Math 635 Lecture 1

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The overall goal of this course is to build a notion of distance on a manifold. But in order to have such a notion, we need additional structure. Specifically, we need a Riemannian metric: a C^∞ assignment of an inner product to each tangent space T_pM, $\forall p \in M$.

Ex: Say $S \subset \mathbb{R}^N$ is a submanifold. $\forall p \in S$, $T_pS \hookrightarrow \mathbb{R}^N$, so it inherits the Euclidean inner product of $\mathbb{R}^N$. We can use this to define the length of a curve $\gamma : (a,b) \to S$ – $\int_a^b ||\dot\gamma(s)||\ ds$, where $||\dot\gamma|| = \sqrt{\langle \dot\gamma, \dot\gamma\rangle}$.

For a systematic study of these ideas, we need to review:

- Vector bundles
- Connections on vector bundles

The basic idea of a connection is that a Riemannian metric allows us to talk about parallel transport of vectors along curves. Specifically, a connection allows us to differentiate sections of a vector bundle along curves in the base.

Vector Bundles (Review)

Defn: A vector bundle (of rank r) over M is a surjective submersion $\pi : \mathcal{E} \to M$ s.t.

a) For each $p \in M$, the fiber $\mathcal{E}_p \overset{\text{def}}{=} \pi^{-1}(p)$ is a vector space (of rank r).

b) There's an (open[1]) covering $\{U_\alpha\}$ of M, and diffeomorphisms $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$ called local trivializations, such that

   i) $\forall \alpha$, the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

   ii) $\forall p \in U,\ \chi_\alpha|_{\mathcal{E}_p} : \mathcal{E}_p \to \{p\} \times \mathbb{R}^r \cong \mathbb{R}^r$ is a linear isomorphism.

Colloquially, we think of $\mathcal{E}$ as a family $\{\mathcal{E}_p\}$ of vector spaces, parameterized by M.

Ex: TM and T^*M are examples of vector bundles.

Defn: Given a vector bundle $\pi : \mathcal{E} \to M$, let $\Gamma(\mathcal{E}) \overset{\text{def}}{=} \{s : M \to \mathcal{E} \mid s \text{ smooth and } \pi \circ s \equiv I_M\}$, the set of mooth sections of $\mathcal{E}$.

Defn: The zero section is s s.t. $\forall p \in M$, $s(p) = 0 \in \mathcal{E}_p$.

Ex: If $\mathcal{E} = TM$, then $\mathcal{E}_p = T_pM$, so

- $\Gamma(TM) = \mathfrak{X}(M)$, the set of all smooth vector fields on M.
- $\Gamma(T^*M)$ is the set of all smooth 1-forms on M.

Observe: $\Gamma(\mathcal{E})$ is a module over $C^\infty(M)$ – $\forall s,t \in \Gamma(\mathcal{E}), f \in C^\infty(M)$, $(fs+t)(p) = f(p)s(p) + t(p)$.

Defn: A moving frame is a collection of sections $(E_1, \ldots, E_r)$ over U s.t. $\forall p \in U$, $(E_1(p), \ldots, E_r(p))$ is a basis of $\mathcal{E}_p$.

[1] In the future, assume that any covering is open, unless explicitly stated otherwise.

Observe: Using our notion of modules, a local trivialization $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ defines a moving frame on U. If $(x^1, \dots, x^n)$ are coordinates on $U \subseteq M$, then $\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$ is a moving frame of TM on U.

Given χ, define $E_j(p) \in \mathcal{E}_p$ such that $\chi(p, E_j(p)) = (p, \langle 0, \dots, 1, \dots, 0\rangle)$ (with a 1 in the jth entry). Then we have

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, shift left, "\chi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[ll, shift left, "\chi^{-1}"] \arrow[dl] \\
& U \arrow[ur, bend right, "p \mapsto (p, \langle 0, \dots, 1, \dots, 0\rangle)"'] &
\end{tikzcd}$$

Then $\forall s \in \Gamma(\mathcal{E}|_U)$, $\exists f_j : U \to \mathbb{R}$ such that $\forall p \in U$,

$$s(p) = \sum_{j=1}^{r} f_j(p) E_j(p)$$

This is true for any section $s : U \to \pi^{-1}(U) = \mathcal{E}|_U$, and, in fact, these f_j's are unique, simply because $s(p) \in \mathcal{E}_p$, and $\{E_j(p)\}$ is a basis of $\mathcal{E}_p$.

Lemma: s is smooth iff $\forall j$, f_j is smooth.

Defn: The <u>trivial bundle</u> of rank r is $\pi : M \times \mathbb{R}^r \to M$.

Operations with Vector Bundles

Principle: Any operation or construction that one can do with vector spaces, that is natural with respect to linear isomorphisms, can be extended to vector bundles, by doing such an operation or construction fiber-wise.

Ex:

- $V \rightsquigarrow V^*$
- $V, W \rightsquigarrow V \oplus W$
- $V, W \rightsquigarrow V \otimes W$
- $V, W \rightsquigarrow \mathrm{Hom}(V, W)$
- $V \subset w \rightsquigarrow W/V$

Idea: Given any two bundles $\mathcal{E}, \mathcal{F} \to M$ over M, we can form the "Whitney direct sum", $\mathcal{E} \oplus \mathcal{F} \to M$, such that $\forall p \in M$, $(\mathcal{E} \oplus \mathcal{F})_p = \mathcal{E}_p \oplus \mathcal{F}_p$. This already defines what $\mathcal{E} \oplus \mathcal{F}$ has to be as a set:

$$\mathcal{E} \oplus \mathcal{F} = \bigsqcup_{p \in M} \mathcal{E}_p \oplus \mathcal{F}_p$$

The question is: how do we give such a set the topology and C^∞ structure of a vector bundle?