Math 635 Lecture 10

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Recall from last time:

Defn: Given a vector bundle $\mathcal{E} \to M$ with connection ∇, $X, Y \in \mathfrak{X}(M)$, the <u>curvature operator</u> $\mathcal{R}$ of ∇, evlaulated on (X, Y), is

$$\begin{aligned} \mathcal{R}(X,Y) &: \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}) \\ (X,Y) &\mapsto [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \end{aligned}$$

Observe: When Do Carmo defines the curvature operator (Chapter 4, Definition 2.1, in the case where $\mathcal{E} = TM$), they use the opposite sign.

Observe: $\mathcal{R}$ is given by a tensor! What does that mean? Last time, using the second approach, we computed locally in a moving frame $(E_1, \dots, E_r)$ (with associated connection matrix ϑ) that $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + d\vartheta(X,Y) + [\vartheta(X), \vartheta(Y)]$. So $\mathcal{R}$ has, for its components in the given frame, the components of the vector

$$(d\vartheta(X,Y) + [\vartheta(X), \vartheta(Y)])\vec{f} \qquad s = f^i E_i, \vec{f} = \begin{pmatrix} f^1 \\ \vdots \\ f^r \end{pmatrix}$$

Defn: $\Omega \overset{\text{def}}{=} d\vartheta + \vartheta \wedge \vartheta$ is the <u>curvature matrix</u> of ∇ with respect to the moving frame $(E_1, \dots, E_r)$.

(This is true because we observed $(\vartheta \wedge \vartheta)(X,Y) = [\vartheta(X), \vartheta(Y)]$.)

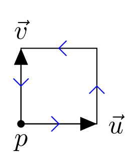
In fact, $\forall p \in U = \mathrm{dom}(E_i)$, $\mathcal{R}(X,Y)(s)(p) \in \mathcal{E}_p$ is the image of $s(p)$ by the linear transformation $\mathcal{E}_p \to \mathcal{E}_p$ whose matrix (in the basis $(E_1(p), \dots, E_r(p))$) is $\Omega_p(X_p, Y_p)$.

The virtue of this definition is that it's a well-defined global object! But it turns out to be a differential operator of order 0, meaning there's no derivatives, so it's just multiplication. At each point it's given by a linear transformation of the fibers, with the matrix determined by X_p and Y_p.

Observe: The dependence on X and Y is punctual! $\forall p \in M$, $\mathcal{R}(X,Y)(s)(p)$ depends only on $X_p, Y_p \in T_pM$ and $s(p) \in \mathcal{E}_p$.

$\mathcal{R}$, as an object, is "an End-$\mathcal{E}$ valued 2-form on M". That is, $\forall p \in M$, $\forall u, v \in T_pM$, $\mathcal{R}_p(u,v) : \mathcal{E}_p \to \mathcal{E}_p$ is a linear map, and $\mathcal{R}(\cdot, \cdot)$ is bilinear and skew-symmetric.

Intuition: $\mathcal{R}$ is given by infinitesimal holonomy. Given a tiny loop at p below, the holonomy of the path is approximately $\exp(\mathcal{R}_p(u,v))$ (using the matrix exponential).



Even though we're taling about an operator, it's given by a tensor. $\mathcal{R}$ itself is a section of

$$\underbrace{T^*M \otimes T^*M}_{\text{2-form part}} \otimes \underbrace{\mathcal{E}_p \otimes \mathcal{E}_p}_{\text{Endomorphism part}}$$

We're well on our way to defining the Levi-Civita connection!

Consider a vector bundle $\mathcal{E} \to M$, now with a positive definite inner product on each fiber. (In the case where $\mathcal{E} = TM$, this exactly is a Riemannian metric.)

Defn: A connection ∇ on $\mathcal{E}$ (with $\langle\ ,\ \rangle$) is said to preserve $\langle\ ,\ \rangle$ iff $\forall \gamma : [a,b] \to M$, parallel transport $\mathcal{P}_\gamma : \mathcal{E}_{\gamma(a)} \to \mathcal{E}_{\gamma(b)}$ is an isometry, i.e., $\forall u, v \in \mathcal{E}_{\gamma(a)}$, $\langle \mathcal{P}_\gamma(u), \mathcal{P}_\gamma(v)\rangle_{\gamma(B)} = \langle u, v\rangle_{\gamma(a)}$.

Prop: Given a vector bundle $\mathcal{E} \to M$, inner product $\langle\ ,\ \rangle$ on each fiber, and a connection ∇, the following are equivalent:

(a) ∇ preserves $\langle\ ,\ \rangle$.
(b) $\forall s, t \in \Gamma(\mathcal{E})$, $\forall X \in \mathfrak{X}(M)$, $X(\langle s, t\rangle) = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$. Note that $\langle s, t\rangle$ is a function on M, which we can differentiate with respect to X. We can think of this as a sort of "product rule".
(c) $\forall (E_1, \dots, E_r)$ local orthonormal frame (which exists by Gram-Schmidt), the connection matrix ϑ is skew symmetric, i.e., $\forall i, j$, $\theta^i_j = -\theta^j_i$.

Proof: First, we show that (b) $\Leftrightarrow$ (c). Let $(E_1, \dots, E_r)$ be our local orthonormal frame. Then there are functions f^i, g^j such that $s = f^i E_i$ and $t = g^j E_j$. Thus, we can form $\vec{f}, \vec{g}$, and by orthonormality of the frame

$$\langle s, t\rangle = \sum_{i=1}^r \sum_{j=1}^r f^i g^j \underbrace{\langle E_i, E_j\rangle}_{=\delta_{ij}} = \sum_{i=1}^r f^i g^i = \vec{f} \cdot \vec{g}$$

Thus, with a slight abuse of notation,

$$\langle \nabla_X s, t\rangle \text{ “$=$” } (\nabla_X \vec{f}) \cdot \vec{g} = (X(\vec{f}) + \vartheta(X)\vec{f}) \cdot \vec{g}$$

And

$$\langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle = \underbrace{X(\vec{f}) \cdot \vec{g} + \vec{f} \cdot X(\vec{g})}_{=X(\vec{f}\cdot\vec{g}) = X(\langle s,t\rangle)} + (\vartheta(X)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(X)\vec{g})$$

So the product rule holds iff $\forall s, t \ /\ \forall \vec{f}, \vec{g}$, $(\vartheta(X)\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(X)\vec{g}) = 0$, which is true iff $\vartheta(X)$ is skew-symmetric.

In order to show (a), we just change the setting a bit. Let $\gamma : [a,b] \to M$ be a smooth curve. Take $V, W \in \Gamma_\gamma(\mathcal{E})$. We claim that, just as above, we get

$$\underbrace{\left\langle \frac{DV}{dt}, W \right\rangle}_{\text{a function of } t} + \left\langle V, \frac{DW}{dt} \right\rangle - \frac{d}{dt}\langle V, W\rangle = (\vartheta(\dot{\gamma})\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot{\gamma})\vec{g})$$

Assume V and W are parallel along γ. By definition, this means $\frac{DV}{dt} = \frac{DW}{dt} = 0$. Then

$$-\frac{d}{dt}\langle V, W\rangle = (\vartheta(\dot{\gamma})\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot{\gamma})\vec{g})$$

Well,

$$\begin{aligned} \nabla \text{ preserves } \langle\ ,\ \rangle &\Leftrightarrow \frac{d}{dt}\langle V, W\rangle = 0,\ \forall V, W \text{ parallel} \\ &\Leftrightarrow (\vartheta(\dot{\gamma})\vec{f}) \cdot \vec{g} + \vec{f} \cdot (\vartheta(\dot{\gamma})\vec{g}) = 0 \text{ in all instances} \\ &\Leftrightarrow \vartheta \text{ is skew symmetric} \end{aligned}$$

$\square$

Thm: Let M be a Riemannian manifold. Then $\exists! \nabla$ on $\mathcal{E} = TM \to M$ such that

(a) ∇ preserves the Riemannian metric. *(This depends on the choice of Riemannian metric.)*
(b) $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y - \nabla_Y X = [X, Y]$. *(This does not depend on the choice of Riemannian metric.)*

Defn: This ∇ is called the <u>Levi-Civita connection</u> on M.