

# Math 635 Lecture 32

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Let  $M \subset \bar{M}$  be a Riemannian submanifold. Continuing from last time:

**Thm:** (Gauss' Formula) Given  $W, X, Y, Z \in \mathfrak{X}(\bar{M})$  tangent to  $M$ ,  $\forall p \in M$ ,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z) \rangle - \langle B(W, Z), B(X, Y) \rangle$$

Proof: Well, the left hand side is

$$\bar{R}(W, X, Y, Z) = \langle (\bar{\nabla}_W \bar{\nabla}_X - \bar{\nabla}_X \bar{\nabla}_W - \bar{\nabla}_{[W, X]})Y, Z \rangle = \langle \bar{\nabla}_W \bar{\nabla}_X Y, Z \rangle + \langle -\bar{\nabla}_X \bar{\nabla}_W Y, Z \rangle + \langle -\bar{\nabla}_{[W, X]} Y, Z \rangle$$

We first work with the first term:

$$\langle \bar{\nabla}_W \bar{\nabla}_X Y, Z \rangle = \langle \bar{\nabla}_W (\nabla_X Y + B(X, Y)), Z \rangle = \langle \nabla_W \nabla_X Y, Z \rangle + \langle \bar{\nabla}_W B(X, Y), Z \rangle$$

because  $\nabla = \bar{\nabla}^T$ , and  $Z$  is tangent to  $M$ . The first term will eventually contribute to  $R(W, X, Y, Z)$ . As for the second term, note that  $B(X, Y)$  is normal to  $M$  and  $Z$  is tangent to  $M$ , so  $\langle B(X, Y), Z \rangle = 0$ . So we can use the compatibility of  $\bar{\nabla}$  with the metric and differentiation (specifically, the product rule) to get

$$\langle \bar{\nabla}_W B(X, Y), Z \rangle = -\langle B(X, Y), \bar{\nabla}_W Z \rangle = -\langle B(X, Y), B(W, Z) \rangle$$

By performing the analogous computations on the second term, we get

$$\langle -\bar{\nabla}_X \bar{\nabla}_W Y, Z \rangle = \langle -\nabla_X \nabla_W Y, Z \rangle + \langle B(W, Y), B(X, Z) \rangle$$

As for the third term,

$$\langle -\bar{\nabla}_{[W, X]} Y, Z \rangle = \langle -\nabla_{[W, X]} Y, Z \rangle$$

□

Recall: This implies that if  $\bar{M}$  is flat, we have an orthonormal eigenbasis of the shape operator  $e_i$ , with corresponding eigenvalues  $\kappa_i$ . We get  $\forall i \neq j$ ,  $K(e_i, e_j) = \kappa_i \kappa_j$ .

Recall: For  $M \subseteq \mathbb{R}^{n+1}$  a hypersurface ( $\dim M = n$ ), if  $M$  is oriented by a unit normal field  $m \ni p \mapsto N_p \in (T_p M)^\perp$ , we can interpret the unit normal field as a map  $N : M \rightarrow S^n$ .  $\forall p \in M$ , the shape operator at  $p$  with respect to  $N_p$ , denoted  $S_p$ , satisfies  $S_p = -(dN)_p$  (with  $N$  as a map). We call  $N$  the Gauss spherical map. In this case, we're implicitly identifying  $T_p M \cong T_p S^n$ .

**Cor:** For  $M \subset \mathbb{R}^3$  a surface,  $K(p) = \det(dN)_p$ , where  $K(p)$  is the sectional curvature of  $M$  at  $p$ . (Because  $\dim M = 2$ , there's only one tangent plane at a given  $p \in M$ .)

We want to explore the global implications of this, when  $\dim M = 2$ . Let  $N = M \rightarrow S^2$ .

Observe: If  $K(p) \neq 0$ , then  $N$  is a local diffeomorphism at  $p$ . So if  $K$  is non-vanishing, and if  $M$  is compact and connected, then  $N$  is a covering map! Why is this true? Well, fix  $q \in S^2$ . Then  $N^{-1}(q) = \bigsqcup_{i=1}^I \{p_i\}$ . (This is finite because  $M$  is compact.)  $\forall i$ ,  $p_i$  has a neighborhood  $U_i$  such that  $N|_{U_i} : U_i \rightarrow N(U_i) = V_i$  is a diffeomorphism. Now, let  $V = \bigcap_{i=1}^I V_i$ . This is a neighborhood of  $q$ , and we claim  $V$  is evenly covered; to see this, just let  $\tilde{U}_i = N^{-1}(V_i) \cap U_i$ .

**Cor:** If  $K$  is non-vanishing, and  $M$  is compact and oriented, then  $M$  is diffeomorphic to  $S^2$ .

**Ex:** Consider  $M$  an ellipsoid,  $K > 0$ .

Question: Is there a metric on  $S^2$  where  $K < 0$ ? Answer: No, purely by topology.

**Thm:** (Gauss-Bonnet) Let  $n = 2m$  be even. Let  $M \subset \mathbb{R}^{n+1}$  be a compact, oriented, connected manifold. Let  $N : M \rightarrow S^n$  be the Gauss map. Define  $\mathcal{K} : M \rightarrow \mathbb{R}$ , the Gaussian curvature, by  $N^*(dV_{S^n}) = \mathcal{K} dV_M$ , where  $dV_{S^n}$  is the volume form on  $S^n$  and  $dV_M$  is the volume form on  $M$ . Then

$$\int_M \mathcal{K} dV_M = \frac{1}{2} \text{Vol}(S^n) \chi(M)$$

where  $\chi(M)$  is the Euler characteristic of  $M$ . Also,  $\mathcal{K}$  is intrinsic – it depends only on the induced metric on  $M$ .

**Defn:** The Euler characteristic of  $M$  is  $\chi(M) = \sum_{i=0}^n (-1)^i \dim H^i(M)$ .

Note:  $\mathcal{K}(p) = \det(dN)_p = \prod_{i=1}^n \kappa_i$ . Why is this product intrinsic? Well, since  $n = 2m$  is even, we can write

$$\mathcal{K}(p) = \prod_{i=1}^m \kappa_{2m-1} \kappa_{2m}$$

$\kappa_{2m-1} \kappa_{2m}$  is a specific sectional curvature, which we know to be intrinsic.

Plan for how we'll prove Gauss-Bonnet:

1. Show  $\mathcal{K}$  is intrinsic (for  $n \geq 4$ ).
2. Show  $\mathcal{K} dV_M = N^*(dV_{S^n})$ . This implies

$$\int_M \mathcal{K} dV_M = \int_M N^*(dV_{S^n}) = \deg(N) \int_{S^n} dV_{S^n} = \deg(N) \text{Vol}(S^n)$$

3.  $\deg(N) = \frac{1}{2} \chi(M)$

Steps 2 and 3 are just pure differential topology.

“Degree theory”:  $N : M \rightarrow S^n$  induces

$$\begin{array}{ccc} N^* : H^n(S^n) & \longrightarrow & H^n(M) \\ \parallel & & \parallel \\ \mathbb{R} & & \mathbb{R} \end{array}$$

So  $N^*$  is just multiplication by a number. In fact, that number is an integer, and is called the degree of  $N$ .