

Math 635 Lecture 25

Professor Alejandro Uribe-Ahumada

Transcribed by Thomas Cohn

3/19/21

Space of Constant Curvature

Such spaces are locally isometric to a sphere (if the curvature is positive), a Euclidean space (if the curvature is zero), or a hyperbolic space (if the curvature is negative).

Lemma: M has constant sectional curvature $K_0 \in \mathbb{R}$ iff $\forall X, Y, W, Z \in \mathfrak{X}(M)$,

$$\langle X, Y, W, Z \rangle = K_0 (\langle Y, W \rangle \langle X, Z \rangle - \langle X, W \rangle \langle Y, Z \rangle)$$

Proof: Note that by definition of K ,

$$\langle X, Y, Y, X \rangle = K(X, Y) \underbrace{(\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2)}_{=|X \wedge Y|^2}$$

This is a special case of the identity we want! Proving \Leftarrow is trivial. As for proving \Rightarrow , we show that the right hand side has the same symmetry properties as the left hand side. (This is just a messy computation.) Then we use the fact that we know K can determine \mathcal{R} if it's applied everywhere. \square

Cor: Let M have constant curvature, γ a geodesic of speed 1, and J a normal Jacobi field. Then the Jacobi equation becomes $J'' + KJ = 0$. (Recall the notation: $J' = \frac{DJ}{dt}$, $J'' = \frac{D^2J}{dt^2}$.)

Proof: By the lemma, we have, $\forall X \in \mathfrak{X}(M)$,

$$\langle \mathcal{R}(J, \dot{\gamma}) \dot{\gamma}, X \rangle = \langle J, \dot{\gamma}, \dot{\gamma}, X \rangle = K(|\dot{\gamma}|^2 \langle J, X \rangle - \langle J, \dot{\gamma} \rangle \langle \dot{\gamma}, X \rangle)$$

This is equal to $K \langle J, X \rangle$ for all X , $|mcR(J, \dot{\gamma})\gamma = KJ$. \square

Now, we consider a generalization of the surface case. Let γ be a geodesic on M , with constant curvature.

Let $N(t) \in \Gamma_\gamma(TM)$ be a unit normal and parallel field, determined by $N(0)$. In the surface case, we call this $\dot{\gamma}^\perp$. Define $J(t) = \varphi(t)N(t)$. Then J is Jacobi iff $\ddot{\varphi} + K\varphi = 0$.

Cor: If $J(0) = 0$, then $J(t) = \begin{cases} A \sin(\sqrt{K}t)N & K > 0 \\ tN & K = 0 \\ A \sinh(\sqrt{-K}t)N & K < 0 \end{cases}$

Recall the differential of the exponential map:

Thm: $d(\exp_p)_v(w) = J(1)$, where J is the Jacobi field along $\gamma : t \mapsto \exp_p(tv)$ s.t. $J(0) = 0$, $J'(0) = w$.

Proof: Define $f(s, t) = \exp_p(t(v + sw))$. Observe: $d(\exp_p)_v(w) = \partial_s f|_{s=0, t=1}$. Let $J(t) = \partial_s f(s=0, t)$. This is a Jacobi field, since the t curves are geodesics. Now, we just need to check that J satisfies the initial conditions.

$$J(0) = 0, \quad J' = \frac{D}{dt} \partial_s f = \frac{D}{ds} \partial_t f, \quad \partial_t f(s, t=0) = d(\exp)_0(v + sw) = v + sw \in T_p M$$

$s \mapsto \partial_t f(s, 0)$ is entirely contained in $T_p M$. Still, it's a field along $s \mapsto f(s, 0) = p$, so it's just a constant "curve". Finally, $\frac{D}{ds} \partial_t f = \frac{d}{ds}(v + sw) = w$. \square

Next, we examine a result of the "rate of spreading of geodesics". As before, define $f(s, t) = \exp_p(t(v + sw))$.

Prop: Take $v, w \in T_p M$ orthonormal, and let $\pi = \text{span}(v, w)$, $\gamma(t) = \exp_p(tv)$. Let $J(t)$ be the Jacobi field s.t. $J(0) = 0$, $J'(0) = w$. (Again, exactly as before.) Then

$$\|J(t)\| = t - \frac{t^3}{6} K_p(\pi) + O(t^3) \quad \text{as } t \rightarrow 0$$

as $t \rightarrow \infty$.

Interpretation: Rate of spreading of the ray $t(v + sw)$ with respect to $t \mapsto tv$. $\frac{\partial}{\partial s} |t(v + sw)| = tw$. We want to measure the same object on the manifold after exponentiation. We do so with respect to infinitesimal change in s .

$$\|J(t)\| \sim \begin{cases} < t & K_p(\pi) > 0 \\ > t & K_p(\pi) < 0 \end{cases}$$

Proof of the proposition: It's true iff

$$\|J(t)\|^2 = t^2 - \frac{t^4}{3} K_p(\pi) + O(t^4)$$

So we need to compute 4 derivatives of $\langle J, J \rangle$ at zero. Define $a_k = \langle J, J \rangle^{(k)}(t=0)$. Then clearly $a_0 = 0$. $\langle J, J \rangle' = 2 \langle J', J \rangle$, so $a_1 = 0$.

$$\frac{1}{2} \langle J, J \rangle'' = \langle J'', J \rangle + \langle J', J' \rangle \Rightarrow a_2 = \|w\|^2 = 1$$

$$\frac{1}{2} \langle J, J \rangle^{(3)} = \langle J^{(3)}, J \rangle + \langle J'', J' \rangle + 2 \langle J'', J' \rangle = \underbrace{\langle J^{(3)}, J \rangle}_{=0} + 3 \underbrace{\langle J'', J' \rangle}_{=-\langle \mathcal{R}(J, \dot{\gamma}) \dot{\gamma}, J' \rangle} = 0$$

because $J(0) = 0$ by the Jacobi equations. Finally, we compute the fourth derivative:

$$\frac{1}{2} \langle J, J \rangle^{(4)} = \cancel{\langle J^{(4)}, J \rangle} \overset{0}{+} 3 \left(\underbrace{\langle J^{(3)}, J' \rangle}_{?} + \underbrace{\langle J'', J'' \rangle}_{=0} \right)$$

We will figure out $\langle J^{(3)}, J' \rangle$ in the next lecture.