Math 635 Lecture 19

Thomas Cohn

3/5/21

The Variantional Point of View of Geodesics

This material is covered in Do Carmo, chapter 9 §2, and in parts of Lee Riemannian Manifolds, chapters 6 and 10.

Throughout these notes, let M be a Riemannian manifold, $p,q \in M$, and $a > 0$.

Defn: $\Omega_{pq}^a = \{c : [0,a] \to M \mid c \text{ is continuous and piecewise } C^1, c(0) = p, c(a) = q\}$. In this case, piecewise C^1 means $\forall c$, there's a partition $0 = t_0 < t_1 < \cdots < t_N = a$ such that $\forall i,\ c|_{[t_i,t_{i+1}]}$ is C^1. In other words, c is C^1 on (t_i, t_{i+1}), and the one sided limits

$$\lim_{t\to t_i^-} \frac{dc}{dt} \qquad \lim_{t\to t_{i+1}^+} \frac{dc}{dt}$$

exist.

Defn: We define two functionals on this space.

1. From Lee: The length functional $L : \Omega_{pq}^a \to \mathbb{R}$
   $$c \mapsto L(c) = \int_0^a \left|\left|\tfrac{dc}{dt}\right|\right| \, dt$$
   Any minima of L, if one exists, corresponds to shortest paths between p and q.
2. From Do Carmo: The energy functional $E : \Omega_{pq}^a \to \mathbb{R}$
   $$c \mapsto E(c) = \tfrac{1}{2} \int_0^a \left|\left|\tfrac{dc}{dt}\right|\right|^2 \, dt$$

Lemma: $\forall c \in \Omega_{pq}^a$, $L(c)^2 \leq 2aE(c)$, with equality iff $||\dot{c}||$ is constant.

Proof: Use the Cauchy-Shwarz inequality for functions on $[0,a]$: If $f,g : [0,1] \to \mathbb{R}$, then

$$\left(\int_0^a fg\, dt\right)^2 \leq \left(\int_0^a f^2\, dt\right)\left(\int_0^a g^2\, dt\right)$$

Given a path c, apply Cauchy-Schwarz to $f = \left|\left|\frac{dc}{dt}\right|\right|$, with $g \equiv 1$. $\square$

Cor: Suppose $\gamma \in \Omega_{pq}^a$ is a minimizing geodesic. Then $\forall c \in \Omega_{pq}^a$, $E(\gamma) \leq E(c)$, with equality iff c is a minimizing geodesic.

Proof: $||\dot{\gamma}||$ is constant, so $E(\gamma) = \frac{1}{2a}L(\gamma)^2 \leq \frac{1}{2a}L(c)^2 \leq E(c)$. Thus, if $E(\gamma) = E(c)$, then everything must be equal, so $L(\gamma) = L(c)$, so c is a minimizing path, so c is a geodesic. $\square$

We want to look for minimizers of E. This is a hard problem, and in fact, the may not exist.

The idea of the calculus of variations is to differentiate E, and then look for critical points. This is crazy, because Ω_{pq}^a is not a manifold. But one can still define variations of $\gamma \in \Omega_{pq}^a$. Colloquially, these are smooth paths in Ω_{pq}^a that pass through γ.

Defn: Let $\gamma \in \Omega_{pq}^a$. A <u>proper variation</u> (or <u>pinned variation</u>) of γ is

$$f : \underset{s}{(-\varepsilon, \varepsilon)} \times \underset{t}{[0,a]} \to M$$

such that

(i) $f(0,t) = \gamma(t)$
(ii) There is a partition $0 = t_0 < t_1 < \cdots < t_N = a$ such that $\forall i,\ f|_{(-\varepsilon,\varepsilon)\times[t_i,t_{i+1}]}$ is C^1.
(iii) $\forall s$, $f(s,0) = p$ and $f(s,a) = q$. That is, $\forall s \in (-\varepsilon, \varepsilon)$, $(t \mapsto f(s,t)) \in \Omega_{pq}^a$. In this sense, $s \mapsto (t \mapsto f(s,t))$ is a "smooth" curve in Ω_{pq}^a, passing through γ at $t = 0$.

Conditions (i) and (ii) are what make it a variation; condition (iii) makes it proper/pinned.

Defn: For f a proper variation of γ, $\forall t$, $V(t) = \partial_s f(0,t) \in T_{\gamma(t)}M$ is the <u>variation field</u> of the variation. So $V \in \Gamma_\gamma(TM)$.

Observe: Because the variation is proper, we must have $V(0) = 0$.

Lemma: Given any $V \in \Gamma_\gamma(TM)$ s.t. $V(0) = 0$, $V(a) = 0$, there exists a proper variation f whose variation field is V.

Proof: Let $f(s,t) = \exp_{\gamma(t)}(sV(t))$. We need $|s| < \varepsilon$ to be nonzero, but because of the compactness of the curve, we can construct a finite subcover, so that $|s| > 0$. $\square$

Idea: Differentiate the energy E w.r.t. a given varation of $\gamma \in \Omega^a_{pq}$.
Computation: Let f as above, a proper variation of $\gamma \in \Omega^a_{pq}$. Define

$$E(s_0) = \frac{1}{2}\int_0^a ||\partial_t f(s,t)||^2 \, dt = E(f|_{s=s_0})$$

Then compute

$$\begin{aligned}
\frac{dE}{ds} &= \frac{1}{2}\int_0^a \frac{d}{ds}\langle \partial_t f, \partial_t f\rangle \, dt \\
&= \frac{\cancel{1}}{\cancel{2}}\int_0^a \cancel{2}\left\langle \frac{D}{ds}\partial_t f, \partial_t f\right\rangle dt \\
&= \int_0^a \left\langle \frac{D}{dt}\partial_s f, \partial_t f\right\rangle dt \\
\text{(integration by parts)} \qquad &= [\langle \partial_s f, \partial_t f\rangle]_{t=0}^{t=a} - \int_0^a \left\langle \partial_s f, \frac{D}{dt}, \partial_t f\right\rangle dt
\end{aligned}$$

Where the integration by parts succeeds because

$$\frac{d}{dt}\langle \partial_s f, \partial_t f\rangle = \left\langle \frac{D}{dt}\partial_s f, \partial_t f\right\rangle + \left\langle \partial_s f, \frac{D}{dt}\partial_t f\right\rangle$$

Note that we have to adjust the term outside of the integral due to corners, but the boundary terms will all appear in a similar form. We conclude with

$$\frac{d}{ds}E(0) = -\int_0^a \left\langle \underbrace{V(t)}_{\partial_s f|_{s=0}}, \frac{D}{dt}\frac{d\gamma}{dt}\right\rangle dt - \underbrace{\sum_i \langle V(t_i), \underbrace{\Delta\dot\gamma(t_i)}_{=\frac{d\gamma}{dt}(t_i^+) - \frac{d\gamma}{dt}(t_i^-)}\rangle}_{\text{corner terms}}$$

which is the first variation formula.

Now, choose

$$V(t) = \begin{cases} \frac{D}{dt}\frac{d\gamma}{dt} & t \in (t_i, t_{i+1}) \text{ for some } i \\ V(t_i) & t = t_i \text{ for some } i \end{cases}$$

We conclude that if $\gamma \in \Omega^a_{pq}$ is a critical point of E in the sense that for all variations f, $E'(0) = 0$, then γ is a geodesic!

Next time, we'll use the second derivative test. This is where curvature will appear!