

Math 635 Lecture 22

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/12/21

Sectional Curvature

Defn: $\forall p \in M, \forall u, v \in T_p M$ linearly independent, the sectional curvature is defined to be

$$K_p(u, v) \stackrel{\text{def}}{=} \frac{\mathcal{R}(u, v, v, u)}{|u \wedge v|^2} = \frac{\langle \mathcal{R}(u, v)v, u \rangle}{\|u\|^2 \|v\|^2 - \langle u, v \rangle^2}$$

Last time, we claimed that $K_p(u, v)$ only depends on the plane $\pi = \text{span}(u, v) \in \text{Gr}(2, T_p M)$. So really, K_p is a function $K_p : \text{Gr}(2, T_p M) \rightarrow \mathbb{R}$.

Proof of this claim: We want to show that the right-hand side is invariant under the following “moves”:

- $(u, v) \rightsquigarrow (\lambda u, v)$ (for $\lambda \neq 0$)
- $(u, v) \rightsquigarrow (v, u)$
- $(u, v) \rightsquigarrow (u + \lambda v, v)$

This is easy to do using known properties of \mathcal{R} .

Observe: If $\dim M = 2$, then think about \mathcal{R} and its symmetries. How many independent components does it have? Well, for $\mathcal{R}(u, v, u', v')$ (with u and v linearly independent), we can write (u', v') in terms of (u, v) via a change of basis:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_A \begin{bmatrix} u \\ v \end{bmatrix}$$

So $\mathcal{R}(u, v, u', v') = \det(A)\mathcal{R}(u, v, u, v) = -\det(A)\mathcal{R}(u, v, v, u)$. So we conclude that in dimension 2, there's only one degree of freedom! All the information that's contained in \mathcal{R} reduces to knowing the function $K(p) = \mathcal{R}_p(e, f, f, e)$, where (e, f) is an orthonormal basis of $T_p M$ ($K : M \rightarrow \mathbb{R}$).

Prop: K from above is the Gaussian curvature.

In dimension n , there are $\frac{n^2(n^2-1)}{12}$ degrees of freedom.

Prop: The function $K : \bigcup_{p \in M} \text{Gr}(2, T_p M) \rightarrow \mathbb{R}$ completely determines \mathcal{R} .

Proof: This is just an algebraic exercise. As a preliminary, start with a bilinear map $b : V \times V \rightarrow \mathbb{R}$. Note that b is the sum of a symmetric and antisymmetric bilinear form; this can be seen by writing $b(x, y) = x^T M y$ for a unique matrix M , and then noting that we can write $M = \frac{1}{2}(M + M^T) + \frac{1}{2}(M - M^T)$. $M + M^T$ is symmetric, and $M - M^T$ is antisymmetric.

Now, if b is symmetric, then b is uniquely determined by the mapping $x \mapsto q(x) = b(x, x)$, because $q(x + y) = q(x) + q(y) + 2b(x, y)$. We can then solve for $b(x, y)$. (This is known as the “polarization identity”.)

Cor: If $b(x, x) = 0, \forall x \in V$, then b must be skew-symmetric.

We now continue the proof of the proposition. Let V be a vector space (e.g. $T_p M$). Let $\mathcal{R}, \mathcal{R}' : V \times V \times V \times V \rightarrow \mathbb{R}$ be multilinear maps, with the symmetries of the Riemannian curvature. Then define $D = \mathcal{R} - \mathcal{R}'$, and it has the same symmetries. Assume $\forall v, w, x, D(v, w, w, v) = 0$ (x will be used later). We want to show $D \equiv 0$.

Why is this true? Well,

$$\begin{aligned}
? &= D(v+w, x, x, v+w) \\
&= \underbrace{D(v, x, x, v)}_{=0} + D(v, x, x, w) + D(w, x, x, v) + \underbrace{D(w, x, x, w)}_{=0} \\
&= D(v, x, x, w) - D(v, x, x, w) \\
&= 0
\end{aligned}$$

Finally, we use the Bianchi identity, to show that

$$0 = D(u, v, w, t) + D(w, u, v, t) + D(v, w, u, t) = 3D(u, v, w, t)$$

□

Now, we return to studying the second variation formula. Recall: For a geodesic γ , and V the variation field of a proper variation of γ , we have

$$E''(0) = - \int_0^a \left\langle V, \frac{D^2}{dt^2} V + \mathcal{R}(V, \dot{\gamma}) \dot{\gamma} \right\rangle dt$$

Thm: (Bonnet-Myers V1) If sectional curvature K satisfies $K > \left(\frac{\pi}{\ell}\right)^2 > 0$ for some $\ell > 0$, then no geodesic of length ℓ is minimizing.

Proof: Let γ be a geodesic of length ℓ . We need to show the energy of γ is not a minimum. Let $\gamma : [0, \ell] \rightarrow M$, with $\gamma(0) = p$, $\gamma(\ell) = q$, and $\|\dot{\gamma}\| = 1$. Pick $N \in T_p M$ a unit vector such that $\langle N, \dot{\gamma}(0) \rangle = 0$. Let $E(t)$ be the parallel transport of N along γ , so then $\frac{D}{dt} E \equiv 0$. Observe that $\forall t$, $\langle E(t), \dot{\gamma}(t) \rangle = 0$. So if we define

$$V(t) \stackrel{\text{def}}{=} \sin\left(t \frac{\pi}{\ell}\right) E(t)$$

then $V(0) = 0$ and $V(\ell) = 0$, so V is the variation field of a pinned variation. So now, we can substitute into the second variation formula. We compute

$$\frac{D}{dt} V(t) = \frac{\pi}{\ell} \cos\left(t \frac{\pi}{\ell}\right) E(t) + 0 \quad \Rightarrow \quad \frac{D^2}{dt^2} V(t) = -\left(\frac{\pi}{\ell}\right)^2 \sin\left(t \frac{\pi}{\ell}\right) E(t)$$

Thus,

$$\left\langle V, \frac{D^2}{dt^2} V \right\rangle = -\left(\frac{\pi}{\ell}\right)^2 \sin^2\left(t \frac{\pi}{\ell}\right) \underbrace{\|v\|^2}_{=1}$$

Also,

$$\langle V, \mathcal{R}(V, \dot{\gamma}) \dot{\gamma} \rangle = \sin^2\left(t \frac{\pi}{\ell}\right) \underbrace{\langle E, \mathcal{R}(E, \dot{\gamma}) \dot{\gamma} \rangle}_{=K(E, \dot{\gamma})}$$

So for a proper variation of γ with variation field $V = f_s|_{s=0}$, we have

$$E''(0) = \int_0^\ell \underbrace{\sin^2\left(t \frac{\pi}{\ell}\right)}_{\geq 0} \underbrace{\left(\left(\frac{\pi}{\ell}\right)^2 - K(E, \dot{\gamma})\right)}_{< 0} dt$$

So $E''(0) < 0$, so for $|s| \ll 1$, $E(s) < E(0)$, so γ is not minimizing. □