Math 635 Lecture 4

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Recall: A Riemannian metric g on M is a covariant 2-tensor (i.e. a tensor of type (0,2)), such that, at each point, g_p is symmetric and positive definite.

Prop: Any C^{∞} manifold M has infinitely many Riemannian metrics.

Proof 1: We can embed $M \hookrightarrow \mathbb{R}^N$ (for some N), and use the induced metric. \square

Proof 2: We can use a partition of unity. Let $\{U_{\alpha}, \phi_{\alpha}\}$ be an atlas of M, and $\{\chi_{\alpha}\}$ a subordinate partition of unity. (Recall: A subordinate partition of unity means $\forall \alpha, \chi_{\alpha} \in C_{0}^{\infty}(M)$, $\{\operatorname{supp} \chi_{\alpha}\}$ is a locally finite cover of M, and $\forall p \in M, \sum_{\alpha} \chi_{\alpha}(p) = 1$.) $\forall \alpha$, let g^{α} be the Riemannian metric on U_{α} s.t. ϕ_{α} is an isometry (to the standard flat metric on \mathbb{R}^{n}). That is, $(g^{\alpha})_{ij} = \delta_{ij}$. Define, $\forall p \in M, \forall u, v \in T_{p}M$,

$$g_p(u, v) = \sum_{\alpha} \underbrace{\chi_{\alpha}(p) g_p^{\alpha}(u, v)}_{\text{Interpret this as 0 if } p \notin U_{\alpha}}$$

Basically, we're extending $\chi_{\alpha}g^{\alpha}$ to 0 outside U_{α} . Now that we've define g, we need to check that it satisfies the appropriate properties. Symmetric matrices form a closed group under addition, so clearly, every g_p is symmetric. And because $\chi_{\alpha} \geq 0$, we can also assert that g_p is positive semidefinite. Finally we have to check that $g_p(u,u)=0 \Rightarrow u=0$. Well, $\exists \alpha$ s.t. $\chi_{\alpha}(p)>0$, so $p\in U_{\alpha}$. Thus, we must have $g_p^{\alpha}(u,u)=0$, so we must have u=0 by the positive definiteness of g_p^{α} . \square

Observe: An analogous proof shows that any vector bundle has infinitely many C^{∞} sections, where we use local trivializations instead of coordinates.

Note that positive-definiteness is important! It's not true that M has a non-degenerate symmetric covariant 2-tensor of any arbitrary signature. For example, general relativity cares about the signature (-,+,+,+), and such a tensor (with all relevant properties) doesn't always exist.

Defn: Let G be a Lie group. (Recall that this means G is a manifold, and a group, with the group operation and inversion C^{∞} functions with respect to the smooth structure.) $\forall k \in G$, we have the left translation map $L_k : G \to G$, where $L_k(h) = kh$. A Riemannian metric on G is said to be <u>left-invariant</u> iff $\forall k \in G$, L_k is an isometry.

Prop: Let $\mathfrak{g} = T_e G$, the Lie algebra (where $e \in G$ is the identity element of the group). Then given any positive definite inner product $\langle \ , \ \rangle_e$ on \mathfrak{g} , there exists a unique left-invariant Riemannian metric g on G s.t. $g_e = \langle \ , \ \rangle_e$.

Proof: $\forall k \in G$, we require (by left-invariantness) that $d(L_k)_e : (\mathfrak{g}, \langle , \rangle_e) \to (T_k G, g_k)$ is an isometry. We define g this way $-g_k$ is the "pushforward" of \langle , \rangle_e by $d(L_k)_e$. Specifically, $\forall u, v \in T_k G$,

$$g_k(u,v) = \langle d(L_k^{-1})(u), d(L_k^{-1})(v) \rangle_e = \langle d(L_{-k})(u), d(L_{-k})(v) \rangle_e$$

We claim that this is smooth, using the smoothness of the map $G \times G \to G$.

$$(k,h) \mapsto kh$$

Question: We can also define everything in terms of right-invariant metrics. Do bi-invariant metrics exist? Answer (which we will show in HW): It depends on G!

If M and N are Riemannian manifolds, then $M \times N$ has a natural Riemannian metric. $\forall (p,q) \in M \times N$, $T_{(p,q)}(M \times N) = (T_p M) \oplus (T_q N)$. The product metric is defined as follows: Declare $T_p M$ and $T_q N$ to be orthogonal. With g^M, g^N the Riemannian metrics for M, N (respectively), we have the product metric on $M \times N$ (as a block matrix):

$$g^{M \times N} = \left(\begin{array}{c|c} g^M & 0 \\ \hline 0 & g^N \end{array}\right)$$

Defn: Let M, N be Riemannian manifolds. A smooth map $F: M \to N$ is a local isometry iff $\forall p \in M$,

$$dF_p: (T_p M, g_p^M) \to (T_{F(p)} N, g_{F(p)}^N)$$

is a linear isometry.

Note that local isometries are local diffeomorphisms, by the inverse function theorem.

Defn: A surjective C^{∞} map $\pi: M \to N$ is a smooth covering map iff $\forall q \in N$, there's a neighborhood V of q s.t. π maps the connected components of $\pi^{-1}(U)$ diffeomorphically onto V. That is,

$$\pi^{-1}(U) = \bigsqcup_{j} U_{j}, \qquad \forall j, \, \pi|_{U_{j}} : U_{j} \xrightarrow{\sim} V$$

Defn: Let π be a smooth covering map. The <u>automorphism group</u> is $\operatorname{Aut}(\pi) = \{F : M \to N \mid \pi = F \circ \pi\}$. We say that such an F "shuffles the fibers". $\operatorname{Aut}(\pi)$ is a group under composition.

Defn: π is <u>normal</u> iff Aut(π) acts transitively on the fibers.

Ex: $S^n \stackrel{\pi}{\to} \mathbb{RP}^n$. Aut $(\pi) = \{\text{identity map, antipodal map}\} \cong \mathbb{Z}_2$. $\mathbb{R}^n \stackrel{\pi}{\to} \mathbb{R}^n / \mathbb{Z}^n$.

Ex: (HW) $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$. For any genus g surface G, with g > 1, $\exists \pi : \mathcal{H} \to G$.

Thm: Let $\pi: M \to N$ be a smooth normal covering map. Let g^M be a Riemannian metric on M s.t. $\forall F \in \operatorname{Aut}(\pi)$ is an isometry. Then there's a unique Riemannian metric g_N on N s.t. π is a local isometry.

Proof: Let $q \in N$, and V a neighborhood of q. Then $\pi^{-1}(V) = \bigsqcup_j U_j$ (where the U_j are the connected components) such that $\pi|_{U_j}: U_j \overset{\sim}{\to} V$. We want to choose a metric on V s.t. $\forall j, \ \pi|_{U_j}$ is an isometry. Well, choose a j_0 , and put such a metric on V. $\forall j \neq j_0, \ \exists F \in \operatorname{Aut}(\pi)$ s.t. $F(U_j) = F(U_{j_0})$. F is an isometry, by assumption, so π is an isometry on U_j . \square