Math 635 Lecture 25

Thomas Cohn

3/19/21

Space of Constant Curvature

Such spaces are locally isometric to a sphere (if the curvature is positive), a Euclidean space (if the curvature is zero), or a hyperbolic space (if the curvature is negative).

Lemma: M has constant sectional curvature $K_0 \in \mathbb{R}$ iff $\forall X, Y, W, Z \in \mathfrak{X}(M)$,

$$(X, Y, W, Z) = K_0(\langle Y, W\rangle \langle X, Z\rangle - \langle X, W\rangle \langle Y, Z\rangle)$$

Proof: Note that by definition of K,

$$(X, Y, Y, X) = K(X, Y)\underbrace{(\langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2)}_{=|X\wedge Y|}$$

This is a special case of the identity we want! Proving $\Leftarrow$ is trivial. As for proving $\Rightarrow$, we show that the right hand side has the same symmetry properties as the left hand side. (This is just a messy computation.) Then we use the fact that we know K can determine $\mathcal{R}$ if it's applied everywhere. $\square$

Cor: Let M have constant curvature, γ a geodesic of speed 1, and J a normal Jacobi field. Then the Jacobi equation becomes $J'' + KJ = 0$. (Recall the notation: $J' = \frac{DJ}{dt}$, $J'' = \frac{D^2J}{dt^2}$.)

Proof: By the lemma, we have, $\forall X \in \mathfrak{X}(M)$,

$$\langle \mathcal{R}(J, \dot\gamma)\dot\gamma, X\rangle = (J, \dot\gamma, \dot\gamma, X) = K(||\dot\gamma||^2 \langle J, X\rangle - \langle J, \dot\gamma\rangle \langle \dot\gamma, X\rangle)$$

This is equal to $K\langle J, X\rangle$ for all X, $|mcR(J, \dot\gamma)\gamma = KJ$. $\square$

Now, we consider a generalization of the surface case. Let γ be a geodesic on M, with constant curvature.
Let $N(t) \in \Gamma_\gamma(TM)$ be a unit normal and parallel field, determined by $N(0)$. In the surface case, we call this $\dot\gamma^\perp$. Define $J(t) = \varphi(t)N(t)$. Then J is Jacobi iff $\ddot\varphi + K\varphi = 0$.

Cor: If $J(0) = 0$, then $J(t) = \begin{cases} A\sin(\sqrt{K}t)N & K > 0 \\ tN & K = 0 \\ A\sinh(\sqrt{-K}t)N & K < 0 \end{cases}$

Recall the differential of the exponenial map:

Thm: $d(\exp_p)_v(w) = J(1)$, where J is the Jacobi field along $\gamma : t \mapsto \exp_p(tv)$ s.t. $J(0) = 0$, $J'(0) = w$.

Proof: Define $f(s, t) = \exp_p(t(v + sw))$. Observe: $d(\exp_p)_v(w) = \partial_s f|_{s=0,t=1}$. Let $J(t) = \partial_s f(s = 0, t)$. This is a Jacobi field, since the t curves are geodesics. Now, we just need to check that J satisfies the initial conditions.

$$J(0) = 0, \qquad J' = \frac{D}{dt}\partial_s f = \frac{D}{ds}\partial_t f, \qquad \partial_t f(s, t = 0) = d(\exp)_0(v + sw) = v + sw \in T_pM$$

$s \mapsto \partial_t f(s, 0)$ is entirely contained in T_pM. Still, it's a field along $s \mapsto f(s, 0) = p$, so it's just a constant "curve". Finally, $\frac{D}{ds}\partial_t f = \frac{d}{ds}(v + sw) = w$. $\square$

Next, we examine a result of the "rate of spreading of geodesics". As before, define $f(s, t) = \exp_p(t(v + sw))$.

Prop: Take $v, w \in T_pM$ orthonormal, and let $\pi = \mathrm{span}(v, w)$, $\gamma(t) = \exp_p(tv)$. Let $J(t)$ be the Jacobi field s.t. $J(0) = 0$, $J'(0) = w$. (Again, exactly as before.) Then

$$||J(t)|| = t - \frac{t^3}{6} K_p(\pi) + O(t^3) \qquad \text{as} \qquad t \to 0$$

as $t \to \infty$.

Interpretation: Rate of spreading of the ray $t(v + sw)$ with respect to $t \mapsto tv$. $\frac{\partial}{\partial s} |t(v + sw)| = tw$. We want to measure the same object on the manifold after exponentiation. We do so with respect to infinitesimal change in s.

$$||J(t)|| \sim \begin{cases} < t & K_p(\pi) > 0 \\ > t & K_p(\pi) < 0 \end{cases}$$

Proof of the proposition: It's true iff

$$||J(t)||^2 = t^2 - \frac{t^4}{3} K_p(\pi) + O(t^4)$$

So we need to compute 4 derivatives of $\langle J, J \rangle$ at zero. Define $a_k = \langle J, J \rangle^{(k)} (t = 0)$. Then clearly $a_0 = 0$. $\langle J, J \rangle' = 2 \langle J', J \rangle$, so $a_1 = 0$.

$$\frac{1}{2} \langle J, J \rangle'' = \langle J'', J \rangle + \langle J', J' \rangle \quad \Rightarrow \quad a_2 = ||w||^2 = 1$$

$$\frac{1}{2} \langle J, J \rangle^{(3)} = \left\langle J^{(3)}, J \right\rangle + \langle J'', J' \rangle + 2 \langle J'', J' \rangle = \underbrace{\left\langle J^{(3)}, J \right\rangle}_{=0} + 3 \underbrace{\langle J'', J' \rangle}_{= -\langle \mathcal{R}(J, \dot{\gamma}) \dot{\gamma}, J' \rangle} = 0$$

because $J(0) = 0$ by the Jacobi equations. Finally, we compute the fourth derivative:

$$\frac{1}{2} \langle J, J \rangle^{(4)} = \cancelto{0}{\left\langle J^{(4)}, J \right\rangle} + 3 \left(\underbrace{\left\langle J^{(3)}, J' \right\rangle}_{?} + \underbrace{\left\langle J'', J'' \right\rangle}_{=0} \right)$$

We will figure out $\left\langle J^{(3)}, J' \right\rangle$ in the next lecture.