Stats 426 Lecture 6

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Functions of Random Variables

Suppose $(Y_1, \ldots, Y_m) = g(X_1, \ldots, X_n)$. We want to find the joint pdf of $(Y_1, \ldots, Y_m)$, using the pdf of $(X_1, \ldots, X_n)$, denoted $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$. In the discrete case, we have

$$f_{Y_1,\ldots,Y_m}(y_1, \ldots, y_m) = \sum_{\substack{(x_1,\ldots,x_n): \\ g(x_1,\ldots,x_n)=(y_1,\ldots,y_m)}} f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$

In the continuous case, we can find the cdf, and then differentiate to get the pdf. Alternatively, if $m = n$, we can use the change of variables rule:

$$dy_1 \cdots dy_n = |\det J_g(x_1, \ldots, x_n)| \; dx_1 \cdots dx_n$$

where $J_g(x_1, \ldots, x_n)$ is the Jacobian, the matrix of partial derivatives

$$J_g(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial g_1}{\partial x_n}(x_1, \ldots, x_n) \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(x_1, \ldots, x_n) & \cdots & \frac{\partial g_n}{\partial x_n}(x_1, \ldots, x_n) \end{bmatrix}$$

Suppose X and Y are two i.i.d. random variables, and $Z = X + Y$. In the discrete case, with probability mass functions p, p_X, p_Y, then

$$p_Z(z) = P(X + Y = z) = \sum_{(x,y):x+y=z} p(x, y) = \sum_{x \in X} p(x, z - x) \overset{*}{=} \sum_{x \in X} p_X(x) p_Y(z - x)$$

*only if X and Y are independent. In the continuous case, with probability mass functions f, f_X, f_Y, we have

$$f_Z(z) = \int_X f(x, z - x)\, dx \overset{*}{=} \int_X f_X(x) f_Y(z - x)\, dx$$

*only if X and Y are independent.

Ex: Let $T_1, T_2 \sim \text{Exp}(\lambda)$ be independent, and $S = T_1 + T_2$. Then

$$f_S(s) = \int_{\mathbb{R}} f_{T_1}(x) f_{T_2}(s - x)\, dx = \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\, dx = \int_0^s \lambda^2 e^{-\lambda s}\, dx = s\lambda^2 e^{-\lambda s}$$

Thus, $S \sim \text{Gamma}(\lambda, z)$ for some constant z, that depends on λ.

Ex: Let $S^{[n]} = \sum_{i=1}^n T_i$, where the $T_i \sim \text{Exp}(\lambda)$ are i.i.d. We just showed that $S^{[2]} = S^{[1]} + T$, where $T \sim \text{Gamma}(\lambda, z)$, and $T \perp S^{[1]}$. Suppose for some k, $S^{[k]} = S^{[k-1]} + T \sim \text{Gamma}(\lambda, k)$. Then

$$\int_0^s \frac{\lambda^k}{\Gamma(k)} x^{k-1} e^{-\lambda k} \cdot \lambda e^{-\lambda(s-x)}\, dx = \int_0^s \frac{\lambda^{k+1}}{\Gamma(k)} x^{k-1} e^{-\lambda s}\, dx = \left[\frac{\lambda^{k+1}}{k\Gamma(k)} e^{-\lambda s} x^k\right]_{x=0}^{x=s} = \frac{\lambda^{k+1}}{\Gamma(k+1)} s^k e^{-\lambda s}$$

Thus, $S^{[k+1]} = S^{[k]} + T \sim \text{Gamma}(\lambda, k + 1)$.

Transformation

Consider $(Y_1, Y_2) = g(X_1, X_2) = (g_1(X_1, X_2), g_2(X_1, X_2))$, where g is differentiable and invertible. Also, suppose the Jacobian is nonzero, $\forall (x_1, x_2) \in \text{supp } f_{X_1,X_2}$, and that g maps supp f_{X_1,X_2} to supp f_{Y_1,Y_2}. Then the joint probability density function of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(h_1(y_1, y_2), h_2(y_1, y_2))J^{-1}(h_1(y_1, y_2), h_2(y_1, y_2))$$

for $(y_1, y_2) \in \text{supp } f_{Y_1,Y_2}$ and 0 otherwise, where $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$, and

$$J(x_1, x_2) = \det \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

Ex: The pdf of a ratio. Suppose X and Y are continuous random variables with joint pdf f, and maginal pdfs f_X and f_Y. Let $Z = Y/X$. We can find the marginal pdf of Z in two ways.

1: Change (X, Y) to (X, XZ). Then the Jacobian is

$$\begin{bmatrix} \frac{\partial X}{\partial X} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial X} & \frac{\partial Y}{\partial Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Z & X \end{bmatrix}$$

So

$$f_{X,Z}(x, z) = f_{X,Y}(x, xz)\,|x| \qquad f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, xz)\,|x|\;dx$$

2: Change $(X, Y/X)$ to (X, Z). Then the Jacobian is

$$\begin{bmatrix} \frac{\partial X}{\partial X} & \frac{\partial X}{\partial Y} \\ \frac{\partial Z}{\partial X} & \frac{\partial Z}{\partial Y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -YX^2 & 1/X \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -Z/X & 1/X \end{bmatrix}$$

So

$$f_{X,Z}(x, z) = f_{X,Y}(x, xz) \left|\frac{1}{x}\right|^{-1} = f_{X,Y}(x, xz)\,|x|$$

We get the same thing – both methods work!

In general, if X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} |x|\, f(x, xz)\, dx = \int_{-\infty}^{\infty} |x|\, f_X(x) f_Y(xz)\, dx$$

Ex: If $X, Y \sim \text{N}(0, 1)$ are i.i.d., show that the pdf of $Z = Y/X$ is $\frac{1}{\pi(1+z^2)}$. Well,

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2z^2/2}\, dx \\
&= \int_{-\infty}^{0} \frac{-x}{2\pi} e^{-(1+z^2)x^2/2}\, dx + \int_{0}^{\infty} \frac{x}{2\pi} e^{-(1+z^2)x^2/2}\, dx \qquad {\scriptstyle s=x^2 \atop ds=2x\,dx} \\
&= \int_{0}^{\infty} \frac{1}{4\pi} e^{-(1+z^2)z/2}\, dx + \int_{0}^{\infty} \frac{1}{4\pi} e^{-(1+z^2)s/2}\, ds \\
&= \frac{1}{2\pi} \int_{0}^{\infty} e^{-(1+z^2)s/2}\, ds \\
&= \frac{1}{2\pi} \left[\frac{-2}{1+z^2} e^{-(1+z^2)s/2} \right]_{s=0}^{s=\infty} \\
&= \frac{1}{\pi(1+z^2)}
\end{aligned}$$

□

Ex: Suppose X and Y are i.i.d. standard normal random variables. Consider the polar coordinate system $X = R\cos\Theta$, $Y = R\sin\Theta$, for $R \geq 0$ and $-\pi < \Theta < \pi$. Find the joint probability density function of (R, Θ). Well, we change (X, Y) to $(R\cos\Theta, R\sin\Theta)$. So the Jacobian is

$$\begin{bmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \Theta} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \Theta} \end{bmatrix} = \begin{bmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{bmatrix}$$

So $\det J = R\cos^2\Theta - (-R\sin^2\Theta) = R\cos^2\Theta + R\sin^2\Theta = R$. Therefore,

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}\,|r|\,e^{-r^2/2} = \frac{r}{2\pi}e^{-r^2/2} \qquad \text{for } r \geq 0, -\pi < \theta < \pi$$

Exer: Use the other method – change $(\sqrt{X^2+Y^2}, \arctan(Y/X))$ to (R, Θ). Check that you get the same result.

Ex: Show that R, Θ are independent. Well,

$$f_R(r) = \int_{-\pi}^{\pi} \frac{r}{2\pi}e^{-r^2/2}\,d\theta = re^{-r^2/2} \qquad \text{for } r \geq 0$$

$$f_\Theta\theta = \int_0^\infty \frac{r}{2\pi}e^{-r^2/2}\,dr = \cdots = \left[\frac{-1}{2\pi}e^{-r^2/2}\right]_{r=0}^{r=\infty} = \frac{1}{2\pi}$$

So

$$f_R(r)f_\Theta(\theta) = \frac{r}{2\pi}e^{-r^2/2} = f_{R,\Theta}(r,\theta) \qquad \Rightarrow \qquad R \perp \Theta$$

□

Expected Value

Defn: Suppose X is a discrete random variable, with pmf p. Then the expected value of X is given by $E(X) = \sum_{x\in X} xp(x)$ if $\sum_{x\in X} |x|\,p(x) < \infty$. If this sum is infinite, then the expected value is undefined.

Defn: Suppose X is a continuous random variable, with pdf f. Then the expected value of X is given by $E(X) = \int_X xp(x)\,dx$ if $\int_X |x|\,f(x)\,dx < \infty$. If this integral is infinite, then the expected value is undefined.

For constants a, b, $E(aX+b) = aE(X) + b$.

Exer: Derive the following:

- $X \sim \text{Ber}(p) \rightsquigarrow E(X) = p$
- $X \sim \text{Bin}(n,p) \rightsquigarrow E(X) = np$
- $X \sim \text{Geom}(p) \rightsquigarrow E(X) = 1/p$
- $X \sim \text{Poisson}(\lambda) \rightsquigarrow E(X) = \lambda$
- $X \sim \text{Unif}(a,b) \rightsquigarrow E(X) = (a+b)/2$
- $X \sim \text{Exp}(\lambda) \rightsquigarrow E(X) = 1/\lambda$
- $X \sim \text{Gamma}(\lambda,\alpha) \rightsquigarrow E(X) = \alpha/\lambda$
- $X \sim \text{Beta}(a,b) \rightsquigarrow E(X) = a/(a+b)$
- $X \sim \text{N}(\mu,\sigma^2) \rightsquigarrow E(X) = \mu$