Math 635 Lecture 32

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Let $M \subset \bar{M}$ be a Riemannian submanifold. Continuing from last time:

Thm: (Gauss' Formula) Given $W, X, Y, Z \in \mathfrak{X}(\bar{M})$ tangent to M, $\forall p \in M$,

$$\bar{R}(W, X, Y, Z) = R(W, X, Y, Z) + \langle B(W, Y), B(X, Z)\rangle - \langle B(W, Z), B(X, Y)\rangle$$

Proof: Well, the left hand side is

$$\bar{R}(W, X, Y, Z) = \left\langle (\bar{\nabla}_W \bar{\nabla}_X - \bar{\nabla}_X \bar{\nabla}_W - \bar{\nabla}_{[W,X]})Y, Z\right\rangle = \left\langle \bar{\nabla}_W \bar{\nabla}_X Y, Z\right\rangle + \left\langle -\bar{\nabla}_X \bar{\nabla}_W Y, Z\right\rangle + \left\langle -\bar{\nabla}_{[W,X]} Y, Z\right\rangle$$

We first work with the first term:

$$\left\langle \bar{\nabla}_W \bar{\nabla}_X Y, Z\right\rangle = \left\langle \bar{\nabla}_W \left(\nabla_X Y + B(X, Y)\right), Z\right\rangle = \langle \nabla_W \nabla_X Y, Z\rangle + \left\langle \bar{\nabla}_W B(X, Y), Z\right\rangle$$

because $\nabla = \bar{\nabla}^T$, and Z is tangent to M. The first term will eventually contribute to $R(W, X, Y, Z)$. As for the second term, note that $B(X, Y)$ is normal to M and Z is tangent to M, so $\langle B(X, Y), Z\rangle = 0$. So we can use the compatibility of $\bar{\nabla}$ with the metric and differentiation (specifically, the product rule) to get

$$\left\langle \bar{\nabla}_W B(X, Y), Z\right\rangle = -\left\langle B(X, Y), \bar{\nabla}_W Z\right\rangle = -\langle B(X, Y), B(W, Z)\rangle$$

By performing the analogous computations on the second term, we get

$$\left\langle -\bar{\nabla}_X \bar{\nabla}_W Y, Z\right\rangle = \langle -\nabla_X \nabla_W Y, Z\rangle + \langle B(W, Y), B(X, Z)\rangle$$

As for the third term,

$$\left\langle -\bar{\nabla}_{[W,X]} Y, Z\right\rangle = \left\langle -\nabla_{[W,X]} Y, Z\right\rangle$$

$\square$

Recall: This implies that if $\bar{M}$ is flat, we have an orthonormal eigenbasis of the shape operator e_i, with corresponding eigenvalues κ_i. We get $\forall i \neq j$, $K(e_i, e_j) = \kappa_i \kappa_j$.

Recall: For $M \subseteq \mathbb{R}^{n+1}$ a hypersurface ($\dim M = n$), if M is oriented by a unit normal field $m \ni p \mapsto N_p \in (T_pM)^\perp$, we can interpret the unit normal field as a map $N : M \to S^n$. $\forall p \in M$, the shape operator at p with respect to N_p, denoted S_p, satisfies $S_p = -(dN)_p$ (with N as a map). We call N the Gauss spherical map. In this case, we're implicitly identifying $T_pM \cong T_pS^n$.

Cor: For $M \subset \mathbb{R}^3$ a surface, $K(p) = \det(dN)_p$, where $K(p)$ is the sectional curvature of M at p. (Because $\dim M = 2$, there's only one tangent plane at a given $p \in M$.)

We want to explore the global implications of this, when $\dim M = 2$. Let $N = M \to S^2$.

Observe: If $K(p) \neq 0$, then N is a local diffeomorphism at p. So if K is non-vanishing, and if M is compact and connected, then N is a covering map! Why is this true? Well, fix $q \in S^2$. Then $N^{-1}(q) = \bigsqcup_{i=1}^{I} \{p_i\}$. (This is finite because M is compact.) $\forall i$, p_i has a neighborhood U_i such that $N|_{U_i} : U_i \to N(U_i) = V_i$ is a diffeomorphism. Now, let $V = \bigcap_{i=1}^{I} V_i$. This is a neighborhood of q, and we claim V is evenly covered; to see this, just let $\tilde{U}_i = N^{-1}(V_i) \cap U_i$.

Cor: If K is non-vanishing, and M is compact and oriented, then M is diffeomorphic to S^2.

Ex: Consider M an ellipsoid, $K > 0$.

Question: Is there a metric on S^2 where $K < 0$? Answer: No, purely by topology.

Thm: (Gauss-Bonnet) Let $n = 2m$ be even. Let $M \subset \mathbb{R}^{n+1}$ be a compact, oriented, connected manifold. Let $N : M \to S^n$ be the Gauss map. Define $\mathscr{K} : M \to \mathbb{R}$, the Gaussian curvature, by $N^*(dV_{S^n}) = \mathscr{K}\, dV_M$, where dV_{S^n} is the volume form on S^n and dV_M is the volume form on M. Then

$$\int_M \mathscr{K}\, dV_M = \frac{1}{2}\operatorname{Vol}(S^n)\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M. Also, $\mathscr{K}$ is intrinsic – it depends only on the induced metric on M.

Defn: The Euler characteristic of M is $\chi(M) = \sum_{i=0}^n (-1)^i \dim H^i(M)$.

Note: $\mathscr{K}(p) = \det(dN)_p = \prod_{i=1}^n \kappa_i$. Why is this product intrinsic? Well, since $n = 2m$ is even, we can write

$$\mathscr{K}(p) = \prod_{i=1}^m \kappa_{2m-1}\kappa_{2m}$$

$\kappa_{2m-1}\kappa_{2m}$ is a specific sectional curvature, which we know to be intrinsic.

Plan for how we'll prove Gauss-Bonnet:

1. Show $\mathscr{K}$ is intrinsic (for $n \geq 4$).
2. Show $\mathscr{K}\, dV_M = N^*(dV_{S^n})$. This implies

$$\int_M \mathscr{K}\, dV_M = \int_M N^*(dV_{S^n}) = \deg(N)\int_{S^n} dV_{S^n} = \deg(N)\operatorname{Vol}(S^n)$$

3. $\deg(N) = \frac{1}{2}\chi(M)$

Steps 2 and 3 are just pure differenial topology.

"Degree theory": $N : M \to S^n$ induces

$$\begin{array}{ccc} N^* \colon H^n(S^n) & \longrightarrow & H^n(M) \\ \wr\| & & \wr\| \\ \mathbb{R} & & \mathbb{R} \end{array}$$

So N^* is just multiplication by a number. In fact, that number is an integer, and is called the degree of N.