Math 635 Lecture 12

Thomas Cohn

2/15/21

Recall:
Defn: $\gamma : [a, b] \to M$ (for M a Riemannian manifold) is a <u>geodesic</u> iff $\frac{D}{dt}\dot{\gamma} = 0$.

Recall: $\dot{\gamma}$ is the natural lift of γ along γ. We say $\dot{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$, so there's some ambiguity in the notation.

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

Review: In coordinates on $U \subset M$, we write $\gamma(t) = (x^1(t), \dots, x^n(t))$, with each $x^i \in C^\infty([a, b], M)$. Then γ is a geodesic iff $\ddot{x}^k(t) = -\dot{x}^i(t)\dot{x}^j(t)\Gamma^k_{ij}(\gamma(t))$, where $\Gamma^k_{ij} : U \to \mathbb{R}$ are the Christoffel symbols.

Observe: If ∇ is trivial, i.e., the "flat case", then $\Gamma^k_{ij} = 0$. So $\ddot{x}^k = 0$, and $\forall k$, $x^k(t) = tv^k(0) + x^k(0)$. See Do Carmo, Chapter 3, §2 for more details.

We want to rewrite the geodesic equations, locally, as a first order system in twice as many unknowns. We introduce $v^1, \dots, v^n$, which we call the "velocities", such that $v_k \stackrel{\text{def}}{=} \dot{x}^k$, and $\dot{v}^k = -\Gamma^k_{ij}(\gamma(t))v^i v^j$ are the "accelerations".

Note that time derivatives have been solved in all cases, so there is a unique solution (for a small time interval) given $x^k(0)$ and $v^k(0)$, for $k = 1, \dots, n$.

Lemma: (Do Carmo 2.3) $\exists! G \in \mathfrak{X}(TM)$ s.t. the integral curves of G are precisely of the form $\dot{\gamma}(t) = (\gamma(t), \frac{d\gamma}{dt}(t))$, where γ is a geodesic. In other words, the integral curves of G are precisely the lifts to TM of geodesics on M. (Integral curves of G are locally solutions to the above system of differential equations.)

Proof: First, we'll prove local existence and uniqueness of G in coordinates. Let $V \subset M$ be a coordinate neighborhood, with coordinates $(x^1, \dots, x^n)$, inducing coordinates $(x^1, \dots, x^n, v^1, \dots, v^n)$ on TV by $v = \sum_{i=1}^n v^i \, \partial_{x^i}|_p$ for $(p, v) \in TV$. Then $G = \sum_{i=1}^n a_i \partial_{x^i} + b_i \partial_{v^i}$, for some $a_i, b_i \in C^\infty(TV)$ (note that this is true for any vector field on TV).

Now, comparing with the system of differential equations, we can see that we must have $a_i = v^i = \dot{x}^i$, $\forall i$. So $G(x, v) = v^i \partial_{x^i} - \Gamma^k_{ij}(x)v^i v^j \partial_{v^k}$ iff the integral curves of V solve the system of equations.

Finally, local existence and uniqueness implies global existence and uniqueness by covering M with coordinate charts. $\square$

Observe: The vector field G can be described using T^*M and its symplectic form, and $TM \to T^*M$ by $T_pM \to T^*_pM$ using $\langle \ , \ \rangle_p$. In the future, we'll also consider the Hamiltonian picture...

Now, we want to think about the flow of G on TM. Let $X \in \mathfrak{X}(M)$. Given any $m \in M$, there's a neighborhood $\mathcal{U} \subseteq M$ of m, $\delta > 0$, and $\varphi : (-\delta, \delta) \times \mathcal{U} \to M$ smooth such that $\forall \mu \in \mathcal{U}$, $t \mapsto \varphi(t, \mu)$ is the integral curve of X s.t. $\varphi(0, \mu) = X_\mu$, and $\forall t$, $\frac{d}{dt}\varphi(t, \mu) = X_{\varphi(t,\mu)}$.

Now, apply this to $\mathcal{M} = TM$, $X = G$, and $m = (p, 0)$ for $p \in M$. Then $\exists \mathcal{U} \subseteq \mathcal{M}$ and $\delta > 0$ as in the theorem. So we have $\{(q, 0) \in TM : q \in M\} \cong M$.

Claim: $\exists V \subseteq M$, a neighborhood of p, and $\varepsilon > 0$ s.t. $\{(q, v) \in TM \mid q \in V, ||v|| < \varepsilon\} \subseteq \mathcal{U}$.

We get

$$\begin{tikzcd}
(-\delta,\delta) \times \{(q,v) \mid q \in V, ||v|| < \varepsilon\} \arrow[r, "\varphi"] \arrow[dr, "\gamma \overset{\text{def}}{=} \pi \circ \varphi"'] & TM \arrow[d, "\pi"] \\
& M
\end{tikzcd}$$

An important property of γ is that $\forall (q,v)$, $t \mapsto \gamma(t,q,v)$ is *the unique* geodesic s.t. $\gamma(0,q,v) = q$ and $\left.\frac{d}{dt}\gamma(t,q,v)\right|_{t=0} = v$.

Lemma: By reparameterizing geodesics by a constant factor in time, one can show (keeping the notation from our previous discussion) that, for $a > 0$, then $\gamma(t,q,av) = \gamma(at,q,v)$, provided that both sies are defined.

Proof: Check that both sides are geodesics, with the same initial conditions. Then by uniqueness of geodesics, they're equivalent. $\square$