Stats 426 Lecture 9

Thomas Cohn

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Ex: Suppose $X \sim \mathrm{N}(0,1)$. The mgf of X is

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(x-t)^2/2} e^{t^2/2}\,dx = e^{t^2/2}$$

The mgf of $Y = \sigma X + \mu$ is $M_Y(t) = e^{(\sigma t)^2/2 + \mu t}$.
$M_Y^{(1)}(0) = \mu$; $M_Y^{(2)}(0) = \mu^2 + \sigma^2$.

Ex: If $X_1 \perp X_2$, $X_1 \sim \mathrm{N}(\mu_1, \sigma)$, $X_2 \sim \mathrm{N}(\mu_2, \sigma)$, then

$$M_{X_1+X_2}(t) \overset{X_1 \perp X_2}{=} M_{X_1}(t) M_{X_2}(t) = e^{\mu_1 t} e^{(\sigma t)^2/2} e^{\mu_2 t} e^{(\sigma t)^2/2} = e^{(\mu_1+\mu_2)t} e^{(\sqrt{2}\sigma t)^2/2}$$

So $X_1 + X_2 \sim \mathrm{N}(\mu_1 + \mu_2, 2\sigma^2)$.

Limit Theorems

We are interested in learning the limiting behavior of the sum (or the average) of independent random variables when the number of summands becomes large.

Consider $X_1, \ldots, X_n$ i.i.d. with mean μ and standard deviation σ, with $\bar{X}_n = n^{-1} \sum_i X_i$. Recall:

- $E(\bar{X}_n) = \mu$, $\mathrm{Var}(\bar{X}_n) = \sigma^2/n$
- $\mathrm{Var} \to 0$ as $n \to \infty$
- The pdf of $\bar{X}_n$ has mass concentrated around μ

We will discuss the weak law of large numbers (WLLN) and the central limit theorem (CLT).

Weak Law of Large Numbers

Thm: (WLLN) Consider $X_1, \ldots, X_n$ i.i.d., with mean μ and variance σ^2. Let $\bar{X}_n = n^{-1} \sum_i X_i$.
Then $\forall \varepsilon > 0$, $P(|\bar{X}_n - \mu| > \varepsilon) \to 0$ as $n \to \infty$.

Proof: This is a direct result of the Chebyshev inequality.

$$P(|\bar{X}_n - \mu| > \varepsilon) \le \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \to 0 \text{ as } n \to \infty$$

□

We also denote this by $\bar{X}_n \xrightarrow{P} \mu$.

Ex: Suppose $Z_1, \ldots, Z_n \sim \mathrm{Ber}(p)$ are i.i.d., with $p \in (0,1)$. Then $Y_n = \sum_i Z_i \sim \mathrm{Bin}(n,p)$.
$EZ_i = p$. $\frac{Y_n}{n} \xrightarrow{P} p$.

Ex: Suppose $Z_1, \ldots, Z_n \sim \mathrm{Exp}(\lambda)$ are i.i.d. Then $Y_n = \sum_i Z_i \sim \mathrm{Gamma}(\lambda, n)$.
$EZ_i = \frac{1}{\lambda}$. $\frac{Y_n}{n} \xrightarrow{P} \frac{1}{\lambda}$.

Convergence in Distribution

Defn: Let $Y_1, \ldots, Y_n$ be a sequence of random variables, with cdfs F_i. Let Y be a random variable with cdf F. We say that Y_n <u>converges in distribution</u> to Y if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

at every point x at which F is continuous. We denote this by $Y_n \xrightarrow{D} Y$.

Thm: (Continuity Theorem) Let F be a cdf with mgf M. If $M_n(t) \to M(t)$ for all t in an open interval containing 0, then $F_n(x) \to F(x)$ at all points x for which F is continuous.

Ex: Suppose $Z_1, \ldots, Z_n \sim \text{Exp}(\lambda)$ are i.i.d., then $X = \sum_i Z_i \sim \Gamma(\lambda, n)$. $M_X(t)$ is $(1-\lambda^{-1}t)^{-n}$. Let $Y = \frac{\lambda}{\sqrt{n}}X - \sqrt{n}$. What does the mgf of Y converge to as $n \to \infty$? Well,

$$M_Y(t) = e^{-\sqrt{n}t}(1 - (\lambda^{-1}\lambda/\sqrt{n})t)^{-n} = e^{-\sqrt{n}t}(1 - t/\sqrt{n})^{-n}$$

And

$$\begin{aligned}
\ln(M_Y(t)) &= -\sqrt{n}t - n\log(1 - \tfrac{t}{\sqrt{n}}) \\
&= -\sqrt{n}t - n(-t/\sqrt{n} - (1/2)t^2/n - O(n^{-3/2})) \\
&= \tfrac{1}{2}t^2 - O(n^{-1/2}) \\
&\to (1/2)t^2 \text{ as } n \to \infty
\end{aligned}$$

Thus, $\lim_{n\to\infty} M_Y(t) = e^{t^2/2}$.

Ex: Suppose $Z_1, \ldots, Z_n \sim \text{Exp}(\lambda)$ are i.i.d., then $X = \sum_i Z_i \sim \Gamma(\lambda, n)$. We want to show $M_X(t) = (1 - \lambda^{-1}t)^{-n}$. Well,

$$M_Z(t) = E(e^{tZ}) = \int_0^\infty e^{tz}\lambda e^{-\lambda z}\,dz = \lambda\int_0^\infty \frac{\lambda - t}{\lambda - t}e^{-(\lambda - t)z}\,dz = \frac{\lambda}{\lambda - t} = (1 - \tfrac{t}{\lambda})^{-n}$$

Because $X = \sum_i Z_i$,

$$M_X(t) = ((1 - \tfrac{t}{\lambda})^{-1})^{-n} = (1 - t\lambda^{-1})^{-n}$$

Also,

$$E(X) = M_X'(t)\big|_{t=0} = -n(1 - \tfrac{t}{\lambda})^{-n-1}(-\tfrac{1}{\lambda})\big|_{t=0} = \tfrac{n}{\lambda}$$