Math 635 Lecture 2

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Lemma: Let M be a C^∞ manifold. Assume $\forall p \in M$, we have a vector space $\mathcal{E}_p$ (of dimension r). Let $\mathcal{E} = \bigsqcup_{p\in M} \mathcal{E}_p$, and $\pi : \mathcal{E} \to M$ the natural projection, where $\mathcal{E}_p \mapsto p \in M$. Assume we're given $\{U_\alpha\}$, a cover of M, plus bijections χ_α such that $\forall \alpha$, the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[rr, "\chi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{R}^r \arrow[dl] \\
& U_\alpha &
\end{tikzcd}$$

i.e. $\chi_\alpha(\mathcal{E}_p) = \{p\} \times \mathbb{R}^r$.

Observe that this gives us $\forall \alpha, \beta$ s.t. $U_\alpha \cap U_\beta \neq \emptyset$, a map $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$, by:

$$\begin{tikzcd}
& \pi^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "\chi_\alpha"'] \arrow[dr, "\chi_\beta"] & \\
(U_\alpha \cap U_\beta) \times \mathbb{R}^r \arrow[rr, "\chi_\beta \circ \chi_\alpha^{-1}"] & & (U_\alpha \cap U_\beta) \times \mathbb{R}^r \\
(p, v) \arrow[rr, mapsto] & & (p, \tau_{\alpha,\beta}(p)v)
\end{tikzcd}$$

$$p \in U_\alpha \cap U_\beta$$
$$v \in \mathbb{R}^r$$

This mapping is a "change of trivialization", like a transition map for vector bundles. The matrix of $\tau_{\alpha,\beta}$ depends on p, but it's always a linear map.

If, $\forall \alpha, \beta$, $\tau_{\alpha,\beta}$ is C^∞, then there is a unique topology on $\mathcal{E}$, and a unique differentiable structure, that makes $\pi : \mathcal{E} \to M$ a smooth vector bundle, and each χ_α is a diffeomorphism, i.e., a smooth local trivialization.

Observe: On triple intersections, $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall p$, one has $\tau_{\alpha,\beta}(p)\tau_{\beta,\gamma}(p) = \tau_{\alpha,\gamma}(p)$. This is called a "cocycle condition".

Now, imageine starting with a cover $\{U_\alpha\}$ and C^∞ maps $\tau_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{R})$ satisfying the cocycle condition. Then, if we choose, $\forall p$, $\mathcal{E}_p = \mathbb{R}^r$, and $\chi_\alpha = I_n$, then we get a vector bundle $\mathcal{E} \to M$. So $\{U_\alpha; \tau_{\alpha,\beta}\}$, called a Čech cocycle, is all we need to put together a vector bundle.

Now, we apply the lemma to construct new bundles from old ones.

Ex: Given $\mathcal{E}', \mathcal{E}'' \to M$ vector bundles, with ranks r' and r'', respectively, define, $\forall p \in M$, $(\mathcal{E}' \oplus \mathcal{E}'')_p = \mathcal{E}'_p \oplus \mathcal{E}''_p$. Consider a cover $\{U_\alpha\}$ of M, and local trivializations for both $\mathcal{E}'$ and $\mathcal{E}''$. We get, $\forall \alpha$,

$$\chi'_\alpha : (\pi')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r'}$$
$$\chi''_\alpha : (\pi'')^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{r''}$$

Now define $\pi : \mathcal{E}' \oplus \mathcal{E}'' \to M$, and

$$\begin{aligned}
\chi_\alpha : \pi^{-1}(U_\alpha) &\to U_\alpha \times (\mathbb{R}^{r'} \oplus \mathbb{R}^{r''}) \\
(p, (v', v'')) &\mapsto (p, (\chi'_\alpha(v'), \chi''_\alpha(v''))) \\
v' &\in \mathcal{E}'_p \\
v'' &\in \mathcal{E}''_p
\end{aligned}$$

We have to check that the $\tau_{\alpha,\beta}$ are smooth. Note that $\tau_{\alpha,\beta}(p)$ will be a block diagonal matrix, and the two diagonal blocks vary smoothly, since χ_α and χ_β are diffeomorphisms. Thus, $\mathcal{E}' \oplus \mathcal{E}'' \to M$ is a natural bundle. This is called the Whitney direct sum or Whitney sum.

Observe:

- $\mathrm{rank}(\mathcal{E}' \oplus \mathcal{E}'') = \mathrm{rank}(\mathcal{E}') + \mathrm{rank}(\mathcal{E}'')$
- $\dim(\mathcal{E}' \oplus \mathcal{E}'') = r' + r'' + n$, where n is the dimension of the total space.

Similarly, we can define $\mathcal{E}' \otimes \mathcal{E}'' \to M$ with fibers $(\mathcal{E}' \otimes \mathcal{E}'')_p = \mathcal{E}'_p \otimes \mathcal{E}''_p$. $\mathrm{rank}(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r''$, and $\dim(\mathcal{E}' \otimes \mathcal{E}'') = r' \cdot r'' + n$.

Review of Tensor Products

Defn: Let V, W be finite-dimensional vector spaces. Their tensor product $V \otimes W$ is the free vector space over $V \times W$, modulo an equivalence relation, $\sim$. The free vector space over $V \times W$ is the set of all formal finite linear combinations of pairs $(v, w) \in V \times W$, and $\sim$ is defined such that

$$\begin{cases} (v_1 + v_2, w) \sim (v_1, w) + (v_2, w) \\ (\lambda v, w) \sim \lambda(v, w) \end{cases}$$

Notation: $\forall (v, w) \in V \times W$, $v \otimes w = [(v, w)]$.

Claim: If $(e_1, \ldots, e_k)$, $(f_1, \ldots, f_\ell)$ are bases of V, W, then $\{e_i \otimes f_j \mid i \in \{1, \ldots, k\}, j \in \{1, \ldots, \ell\}\}$ is a basis of $V \otimes W$.

Cor: $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

The universal property of $V \otimes W$: We have a bilinear map

$$\begin{aligned} V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

It's "universal" in that for any bilinear map, there's a unique linear map such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[rr, "(v,w)\mapsto v\otimes w"] \arrow[dr, "\text{bilinear map}"'] & & V \otimes W \arrow[dl, "\text{linear map}"] \\ & Z & \end{tikzcd}$$

There are other realizations of $V \otimes W$.

$$V \otimes W \cong \mathrm{Hom}(V^*, W) \qquad \text{where} \qquad v \otimes w \mapsto \begin{pmatrix} V^* & \to & W \\ \alpha & \mapsto & \alpha(v)w \end{pmatrix}$$

This is completely natural – we don't need to choose a basis.

Cor: $V^* \otimes V^* \cong \mathrm{Hom}((V^*)^*, V^*) = \mathrm{Hom}(V, V^*) \overset{\text{claim}}{\cong} \mathrm{Bil}(V \times V, \mathbb{R})$, by

$$\begin{aligned} \mathrm{Hom}(V, V^*) &\cong \mathrm{Bil}(V \times V, \mathbb{R}) \\ (f : V \to V^*) &\mapsto \begin{pmatrix} V \times V \to \mathbb{R} \\ (v_1, v_2) \mapsto (f(v_1))(v_2) \end{pmatrix} \end{aligned}$$

(Note that $f(v_1) \in V^*$.)

Thus, $V \otimes V \cong \mathrm{Bil}(V^* \times V^*, \mathbb{R})$. This is the realization we will use!

Note: When taking multiple tensor products:

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{\ell} \cong \{\text{multilinear maps } \underbrace{V^* \times \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{\ell} \to \mathbb{R}\}$$

Now, we put everything together, and define tensor bundles.

Defn: Given a smooth manifold M, $\forall p \in M$, we define

$$T^{(k,\ell)}(T_pM) = \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{\ell}$$

k is called the contravariant degree, and ℓ is called the covariant degree. Using the lemma from before, we get tensor bundles

$$\bigsqcup_{p \in M} T^{(k,\ell)}(T_pM) = T^{(k,\ell)}(TM) \to M$$

Exer: Compute the rank of this bundle.

In coordinate $(x^1, \dots, x^n)$ on $U \overset{\text{open}}{\subseteq} M$, we have a moving frame of $T^{(k,\ell)}(TM)$:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \,\middle|\, i_a, j_b \in \{1, \dots, n\} \right\}$$

Defn: Any smooth section of a tensor bundle is called a <u>tensor</u>.

Note that any tensor is a combination of these basis elements with C^∞ functions as coefficients:

$$\sum A^{i_1 \cdots i_k}_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell}$$

Defn: A <u>Riemannian metric</u> is a symmetric, positive-definite $(0,2)$ tensor $g_p : T_pM \times T_pM \to \mathbb{R}$.