t_n)$. Note that $d(x_n, x_m) \leq |t_n - t_m|$, $\forall n,m$. Since (t_n) converges, (x_n) is Cauchy, so $\exists w \in M$ such that

$\lim_{n\to\infty} x_n = w$. Let W be a totally normal neighborhood of w, and $\delta > 0$ such that $\forall x \in W$, the geodesic ball $B(x,\delta) \supseteq W$. Then $\exists N \in \mathbb{N}$, such that $\forall n, m > N$, $|t_n - t_m| < \delta/2$, and $\gamma(t_m) \in W$. Pick $n > N$. Use $\exp_{x_n}$ to "relaunch" the geodesic. $s \mapsto \exp_{x_n}(s\dot\gamma(t_n))$ exists for $|s| < \delta$. This extends γ past T, since $t_n + \delta > T$. Oops!

(c) $\Rightarrow$ assumption: Trivial. $\square$

Defn: If M satisfies these properties, we call M a complete Riemannian manifold.

We obtain yet another version of Bonnet-Myer as a corollary:

Thm: (Bonnet-Myer III) Let M be a compact, connected Riemannian manifold, and assume Ric $> \left(\frac{\pi}{\ell}\right)^2$. Then M is compact and the diameter of M is no more than ℓ.

Proof: We already know that any geodesic with a length of at least ℓ is not minimizing. Also, any pair of points can be joined by a minimizing geodesic. Thus, the diameter of M is at most L, so M is bounded and compact. $\square$

Math 635 Lecture 29

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/29/21

Today's goal is to prove Cartan-Hadamard: If M is a complete, connected Riemannian manifold with $K \leq 0$, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Defn: A C^∞ map $F : \tilde{M} \to M$ is a smooth covering map iff $\forall p \in M$, there's a neighborhood V of p such that $F^{-1}(V) = \bigcup_\alpha U_\alpha$, where $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset$, and $\forall \alpha$, $F|_{U_\alpha}^{V}$ is a diffeomorphism. We say that V is evenly covered.

Observe: A smooth covering map $F : \tilde{M} \to M$ is always a local diffeomorphism, as the definition of local diffeomorphism is $\forall \tilde{p} \in \tilde{M}$, there's a neighborhood U of $\tilde{p}$ and V of $F(\tilde{p})$ such that $F|_U^V$ is a diffeomorphism.

Defn: A smooth map $F : \tilde{M} \to M$ between Riemannian manifolds is a local isometry iff $\forall \tilde{p} \in M$, there's a neighborhood U of $\tilde{p}$ and V of $F(\tilde{p})$, $F|_U^V$ is an isometry.

Some properties of a local isometry $F : \tilde{M} \to M$:

- F is a local diffeomorphism.
- If $\tilde{\gamma} : I \to \tilde{M}$ is a geodesic on $\tilde{M}$, then $\gamma = F \circ \tilde{\gamma}$ is a geodesic on M.
- If $c : [a, b] \to \tilde{M}$ is any path, $\ell(F \circ c) = \ell(c)$.

Lemma: Let $F : \tilde{M} \to M$ be a local isometry, where $\tilde{M}$ and M are connected, complete Riemannian manifolds. Then F is a surjective covering map.

Proof: The main property of F is $\forall p \in M$, $\tilde{p} \in F^{-1}(p)$, $\forall v \in T_pM$, $\exists! \tilde{v} \in T_{\tilde{p}}\tilde{M}$ such that $F_{*,\tilde{p}}(\tilde{v}) = v$, and also $F \circ (\exp_{\tilde{p}}(t\tilde{v})) = \exp_p(tv)$. We obtain the existence and uniqueness of $\tilde{v}$ because $dF_{\tilde{p}}$ is a bijection. And the equality with the exponential map is true because both sides are geodesics on M with the same initial conditions. We'll say that we can "lift" geodesics from M to $\tilde{M}$: choose $\tilde{p} \in F^{-1}(p)$. Then $\exists! \tilde{\gamma}$ geodesic on $\tilde{M}$ such that $(F \circ \gamma)(t) = \exp_p(tv)$ and $\tilde{\gamma}(0) = \tilde{p}$.

To show the map is surjective, let $\tilde{p} \in \tilde{M}$, and define $p = F(\tilde{p})$. Let $q \in M$. Then by completeness, there is a geodesic γ on M joining p to q – $\gamma(0) = p$ and $\gamma(T) = q$. Let $\tilde{\gamma}$ be the lift of γ to $\tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{p}$. $\forall t$, $(F \circ \tilde{\gamma})(t) = \gamma(t)$. So $F(\tilde{\gamma}(T)) = \gamma(T) = q$, so $q \in \operatorname{im} F$.

Next, we show the map is a covering map. Let $p \in M$. Since F is a local isometry, F_* is always bijective, so p is a regular value. Thus, $F^{-1}(p) = \bigsqcup_\alpha \{\tilde{p}_\alpha\}$ is the disjoint union of (at most) countably many points. Let $\varepsilon > 0$ such that there's an open geodesic all $B_\varepsilon(p) \subset M$, centered at p with radius ε. $\forall \alpha$, define the open metric ball $U_\alpha = \left\{\tilde{q} \in \tilde{M} \mid \tilde{d}(\tilde{p}_\alpha, \tilde{q}) < \varepsilon\right\}$. We claim that $F^{-1}(B_\varepsilon(p)) = \bigcup_\alpha U_\alpha$, and the conditions of being a covering map are satisfied by the U_α.

Claim 1: $\forall \alpha$, F maps U_α into $B_\varepsilon(p)$, and $F|_{U_\alpha}$ is a bijection (so as a result, the restriction of F is a diffeomorphism). Proof: Pick $\tilde{q} \in U_\alpha$. Let $\tilde{\gamma}$ be a geodesic segment in $\tilde{M}$ joining $\tilde{p}_\alpha$ to $\tilde{q}$. Then $\ell(\tilde{\gamma}) < \varepsilon$. Consider $\gamma = F \circ \tilde{\gamma}$, a geodesic of the same length, $\ell(\gamma) = \ell(\tilde{\gamma}) < \varepsilon$. Then $\operatorname{im}\gamma \subseteq B_\varepsilon(p)$, so $F(\tilde{q}) \in B_\varepsilon(p)$. Now, we construct the inverse of $F|_{U_\alpha}^{B_\varepsilon(p)}$. Start with some $q \in B_\varepsilon(p)$. Lift the radial geodesic from p to q u to$\tilde{}$, starting at $\tilde{p}_\alpha$. Then its endpoint is the inverse of $q \in U_\alpha$.

Claim 2: $\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset$. Proof: We will show $\tilde{d}(\tilde{p}_\alpha, \tilde{b}_\beta) > 2\varepsilon$. By the triangle inequality, this suffices. Let $\tilde{\gamma}$ be he minimizing geodesic from $\tilde{p}_\alpha$ to $\tilde{p}_\beta$. Consider $\gamma = F \circ \tilde{\gamma}$. We claim that γ must exit $B_\varepsilon(p)$, because any geodesic contained in $B_\varepsilon(p)$ and passing through p is a radial geodesic, so it must be minimizing. It's not, so thus, $\ell(\gamma) > 2\varepsilon$.

Claim 3: $F^{-1}(B_\varepsilon) = \bigcup_\alpha U_\alpha$. Proof: $\supseteq$ is part of claim 1. For $\subseteq$, let $\tilde{q} \in F^{-1}(B_\varepsilon(p))$, so $F(q) \in B_\varepsilon(p)$. Then let γ be the radial geodesic from $F(\tilde{q})$ back to p. Let $\tilde{\gamma}$ be the lift of γ, starting at $\tilde{q}$. $\tilde{\gamma}$ ends at$\tilde{}$ such that $F(\tilde{p}) =$, so

$\tilde{p} \in F^{-1}(p)$. Thus, $\exists \alpha$ s.t. $\tilde{p} = \tilde{p}_\alpha$, and $\ell(\tilde{\gamma}) = \ell(\gamma) < \varepsilon$. $\square$

Lemma: If M is such that $K \leq 0$ everywhere, then there are no conjugate points.

Proof: HW

Thm: (Cartan-Hadamard) Let M be a complete Riemannian manifold, with $K \leq 0$ everywhere, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Note that the second lemma implies $\exp_p$ has no critical points. The idea of the proof is we put a metric on T_pM that makes $\exp_p$ a local isometry. Then we have to check that this metric is complete. It is, because rays $t \mapsto tv$ are geodesics in this (crazy) metric, and they exist $\forall t$.

Math 635 Lecture 30

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/31/21

Last time, we proved Cartan-Hadamard. To review:

Thm: (Cartan-Hadamard) If M is a complete, connected Riemannian manifold with $K \leq 0$, then $\forall p \in M$, $\exp_p : T_pM \to M$ is a smooth covering map.

Proof (sketch): From homework, we know $\exp_p$ is a local diffeomorphism. Then we show the following proposition: If $F : \tilde{M} \to M$ is a local isometry, with $\tilde{M}$ and M complete, then F is a covering map. Finally, we take $\tilde{M} = T_pM$, with the "pull-back" metric such that $\exp_p$ is a local isometry. Now, we apply the proposition. We just need the metric to be complete, and it is, because rays $t \mapsto tv$ (for $v \in T_pM \setminus \{0\}$) are geodesics with respect to the pull-back metric, and because $\exp_p$ is a local isometry, they map to geodesics on M. So T_pM with this metric has the property that all geodesics through $0 \in T_pM$ can be continued, $\forall t \in \mathbb{R}$, so T_pM is complete. $\square$

Why is this such a big deal?

Cor: If M is as in Cartan-Hadamard, and simply connected, then M is diffeomorphic to $\mathbb{R}^n$.

In general, for M complete with $K \leq 0$, M is said to be aspherical: $\forall n > 1$, $\pi_n(M) = 0$.

Observe: If M is complete, then M cannot be isometrically embedded as a proper open set of some Riemannian manifold W. This is true because covering maps are surjective.

Prop: If $M \subsetneq W$, and M is open, then M is not complete.

Proof: Let $p \in \partial M$, and U a normal neighborhood of p in W. Let $q \in M \cap U$. Let γ be a radial geodesic from p to q. Reverse t, so we have $\gamma^- : q \rightsquigarrow p$. For small t, γ^- is a geodesic in M. But for larger t, $\gamma^-(t) \notin M$, so M is not complete. $\square$

The Second Fundamental Form

(Refer to Chapter 6 of Do Carmo.)

Note: The first fundamental form is just the metric itself.

The setting and notation we'll consider is $\bar{M}$ a Riemannian manifold (known as the "ambient space"), $M \subseteq \bar{M}$ a submanifold with the induced metric, $\bar{\nabla}$ the Levi-Civita connection on $\bar{M}$, and ∇ the Levi-Civita connection on M. Further, if $p \in M$, $v \in T_p\bar{M}$, we'll write $v = v^T + v^\perp = v^T + v^N$, where v^T is the orthogonal projection of v onto T_pM (known as the "tangential component") and $v^\perp$ (or v^N) is its orthogonal complement in $(T_pM)^\perp$ (known as the "normal component").

Recall (from homework): $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^T$, where $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$ are (arbitrary) extensions of X and Y.

Defn: $\forall X, Y \in \mathfrak{X}(M)$, $B(X,Y) = (\bar{\nabla}_{\bar{X}} \bar{Y})^\perp$. So $\forall p \in M$, $B(X,Y)(p) = (\bar{\nabla}_{\bar{X}} \bar{Y})(p) - (\nabla_X Y)(p)$.

Note: $B(X,Y)(p) \in (T_pM)^\perp$.

Prop: $B(X,Y)(p)$ depends only on X_p and Y_p, and is therefore equivalent to tensors (maps) $\forall p \in M$:

$$B_p : T_pM \times T_pM \to (T_pM)^\perp$$

Moreover, $\forall p \in M$, B_p is bilinear and symmetric: $\forall x, y \in T_pM$, $B_p(x,y) = B_p(y,x)$.

Proof: We'll first show symmetry. Let $X, Y \in \mathfrak{X}(M)$. Then

$$B(X,Y) - B(Y,X) = \left(\bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}\right)\big|_M - (\nabla_X Y - \nabla_Y X) = \left[\bar{X}, \bar{Y}\right]\big|_M - [X,Y]$$

But we proved in homework (in Math 591) that on $\left[\bar{X}, \bar{Y}\right]\big|_M = [X,Y]$, so we have $B(X,Y) - B(Y,X) = 0$, and thus, B is symmetric.

Now, at p, we know that any connection depends pointwise in the "lower" entry. So $B(X,Y)(p)$ depends (with respect to X) only on X_p. From symmetry, the same is true for Y, so B is bilinear. $\square$

To get information from B, one chooses $\nu_p \in (T_pM)^\perp$ with $||\nu_p|| = 1$.

Defn:

- $H_\nu : T_pM \times T_pM \to \mathbb{R}$ by taking $H_\nu(x,y) = \langle B_p(x,y), \nu \rangle$.
- $I\!I_p(x) = H_\nu(x,x)$.
- $S_\nu : T_pM \to T_pM$, the shape operator, also known as the Weingarten map, is defined by $\forall x, y \in T_pM$,

$$\langle S_\nu(x), y \rangle = H_\nu(x,y) = \langle B(x,y), \nu \rangle$$

  S_ν exists and is well-defined because $\langle \ , \ \rangle$ is non-degenerate.

Observe: Because B is symmetric, H_ν is symmetric, so S_ν is self-adjoint. Thus, S_ν can be diagonalized, i.e., three's a basis $(e_1, \ldots, e_n)$ of T_pM such that $S_\nu(e_i) = \kappa_i e_i$, for $\kappa_i \in \mathbb{R}$.

Defn: The κ_i are the principal curvatures of M at p, and the e_i are the principal directions.

Looking ahead, we will prove if $\bar{M}$ is flat (e.g. $\bar{M} = \mathbb{R}^n$), then the intrinsic sectional curvature of M satisfies $K(e_i, e_j) = \kappa_i \kappa_j$, $\forall i \neq j$. When $\dim M = 2$, this is Gauss' "Theorem Egregium".

Math 635 Lecture 31

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/2/21

Recall our definitions: $\bar{M}$ is a Riemannian manifold, $M \subseteq \bar{M}$ is a submanifold with the induced metric. $\forall X, Y \in \mathfrak{X}(M)$, $B(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y \in (T_pM)^\perp$, where $\bar{X}$ and $\bar{Y}$ are extensions of X and Y (respectively) to $\bar{M}$. (This is normal to M.) We can also think of it as $\bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_X Y + B(X,Y)$.

- B is a symmetric tensor – $\forall p \in M$, $B_p : T_pM \times T_pM \to (T_pM)^\perp \subseteq T_p\bar{M}$.
- If we define, $\forall p \in M$, $\nu_p \in (T_pM)^\perp$ a unit normal vector, then $S_\nu : T_pM \to T_pM$ is the shape operator, with the defining property $\forall x, y \in T_pM$, $\langle S_\nu(x), y\rangle = \langle B(x,y), \nu\rangle$.

Because B is symmetric, S_ν is self-adjoint, so there's an orthonormal eigenbasis $(e_1, \ldots, e_n)$ of T_pM and $\kappa_1, \ldots, \kappa_n$ such that $S_\nu(e_i) = \kappa_i e_i$. The e_i are the principal directions, and the κ_i are the principal curvatures.

Observe: If $\operatorname{codim} M = 1$, then $\delta_{ij}\kappa_i = \langle S_\nu(e_i), e_j\rangle = \langle B(e_i, e_j), \nu\rangle$ by orthonormality. This is just the ν-component of $B(e_i, e_j)$, so $B(e_i, e_j) = \delta_{ij}\kappa_i\nu$.

Thm: (Weingarten Formula) Pick $p \in M$ and $\nu \in (T_pM)^\perp$. Extend ν to a unit normal vector field N on M. Then $\forall x \in T_pM$, the Weingarten Formula says

$$S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

(This is computed via a curve in M with velocity x.)

Proof: Let $x, y \in T_pM$, and extend these to vector fields $\bar{X}$ and $\bar{Y}$ tangent to M. Then

$$\langle S_\nu(x), y\rangle = \langle B(x,y), N_p\rangle(p) = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y, N\right\rangle(p) = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y}, N\right\rangle(p)$$

because $\langle \nabla_X Y, N\rangle(p) = 0$, as $(\nabla_X Y)_p \in T_pM$ and $N \in (T_pM)^\perp$ are orthogonal. On the other hand, $0 = \bar{Y}, N\big|_M$, so

$$0 = \bar{X}\left\langle \bar{Y}, N\right\rangle = \left\langle \bar{\nabla}_{\bar{X}}\bar{Y}, N\right\rangle + \left\langle \bar{Y}, \bar{\nabla}_{\bar{X}}N\right\rangle$$

The first term in the last expression is $\langle S_\nu(x), y\rangle$ by the above. So we have

$$\langle S_\nu(x), y\rangle = -\left\langle \bar{Y}, \bar{\nabla}_{\bar{X}}N\right\rangle = -\left\langle y, \bar{\nabla}_x N\right\rangle = \left\langle -\bar{\nabla}_x N, y\right\rangle \qquad \Rightarrow \qquad S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

$\square$

Consider the case where $\bar{M} = \mathbb{R}^{n+1}$, and $\dim M = n$. We identify tangent spaces with subspaces. Let U be the domain of N above. Then N is a map $N : U \to \mathbb{S}^n$, the unit sphere. We have

$$T_pM \cong T_{N(p)}\mathbb{S}^n$$
$$S_{N_p} : T_pM \to T_pM \cong T_{N(p)}\mathbb{S}^n$$

Claim: $S_{N_p} : T_pM \to T_{N(p)}\mathbb{S}^n$ is equal to $-dN_p$.
Check this: Let $c(t)$ be a curve on M with $p = c(0)$, $x = \dot{c}(0)$. Then

$$S_{N_p}(x) = -\bar{\nabla}_x N = -\frac{d}{dt}N_{x(t)}\bigg|_{t=0} = dN_p(x)$$

Ex:

a) Let $M = \mathbb{S}^n_R$, the sphere of radius R in $\mathbb{R}^{n+1}$. Let N be the outward pointing unit normal.

$$\begin{aligned} N : \mathbb{S}^n_R &\to \mathbb{S}^n \\ p &\mapsto \tfrac{1}{R}p \end{aligned} \qquad \Rightarrow \qquad S_N = \frac{1}{R}\mathrm{Id}$$

In this case, every direction is principal, with corresponding principal curvature $\kappa = -\frac{1}{R}$.

b) Let M be a cylinder of radius R in $\mathbb{R}^3$. Specifically, with parameterization

$$\vec{r}(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

Let $N(\theta, z) = (\cos\theta, \sin\theta, 0)$. We have the basis of T_pM:

$$\begin{cases} \frac{\partial}{\partial\theta} = (-R\sin\theta, R\cos\theta, 0) \\ \frac{\partial}{\partial z} = (0, 0, 1) \end{cases}$$

Then we have $T_pM \cong T_{N_p}\mathbb{S}^2$. The matrix of dN with respect to the above basis is

$$\begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

So we conclude that $\frac{\partial}{\partial\theta}$ is a principal direction, with corresponding principal curvature $\kappa_1 = -\frac{1}{R}$. $\frac{\partial}{\partial z}$ is the other principal direction, with corresponding principal curvature $\kappa_2 = 0$.

Now, back to the general $M \subseteq \bar{M}$.

Thm: Let W, X, Y, Z be vector fields on $\bar{M}$ which are tangent to M. Then $\forall p \in M$,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z)\rangle - \langle B(W, Z), B(X, Y)\rangle$$

Proof: Next time...

Cor: If $x, y \in T_pM$ are orthonormal, then

$$\bar{K}(x, y) = K(x, y) + ||B(x, y)||^2 - \langle B(x, x), B(y, y)\rangle$$

Proof: Simply use $\bar{K}(x, y) = R(x, y, y, x)$. $\square$

Cor: If $\bar{M}$ is flat, and M has codimension 1, i.e., it's a hypersurface, then $\forall i \neq j$, $K(e_i, e_j) = \kappa_i\kappa_j$.

Ex: $\mathbb{S}^n_R$ has constant sectional curvature $\frac{1}{R^2}$.

Math 635 Lecture 32

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/5/21

Let $M \subset \bar{M}$ be a Riemannian submanifold. Continuing from last time:

Thm: (Gauss' Formula) Given $W, X, Y, Z \in \mathfrak{X}(\bar{M})$ tangent to M, $\forall p \in M$,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z)\rangle - \langle B(W, Z), B(X, Y)\rangle$$

Proof: Well, the left hand side is

$$\bar{R}(W, X, Y, Z) = \left\langle(\bar{\nabla}_W\bar{\nabla}_X - \bar{\nabla}_X\bar{\nabla}_W - \bar{\nabla}_{[W,X]})Y, Z\right\rangle = \left\langle\bar{\nabla}_W\bar{\nabla}_X Y, Z\right\rangle + \left\langle-\bar{\nabla}_X\bar{\nabla}_W Y, Z\right\rangle + \left\langle-\bar{\nabla}_{[W,X]}Y, Z\right\rangle$$

We first work with the first term:

$$\left\langle\bar{\nabla}_W\bar{\nabla}_X Y, Z\right\rangle = \left\langle\bar{\nabla}_W\left(\nabla_X Y + B(X, Y)\right), Z\right\rangle = \langle\nabla_W\nabla_X Y, Z\rangle + \left\langle\bar{\nabla}_W B(X, Y), Z\right\rangle$$

because $\nabla = \bar{\nabla}^T$, and Z is tangent to M. The first term will eventually contribute to $R(W, X, Y, Z)$. As for the second term, note that $B(X, Y)$ is normal to M and Z is tangent to M, so $\langle B(X, Y), Z\rangle = 0$. So we can use the compatibility of $\bar{\nabla}$ with the metric and differentiation (specifically, the product rule) to get

$$\left\langle\bar{\nabla}_W B(X, Y), Z\right\rangle = -\left\langle B(X, Y), \bar{\nabla}_W Z\right\rangle = -\langle B(X, Y), B(W, Z)\rangle$$

By performing the analogous computations on the second term, we get

$$\left\langle-\bar{\nabla}_X\bar{\nabla}_W Y, Z\right\rangle = \langle-\nabla_X\nabla_W Y, Z\rangle + \langle B(W, Y), B(X, Z)\rangle$$

As for the third term,

$$\left\langle-\bar{\nabla}_{[W,X]}Y, Z\right\rangle = \left\langle-\nabla_{[W,X]}Y, Z\right\rangle$$

□

Recall: This implies that if $\bar{M}$ is flat, we have an orthonormal eigenbasis of the shape operator e_i, with corresponding eigenvalues κ_i. We get $\forall i \neq j$, $K(e_i, e_j) = \kappa_i\kappa_j$.

Recall: For $M \subseteq \mathbb{R}^{n+1}$ a hypersurface ($\dim M = n$), if M is oriented by a unit normal field $m \ni p \mapsto N_p \in (T_pM)^\perp$, we can interpret the unit normal field as a map $N : M \to S^n$. $\forall p \in M$, the shape operator at p with respect to N_p, denoted S_p, satisfies $S_p = -(dN)_p$ (with N as a map). We call N the Gauss spherical map. In this case, we're implicitly identifying $T_pM \cong T_pS^n$.

Cor: For $M \subset \mathbb{R}^3$ a surface, $K(p) = \det(dN)_p$, where $K(p)$ is the sectional curvature of M at p. (Because $\dim M = 2$, there's only one tangent plane at a given $p \in M$.)

We want to explore the global implications of this, when $\dim M = 2$. Let $N = M \to S^2$.

Observe: If $K(p) \neq 0$, then N is a local diffeomorphism at p. So if K is non-vanishing, and if M is compact and connected, then N is a covering map! Why is this true? Well, fix $q \in S^2$. Then $N^{-1}(q) = \bigsqcup_{i=1}^I \{p_i\}$. (This is finite because M is compact.) $\forall i$, p_i has a neighborhood U_i such that $N|_{U_i} : U_i \to N(U_i) = V_i$ is a diffeomorphism. Now, let $V = \bigcap_{i=1}^I V_i$. This is a neighborhood of q, and we claim V is evenly covered; to see this, just let $\tilde{U}_i = N^{-1}(V_i) \cap U_i$.

Cor: If K is non-vanishing, and M is compact and oriented, then M is diffeomorphic to S^2.

Ex: Consider M an ellipsoid, $K > 0$.

Question: Is there a metric on S^2 where $K < 0$? Answer: No, purely by topology.

Thm: (Gauss-Bonnet) Let $n = 2m$ be even. Let $M \subset \mathbb{R}^{n+1}$ be a compact, oriented, connected manifold. Let $N : M \to S^n$ be the Gauss map. Define $\mathscr{K} : M \to \mathbb{R}$, the Gaussian curvature, by $N^*(dV_{S^n}) = \mathscr{K}\, dV_M$, where dV_{S^n} is the volume form on S^n and dV_M is the volume form on M. Then

$$\int_M \mathscr{K}\, dV_M = \frac{1}{2}\operatorname{Vol}(S^n)\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M. Also, $\mathscr{K}$ is intrinsic – it depends only on the induced metric on M.

Defn: The <u>Euler characteristic</u> of M is $\chi(M) = \sum_{i=0}^{n}(-1)^i \dim H^i(M)$.

Note: $\mathscr{K}(p) = \det(dN)_p = \prod_{i=1}^{n} \kappa_i$. Why is this product intrinsic? Well, since $n = 2m$ is even, we can write

$$\mathscr{K}(p) = \prod_{i=1}^{m} \kappa_{2m-1}\kappa_{2m}$$

$\kappa_{2m-1}\kappa_{2m}$ is a specific sectional curvature, which we know to be intrinsic.

Plan for how we'll prove Gauss-Bonnet:

1. Show $\mathscr{K}$ is intrinsic (for $n \geq 4$).
2. Show $\mathscr{K}\, dV_M = N^*(dV_{S^n})$. This implies

$$\int_M \mathscr{K}\, dV_M = \int_M N^*(dV_{S^n}) = \deg(N)\int_{S^n} dV_{S^n} = \deg(N)\operatorname{Vol}(S^n)$$

3. $\deg(N) = \frac{1}{2}\chi(M)$

Steps 2 and 3 are just pure differenial topology.

"Degree theory": $N : M \to S^n$ induces

$$\begin{array}{ccc} N^*\colon H^n(S^n) & \longrightarrow & H^n(M) \\ \wr\parallel & & \wr\parallel \\ \mathbb{R} & & \mathbb{R} \end{array}$$

So N^* is just multiplication by a number. In fact, that number is an integer, and is called the degree of N.

Math 635 Lecture 33

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/7/21

Today, we'll take the first step towards proving Gauss-Bonnet. Let $M \subseteq \mathbb{R}^{n+1}$ be a compact, oriented manifold of even dimension $\dim M = n = 2m$, with $N : M \to S^n$ given by the orientation. Define the Gaussian curvature $\mathscr{K} : M \to \mathbb{R}$ by $\mathscr{K}\, dV_M = N^* dV_{S^n}$.

Thm: $\mathscr{K}$ is intrinsic to the Riemannian metric of N.

Recall the Weingarten formula: $\forall p \in M$, we have the commutative diagram

$$\begin{tikzcd} T_pM \arrow[r, "-dN"] \arrow[dr, "S_{N(p)}"'] & T_{N(p)}S^n \arrow[d, phantom, "\cong" rotate=90] \\ & T_pM \end{tikzcd}$$

where $T_{N(p)}S^n \cong T_pM$ isometrically by translation. S_{N_p} is the shape operator by the Weingarten formula, so

$$\mathscr{K}\, dV_M\big|_p = S^*_{N(p)} dV_M\Big|_p \qquad \Rightarrow \qquad \mathscr{K} = \det S_{N(p)} = \prod_{i=1}^n \kappa_i$$

To prove this, we use orthonormal moving frames on M. Let $(E_1, \ldots, E_n)$ be a positive orthonormal moving frame. We get the curvature matrix (Ω^i_j), a matrix of 2-forms.

$$\forall X, Y \in \mathfrak{X}(M), \quad R(X,Y)(E_j) = \Omega^i_j(X,Y)E_i \quad \Rightarrow \quad \Omega^i_j(X,Y) = R(X,Y,E_j,E_i)$$

Also, we have Gauss' formula:

$$0 = R(W,X,Y,Z) + \langle B(W,Y), B(X,Z)\rangle - \langle B(W,Z), B(X,Y)\rangle$$

So

$$\Omega^i_j(E_k, E_\ell) = \langle B(E_i,E_k), B(E_j,E_\ell)\rangle - \langle B(E_i,E_\ell), B(E_j,E_k)\rangle$$

Recall: $S_{ki} = S_{ik} \overset{\text{def}}{=} \langle S(E_i), E_k\rangle = \langle B(E_i,E_k), N\rangle$ is the N-component of $B(E_i,E_k)$ (S_{ij}) is the matrix of S. Written all together, we have

$$\Omega^i_j(E_k,E_\ell) = S_{ik}S_{j\ell} - S_{i\ell}S_{jk}$$

Prop: Let $n = 2m$. Then with σ_n the symmetric group,

$$\mathscr{K}\, dV = \frac{1}{n!} \sum_{\alpha \in \sigma_n} (-1)^\alpha \bigwedge_{i=1}^m \Omega^{\alpha(2i-1)}_{\alpha(2i)} \overset{\text{def}}{=} \text{Pf}(\Omega)$$

Defn: $\text{Pf}(\Omega)$ is called the <u>Pfaffian</u> of Ω.

We'll prove that the Pfaffian is independent of choice of moving frame. It's enough to show $\text{Pf}(\Omega)(E_1,\ldots,E_n) = \mathscr{K}$. Well, we introduce

$$Q = \{\varphi \in \sigma_n \mid \forall i \in \{1,\ldots,n\}, \varphi(2i-1), \varphi(2i) \in \{2i-1, 2i\}\}$$

We then begin to compute

$$\begin{aligned} \text{Pf}(\Omega)(E_1,\ldots,E_n) &= \frac{1}{n!2^m} \sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta \prod_{i=1}^m \Omega^{\alpha(2i-1)}_{\alpha(2i)}(E_{\beta(2i-1)}, E_{\beta(2i)}) \\ &= \frac{1}{n!2^m} \sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta \sum_{\varphi\in Q} (-1)^\varphi \prod_{i=1}^m S_{\alpha(2i-1)\beta\varphi(2i-1)} S_{\alpha(2i)\beta\varphi(2i)} \\ &= \frac{1}{n!2^m} \sum_{\varphi\in Q} \sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta(-1)^\varphi \prod_{i=1}^m S_{\alpha(2i-1)\beta\varphi(2i-1)} S_{\alpha(2i)\beta\varphi(2i)} \end{aligned}$$

Eventualy, this computation will give us $\det S$.

Defn: (Official definition) For $X = (x^i_j)$ an $n \times n$ matrix of commuting variables (with $n = 2m$ even), we define the Pfaffian

$$\mathrm{Pf}(X) = \frac{1}{n!} \sum_{\alpha \in \sigma_n} (-1)^\alpha \prod_{i=1}^{m} X^{\alpha(2i-1)}_{\alpha(2i)}$$

Lemma: $\forall X, Y$, $\mathrm{Pf}(Y^T X Y) = \det(Y)\,\mathrm{Pf}(X)$.

Proof: Just another direct computation.

Now, go back to moving frames and curvature matrices. Considering local frames, write $E_i = a^j_i F_j$ on $U \subseteq M$. Then $A(p) = (a^j_i(p)) \in \mathrm{SO}(n)$, and we konw that $\Omega_F = A^{-1} \Omega_E A$. So $\mathrm{Pf}(\Omega_F) = \det(A)\,\mathrm{Pf}(\Omega_E) = \mathrm{Pf}(\Omega_E)$.

Thus, $\mathrm{Pf}(\Omega_F) = \mathrm{Pf}(\Omega_E)$. So by the usual arguments, there's a unique global top-degree form on M such that for any moving frame on U, it agrees with $\mathrm{Pf}(\Omega)$. Therefore, by our proposition, $\mathscr{K}\,dV$ is of that form. $\square$

Question: Are there other combinations of the Ω^i_j's that give global forms on M? We need some polynomial $P : \mathrm{so}(n) \to \mathbb{R}$ such that $\forall A \in \mathrm{SO}(n)$, $\forall X \in \mathrm{so}(n)$, $P(A^{-1} X A) = P(X)$. Given such a P, repeat the previous argument to show that there's a global form ϖ such that on any U with a moving frame $E_1, \ldots, E_n$, $\varpi|_U = P(\Omega)$.

$P(\Omega_E) = P(\Omega_F)$, so $dP(\Omega) = 0$ always ($\forall P$ invariant). We get the Chern-Weil morphism:

$$\{\mathrm{Ad}\text{-invariant polynomials on } \mathrm{so}(n)\} \to H^* M$$

Math 635 Leture 34

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/9/21

Pontryagin Classes

Observe: $\mathrm{Pf}|_{\mathrm{so}(n)} : \mathrm{so}(n) \to \mathbb{R}$ (for n even) is $\mathrm{Ad}_{\mathrm{SO}(n)}$-invariant. So $\mathrm{Pf}(\Omega)$ is independent of choice of frame, where Ω is the curvature matrix with respect to that frame. We also have the Chern-Weil morphism:

$$\underbrace{\left\{\mathrm{Ad}_{\mathrm{SO}(n)}\text{invariant polynomials on so}(n)\right\}}_{I(\mathrm{so}(n))} \to \Omega^*(M)$$

where $p \mapsto p(\Omega)$. An amazing fact is that $p(\Omega)$ is always closed! So we get $I(\mathrm{so}(n)) \to H^*_{dR}(M)$, the deRham cohomology.

Ex: Elements of $I(\mathrm{so}(r))$ are Pontryagin polynomials. Let $A \in \mathrm{so}(r)$ be skew-symmetric (as usual, with r even). We claim that $A^T = -A$ implies the characteristic polynomial is even.

$$\det(\lambda I - A) = \sum_{k=0}^{r/2} \lambda^{r-2k} P_k(A)$$

where $P_k(A)$ is homogeneous, and of degree $2r$.

We can apply this idea to a rank-r vector bundle $\mathcal{E} \to M$. The idea is to use a metric on each fiber of $\mathcal{E}$ to get an orthonormal frame, and then the connection to get curvature forms. We get $P_r(\Omega)$, a differential form of degree $4r$. (Ω is the curvature matrix w.r.t. the orthonormal moving frame.) $[P_r(\Omega)] \in H^{4r}(M)$.

Thm: The cohomology classes are independent of the connection chosen – they're purely topological, and associated to $\mathcal{E}$.

So Gauss-Bonnet implies that $[\mathscr{K}\, dV] \in H^n(M)$ is independent of the metric.

Now, back to Gauss-Bonnet. We want to show

$$\int_M \mathscr{K}\, dV = \frac{\mathrm{Vol}(S^n)}{2}\chi(M) \qquad \text{using} \qquad \int_M \mathscr{K}\, dV = \int_M N^*(dV_{S^n})$$

where N is the Gauss spherical map.

Degree Theory: *What happens when you pullback a top-degree form.* (See Lee Differentiable Manifolds page 457.)

Preliminary (but still important) result:

Thm: Let M be a compact, connected, oriented manifold. (Note: It must have empty boundary.) Then the integration map

$$\begin{aligned} \textstyle\int_M : H^n(M) &\to \mathbb{R} \\ [\omega] &\mapsto \textstyle\int_M \omega \end{aligned}$$

is an isomorphism! (We know it's well-defined by Stokes' theorem.) As a result, $\dim H^n(M) = 1$.

Proof: We'll work with compactly-supported forms in open sets. Observe that $\int_M$ is nonzero – $\int_M d\,\mathrm{Vol} > 0$. We know $\int_M$ is a linear map. So we need to show $\forall \omega \in \Omega^n(M)$ such that $\int_M \omega = 0$, $\exists \eta \in \Omega^{n-1}(M)$ such that $d\eta = \omega$.

Step 1: Assume $\omega \in \Omega^n_0(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \omega = 0$. Then we claim $\exists \eta \in \Omega^{n-1}_0(\mathbb{R}^n)$ such that $d\eta = \omega$. Observe that the homotopy axiom implies $H^k(\mathbb{R}^n) = 0$ for $k > 0$, so such an η exists, and the claim is that η can be chosen to have compact support. For this, see Lemma 17.27 in Lee.

Now, back to the manifold case. Let $\{U_i\}$ be a finite cover M (possible by compactness) such that $\forall i$, $U_i \cong \mathbb{R}^n$ diffeomorphically. WOLOG if $M_k = U_1 \cup \cdots \cup U_k$, then $M_k \cap U_{k+1} = \emptyset$. (Use M's connectedness, and renumber the U_i if necessary. If no such U_i existed, then union all of them, and we would have two disjoint open sets that cover M, making it disconnected. Oops!)

Introduction: If $\omega \in \Omega_0^n(M_k)$ is such that $\int_{M_k} \omega = 0$, then $\exists \eta \in \Omega_0^{n-1}(M_k)$ such that $d\eta = \omega$. For $k = 1$, see the previous claim. Then use induction and a partition of unity to complete the proof. (See Lee for the full details.) □

Defn: Let $F : M_1 \to M_2$ be smooth, where M_1 and M_2 are compact, connected, oriented manifolds of the same dimension, $\dim M_1 = \dim M_2 = n$. Consider $F^* : H^n(M_2) \to H^n(M_1)$. By the previous result, we know that $\dim H^n(M_2) = \dim H^n(M_1) = 1$. Thus, F^* is multiplication by a scalar, and that number is called the degree of F.

$\forall c \in H^n(M_2)$, $\int_{M_1} F^*(c) = \deg F \int_{M_2} c$. That is,

$$\begin{array}{ccc} H^n(M_2) & \xrightarrow{F^*} & H^n(M_1) \\ \downarrow{\scriptstyle \int_{M_2}} & & \downarrow{\scriptstyle \int_{M_1}} \\ \mathbb{R} & \xrightarrow[\text{mult. by } \deg F]{} & \mathbb{R} \end{array}$$

Thm: Let $q \in M_2$ be a regular value of F. Write $F^{-1}(q) = \bigcup_{i=1}^N \{p_i\}$. This is a zero-manifold and compact, so it's the finite disjoint union of points. Define

$$(-1)^{p_i} = \begin{cases} 1 & dF_p \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

Then

$$\deg(F) = \sum_{i=1}^{n} (-1)^{p_i} \in \mathbb{Z}$$

Proof: F is a local diffeomorphism at each p_i. So we can argue that $\exists V$ a neighborhood of q and U_i a neighborhood of each p_i such that $F|_{U_i}^V$ is a diffeomorphism. That is, F is evenly covered at q. Let $\omega \in \Omega_0^n(V)$ be a bump function such that $\int_V \omega = \int_{M_2} \omega = 1$ (by extending ω to 0 on M_2 outside of V). What is $\int_{M_1} F^*\omega$? Well, it's equal to $\deg(F) \cdot 1 = \deg(F)$. But $F^{-1}(V)$ is the union of the U_i's, so

$$\int_{M_1} F^*(\omega) = \sum_{i=1}^{N} \underbrace{\int_{U_i} (F|_{U_i})^* \omega}_{=\pm 1 \text{ by diffeo invariance of integrals}} = \sum_{i=1}^{N} (-1)^{p_i}$$

□

Cor: Gauss-Bonnet reduces to the (purely topological) statement $\deg(N) = \frac{1}{2}\chi(M)$ ($N : M \to S^n$ is the Gauss map).

Observe:

1. If $F, F' : M_1 \to M_2$ are homotopic, then $\deg(F) = \deg(F')$, because $F^* = (F')^*$.
2. If $M_1 = \partial W$, and $F : M_1 \to M_2$ extends to $\tilde{F} : W \to M_2$, hen the degree of F is 0.

$$\begin{array}{ccc} M_1 & \xrightarrow{F} & M_2 \\ \downarrow & \nearrow{\scriptstyle \tilde{F}} & \\ W & & \end{array}$$

Prove this by using Stokes theorem to show $F^* = 0$.

Math 635 Lecture 35

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/12/21

Degree Theory

Given $F : M_1 \to M_2$, where M_1 and M_2 are compact, connected, oriented manifolds of dimension $n = \dim M_1 = \dim M_2$. Then $\exists \deg(F) \in \mathbb{Z}$ such that

a) $F^* : H^n(M_2) \to H^n(M_1)$ is "multiplication by $\deg F$".
b) $\forall a \in \Omega^n(M_2)$, $\int_{M_1} F^*\alpha = \deg(F) \int_{M_2} \alpha$.
c) If $q \in M_2$ is a regular value of F, so $F^{-1}(q) = \bigcup_{i=1}^N \{p_i\}$, then $\deg(F) = \sum_{i=1}^N (-1)^{p_i}$, where

$$(-1)^{p_i} = \begin{cases} 1 & dF_p \text{ preserves orientation} \\ -1 & \text{otherwise} \end{cases}$$

Application:

Thm: If $M_1 = \partial W$, and $f : M_1 \to M_2$ extends to a smooth function $\tilde{F} : W \to M_2$, then $\deg(F) = 0$.

$$\begin{tikzcd} M_1 \arrow[r, "F"] \arrow[d, hook, "\iota"'] & M_2 \\ W \arrow[ur, "\tilde{F}"'] & \end{tikzcd}$$

Proof: Let $\alpha \in \Omega^p(M_2)$. Then

$$\int_{M_1} F^*\alpha = \int_{M_1} \iota^*\tilde{F}^*\alpha \overset{(1)}{=} \int_W d\tilde{F}^*\alpha = \int_W \tilde{F}^* d\alpha = 0$$

because α is a top-degree form, and (1) because of Stokes' theorem. $\square$

Cor: There is no C^∞ map $\overline{B}^n \to S^n$ (where $\overline{B}^n$ is the closed unit ball in $\mathbb{R}^n$) which is the identity on $S^n = \partial\overline{B}^n$.

Proof: $\deg(I_{S^n}) = 1 \neq 0$. $\square$

(We presented some examples of Gauss-Bonnet and degree theory, but I'm not yet talented enough with TikZ to reproduce them.)

The Laplacian and Hodge Theory

Let (M, g) be a Riemannian manifold.

Defn: The gradient is

$$\begin{aligned} \nabla : C^\infty(M) &\to \mathfrak{X}(M) \\ f &\mapsto \nabla f \end{aligned}$$

where ∇f is metric-dual to df. That is, $\forall v \in T_pM$, $\langle \nabla f(p), v\rangle = df_p(v)$.

∇ has a product rule: $\nabla(fg) = f\nabla g + g\nabla f$.

Defn: The divergence $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ is defined by $\forall X \in \mathfrak{X}(M)$, $\mathcal{L}_X \operatorname{Vol} = (\operatorname{div} X) \operatorname{Vol}$, where $\operatorname{Vol} \in \Omega^n(M)$ is a volume form.

Observe: This is really a local definition, and div is independent of orientation.

Properties:

(a) In coordinate, $X = f^i \frac{\partial}{\partial x^i}$, then $\operatorname{div} X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} f^i)$. (Note: we're using the shorthand $\sqrt{g} = \sqrt{\det(g_{ij})}$).
(b) $\operatorname{div}(fX) = f \operatorname{div} X + X(f)$.
(c) $(\operatorname{div} X)(p) = \operatorname{tr}(T_pM \ni v \mapsto (\nabla_v X)(p) \in T_pM)$.

Thm: (Divergence Theorem) Let M be a manifold-with-boundary, with the boundary ∂M oriented by ν, an outward point unit normal vector field so that $\mathrm{Vol}_{\partial M} = \iota_\nu \mathrm{Vol}_M$. Then $\forall X \in \mathfrak{X}(M)$,

$$\int_M (\operatorname{div} X) \mathrm{Vol}_M = \int_{\partial M} \langle X, \nu \rangle \mathrm{Vol}_{\partial M}$$

The right-hand side is the flux of X out of M.

Cor: $$(\operatorname{div} X)(p) = \lim_{\varepsilon \searrow 0} \frac{1}{\mathrm{Vol}\, B_\varepsilon(p)} \int_{B_\varepsilon(p)} (\operatorname{div} X) d\,\mathrm{Vol}_M \overset{\text{Thm}}{=} \lim_{\varepsilon \searrow 0} \frac{1}{\mathrm{Vol}\, B_\varepsilon(p)} \int_{S_\varepsilon(p)} \langle X, \nu \rangle \mathrm{Vol}_{S_\varepsilon(p)}$$

"$(\operatorname{div} X)(p)$ is the infinitesimal flux per unit volume at p."

Observe: If $\partial M = \emptyset$, then we get $\int_M (\operatorname{div} X) \mathrm{Vol}_M = 0$. Proof: The left-hand side is simply

$$\int_M \mathcal{L}_X \mathrm{Vol}_M = \int_M d(\iota_X \mathrm{Vol}_M) = \int_{\partial M} j^*(\iota_X \mathrm{Vol}_M)$$

where $j : \partial M \hookrightarrow M$. Then, we check that $j^*(\iota_X \mathrm{Vol}_M) = \langle X, \nu \rangle \mathrm{Vol}_{\partial M}$. Use that $X = \langle X, \nu \rangle + \eta$, where η is tangent to ∂M. $\square$

Defn: The <u>laplacian</u> on functions is

$$\begin{aligned} \Delta : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto \Delta f = -\operatorname{div}(\nabla f) \end{aligned}$$

Note that some people don't use the negative sign.

Combining the previous formulas, in coordinates, we get

$$\begin{aligned} \Delta f &= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) \\ &= -\frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \text{lower order terms} \right) \\ &= -g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \text{lower order terms} \end{aligned}$$

So the highest order term only depend son the metric, not derivatives of the metric.

Next time, we'll consider the following commutative diagram:

$$\begin{array}{ccc} C^\infty(M) & \underset{-\operatorname{div}}{\overset{\nabla}{\rightleftarrows}} & \mathfrak{X}(M) \\ \| & & \|\wr \ \text{metric dual} \\ C^\infty(M) & \underset{\delta}{\overset{d}{\rightleftarrows}} & \Omega^1(M) \end{array}$$

We'll see that $\delta = d^*$ (the adjoint) if we use ℓ^2 inner products on $C^\infty(M)$ and $\Omega^1(M)$. On functions, $\Delta = \delta d = d^* d$.

Math 635 Lecture 36

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/14/21

Recall from last time: Let M be a Riemannian manifold. We defined the differential operators ∇ (gradient) and div (divergence), and we have

$$\begin{tikzcd} C^\infty(M) \arrow[r, shift left, "\nabla"] \arrow[d, equal] & \mathfrak{X}(M) \arrow[l, shift left, "\substack{\mathrm{div} \\ \text{or} \\ -\mathrm{div}}"] \arrow[d, "\wr", "\text{(metric dual)}"'] \\ C^\infty(M) \arrow[r, shift left, "d"] & \Omega^1(M) \arrow[l, shift left, "\substack{-\delta \\ \text{or} \\ \delta}"] \end{tikzcd}$$

"$\delta = -\operatorname{div}$ on the differential form side"

We also defined the Laplacian on functions $\Delta : C^\infty(M) \to C^\infty(M)$ by $\Delta = \delta \circ d$ iff $\Delta f = -\operatorname{div}(\nabla f)$.

ℓ^2 Inner Products

Defn: Assume M is oriented. $\forall f, g \in C^\infty(M)$, we define the $\underline{\ell^2 \text{ inner product}}$ by

$$\langle f, g\rangle_{\ell^2} = \int_M fg \, d\operatorname{Vol}$$

We can extend this to sections of real vector bundles over M, $\mathcal{E} \xrightarrow{\pi} M$. Put a Euclidean structure on the fibers of $\mathcal{E}$: $\forall p \in M$, $\langle\ ,\ \rangle_p$ is a Euclidean inner product on $\mathcal{E}_p = \pi^{-1}(p)$, varying smoothly with p.

Defn: $\forall s, t \in \Gamma_0(\mathcal{E})$ (compactly supported sections). Then we define the $\underline{\ell^2 \text{ inner product}}$ by

$$\langle s, t\rangle_{\ell^2} = \int_M \underbrace{\langle s(p), t(p)\rangle_p}_{\text{function of } p} d\operatorname{Vol}$$

Consider the case $\mathcal{E} = \bigwedge^k(T^*M)$. Then the Euclidean structure on $\bigwedge^k(T^*M)$ is induced by the Riemannian metric. For $k = 1$, we simply have $T^*M \cong TM$ by the metric dual. For general k, $\forall p \in M$, let $V = T_p^*M$. We define an inner product on $\bigwedge^k V$:

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) \overset{\text{def}}{=} \det\left(\langle v_i, w_j\rangle\right)_{ij}$$

Check: If $(e_1, \ldots, e_n)$ is an orthonormal basis of V, then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid i_1 < \cdots < i_k\}$ is an orthonormal basis of $\bigwedge^k V$.

In this way, we get the notion of an ℓ^2 inner product of any two k-forms $\alpha, \beta \in \Omega^k(M)$ by

$$\langle \alpha, \beta\rangle_{\ell^2} = \int_M \langle \alpha_p, \beta_p\rangle_p \, d\operatorname{Vol}$$

Prop: $\forall f \in C^\infty(M), X \in \mathfrak{X}(M)$, one has

$$\langle \nabla f, X\rangle_{\ell^2} = -\langle f, \operatorname{div} X\rangle_{\ell^2}$$

That is, $\forall f \in \Omega^0(M), \alpha \in \Omega^1(M)$,

$$\langle df, \alpha\rangle_{\ell^2} = \langle f, \delta\alpha\rangle_{\ell^2}$$

That is, $\delta = d^*$, the adjoint of d, so $\Delta = d^*d$.

Proof: Start with $\mathcal{L}_{fX}(d\,\mathrm{Vol}) = f\mathcal{L}_X(d\,\mathrm{Vol}) + Xf$. Now integrate:

$$\int_M \mathcal{L}_{fX}(d\,\mathrm{Vol}) = \int_M \mathrm{div}(fX) d\,\mathrm{Vol} = 0$$

because $\partial M = \emptyset$. So we have

$$0 = \int_M f\,\mathrm{div}(X) d\,\mathrm{Vol} + \int_M \underbrace{\langle X, \nabla F\rangle}_{X(f)=df(X)=\langle \nabla f, X\rangle} d\,\mathrm{Vol}$$

So $0 = \langle f, \mathrm{div}\, X\rangle_{\ell^2} + \langle X, \nabla f\rangle_{\ell^2}$. $\square$

Cor: $\langle \Delta\, f, g\rangle_{\ell^2} = \langle f, \Delta\, g\rangle_{\ell^2}$.

Now, generalize to Ω^k. (The previous discussion was for $k = 0$.)

$$\Omega^k \underset{\delta = d^* = ?}{\overset{d}{\rightleftarrows}} \Omega^{k+1}$$

Is there a δ? What is it?

In local coordinates, δ is *also* a differential operator of degree 1. Try integrating by parts!

Preliminary linear algebra: the Hodge star operator. Let V be an n-dimensional vector space, oriented, with an inner product. We claim that $\forall k$, there exists $\star : \bigwedge^k V \to \bigwedge^{n-k} V$ linear such that for any positive oriented basis $(e_1, \ldots, e_n)$ of V, $\star(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n$.

Ex: For $V = \mathbb{R}^3$ with the standard orientation,

$$\begin{aligned} \star : \textstyle\bigwedge^2 V &\to \textstyle\bigwedge^1 V \\ dx^1 \wedge dx^2 &\mapsto dx^3 \end{aligned}$$

(Now do it cyclically.)

Note: $\dim \bigwedge^k = \binom{n}{k} = \binom{n}{n-k} = \dim \bigwedge^{n-k}$.

Observe: On $\mathbb{R}^3$ in the calc 3 context, for $X \in \mathfrak{X}(\mathbb{R}^3)$, we define

$$\mathrm{curl}\, X = \nabla \times X \in \mathfrak{X}(M)$$

What is this object? Well,

$$\begin{tikzcd} \mathfrak{X}(\mathbb{R}^3) \arrow[rrrr, bend right, "\mathrm{curl}"'] & \cong \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "\star"] & \Omega^1(\mathbb{R}^3) \cong & \mathfrak{X}(\mathbb{R}^3) \end{tikzcd}$$

Note that this only works for dim $= 3$.

Some properties of $\star$:

1. We have

$$\begin{tikzcd} \bigwedge^k \arrow[r, "\star"] \arrow[rr, bend right, "(-1)^{k(n-k)}\mathrm{Id}"'] & \bigwedge^{n-k} \arrow[r, "\star"] & \bigwedge^k \end{tikzcd}$$

   because

$$e_1 \wedge \cdots \wedge e_k \overset{\star}{\mapsto} e_{k+1} \wedge \cdots \wedge e_n \overset{\star}{\mapsto} (-1)^\sigma e_1 \wedge \cdots \wedge e_k$$

   "$n-k$ signs, k times".
2. $\star : \bigwedge^n V \to \bigwedge^0 V = \mathbb{R}$ has $\star(\mathrm{Vol}) = 1$.
3. $\forall \alpha, \beta \in \bigwedge^k V$, $\langle \alpha, \beta\rangle = \star(\alpha \wedge (\star\beta)) \in \mathbb{R}$.

Cor: Apply/extend $\star$ to forms on a compact, oriented, Riemannian manifold M (with $\dim M = n$), $\Omega^k(M)$, by acting pointwise: $\star : \Omega^k(M) \to \Omega^{n-k}(M)$. Note: $\forall \alpha, \beta \in \Omega^k(M)$, $\langle \alpha, \beta\rangle_{\ell^2} = \int_M \alpha \wedge (\star\beta)$.

Back to our main question:

Prop: The adjoint δ of $d : \Omega^k \to \Omega^{k+1}$ is $\delta = (-1)^{nk+1} \star d\star$.

Note: If $\beta \in \Omega^{k+1}$, $\star\beta \in \Omega^{n-k-1}$, so $d \star \beta \in \Omega^{n-k}$, so $\star d \star \beta \in \Omega^k$. Superficially, $\delta = \star d\star : \Omega^{k+1} \to \Omega^k$. Now, we prove it:

Proof: Let $\alpha \in \Omega^k, \beta \in \Omega^{k+1}$. We want to show $\langle d\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \delta\beta\rangle_{\ell^2}$. We'll use integration by parts. Starting with the fact that $0 = \int_M d(\alpha \wedge \star\beta)$, because $\alpha \wedge \star\beta$ is a $n-1$ form, so $d(\alpha \wedge \star\beta)$ is a top-degree form. By Stokes' theorem, since we have an empty boundary, this integral is 0. Well,

$$0 = \int_M d(\alpha \wedge \star\beta) = \underbrace{\int_M d\alpha \wedge \star\beta}_{=\langle d\alpha, \beta\rangle_{\ell^2}} + (-1)^k \int_M \alpha \wedge (d \star \beta)$$

So

$$\langle d\alpha, \beta\rangle_{\ell^2} = (-1)^? \int_M \alpha \wedge (d \star \beta) = (-1)^? \int_M \alpha \wedge (\star\star) d \star \beta = (-1)^? \langle \alpha, \star d \star \beta\rangle_{\ell^2} = (-1)^? \langle \alpha, \delta\beta\rangle$$

(We didn't do the sign computations, but they do work out.) $\square$

Defn: The Laplacian on forms $\Delta : \Omega^k(M) \to \Omega^k(M)$ is $\Delta = \delta d + d\delta$.

Math 635 Lecture 37

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/16/21

Recall from last time: Let M be a compact, oriented, Riemannian manifold. Then the Laplacian on forms $\Delta : \Omega^k \to \Omega^k$ is $\Delta = \delta d + d\delta$. (Note that these are technically different d's and different δ's, because d increases the degree and δ reduces it.) And we have the deRham complex

$$\cdots \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{k-1} \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^k \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{k+1} \underset{\delta}{\overset{d}{\rightleftarrows}} \cdots$$

where $d^* = \delta$, i.e., $\forall \alpha \in \Omega^{k-1}, \beta \in \Omega^k$, $\langle d\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \delta\beta\rangle_{\ell^2}$.

Lemma:

(1) $\delta^2 = 0$ (because $\delta^2 = (d^2)^*$)
(2) $\Delta^* = \Delta$: $\forall \alpha, \beta \in \Omega^k$, $\langle \Delta\,\alpha, \beta\rangle_{\ell^2} = \langle \alpha, \Delta\,\beta\rangle_{\ell^2}$
(3) $[\Delta, d] = 0$, $[\Delta, \delta] = 0$
(4) $\Delta\,\alpha = 0$ iff $d\alpha = 0$ and $\delta\alpha = 0$

Proof (3): $(\delta d + d\delta)d = \cancel{\delta dd} + d\delta d$ and $d(\delta d + d\delta) = d\delta d + \cancel{dd\delta} = d\delta d$. So $[\Delta, d] = 0$. (Identical proof for $[\Delta, \delta] = 0$.) $\square$
(4): $\Leftarrow$ is obvious. For $\Rightarrow$, note that $\Delta\,\alpha = 0$ implies

$$0 = \langle \Delta\,\alpha, \alpha\rangle = \langle \delta d\alpha, \alpha\rangle + \langle d\delta\alpha, \alpha\rangle = \langle d\alpha, d\alpha\rangle + \langle \delta\alpha, \delta\alpha\rangle = ||d\alpha||^2_{\ell^2} + ||\delta\alpha||^2_{\ell^2}$$

So $d\alpha = 0$ and $\delta\alpha = 0$. $\square$

General Things about Linear Differential Operators

Let $U \subseteq \mathbb{R}^n$, consider $C^\infty(U, \mathbb{C})$.

Defn: A differential operator P on $C^\infty(U, \mathbb{C})$ is of the form $\forall f \in C^\infty(U, \mathbb{C})$,

$$P(f) = \sum_\alpha c_\alpha(x)(D^\alpha f)(x) \quad \alpha \text{ multi-index with } |\alpha| \le n \qquad D^\alpha = \frac{1}{i}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \frac{1}{i}\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

(here, we're using $i = \sqrt{-1}$). Note that P is local – $P(f)(x)$ only depends on f in a neighborhood of x.

On manifolds, on $C^\infty(M, \mathbb{C})$, linear differential operators are generated as a ring by multiplying by functions. For vector bundles/systems, we have $P : C^\infty(U, \mathbb{C}^r) \to C^\infty(U, \mathbb{C}^s)$, where $P = (P_{ij})_{s\times r}$, and P_{ij} is a scalar differential operator.

In the manifold setting, suppose we have two bundles $\mathcal{E}$ and $\mathcal{F}$:

$$\begin{tikzcd} \mathcal{E} \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & M & \end{tikzcd}$$

Consider $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$. Under local trivializations on the same $U \subseteq M$, we have $\Gamma(\mathcal{E}|_U) \cong C^\infty(U, \mathbb{C}^r) \cong \Gamma(\mathcal{F}|_U)$. So P is locally the Euclidean case.

We already know a bunch of examples!

Ex: $d : \Omega^k \to \Omega^{k+1}$ and $\nabla : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E} \otimes T^*M)$ are differential operators of order 1.

The main invariant associated to a differential operator P is called its symbol. The symbol captures the top degree part of the operator. For an operator on $C^\infty(U, \mathbb{C})$, think of computing: take $x_0 \in U, \xi_0 \in T^*_{x_0}U$ (i.e. $(x_0, \xi_0) \in T^*U$). Pick χ, ρ functions, both $C_0^\infty(U, \mathbb{R})$ such that $\chi \equiv 1$ near x_0 and $d\rho_{x_0} = \xi_0$. Consider $P(\chi e^{i\tau\rho})(x_0)$ for $\tau \gg 1$. We get

$$P(\chi e^{i\tau\rho})(x_0) = \sum_{|\alpha| = m \overset{\text{def}}{=} \deg P} c_\alpha(x_0) D^\alpha(\chi e^{i\tau\rho}) + \text{lower degree terms}$$

Look for the highest power of τ on the right-hand side.

$$\tau^m e^{i\tau\rho(x_0)} \sum_{|\alpha|=m} c_\alpha(x_0) \underbrace{(\nabla\rho(x_0))^\alpha}_{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}} + \text{lower order terms in } \tau$$

Now, forget the τ and forget the exponential.

Defn: The <u>symbol</u> of a differential operator P is

$$\sigma_P(x_0, \xi_0) = \sum_{|\alpha|=m} c_\alpha(x_0)\xi^\alpha = \lim_{\tau\to\infty} e^{-i\tau\rho(x_0)} \frac{1}{\tau^m} P(\chi e^{i\tau\rho})(x_0)$$

Conclusion: On manifolds, $P : C^\infty(M, \mathbb{C})\circlearrowleft$, $\sigma_P : T^*M \to \mathbb{C}$ is well-defined. Specifically, it's a homogeneous polynomial in χ of degree M on each fiber.

Ex: Let $X \in \mathfrak{X}(M)$. Then $\sigma_{\mathcal{L}_X} : T^*M \to \mathbb{C}$ is $\sigma_{\mathcal{L}_X}(p, \xi) = i\,\langle \xi, X_p \rangle$.

Ex: $\Delta : C^\infty(M)\circlearrowleft$, $\sigma_\Delta : T^*M \to \mathbb{C}$ is $\sigma_\Delta(p, \xi) = g^{ij}(p)\xi_i\xi_j = ||\xi||^2\,\text{Id}$.
(Recall: this is from the definition of Δ in coordinates.)

If $P : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$,

$$\begin{tikzcd} \mathcal{E} \arrow[dr] & & \mathcal{F} \arrow[dl] \\ & M & \end{tikzcd}$$

Then for $(p, \xi) \in T^*M$, $\sigma_P(p, \xi) : \mathcal{E}_p \to \mathcal{F}_p$ is a linear map between the fibers.

Prop: If P and Q are differential operators such that $P \circ Q$ makes sense, then $\sigma_{P\circ Q} = \sigma_P \circ \sigma_Q$.

Defn: P is an <u>elliptic operator</u> iff $\forall(p, \xi)$ with $\xi \neq 0$, $\sigma_P(p, \xi)$ is invertible.

Ex: $P = \Delta$ is elliptic. $\sigma_\Delta = ||\xi||^2\,\text{Id}$, so it's invertible everywhere. Δ has an approximate inverse G – "$\sigma_G(p, \xi) = \frac{\text{Id}}{||\xi||^2}$".
$\Delta \circ G - I$ and $G \circ \Delta - I$ are smoothing operators – they're very small.

We're skipping a lot of stuff here, but...

Thm: (Spectral Theorem of the Laplacian) Consider $\Delta : \Omega^k(M) \to \Omega^k(M)$, where M is a compact, oriented, Riemannian manifold. Then

(1) $\mathcal{U}^k = \ker \Delta$ has finite dimension.
(2) There's an orthonormal basis (in the ℓ^2 sense) of Ω^k, $\{\alpha_j\}$, and $0 \le \lambda_0 \le \lambda_1 \le \cdots \to +\infty$ s.t. $\Delta\,\alpha_j = \lambda_j\alpha_j$. In other words, we can think of Δ as an infinite matrix, that can be diagonalized by α_j and λ_j. We'll write $\forall\alpha \in \Omega^k$, $\alpha = \alpha^H + \sum_{\lambda>0 \text{ distinct}} \alpha_\lambda$. α^H is the harmonic piece – $\Delta\,\alpha^H = 0$, $\Delta\,\alpha_\lambda = \lambda\alpha_\lambda$. We define Green's operator $G(\alpha) = \sum_{\lambda>0} \frac{1}{\lambda}\alpha_\lambda$. So $(I - \Delta \circ G)(\alpha) = G^H$.

Ex: For Δ on $C^\infty(S^2)$, we have $\lambda = k(k+1)$ with multiplicity $2k+1$ (for $k \in \mathbb{Z}_{\geq 0}$).

Math 635 Lecture 38

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

4/19/21

Recall: The Laplace-Beltrami operator on forms: Δ on Ω^k, where $\Delta = d\delta + \delta d$.

Ex: On $\mathbb{R}^n$, Δ on functions (i.e. $\Omega^0 = C^\infty(\mathbb{R}^n, \mathbb{R})$). Then $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. (This is using our sign convention.)

For $\omega \in \mathbb{R}^n$, $\Delta(e^{i\omega\cdot x}) = ||\omega||^2 e^{i\omega\cdot x}$.

For forms $f_I dx^I$, where $I = \{i_1 < \cdots < i_k\}$ is a multi-index, the Laplacian is $\Delta(f_I dx^I) = (\Delta f_I)dx^I$.

Throughout our discussion today, we'll assume M is compact and oriented.

Ex: $\mathbb{T} = \mathbb{R}^n/\Lambda$, where $\Lambda = \{\sum k_i e_i, k_i \in \mathbb{Z}, e_1, \ldots, e_n \text{ linearly independent}\}$. $\Delta_{\mathbb{R}^n}$ makes sense on $C^\infty(\mathbb{T})$. If $\omega \in \mathbb{R}^n$ is such that $\forall k \in \Lambda$, $\omega \cdot k \in 2\pi\mathbb{Z}$, then $x \mapsto e^{i\omega\cdot x}$ is periodic w.r.t. Λ, so it's $C^\infty(\mathbb{T})$.

Thm: The eigenvalues of $\Lambda_{\mathbb{T}}$ on functions are $||\omega||^2$ for ω satisfying the above condition. Such ω constitute the *dual lattice* Λ^*.

Thm: For S^2, distinct eigenvalues are $k(k+1)$ for $k \in \{0, 1, 2, \ldots\}$, with corresponding muliplicities $2k+1$. That is, listed, with multiplicities, we have eigenvalues

$$\underbrace{0}_{k=0}, \underbrace{2, 2, 2}_{k=1}, \underbrace{6, 6, 6, 6, 6}_{k=2}, \ldots$$

In the general case, we have

Thm: (Spectral Theorem for the Laplacian on Forms) Let M be a compact, oriented manifold. There exists an orthonormal basis (in the ℓ^2 sense) of Ω^k, $\{\alpha_j : j = 0, 1, 2, \ldots\}$ an real eigenvalues $\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to +\infty$ such that $\Delta \alpha_j = \lambda_j \alpha_j$. Moreover, $\lim_{j\to\infty} \lambda_j = +\infty$, i.e., the multiplicities are all finite.

This yields the spectral decomposition of $\alpha \in \Omega^k$:

$$\alpha = \alpha_H + \sum_{\lambda > 0} \alpha_\lambda \qquad \Delta \alpha_\lambda = \lambda\alpha_\lambda, \Delta \alpha_H = 0$$

α_H is the "harmonic part", and the λ's are all distinct eigenvalues. This decomposition is unique and orthogonal.

Note: Because this is an infinite-dimensional space, there's a lot of analysis happening behind the scenes.

Observe: α is smooth iff $||\alpha_\lambda||_{\ell^2} = 0$ (really $\lambda^{-\infty}$). In other words, it decays rapidly as λ grows.

Defn: $G : \Omega^k \to \Omega^k$ is defined by $G(\alpha) = \sum_{\lambda>0} \frac{1}{\lambda}\alpha_\lambda$.

Observe: G is smooth.

By definition,

$$\Delta G(\alpha) = \sum_{\lambda>0} \frac{1}{\lambda} \Delta \alpha_\lambda = \sum_{\lambda>0} \frac{1}{\lambda}\lambda\alpha_\lambda = \sum_{\lambda>0} \alpha_\lambda = \alpha - \alpha_H$$

Lemma: $[G, d] = 0$ and $[G, \delta] = 0$.

Why should this be true? We've seen that $[\Delta, d] = 0$ and $[\Delta, \delta] = 0$. So $\forall \alpha_\lambda$, $\Delta \alpha_\lambda = \lambda\alpha_\lambda$. $d\alpha_\lambda$ is either 0 or an eigenform of Δ. So we get the commutative diagram

$$\begin{array}{ccc} \Omega^k & \xrightarrow{d} & \Omega^{k+1} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ \Omega^k & \xrightarrow{d} & \Omega^{k+1} \end{array}$$

Cor: (Hodge Decomposition Theorem) $\Omega^k = \mathcal{H}^k \oplus d(\Omega^{k-1}) \oplus \delta(\Omega^{k+1})$, where $\mathcal{H}^k = \ker(\Delta : \Omega^k \to \Omega^k)$ is the set of harmonic forms.

This decomposition is an orthogonal direct sum with respect to the ℓ^2 inner prouct. In fact, if $\alpha \in \Omega^k$,

$$\alpha = \alpha_H + \Delta\, G(\alpha) = \alpha_H + \underbrace{d(\delta G(\alpha))}_{\in d(\Omega^{k-1})} + \underbrace{\delta(\overbrace{d(G(\alpha))}^{G(d\alpha)})}_{\in \delta(\Omega^{k+1})}$$

Cor: If $\alpha \in \Omega^k$ is such that $d\alpha = 0$, then $[\alpha] = [\alpha_H]$ (where $[\,\cdot\,]$ is the equivalence class in deRham cohomology).

Proof:

$$\alpha = \alpha_H + d(\delta G(\alpha)) + \delta(\underbrace{dG(\alpha)}_{G(d\alpha)=0}) = \alpha_H + d(\delta G(\alpha))$$

□

Recall: $\Delta\,\alpha = 0$ iff $d\alpha = 0$ and $\delta\alpha = 0$. In fact, $\forall c \in H^k_{\mathrm{dR}}(M)$, $\exists! \alpha_H \in [c]$ such that $\Delta\,\alpha_H = 0$.

Cor: The Betti numbers of a manifold are finite.

Proof: $b_k = \dim H^k$. This is the multiplicity of k, which we know to be finite. □

Cor: (Poincaré Duality) The $\star$ operator induces $H^k(M) \cong H^{n-k}(M)$, where $n = \dim M$.

Proof: Follows from the fact that $[\Delta, \star] = 0$. □

Applications of Hodge Theory

Recall that $\forall p \in M$, $\forall u, v \in T_pM$, $\mathrm{Ric}^B_p(u,v) = \mathrm{tr}(T_pM \ni x \mapsto R_p(x,u)v \in T_pM)$. We saw that Ric^B is symmetric, and $\mathrm{Ric}(u) = \mathrm{Ric}^B(u,u)$.

Thm: (Bochner's Theorem) Assume M is connected, compact, and oriented.

(a) If $\mathrm{Ric} \geq 0$, $b_1 \leq \dim M$.
(b) If $\mathrm{Ric} \geq 0$ and $\exists p_0 \in M$ such that $\mathrm{Ric}^B_{p_0}$ is positive definite, then $b_1 = 0$.

Proof: Introduce a "rough Laplacian" associated to ∇. (Recall that $T^*M \cong TM$, and think of the connection as $\nabla : \gamma(T^*M) \to \Gamma(T^*M \otimes T^*M)$.) The rough Laplacian is $\nabla^*\nabla : \Omega^1 \to \Omega^1$. This works because we have $\nabla^* : \Gamma(T^*M \otimes T^*M) \to \Gamma(T^*M)$, using ℓ^2 inner products associated with the Euclidean structures on the bundles.

It turns out this rough Laplacian has the same ymbol as the ordinary Laplacian. In fact, hey only differ by a term of order 0! A calculation show that $\forall \alpha \in \Omega^1$, $\Delta\,\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}^B(\alpha^\sharp, \cdot)$, where $\alpha^\sharp$ is the metric dual to α. Well,

$$\begin{aligned} \Delta\,\alpha = 0 \Rightarrow 0 &= \int_M \langle \Delta\,\alpha, \alpha \rangle \, d\,\mathrm{Vol} \\ \Rightarrow 0 &= \underbrace{||\Delta\,\alpha||^2}_{=\langle \nabla^*\nabla\alpha, \alpha\rangle_{\ell^2}} + \int_M \mathrm{Ric}^B(\alpha^\sharp, \alpha^\sharp) d\,\mathrm{Vol} \\ &\Rightarrow \nabla\alpha = 0 \text{ and } \mathrm{Ric}(\alpha^\sharp) = 0 \\ &\Rightarrow \alpha \text{ is parallel-determined by any value } \alpha_p \in T^*_pM \end{aligned}$$