

Math 635 Lecture 31

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Recall our definitions: \bar{M} is a Riemannian manifold, $M \subseteq \bar{M}$ is a submanifold with the induced metric. $\forall X, Y \in \mathfrak{X}(M)$, $B(X, Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y \in (T_p M)^\perp$, where \bar{X} and \bar{Y} are extensions of X and Y (respectively) to \bar{M} . (This is normal to M .) We can also think of it as $\bar{\nabla}_{\bar{X}} \bar{Y} = \nabla_X Y + B(X, Y)$.

- B is a symmetric tensor – $\forall p \in M$, $B_p : T_p M \times T_p M \rightarrow (T_p M)^\perp \subseteq T_p \bar{M}$.
- If we define, $\forall p \in M$, $\nu_p \in (T_p M)^\perp$ a unit normal vector, then $S_\nu : T_p M \rightarrow T_p M$ is the shape operator, with the defining property $\forall x, y \in T_p M$, $\langle S_\nu(x), y \rangle = \langle B(x, y), \nu \rangle$.

Because B is symmetric, S_ν is self-adjoint, so there's an orthonormal eigenbasis (e_1, \dots, e_n) of $T_p M$ and $\kappa_1, \dots, \kappa_n$ such that $S_\nu(e_i) = \kappa_i e_i$. The e_i are the principal directions, and the κ_i are the principal curvatures.

Observe: If $\text{codim } M = 1$, then $\delta_{ij} \kappa_i = \langle S_\nu(e_i), e_j \rangle = \langle B(e_i, e_j), \nu \rangle$ by orthonormality. This is just the ν -component of $B(e_i, e_j)$, so $B(e_i, e_j) = \delta_{ij} \kappa_i \nu$.

Thm: (Weingarten Formula) Pick $p \in M$ and $\nu \in (T_p M)^\perp$. Extend ν to a unit normal vector field N on M . Then $\forall x \in T_p M$, the Weingarten Formula says

$$S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

(This is computed via a curve in M with velocity x .)

Proof: Let $x, y \in T_p M$, and extend these to vector fields \bar{X} and \bar{Y} tangent to M . Then

$$\langle S_\nu(x), y \rangle = \langle B(x, y), N_p \rangle(p) = \langle \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y, N \rangle(p) = \langle \bar{\nabla}_{\bar{X}} \bar{Y}, N \rangle(p)$$

because $\langle \nabla_X Y, N \rangle(p) = 0$, as $(\nabla_X Y)_p \in T_p M$ and $N \in (T_p M)^\perp$ are orthogonal. On the other hand, $0 = \bar{Y}, N|_M$, so

$$0 = \bar{X} \langle \bar{Y}, N \rangle = \langle \bar{\nabla}_{\bar{X}} \bar{Y}, N \rangle + \langle \bar{Y}, \bar{\nabla}_{\bar{X}} N \rangle$$

The first term in the last expression is $\langle S_\nu(x), y \rangle$ by the above. So we have

$$\langle S_\nu(x), y \rangle = -\langle \bar{Y}, \bar{\nabla}_{\bar{X}} N \rangle = -\langle y, \bar{\nabla}_x N \rangle = \langle -\bar{\nabla}_x N, y \rangle \quad \Rightarrow \quad S_\nu(x) = -(\bar{\nabla}_x N)(p)$$

□

Consider the case where $\bar{M} = \mathbb{R}^{n+1}$, and $\dim M = n$. We identify tangent spaces with subspaces. Let U be the domain of N above. Then N is a map $N : U \rightarrow \mathbb{S}^n$, the unit sphere. We have

$$S_{N_p} : T_p M \rightarrow T_p M \cong T_{N(p)} \mathbb{S}^n$$

Claim: $S_{N_p} : T_p M \rightarrow T_{N(p)} \mathbb{S}^n$ is equal to $-dN_p$.

Check this: Let $c(t)$ be a curve on M with $p = c(0)$, $x = \dot{c}(0)$. Then

$$S_{N_p}(x) = -\bar{\nabla}_x N = -\left. \frac{d}{dt} N_{x(t)} \right|_{t=0} = dN_p(x)$$

Ex:

- a) Let $M = \mathbb{S}_R^n$, the sphere of radius R in \mathbb{R}^{n+1} . Let N be the outward pointing unit normal.

$$\begin{array}{ccc} N : \mathbb{S}_R^n \rightarrow \mathbb{S}^n & \Rightarrow & S_N = \frac{1}{R} \text{Id} \\ p \mapsto \frac{1}{R}p & & \end{array}$$

In this case, every direction is principal, with corresponding principal curvature $\kappa = -\frac{1}{R}$.

- b) Let M be a cylinder of radius R in \mathbb{R}^3 . Specifically, with parameterization

$$\vec{r}(\theta, z) = (R \cos \theta, R \sin \theta, z)$$

Let $N(\theta, z) = (\cos \theta, \sin \theta, 0)$. We have the basis of $T_p M$:

$$\begin{cases} \frac{\partial}{\partial \theta} = (-R \sin \theta, R \cos \theta, 0) \\ \frac{\partial}{\partial z} = (0, 0, 1) \end{cases}$$

Then we have $T_p M \cong T_{N_p} \mathbb{S}^2$. The matrix of dN with respect to the above basis is

$$\begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

So we conclude that $\frac{\partial}{\partial \theta}$ is a principal direction, with corresponding principal curvature $\kappa_1 = -\frac{1}{R}$. $\frac{\partial}{\partial z}$ is the other principal direction, with corresponding principal curvature $\kappa_2 = 0$.

Now, back to the general $M \subseteq \bar{M}$.

Thm: Let W, X, Y, Z be vector fields on \bar{M} which are tangent to M . Then $\forall p \in M$,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z) \rangle - \langle B(W, Z), B(X, Y) \rangle$$

Proof: Next time...

Cor: If $x, y \in T_p M$ are orthonormal, then

$$\bar{K}(x, y) = K(x, y) + \|B(x, y)\|^2 - \langle B(x, x), B(y, y) \rangle$$

Proof: Simply use $\bar{K}(x, y) = R(x, y, y, x)$. \square

Cor: If \bar{M} is flat, and M has codimension 1, i.e., it's a hypersurface, then $\forall i \neq j$, $K(e_i, e_j) = \kappa_i \kappa_j$.

Ex: \mathbb{S}_R^n has constant sectional curvature $\frac{1}{R^2}$.