Math 635 Lecture 33

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Today, we'll take the first step towards proving Gauss-Bonnet. Let $M \subseteq \mathbb{R}^{n+1}$ be a compact, oriented manifold of even dimension $\dim M = n = 2m$, with $N : M \to S^n$ given by the orientation. Define the Gaussian curvature $\mathscr{K} : M \to \mathbb{R}$ by $\mathscr{K}\, dV_M = N^* dV_{S^n}$.

Thm: $\mathscr{K}$ is intrinsic to the Riemannian metric of N.

Recall the Weingarten formula: $\forall p \in M$, we have the commutative diagram

$$\begin{tikzcd} T_pM \arrow[r, "-dN"] \arrow[dr, swap, "S_{N(p)}"] & T_{N(p)}S^n \arrow[d, phantom, "\cong"] \\ & T_pM \end{tikzcd}$$

where $T_{N(p)}S^n \cong T_pM$ isometrically by translation. S_{N_p} is the shape operator by the Weingarten formula, so

$$\mathscr{K}\, dV_M\big|_p = S^*_{N(p)} dV_M\Big|_p \qquad \Rightarrow \qquad \mathscr{K} = \det S_{N(p)} = \prod_{i=1}^n \kappa_i$$

To prove this, we use orthonormal moving frames on M. Let $(E_1, \dots, E_n)$ be a positive orthonormal moving frame. We get the curvature matrix (Ω^i_j), a matrix of 2-forms.

$$\forall X, Y \in \mathfrak{X}(M), \quad R(X,Y)(E_j) = \Omega^i_j(X,Y)E_i \quad \Rightarrow \quad \Omega^i_j(X,Y) = R(X,Y,E_j,E_i)$$

Also, we have Gauss' formula:

$$0 = R(W,X,Y,Z) + \langle B(W,Y), B(X,Z)\rangle - \langle B(W,Z), B(X,Y)\rangle$$

So

$$\Omega^i_j(E_k, E_\ell) = \langle B(E_i,E_k), B(E_j,E_\ell)\rangle - \langle B(E_i,E_\ell), B(E_j,E_k)\rangle$$

Recall: $S_{ki} = S_{ik} \overset{\text{def}}{=} \langle S(E_i), E_k\rangle = \langle B(E_i,E_k), N\rangle$ is the N-component of $B(E_i,E_k)$ (S_{ij}) is the matrix of S. Written all together, we have

$$\Omega^i_j(E_k,E_\ell) = S_{ik}S_{j\ell} - S_{i\ell}S_{jk}$$

Prop: Let $n = 2m$. Then with σ_n the symmetric group,

$$\mathscr{K}\, dV = \frac{1}{n!}\sum_{\alpha \in \sigma_n} (-1)^\alpha \bigwedge_{i=1}^m \Omega^{\alpha(2i-1)}_{\alpha(2i)} \overset{\text{def}}{=} \mathrm{Pf}(\Omega)$$

Defn: $\mathrm{Pf}(\Omega)$ is called the <u>Pfaffian</u> of Ω.

We'll prove that the Pfaffian is independent of choice of moving frame. It's enough to show $\mathrm{Pf}(\Omega)(E_1,\dots,E_n) = \mathscr{K}$. Well, we introduce

$$Q = \{\varphi \in \sigma_n \mid \forall i \in \{1,\dots,n\}, \varphi(2i-1), \varphi(2i) \in \{2i-1, 2i\}\}$$

We then begin to compute

$$\begin{aligned} \mathrm{Pf}(\Omega)(E_1,\dots,E_n) &= \frac{1}{n!2^m}\sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta \prod_{i=1}^m \Omega^{\alpha(2i-1)}_{\alpha(2i)}(E_{\beta(2i-1)}, E_{\beta(2i)}) \\ &= \frac{1}{n!2^m}\sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta \sum_{\varphi\in Q}(-1)^\varphi \prod_{i=1}^m S_{\alpha(2i-1)\beta\varphi(2i-1)}S_{\alpha(2i)\beta\varphi(2i)} \\ &= \frac{1}{n!2^m}\sum_{\varphi\in Q}\sum_{\alpha,\beta\in\sigma_n} (-1)^\alpha(-1)^\beta(-1)^\varphi \prod_{i=1}^m S_{\alpha(2i-1)\beta\varphi(2i-1)}S_{\alpha(2i)\beta\varphi(2i)} \end{aligned}$$

Eventualy, this computation will give us $\det S$.

Defn: (Official definition) For $X = (x^i_j)$ an $n \times n$ matrix of commuting variables (with $n = 2m$ even), we define the Pfaffian

$$\mathrm{Pf}(X) = \frac{1}{n!} \sum_{\alpha \in \sigma_n} (-1)^\alpha \prod_{i=1}^{m} X^{\alpha(2i-1)}_{\alpha(2i)}$$

Lemma: $\forall X, Y$, $\mathrm{Pf}(Y^T X Y) = \det(Y)\,\mathrm{Pf}(X)$.

Proof: Just another direct computation.

Now, go back to moving frames and curvature matrices. Considering local frames, write $E_i = a^j_i F_j$ on $U \subseteq M$. Then $A(p) = (a^j_i(p)) \in \mathrm{SO}(n)$, and we konw that $\Omega_F = A^{-1} \Omega_E A$. So $\mathrm{Pf}(\Omega_F) = \det(A)\,\mathrm{Pf}(\Omega_E) = \mathrm{Pf}(\Omega_E)$.

Thus, $\mathrm{Pf}(\Omega_F) = \mathrm{Pf}(\Omega_E)$. So by the usual arguments, there's a unique global top-degree form on M such that for any moving frame on U, it agrees with $\mathrm{Pf}(\Omega)$. Therefore, by our proposition, $\mathscr{K}\,dV$ is of that form. $\square$

Question: Are there other combinations of the Ω^i_j's that give global forms on M? We need some polynomial $P : \mathrm{so}(n) \to \mathbb{R}$ such that $\forall A \in \mathrm{SO}(n)$, $\forall X \in \mathrm{so}(n)$, $P(A^{-1} X A) = P(X)$. Given such a P, repeat the previous argument to show that there's a global form ϖ such that on any U with a moving frame $E_1, \ldots, E_n$, $\varpi|_U = P(\Omega)$.

$P(\Omega_E) = P(\Omega_F)$, so $dP(\Omega) = 0$ always ($\forall P$ invariant). We get the Chern-Weil morphism:

$$\{\mathrm{Ad}\text{-invariant polynomials on } \mathrm{so}(n)\} \to H^* M$$