How to prove trajectory leaves a region with probability one

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1 2D Case

1.1 Ian's Condition

Given a bounded space $\mathcal{S} \in R^2$, if for any two points $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathcal{S}$, there exits positive constant k_v, k, such that

$$\frac{|y_2 - y_1|}{|x2 - x1|} \leq k_v \quad \Rightarrow \quad \frac{\dot{x}_2 - \dot{x}_1}{x_2 - x_1} \geq k \tag{1a}$$

$$\frac{|y_2 - y_1|}{|x2 - x1|} = k_v \quad \Rightarrow \quad \frac{\dot{y}_2 - \dot{y}_1}{y_2 - y_1} \leq -k \tag{1b}$$

then any trajectory $\phi(t) = (x(t), y(t))$ leaves $\mathcal{S}$ with probability one.

The $\frac{|y_2 - y_1|}{|x2 - x1|} \leq k_v$ defines a cone in the x–y space. Note the vertex of cone will move as $\phi(t)$ moves, as shown in Figure 1.

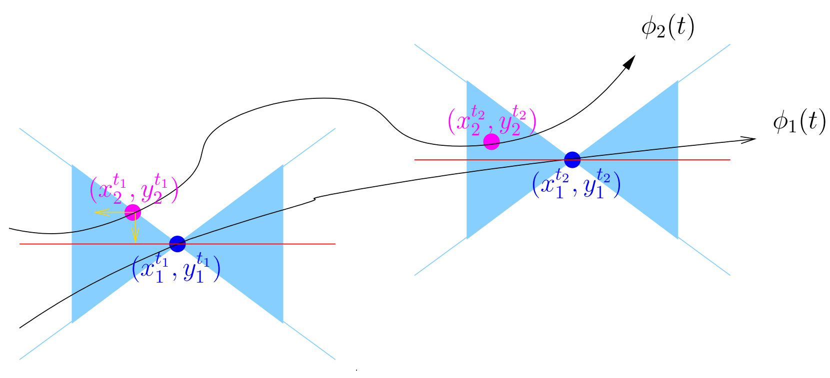

Figure 1: Double Cone

The first condition ensures that all trajectories in the cone diverge from the vertex in the x direction. That is because $x_2 > x_1 \Rightarrow \dot{x_2} > \dot{x_1}$ and $x_2 < x_1 \Rightarrow$

$\dot{x_2} < \dot{x_1}$. The relative flow direction to the vertex is shown as the yellow arrow in Figure 1.

Similarly, the second condition ensures that all trajectories that enter the cone stay in the cone. That is because $y_2 > y_1 \Rightarrow \dot{y_2} < \dot{y_1}$ and $y_2 < y_1 \Rightarrow \dot{y_2} > \dot{y_1}$ for all points on the boundary of the cone.

Then for any two trajectories $\phi_1(t), \phi_2(t)$ that $\phi_1(0) = (x_1, y_0), \phi_2(0) = (x_2, y_0)$, then ϕ_1, ϕ_2 must diverge on x direction by condition 1. Therefore, only one of ϕ_1, ϕ_2 can be trapped in region $\mathcal{S}$. Obviously, at most one value of x for any value of y can lie on a trapped trajectory. Therefore, the mainfold defined by the trapped trajectory in R^d is of maximum dimension $d-1$.

Read DCC96 and Ian's thesis for details.

1.2 Mark's Condition

It is tedious to find the double-cone manually. Thus Mark proposed the new algorithm:

Let $J(x,y)$ be the Jacobian matrix at point (x,y)

$$\begin{aligned} J(x,y) &= \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{bmatrix} \end{aligned}$$

and define

$$\mu = \min(\min_{\mathcal{R}} a(x,y), \min_{\mathcal{R}} -d(x,y)) \quad (2a)$$

$$h_1 = \min_{\mathcal{R}} \left| \frac{a(x,y)}{b(x,y)} \right| \quad (2b)$$

$$h_2 = \max_{\mathcal{R}} \left| \frac{c(x,y)}{d(x,y)} \right| \quad (2c)$$

Then a new sufficient condition is

$$\mu > 0 \quad (3a)$$

$$h_1 > h_2 \quad (3b)$$

Equation 3a implies $a > 0$, and $d < 0$.

Given condition 3a 3b, condition 1a 1b is satisfied with

$$k_v = \sqrt{h_1 \cdot h_2} \quad (4a)$$

$$k = \mu \cdot (1 - \sqrt{\frac{h_2}{h_1}}) \quad (4b)$$

By definition, let's integrate $\dot{x}$ on the line segment from $p_1 = (x_1, y_1)$ to $p_2 =$

(x_2, y_2)[1]

$$
\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{p_1}^{p_2} \frac{\partial \dot{x}(x,y)}{\partial p} dp \\
&= \int_{p_1}^{p_2} \left(\frac{\partial \dot{x}}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial \dot{x}}{\partial y}\frac{\partial y}{\partial p} \right) dp \\
&= \int_{p_1}^{p_2} \left(a(x,y)\frac{\partial x}{\partial p} + b(x,y)\frac{\partial y}{\partial p} \right) dp \\
&\geq \int_{p_1}^{p_2} \left(a(x,y)\frac{\partial x}{\partial p} - \frac{a(x,y)}{h_1}\frac{\partial y}{\partial p} \right) dp \quad (2b : |b| < \frac{a}{h_1}) \\
&= \int_{p_1}^{p_2} \left(a(x,y) - \frac{a(x,y)}{h_1}\frac{\partial y}{\partial p}\frac{\partial p}{\partial x} \right) \frac{\partial x}{\partial p} dp \\
&= \int_{x_1}^{x_2} \left(a(x,y) - \frac{a(x,y)}{h_1}\frac{\partial y}{\partial x} \right) dx \\
&\geq \int_{x_1}^{x_2} \left(a(x,y) - \frac{a(x,y)}{h_1} k_v \right) dx \quad (1a, \frac{dy}{dx} \leq k_v) \\
&= \int_{x_1}^{x_2} a(x,y) \left(1 - \frac{k_v}{h_1} \right) dx \\
&\geq \int_{x_1}^{x_2} \mu \left(1 - \frac{k_v}{h_1} \right) dx \quad (2a, 1 - \frac{k_v}{h_1} > 0) \\
&= \mu(1 - \frac{k_v}{h_1})(x_2 - x_1) \\
&= k(x_2 - x_1) \\
\frac{\dot{x}_2 - \dot{x}_1}{x_2 - x_1} &\geq k
\end{aligned}
$$

[1] assume $x_2 > x_1$ without lost of generality.

Similarly, by integrating $\dot{y}$, we have[2]

$$\begin{aligned}
\dot{y}_2 - \dot{y}_1 &= \int_{p_1}^{p_2} \left(c(x,y)\frac{\partial x}{\partial p} + d(x,y)\frac{\partial y}{\partial p} \right) dp \\
&= \int_{y_1}^{y_2} \left(d(x,y) + c(x,y)\frac{\partial x}{\partial y} \right) dy \\
&\leq \int_{y_1}^{y_2} \left(d(x,y) - d(x,y)h_2\frac{\partial x}{\partial y} \right) dy \quad (2c : |c| < -d \cdot h_2) \\
&\leq \int_{y_1}^{y_2} d(x,y)\left(1 - \frac{h_2}{k_v}\right) dy \quad (1b, \frac{dx}{dy} = \pm k_v, d < 0) \\
&\leq \int_{y_1}^{y_2} -\mu \left(1 - \frac{h_2}{k_v}\right) dy \quad (2a, 1 - \frac{h_2}{k_v} > 0) \\
&= -k(y_2 - y_1) \\
\frac{\dot{y}_2 - \dot{y}_1}{y_2 - y_1} &\leq -k
\end{aligned}$$

1.3 My condition

The sufficient condition without computing k_v, k is:

$$\min_{\mathcal{R}} a > 0 \tag{5a}$$

$$\max_{\mathcal{R}} d < 0 \tag{5b}$$

$$\min_{\mathcal{R}} |a| \cdot \min_{\mathcal{R}} |d| > \max_{\mathcal{R}} |b| \cdot \max_{\mathcal{R}} |c| \tag{5c}$$

Obviously, condition 5a 5b implies condition 3a. Condition 5c implies condition 3b proved as:

$$\begin{aligned}
& \min_{\mathcal{R}} |a| \cdot \min_{\mathcal{R}} |d| > \max_{\mathcal{R}} |b| \cdot \max_{\mathcal{R}} |c| \\
\Leftrightarrow \quad & \frac{\min |a|}{\max |b|} > \frac{\max |c|}{\min |d|} \\
\Rightarrow \quad & \min \left| \frac{a}{b} \right| \geq \frac{\min |a|}{\max |b|} > \frac{\max |c|}{\min |d|} \geq \max \left| \frac{c}{d} \right| \\
\Rightarrow \quad & h_1 > h_2
\end{aligned}$$

1.4 How to find this condition

Let

$$J = \begin{pmatrix} j_{xx} & j_{xy} \\ j_{yx} & j_{yy} \end{pmatrix}$$

[2] assume $y_2 > y_1$ without lost of generality.

and

$$
\begin{aligned}
h_{xy} &= \frac{\max |j_{xy}|}{\min |j_{xx}|} \\
h_{yx} &= \frac{\max |j_{yx}|}{\min |j_{yy}|}
\end{aligned}
$$

Then compute the integral

$$
\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{p_1}^{p_2} \frac{\partial \dot{x}(x,y)}{\partial p} dp \\
&= \int_{p_1}^{p_2} \left(\frac{\partial \dot{x}}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial \dot{x}}{\partial y}\frac{\partial y}{\partial p} \right) dp \\
&= \int_{p_1}^{p_2} j_{xx} dx + j_{xy} dy \\
&= \int_{x_1}^{x_2} j_{xx}(1 + \frac{j_{xy}}{j_{xx}}\frac{dy}{dx}) dx \\
&\geq \int_{x_1}^{x_2} j_{xx}(1 - h_{xy}k_v) dx \quad (|\frac{j_{xy}}{j_{xx}}| \leq h_{xy}, |\frac{dy}{dx}| \leq k_v,\ \textit{requires } j_{xx} \geq 0) \\
&\geq \int_{x_1}^{x_2} k\, dx \quad (k \leq \min |j_{xx}|(1 - h_{xy}k_v)\ \textit{requires } k_v \leq \frac{1}{h_{xy}}) \\
&= k(x_2 - x_1)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{y}_2 - \dot{y}_1 &= \int_{p_1}^{p_2} j_{yx} dx + j_{yy} dy \\
&= \int_{y_1}^{y_2} j_{yy}(1 + \frac{j_{yx}}{j_{yy}}\frac{dx}{dy}) dy \\
&\leq \int_{y_1}^{y_2} j_{yy}(1 - \frac{h_{yx}}{k_v}) dy \quad (|\frac{j_{yx}}{j_{yy}}| \leq h_{yx}, |\frac{dx}{dy}| = \pm\frac{1}{k_v},\ \textit{requires } j_{yy} \leq 0) \\
&\leq \int_{y_1}^{y_2} -k\, dy \quad (k \geq \min |j_{yy}|(1 - \frac{h_{yx}}{k_v})\ \textit{requires } k_v \geq h_{yx}) \\
&= -k(y_2 - y_1)
\end{aligned}
$$

Therefore, the sufficient condition is as shown in equation 5.

If it diverges on both x, y directions, then what is the condition? Now Ian's condition changes to

$$
\frac{|\Delta y|}{|\Delta x|} \leq 1 \quad \Rightarrow \quad \frac{\Delta \dot{x}}{\Delta x} > 0 \tag{6a}
$$

$$
\frac{|\Delta x|}{|\Delta y|} \leq 1 \quad \Rightarrow \quad \frac{\Delta \dot{y}}{\Delta y} > 0 \tag{6b}
$$

This split the space into two cones, it does not require that the trajectory remains in the same cone. But the distance $\max(\Delta x, \Delta y)$ increases no matter which cone this trajectory is in. With is new condition

$$
\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{p_1}^{p_2} j_{xx} dx + j_{xy} dy \\
&= \int_{x_1}^{x_2} j_{xx}(1 + \frac{j_{xy}}{j_{xx}}\frac{dy}{dx}) dx \\
&\geq \int_{x_1}^{x_2} j_{xx}(1 - h_{xy}) dx \quad (|\frac{j_{xy}}{j_{xx}}| \leq h_{xy}, |\frac{dy}{dx}| \leq 1,\ \textit{requires}\ j_{xx} \geq 0) \\
&\geq \int_{x_1}^{x_2} k\, dx \quad (k \leq \min |j_{xx}|(1 - h_{xy})\ \textit{requires}\ h_{xy} \leq 1) \\
&= k(x_2 - x_1) \\
\dot{y}_2 - \dot{y}_1 &= \int_{p_1}^{p_2} j_{yx} dx + j_{yy} dy \\
&= \int_{y_1}^{y_2} j_{yy}(1 + \frac{j_{yx}}{j_{yy}}\frac{dx}{dy}) dy \\
&\geq \int_{y_1}^{y_2} j_{yy}(1 - h_{yx}) dy \quad (|\frac{j_{yx}}{j_{yy}}| \leq h_{yx}, |\frac{dx}{dy}| \leq 1,\ \textit{requires}\ j_{yy} \geq 0) \\
&\geq \int_{y_1}^{y_2} -k\, dy \quad (k \geq \min |j_{yy}|(1 - h_{yx})\ \textit{requires}\ h_{yx} \leq 1) \\
&= -k(y_2 - y_1)
\end{aligned}
$$

Therefore, the sufficient condition for diverging on both directions is

$$
\min_{\mathcal{R}} j_{xx} > \max_{\mathcal{R}} |j_{xy}| \tag{7a}
$$

$$
\min_{\mathcal{R}} j_{yy} > \max_{\mathcal{R}} |j_{yx}| \tag{7b}
$$

2 3D case

Let

$$
J = \begin{pmatrix} j_{xx} & j_{xy} & j_{xz} \\ j_{yx} & j_{yy} & j_{yz} \\ j_{zx} & j_{zy} & j_{zz} \end{pmatrix}
$$

and

$$
h_{uv} = \frac{\max |j_{uv}|}{min |j_{uu}|}
$$

2.1 Diverge on x

The sufficient condition to show that trajectories diverge only on x direction is

$$\frac{|\Delta y|}{|\Delta x|} \leq k_v, \frac{|\Delta z|}{|\Delta x|} \leq k_v \quad \Rightarrow \quad \frac{\Delta \dot{x}}{\Delta x} > 0 \tag{8a}$$

$$\frac{|\Delta y|}{|\Delta x|} = k_v, \frac{|\Delta z|}{|\Delta x|} \leq k_v \quad \Rightarrow \quad \frac{\Delta \dot{y}}{\Delta y} < 0 \tag{8b}$$

$$\frac{|\Delta y|}{|\Delta x|} \leq k_v, \frac{|\Delta z|}{|\Delta x|} = k_v \quad \Rightarrow \quad \frac{\Delta \dot{z}}{\Delta z} < 0 \tag{8c}$$

The first condition 8a ensures that all trajectories in the cone diverge in the x direction. Conditions 8b, 8c ensure that all trajectories remain in the cone.

Let us compute the integral to find the sufficient condition. In the cone, we have

$$\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{p_1}^{p_2} j_{xx} dx + j_{xy} dy + j_{xz} dz \\
&= \int_{x_1}^{x_2} j_{xx} (1 + \frac{j_{xy}}{j_{xx}} \frac{dy}{dx} + \frac{j_{xz}}{j_{xx}} \frac{dz}{dx}) dx \\
&\geq \int_{x_1}^{x_2} j_{xx} (1 - h_{xy} k_v - h_{xz} k_v) dx \quad (|\frac{j_{xy/z}}{j_{xx}}| \leq h_{xy/z}, |\frac{dy/z}{dx}| \leq k_v, \Leftarrow j_{xx} > 0) \\
&\geq \int_{x_1}^{x_2} k\, dx \quad (k \leq \min |j_{xx}| (1 - (h_{xy} + h_{xz}) k_v), \Leftarrow k_v \leq \frac{1}{h_{xy} + h_{xz}}) \\
&= k(x_2 - x_1)
\end{aligned}$$

On the face $\frac{dy}{dx} = k_v$, we have

$$\begin{aligned}
\dot{y}_2 - \dot{y}_1 &= \int_{p_1}^{p_2} j_{yx} dx + j_{yy} dy + j_{yz} dz \\
&= \int_{y_1}^{y_2} (j_{yx} \frac{dx}{dy} + j_{yy} + j_{yz} \frac{dz}{dy}) dy \\
&= \int_{y_1}^{y_2} (j_{yx} \frac{1}{k_v} + j_{yy} + j_{yz} \frac{dz}{k_v dx}) dy \quad (\frac{dy}{dx} = k_v) \\
&= \int_{y_1}^{y_2} j_{yy} (1 + \frac{j_{yx}}{j_{yy}} \frac{1}{k_v} + \frac{j_{yz}}{j_{yy}} \frac{1}{k_v} \frac{dz}{dx}) dy \\
&\leq \int_{y_1}^{y_2} j_{yy} (1 - \frac{h_{yx}}{k_v} - h_{yz}) dx \quad (|\frac{j_{yx/z}}{j_{yy}}| \leq h_{xy/z}, |\frac{dz}{dx}| \leq k_v, \Leftarrow j_{yy} < 0) \\
&\leq \int_{y_1}^{y_2} -k\, dy \quad (k \leq \min |j_{yy}| (1 - h_{yz} - \frac{h_{yx}}{k_v}), \Leftarrow k_v > \frac{h_{yx}}{1 - h_{yz}}) \\
&= k(y_2 - y_1)
\end{aligned}$$

Similarly, on the face $\frac{dy}{dx} = -k_v$, we have

$$
\begin{aligned}
\dot{y}_2 - \dot{y}_1 &= \int_{y_1}^{y_2} (-j_{yx}\frac{1}{k_v} + j_{yy} - j_{yz}\frac{dz}{k_v dx})dy \quad (\frac{dy}{dx} = -k_v) \\
&= \int_{y_1}^{y_2} j_{yy}(1 - \frac{j_{yx}}{j_{yy}}\frac{1}{k_v} - \frac{j_{yz}}{j_{yy}}\frac{1}{k_v}\frac{dz}{dx})dy \\
&\leq \int_{y_1}^{y_2} j_{yy}(1 - \frac{h_{yx}}{k_v} - h_{yz})dy \quad (|\frac{j_{yx/z}}{j_{yy}}| \leq h_{xy/z}, |\frac{dz}{dx}| \leq k_v, \Leftarrow j_{yy} < 0) \\
&\leq \int_{y_1}^{y_2} -k\, dy \quad (k \leq \min |j_{yy}|(1 - h_{yz} - \frac{h_{yx}}{k_v}), \Leftarrow k_v > \frac{h_{yx}}{1 - h_{yz}})
\end{aligned}
$$

For z variables, it is similar.

$$
\begin{aligned}
\dot{z}_2 - \dot{z}_1 &= \int_{p_1}^{p_2} j_{zx}dx + j_{zy}dy + j_{zz}dz \\
&= \int_{z_1}^{z_2} (j_{zx}\frac{dx}{dz} + j_{zy}\frac{dy}{dz} + j_{zz})dz \\
&= \int_{z_1}^{z_2} (\pm j_{zx}\frac{1}{k_v} \pm j_{zy}\frac{dy}{k_v dx} + j_{zz})dz \quad (\frac{dz}{dx} = \pm k_v) \\
&= \int_{z_1}^{z_2} j_{zz}(1 \pm \frac{j_{zx}}{j_{zz}}\frac{1}{k_v} \pm \frac{j_{zy}}{j_{zz}}\frac{1}{k_v}\frac{dy}{dx})dy \\
&\leq \int_{z_1}^{z_2} j_{zz}(1 - \frac{h_{zx}}{k_v} - h_{zy})dz \quad (|\frac{j_{zx/y}}{j_{zz}}| \leq h_{zx/y}, |\frac{dy}{dx}| \leq k_v, \Leftarrow j_{zz} < 0) \\
&\leq \int_{z_1}^{z_2} -k\, dz \quad (k \leq \min |j_{zz}|(1 - h_{zy} - \frac{h_{zx}}{k_v}), \Leftarrow k_v > \frac{h_{zx}}{1 - h_{zy}}) \\
&= k(z_2 - z_1)
\end{aligned}
$$

The sufficient condition is

$$
\begin{aligned}
j_{xx} > 0 \qquad k_v &< \frac{1}{h_{xy} + h_{xz}} && \text{(9a)} \\
j_{yy} < 0 \qquad k_v &> \frac{h_{yx}}{1 - h_{yz}} && \text{(9b)} \\
j_{zz} < 0 \qquad k_v &> \frac{h_{zx}}{1 - h_{zy}} && \text{(9c)}
\end{aligned}
$$

which is the same with

$$
\begin{aligned}
\min_{\mathcal{R}} j_{xx} &> 0 && \text{(10a)} \\
\max_{\mathcal{R}} j_{yy} &< 0 && \text{(10b)} \\
\max_{\mathcal{R}} j_{zz} &< 0 && \text{(10c)} \\
\min_{\mathcal{R}} |j_{xx}|(\min_{\mathcal{R}} |j_{yy}| - \max_{\mathcal{R}} |j_{yz}|) &> \max_{\mathcal{R}} |j_{yx}|(\max_{\mathcal{R}} |j_{xy}| + \max_{\mathcal{R}} |j_{xz}|) && \text{(10d)} \\
\min_{\mathcal{R}} |j_{xx}|(\min_{\mathcal{R}} |j_{zz}| - \max_{\mathcal{R}} |j_{zy}|) &> \max_{\mathcal{R}} |j_{zx}|(\max_{\mathcal{R}} |j_{xy}| + \max_{\mathcal{R}} |j_{xz}|) && \text{(10e)}
\end{aligned}
$$

2.2 Diverge on x, y

By using the similar idea, we split the space into two cones.

$$\frac{|\Delta y|}{|\Delta x|} \le 1, \frac{|\Delta z|}{|\Delta x|} \le k_v \quad \Rightarrow \quad \frac{\Delta \dot{x}}{\Delta x} > 0 \tag{11a}$$
$$\frac{|\Delta y|}{|\Delta x|} \le 1, \frac{|\Delta z|}{|\Delta x|} = k_v \quad \Rightarrow \quad \frac{\Delta \dot{z}}{\Delta z} < 0 \tag{11b}$$
$$\frac{|\Delta x|}{|\Delta y|} \le 1, \frac{|\Delta z|}{|\Delta y|} \le k_v \quad \Rightarrow \quad \frac{\Delta \dot{y}}{\Delta y} > 0 \tag{11c}$$
$$\frac{|\Delta x|}{|\Delta y|} \le 1, \frac{|\Delta z|}{|\Delta y|} = k_v \quad \Rightarrow \quad \frac{\Delta \dot{z}}{\Delta z} > 0 \tag{11d}$$

Conditions 11a, 11c ensure distance $\max(|\Delta x|, |\Delta y|)$ increases. Conditions 11b, 11d ensure trajectories remains in these two cones

Let us compute the integral. In the first cone, we have

$$\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{x_1}^{x_2} j_{xx}\left(1 + \frac{j_{xy}}{j_{xx}}\frac{dy}{dx} + \frac{j_{xz}}{j_{xx}}\frac{dz}{dx}\right)dx \\
&\ge \int_{x_1}^{x_2} j_{xx}(1 - h_{xy} - h_{xz}k_v)dx \quad \left(\left|\frac{j_{xy/z}}{j_{xx}}\right| \le h_{xy/z}, \left|\frac{dz}{dx}\right| \le k_v, \left|\frac{dy}{dx}\right| \le 1 \Leftarrow j_{xx} > 0\right) \\
&\ge \int_{x_1}^{x_2} k\, dx \quad \left(k \le \min|j_{xx}|(1 - h_{xy} - h_{xz}k_v), \Leftarrow k_v \le \frac{1 - h_{xy}}{h_{xz}}\right) \\
\dot{z}_2 - \dot{z}_1 &= \int_{z_1}^{z_2} \left(\pm j_{zx}\frac{1}{k_v} \pm j_{zy}\frac{dy}{k_v dx} + j_{zz}\right)dz \quad \left(\frac{dz}{dx} = \pm k_v\right) \\
&= \int_{z_1}^{z_2} j_{zz}\left(1 \pm \frac{j_{zx}}{j_{zz}}\frac{1}{k_v} \pm \frac{j_{zy}}{j_{zz}}\frac{1}{k_v}\frac{dy}{dx}\right)dz \\
&\le \int_{z_1}^{z_2} j_{zz}\left(1 - \frac{h_{zx} + h_{zy}}{k_v}\right)dz \quad \left(\left|\frac{j_{zx/y}}{j_{zz}}\right| \le h_{zx/y}, \left|\frac{dy}{dx}\right| \le 1, \Leftarrow j_{zz} < 0\right) \\
&\le \int_{z_1}^{z_2} -k\, dz \quad \left(k \le \min|j_{zz}|\left(1 - \frac{h_{zx} + h_{zy}}{k_v}\right), \Leftarrow k_v > h_{zx} + h_{zy}\right)
\end{aligned}$$

In the secon cone, we have

$$
\begin{aligned}
\dot{y}_2 - \dot{y}_1 &= \int_{y_1}^{y_2} j_{yy}(1 + \frac{j_{yx}}{j_{yy}}\frac{dx}{dy} + \frac{j_{yz}}{j_{yy}}\frac{dz}{dy})dy \\
&\geq \int_{y_1}^{y_2} j_{yy}(1 - h_{yx} - h_{yz}k_v)dy \quad (|\frac{j_{xy/z}}{j_{xx}}| \leq h_{xy/z}, |\frac{dz}{dy}| \leq k_v, |\frac{dx}{dy}| \leq 1 \Leftarrow j_{yy} > 0) \\
&\geq \int_{x_1}^{x_2} k\,dx \quad (k \leq \min|j_{yy}|(1 - h_{yx} - h_{yz}k_v), \Leftarrow k_v \leq \frac{1-h_{yx}}{h_{yz}}) \\
\dot{z}_2 - \dot{z}_1 &= \int_{z_1}^{z_2} (\pm j_{zx}\frac{dx}{k_v dy} \pm j_{zy}\frac{1}{k_v} + j_{zz})dz \quad (\frac{dz}{dy} = \pm k_v) \\
&= \int_{z_1}^{z_2} j_{zz}(1 \pm \frac{j_{zx}}{j_{zz}}\frac{1}{k_v}\frac{dx}{dy} \pm \frac{j_{zy}}{j_{zz}}\frac{1}{k_v})dz \\
&\leq \int_{z_1}^{z_2} j_{zz}(1 - \frac{h_{zx}+h_{zy}}{k_v})dz \quad (|\frac{j_{zx/y}}{j_{zz}}| \leq h_{zx/y}, |\frac{dx}{dy}| \leq 1, \Leftarrow j_{zz} < 0) \\
&\leq \int_{z_1}^{z_2} -k\,dz \quad (k \leq \min|j_{zz}|(1 - \frac{h_{zx}+h_{zy}}{k_v}), \Leftarrow k_v > h_{zx} + h_{zy})
\end{aligned}
$$

The sufficient condition is

$$
\begin{aligned}
j_{xx} > 0 \qquad & k_v \leq \frac{1-h_{xy}}{h_{xz}} && \text{(12a)} \\
j_{yy} > 0 \qquad & k_v \leq \frac{1-h_{yx}}{h_{yz}} && \text{(12b)} \\
j_{zz} < 0 \qquad & k_v \geq h_{zx} + h_{zy} && \text{(12c)}
\end{aligned}
$$

which is the same with

$$
\begin{aligned}
\min_{\mathcal{R}} j_{xx} &> 0 && \text{(13a)} \\
\min_{\mathcal{R}} j_{yy} &> 0 && \text{(13b)} \\
\max_{\mathcal{R}} j_{zz} &< 0 && \text{(13c)} \\
\min_{\mathcal{R}} |j_{zz}|(\min_{\mathcal{R}} |j_{xx}| - \max_{\mathcal{R}} |j_{xy}|) &> \max_{\mathcal{R}} |j_{xz}|(\max_{\mathcal{R}} |j_{zx}| + \max_{\mathcal{R}} |j_{zy}|) && \text{(13d)} \\
\min_{\mathcal{R}} |j_{zz}|(\min_{\mathcal{R}} |j_{yy}| - \max_{\mathcal{R}} |j_{yx}|) &> \max_{\mathcal{R}} |j_{yz}|(\max_{\mathcal{R}} |j_{zx}| + \max_{\mathcal{R}} |j_{zy}|) && \text{(13e)}
\end{aligned}
$$

2.3 Diverge on x, y, z

By using the similar idea, we split the space into three cones.

$$\frac{|\Delta y|}{|\Delta x|} \leq 1, \frac{|\Delta z|}{|\Delta x|} \leq 1 \quad \Rightarrow \quad \frac{\Delta \dot{x}}{\Delta x} > 0 \tag{14a}$$

$$\frac{|\Delta x|}{|\Delta y|} \leq 1, \frac{|\Delta z|}{|\Delta y|} \leq 1 \quad \Rightarrow \quad \frac{\Delta \dot{y}}{\Delta y} > 0 \tag{14b}$$

$$\frac{|\Delta x|}{|\Delta z|} \leq 1, \frac{|\Delta y|}{|\Delta z|} \leq 1 \quad \Rightarrow \quad \frac{\Delta \dot{z}}{\Delta z} > 0 \tag{14c}$$

Let us compute the integral. In the first cone, we have

$$\begin{aligned}
\dot{x}_2 - \dot{x}_1 &= \int_{x_1}^{x_2} j_{xx}(1 + \frac{j_{xy}}{j_{xx}}\frac{dy}{dx} + \frac{j_{xz}}{j_{xx}}\frac{dz}{dx})dx \\
&\geq \int_{x_1}^{x_2} j_{xx}(1 - h_{xy} - h_{xz})dx \quad (|\frac{j_{xy/z}}{j_{xx}}| \leq h_{xy/z}, |\frac{dy/z}{dx}| \leq 1, \Leftarrow j_{xx} > 0) \\
&\geq \int_{x_1}^{x_2} k\, dx \quad (k \leq \min |j_{xx}|(1 - h_{xy} - h_{xz}), \Leftarrow h_{xy} + h_{xz} \leq 1)
\end{aligned}$$

It is similar for other two cones.

The sufficient condition is

$$j_{xx} > 0 \qquad h_{xy} + h_{xz} \leq 1 \tag{15a}$$

$$j_{yy} > 0 \qquad h_{yx} + h_{yz} \leq 1 \tag{15b}$$

$$j_{zz} > 0 \qquad h_{zx} + h_{zy} \leq 1 \tag{15c}$$

which is the same with

$$\min_{\mathcal{R}} j_{xx} > 0 \tag{16a}$$

$$\min_{\mathcal{R}} j_{yy} > 0 \tag{16b}$$

$$\min_{\mathcal{R}} j_{zz} > 0 \tag{16c}$$

$$\min_{\mathcal{R}} |j_{xx}| > \max_{\mathcal{R}} |j_{xy}| + \max_{\mathcal{R}} |j_{xz}| \tag{16d}$$

$$\min_{\mathcal{R}} |j_{yy}| > \max_{\mathcal{R}} |j_{yx}| + \max_{\mathcal{R}} |j_{yz}| \tag{16e}$$

$$\min_{\mathcal{R}} |j_{zz}| > \max_{\mathcal{R}} |j_{zx}| + \max_{\mathcal{R}} |j_{zy}| \tag{16f}$$

3 n-D Case

Let

$$J = \begin{pmatrix} j_{11} & \cdots & j_{1n} \\ \cdots & \cdots & \cdots \\ j_{n1} & \cdots & j_{nn} \end{pmatrix}$$

and

$$h_{ij} = \frac{\max |j_{ij}|}{min|j_{ii}|}$$

Given n variables $\{x_1, \cdots, x_n\}$, to prove that trajectories diverge on the first m direction $\{x_1, \cdots, x_m\}$, we split the space into m cones. The i^{th} cone is

$$\left.\begin{array}{ll} \frac{|\Delta x_j|}{|\Delta x_i|} \leq 1 & \forall j \in \{1, \cdots, m\} - \{i\} \\ \frac{|\Delta x_j|}{|\Delta x_i|} \leq k_v & \forall j \in \{m+1, \cdots, n\} \end{array}\right\} \Rightarrow \quad \frac{\Delta \dot{x}_i}{\Delta x_i} > 0 \tag{17a}$$

$$\forall l \in \{m+1, \cdots, n\}$$

$$\left.\begin{array}{ll} \frac{|\Delta x_j|}{|\Delta x_i|} \leq 1 & \forall j \in \{1, \cdots, m\} - \{i\} \\ \frac{|\Delta x_j|}{|\Delta x_i|} \leq k_v & \forall j \in \{m+1, \cdots, n\} - \{l\} \\ \frac{|\Delta x_l|}{|\Delta x_i|} = k_v & \end{array}\right\} \Rightarrow \quad \frac{\Delta \dot{x}_l}{\Delta x_l} < 0 \tag{17b}$$

Conditions 17a ensure distance $\max_{i=1}^{m} |\Delta x_i|$ increases. Conditions 17b ensure trajectories remains in these m cones

Now compute the integral for the i^{th} cone,

$$\begin{aligned} \dot{x}_{i2} - \dot{x}_{i1} &= \int_{p_1}^{p_2} \sum_{j=1}^{n} j_{ij} \, dx_j \\ &= \int_{x_{i1}}^{x_{i2}} \sum_{j=1}^{n} j_{ij} \frac{dx_j}{dx_i} \, dx_i \\ &\geq \int_{x_{i1}}^{x_{i2}} j_{ii} (1 - \sum_{j \in \{1, \cdots, m\} - \{i\}} h_{ij} - \sum_{j=m+1}^{n} h_{ij} k_v) \, dx_i \quad (\Leftarrow j_{ii} > 0) \\ &\geq \int_{x_{i1}}^{x_{i2}} k \, dx_i \quad (k \leq \min |j_{ii}| (1 - \sum_{j \in \{\cdots\}} h_{ij} - \sum_{j \in \{\cdots\}} h_{ij} k_v), \Leftarrow k_v \leq \frac{1 - \sum_{j \in \{1, \cdots, m\} - \{i\}} h_{ij}}{\sum_{j=m+1}^{n} h_{ij}}) \end{aligned}$$

And for all $\forall k \in \{m+1, \cdots, n\}$,

$$
\begin{aligned}
\dot{x}_{l2} - \dot{x}_{l1} &= \int_{p_1}^{p_2} \sum_{j=1}^{n} j_{lj}\, dx_j \\
&= \int_{x_{l1}}^{x_{l2}} \sum_{j=1}^{n} j_{lj} \frac{dx_j}{dx_l}\, dx_l \\
&= \int_{x_{l1}}^{x_{l2}} j_{ll} \pm \sum_{j \in = \{1,\cdots,n\}-\{l\}} j_{lj} \frac{dx_j}{k_v dx_i}\, dx_l \quad \left(\frac{dl}{dx_i} = \pm k_v\right) \\
&= \int_{x_{l1}}^{x_{l2}} j_{ll} \pm \left(\frac{j_{li}}{k_v} + \sum_{j \in \{1,\cdots,m\}-\{i\}} \frac{j_{lj}}{k_v} \frac{dx_j}{dx_i} + \sum_{j \in \{m+1,\cdots,n\}-\{l\}} \frac{j_{lj}}{k_v} \frac{dx_j}{dx_i}\right) dx_l \\
&\leq \int_{x_{l1}}^{x_{l2}} j_{ll} \left(1 - \frac{h_{li}}{k_v} - \sum_{j \in \{1,\cdots,m\}-\{i\}} \frac{h_{lj}}{k_v} - \sum_{j \in \{m+1,\cdots,n\}-\{l\}} h_{lj}\right) dx_l \quad (\Leftarrow j_{ll} < 0) \\
&= \int_{x_{l1}}^{x_{l2}} j_{ll} \left(1 - \sum_{j \in \{1,\cdots,m\}} \frac{h_{lj}}{k_v} - \sum_{j \in \{m+1,\cdots,n\}-\{l\}} h_{lj}\right) dx_l \\
&\leq \int_{x_{l1}}^{x_{l2}} -k\, dx_l \quad \left(\begin{array}{l} k \leq \min |j_{ll}| (1 - \sum_{j \in \{\cdots\}} h_{lj} - \frac{\sum_{j \in \{\cdots\}} h_{lj}}{k_v}) \\ \Leftarrow k_v > \frac{\sum_{j \in \{1,\cdots,m\}} h_{lj}}{1 - \sum_{j \in \{m+1,\cdots,n\}-\{l\}} h_{lj}} \end{array}\right)
\end{aligned}
$$

Therefore, the condition is

$$
\begin{aligned}
j_{ii} &> 0 && \forall i \in \{1, \cdots, m\} && \text{(18a)} \\
j_{ll} &> 0 && \forall l \in \{m+1, \cdots, n\} && \text{(18b)} \\
k_v &< \frac{1 - \sum_{j \in \{1,\cdots,m\}-\{i\}} h_{ij}}{\sum_{j \in \{m+1,\cdots,n\}} h_{ij}} && \forall i \in \{1, \cdots, m\} && \text{(18c)} \\
k_v &> \frac{\sum_{j \in \{1,\cdots,m\}} h_{lj}}{1 - \sum_{j \in \{m+1,\cdots,n\}-\{l\}} h_{lj}} && \forall l \in \{m+1, \cdots, n\} && \text{(18d)}
\end{aligned}
$$

To get conditions 18c, 18d, we have $\forall i \in \{1, \cdots, m\}, \forall l \in \{m+1, \cdots, n\}$

$$
\begin{aligned}
& (18c, 18d) \\
\Leftarrow \quad & 0 < \frac{\sum_{j\in\{1,\cdots,m\}} h_{lj}}{1 - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} h_{lj}} < \frac{1 - \sum_{j\in\{1,\cdots,m\}-\{i\}} h_{ij}}{\sum_{j\in\{m+1,\cdots,n\}} h_{ij}} \\
\Leftrightarrow \quad & 0 < \frac{\sum_{j\in\{1,\cdots,m\}} \max_{\mathcal{R}} |j_{lj}|}{\min_{\mathcal{R}} |j_{ll}| - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} \max_{\mathcal{R}} |j_{lj}|} < \frac{\min_{\mathcal{R}} |j_{ii}| - \sum_{j\in\{1,\cdots,m\}-\{i\}} \max_{\mathcal{R}} |j_{ij}|}{\sum_{j\in\{m+1,\cdots,n\}} \max_{\mathcal{R}} |j_{ij}|} \\
\Leftrightarrow \quad & \left\{ \begin{array}{rcl} (\min_{\mathcal{R}} |j_{ll}| - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} \max_{\mathcal{R}} |j_{lj}|) & & (\sum_{j\in\{1,\cdots,m\}} \max_{\mathcal{R}} |j_{lj}|)* \\ *(\min_{\mathcal{R}} |j_{ii}| - \sum_{j\in\{1,\cdots,m\}-\{i\}} \max_{\mathcal{R}} |j_{ij}|) & > & (\sum_{j\in\{m+1,\cdots,n\}} \max_{\mathcal{R}} |j_{ij}|) \\ \min_{\mathcal{R}} |j_{ll}| - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} \max_{\mathcal{R}} |j_{lj}| & > & 0 \\ \min_{\mathcal{R}} |j_{ii}| - \sum_{j\in\{1,\cdots,m\}-\{i\}} \max_{\mathcal{R}} |j_{ij}| & > & 0 \end{array} \right. \\
\Leftrightarrow \quad & \left\{ \begin{array}{rcl} (\min_{\mathcal{R}} |j_{ll}| - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} \max_{\mathcal{R}} |j_{lj}|) & & \\ *(\min_{\mathcal{R}} |j_{ii}| - \sum_{j\in\{1,\cdots,m\}-\{i\}} \max_{\mathcal{R}} |j_{ij}|) & > & \sum_{p\in\{1,\cdots,m\}}^{q\in\{m+1,\cdots,n\}} (\max_{\mathcal{R}} |j_{lp}| \cdot \max_{\mathcal{R}} |j_{iq}|) \\ \min_{\mathcal{R}} |j_{ll}| - \sum_{j\in\{m+1,\cdots,n\}-\{l\}} \max_{\mathcal{R}} |j_{lj}| & > & 0 \\ \min_{\mathcal{R}} |j_{ii}| - \sum_{j\in\{1,\cdots,m\}-\{i\}} \max_{\mathcal{R}} |j_{ij}| & > & 0 \end{array} \right. \\
\Leftrightarrow \quad & \left\{ \begin{array}{rcl} \sum_{p\in\{m+1,\cdots,n\}}^{q\in\{1,\cdots,m\}} (\min^*_{\mathcal{R}} |j_{lp}| \cdot \min^*_{\mathcal{R}} |j_{iq}|) & > & \sum_{p\in\{1,\cdots,m\}}^{q\in\{m+1,\cdots,n\}} (\max_{\mathcal{R}} |j_{lp}| \cdot \max_{\mathcal{R}} |j_{iq}|) \\ \sum_{p\in\{m+1,\cdots,n\}} \min^*_{\mathcal{R}} |j_{lp}| & > & 0 \\ \sum_{q\in\{1,\cdots,m\}} \min^*_{\mathcal{R}} |j_{iq}| & > & 0 \\ \min^*_{\mathcal{R}} |j_{ij}| & \equiv & \left\{ \begin{array}{rl} -\max_{\mathcal{R}} |j_{ij}| & i \neq j \\ \min_{\mathcal{R}} |j_{ij}| & i = j \end{array} \right. \end{array} \right.
\end{aligned}
$$

4 Proof

Given a n-dimensional bounded region $S \in R^n$, and its *Jacobian matrix*:

$$
J(x)_{x\in S} = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial \dot{x}_n}{\partial x_1} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{pmatrix},
$$

let's define

$$
\begin{aligned}
& j_{uv} = \frac{\partial \dot{x_u}}{\partial x_v} \\
& \underline{j_{uv}} = \min_{x\in S} \frac{\partial \dot{x_u}}{\partial x_v}, \overline{j_{uv}} = \max_{x\in S} \frac{\partial \dot{x_u}}{\partial x_v} \\
& \lfloor j_{uv} \rfloor = \min_{x\in S} \left| \frac{\partial \dot{x_u}}{\partial x_v} \right|, \lceil j_{uv} \rceil = \max_{x\in S} \left| \frac{\partial \dot{x_u}}{\partial x_v} \right| \\
& \langle j_{uv} \rangle = \left\{ \begin{array}{ll} -\lceil j_{uv} \rceil & u \neq v \\ \lfloor j_{uv} \rfloor & u = v \end{array} \right. \\
& \rho_{uv} = \frac{\lceil j_{uv} \rceil}{\lfloor j_{uu} \rfloor},
\end{aligned}
$$

a set of *diverge variables* $\Phi = \{x_1^\Phi, \cdots, x_m^\Phi\}$, a set of *converge variables* $\Psi = \{x_1^\Psi, \cdots, x_{n-m}^\Psi\}$, and their corresponding sets of indices $\phi = \{u|x_u \in \Phi\}$, $\psi = \{v|x_v \in \Psi\}$.

The set Φ generates a diverge subspace and the set Ψ generates a converge subspace. A $(\phi, \psi, \kappa, x_0)$ *multiple cone* is a n-dimensional cone defined as

$$\begin{aligned} \Delta x &= x - x_0 \\ \frac{|\Delta x_q|}{\max_{u\in\phi} |\Delta x_u|} &\leq \kappa \quad \forall q \in \psi. \end{aligned}$$

The cone can be partitioned into m *pieces*. For each $u \in \phi$, the piece of the multiple cone is in the form of

$$\begin{aligned} \frac{|\Delta x_p|}{|\Delta x_u|} &\leq 1 \quad \forall p \in \phi \\ \frac{|\Delta x_q|}{|\Delta x_u|} &\leq \kappa \quad \forall q \in \psi. \end{aligned}$$

A multiple cone divides the diverge subspace into m pieces and limits converge subspace by a cone.

If the dynamics satisfies conditions:

$$\underline{j_{uu}} > 0 \qquad u \in \phi \tag{19a}$$

$$\overline{j_{vv}} < 0 \qquad v \in \psi \tag{19b}$$

$$\sum_{p\in\phi} \langle j_{up} \rangle > 0 \qquad u \in \phi \tag{19c}$$

$$\sum_{q\in\psi} \langle j_{vq} \rangle > 0 \qquad v \in \psi \tag{19d}$$

$$\sum_{p\in\phi} \langle j_{up} \rangle \cdot \sum_{q\in\Psi} \langle j_{vq} \rangle > \sum_{q\in\psi} \lceil j_{uq} \rceil \cdot \sum_{p\in\phi} \lceil j_{vp} \rceil \qquad u \in \phi, v \in \psi, \tag{19e}$$

then there exists positive constant ε and κ

$$\begin{aligned} \mu &= \min_{u\in\{1,\cdots,n\}} \sum_{p\in\phi} \langle j_{up} \rangle \\ h_1 &= \min_{u\in\phi} \frac{\sum_{p\in\phi} \langle j_{up} \rangle}{\sum_{q\in\psi} \lceil j_{uq} \rceil} \\ h_2 &= \max_{v\in\psi} \frac{\sum_{p\in\phi} \lceil j_{vp} \rceil}{\sum_{q\in\psi} \langle j_{vq} \rangle} \end{aligned}$$

$$\kappa = \sqrt{h_1 \cdot h_2} \tag{20a}$$

$$\varepsilon = \mu(1 - \sqrt{\frac{h_2}{h_1}}). \tag{20b}$$

For any two points $p_1, p_2 \in S$ and p_2 is in the $(\phi, \psi, \kappa, p_1)$ multiple cone, the dynamics satisfies the following properties in the u^{th} piece of the multiple cone.

$$\left.\begin{array}{ll} \frac{|\Delta x_p|}{|\Delta x_u|} \leq 1 & \forall p \in \phi \\ \frac{|\Delta x_q|}{|\Delta x_u|} \leq \kappa & \forall q \in \psi \end{array}\right\} \Rightarrow \frac{\Delta \dot{x_u}}{\Delta x_u} \geq \varepsilon \tag{21a}$$

$$\forall l \in \psi \qquad \left.\begin{array}{ll} \frac{|\Delta x_p|}{|\Delta x_u|} \leq 1 & \forall p \in \phi \\ \frac{|\Delta x_q|}{|\Delta x_u|} \leq \kappa & \forall q \in \psi \\ \frac{|\Delta x_l|}{|\Delta x_u|} = \kappa & \end{array}\right\} \Rightarrow \frac{\Delta \dot{x_l}}{\Delta x_l} \leq -\varepsilon \tag{21b}$$

Conditions 21a ensure distance $\max_{p\in\phi} |\Delta x_p|$ increases. Conditions 21b ensure trajectories remains in the multiple cone. If $m \geq 1$, then any trajectory will leave the region S with probability one.

To prove this lama, let us compute the integral for each piece $u \in \phi$

$$\begin{aligned}
&\dot{x_{u2}} - \dot{x_{u1}} \\
&= \int_{p_1}^{p_2} \sum_{v=1}^{n} (j_{uv}\, dx_v) \\
&= \int_{x_{u1}}^{x_{u2}} (\sum_{v=1}^{n} j_{uv} \frac{dx_v}{dx_u})\, dx_u \\
&= \int_{x_{u1}}^{x_{u2}} (j_{uu} + \sum_{p\in\phi-\{u\}} j_{up} \frac{dx_p}{dx_u} + \sum_{q\in\psi} j_{uq} \frac{dx_q}{dx_u})\, dx_u \\
&= \int_{x_{u1}}^{x_{u2}} j_{uu} (1 + \sum_{p\in\phi-\{u\}} \frac{j_{up}}{j_{uu}} \frac{dx_p}{dx_u} + \sum_{q\in\psi} \frac{j_{uq}}{j_{uu}} \frac{dx_q}{dx_u})\, dx_u \\
&\geq \int_{x_{u1}}^{x_{u2}} j_{uu} (1 - \sum_{p\in\phi-\{u\}} \left|\frac{j_{up}}{j_{uu}} \frac{dx_p}{dx_u}\right| - \sum_{q\in\psi} \left|\frac{j_{uq}}{j_{uu}} \frac{dx_q}{dx_u}\right|)\, dx_u \quad {\scriptstyle (j_{uu}>0)} \\
&\geq \int_{x_{u1}}^{x_{u2}} j_{uu} (1 - \sum_{p\in\phi-\{u\}} \rho_{up} - \sum_{q\in\psi} \kappa\rho_{uq})\, dx_u \quad {\scriptstyle (\left|\frac{j_{uv}}{j_{uu}}\right| \leq \rho_{uv}, |\frac{dx_p}{dx_u}| \leq 1, |\frac{dx_q}{dx_u}| \leq \kappa)} \\
&= \int_{x_{u1}}^{x_{u2}} \frac{j_{uu}}{\lfloor j_{uu} \rfloor} (\lfloor j_{uu} \rfloor - \sum_{p\in\phi-\{u\}} \lceil j_{up} \rceil - \kappa \sum_{q\in\psi} \lceil j_{uq} \rceil)\, dx_u \\
&= \int_{x_{u1}}^{x_{u2}} \frac{j_{uu}}{\lfloor j_{uu} \rfloor} (\sum_{p\in\phi} \langle j_{up} \rangle - \kappa \sum_{q\in\psi} \lceil j_{uq} \rceil)\, dx_u \\
&= \int_{x_{u1}}^{x_{u2}} \frac{j_{uu}}{\lfloor j_{uu} \rfloor} \sum_{p\in\phi} \langle j_{up} \rangle (1 - \sqrt{h_1 \cdot h_2} \frac{\sum_{q\in\psi} \lceil j_{uq} \rceil}{\sum_{p\in\phi} \langle j_{up} \rangle})\, dx_u \\
&\geq \int_{x_{u1}}^{x_{u2}} \frac{j_{uu}}{\lfloor j_{uu} \rfloor} \mu (1 - \sqrt{h_1 \cdot h_2} \frac{1}{h_1})\, dx_u \quad {\scriptstyle (\mu \leq \sum_{p\in\phi} \langle j_{up} \rangle, h_1 \leq \frac{\sum_{p\in\phi} \langle j_{up} \rangle}{\sum_{q\in\psi} \lceil j_{uq} \rceil})} \\
&\geq \int_{x_{u1}}^{x_{u2}} \varepsilon\, dx_u \quad {\scriptstyle (j_{uu} = |j_{uu}| \geq \lfloor j_{uu} \rfloor)} \\
&= \varepsilon (x_{u2} - x_{u1}).
\end{aligned}$$

And for all $l \in \psi$, we have

$$
\begin{aligned}
&\dot{x}_{l2} - \dot{x}_{l1} \\
&\quad = \int_{p_1}^{p_2} \sum_{v=1}^{n} (j_{lv}\, dx_v) \\
&\quad = \int_{x_{l1}}^{x_{l2}} (\sum_{v=1}^{n} j_{lv} \frac{dx_v}{dx_l})\, dx_l \\
&\quad = \int_{x_{l1}}^{x_{l2}} (j_{ll} + \sum_{p\in\phi} j_{lp} \frac{dx_p}{dx_l} + \sum_{q\in\psi-\{l\}} j_{lq} \frac{dx_q}{dx_l})\, dx_l \\
&\quad = \int_{x_{l1}}^{x_{l2}} j_{ll}(1 + \sum_{p\in\phi} \frac{j_{lp}}{j_{ll}} \frac{dx_p}{dx_l} + \sum_{q\in\psi-\{l\}} \frac{j_{lq}}{j_{ll}} \frac{dx_q}{dx_l})\, dx_l \\
&\quad = \int_{x_{l1}}^{x_{l2}} j_{ll}(1 \pm \sum_{p\in\phi} \frac{1}{\kappa} \frac{j_{lp}}{j_{ll}} \frac{dx_p}{dx_u} \pm \sum_{q\in\psi-\{l\}} \frac{1}{\kappa} \frac{j_{lq}}{j_{ll}} \frac{dx_q}{dx_u})\, dx_l \quad (\tfrac{dx_l}{dx_u} = \pm\kappa) \\
&\quad \le \int_{x_{l1}}^{x_{l2}} j_{ll}(1 - \sum_{p\in\phi} \frac{1}{\kappa} \left| \frac{j_{lp}}{j_{ll}} \frac{dx_p}{dx_u} \right| - \sum_{q\in\psi-\{l\}} \frac{1}{\kappa} \left| \frac{j_{lq}}{j_{ll}} \frac{dx_q}{dx_u} \right|)\, dx_l \quad (j_{ll} < 0) \\
&\quad \le \int_{x_{l1}}^{x_{l2}} j_{ll}(1 - \frac{1}{\kappa} \sum_{p\in\phi} \rho_{lp} - \sum_{q\in\psi-\{l\}} \rho_{lq})\, dx_l \quad (\left|\tfrac{j_{lv}}{j_{ll}}\right| \le \rho_{lv}, |\tfrac{dx_p}{dx_u}| \le 1, |\tfrac{dx_q}{dx_u}| \le \kappa) \\
&\quad = \int_{x_{l1}}^{x_{l2}} \frac{j_{ll}}{\lfloor j_{ll} \rfloor} (\lfloor j_{ll} \rfloor - \frac{1}{\kappa} \sum_{p\in\phi} \lceil j_{lp} \rceil - \sum_{q\in\psi-\{l\}} \lceil j_{lq} \rceil)\, dx_l \\
&\quad = \int_{x_{l1}}^{x_{l2}} \frac{j_{ll}}{\lfloor j_{ll} \rfloor} (\sum_{q\in\psi} \langle j_{lq} \rangle - \frac{1}{\kappa} \sum_{p\in\phi} \lceil j_{lp} \rceil)\, dx_l \\
&\quad = \int_{x_{l1}}^{x_{l2}} \frac{j_{ll}}{\lfloor j_{ll} \rfloor} \sum_{q\in\psi} \langle j_{lq} \rangle (1 - \frac{1}{\kappa} \frac{\sum_{p\in\phi} \lceil j_{lp} \rceil)}{\sum_{q\in\psi} \langle j_{lq} \rangle}\, dx_l \\
&\quad \le \int_{x_{l1}}^{x_{l2}} \frac{j_{ll}}{\lfloor j_{ll} \rfloor} \mu(1 - \frac{h_2}{\sqrt{h_1 \cdot h_2}})\, dx_l \quad (\mu \le \textstyle\sum_{q\in\psi} \langle j_{lq} \rangle, h_2 \ge \frac{\sum_{p\in\phi} \lceil j_{lp} \rceil}{\sum_{q\in\psi} \langle j_{lq} \rangle}) \\
&\quad \le \int_{x_{l1}}^{x_{l2}} -\varepsilon\, dx_l \quad (j_{ll} = -|j_{ll}| \le -\lfloor j_{ll} \rfloor) \\
&\quad = -\varepsilon(x_{l2} - x_{l1})
\end{aligned}
$$

5 L_2 Norm

The condition is too constrictive, because it requires $\max_{p\in\phi} |\Delta x_p|$ increases. Think of a spiral diverge dynamic system, the L_∞ may decrease. For the Rambus ring oscillator, we found the Jacobian matrix does not satisfy the condition at the metastable point:

$$\begin{aligned} J(x)_{x\in S} &= \begin{pmatrix} b-a & -b & 0 & 0 \\ b & b-a & 0 & 0 \\ 0 & 0 & -(a+b) & -b \\ 0 & 0 & -b & -(a+b) \end{pmatrix} \\ b &> a > 0 \end{aligned}$$

Rather using L_∞ norm, we use L_2 norm instead and find a less constrictive condition. And the cone is similar:

$$\begin{aligned} \Delta x &= x - x^0 \\ \frac{|\Delta x_q|}{\max_{p\in\phi} |\Delta x_p|} &\leq \kappa \quad \forall q \in \psi. \end{aligned}$$

Noting that the cone can be partitioned into m pieces: Δ_{x_i} dominates in the i^{th} pieces:

$$\frac{|\Delta x_p|}{|\Delta x_i|} \leq 1 \quad \forall p \in \phi$$

Apparently, in this piece, the boundary of the cone is in the form of

$$\frac{|\Delta x_q|}{|\Delta x_i|} \leq \kappa \quad \forall q \in \psi.$$

Now, let us define the distance (projected to diverge subspace) between two trajectories.

$$\begin{aligned} y(t) &= ||x(t) - x^0(t)||_2 \\ &= \sqrt{\sum_{x_i\in\phi} \Delta x_i^2} \end{aligned}$$

Here, we only count the diverge variables. To show that the distance of two trajectories always increase, the derivative of y must be positive

$$\begin{aligned} & \frac{dy(t)}{dt} > 0 \\ \Leftrightarrow \quad & \frac{1}{\sqrt{\sum_{i\in\phi} \Delta x_i^2}} \sum_{x_i\in\phi} \Delta x_i \Delta \dot{x}_i > 0 \\ \Leftrightarrow \quad & \sum_{i\in\phi} \Delta x_i \Delta \dot{x}_i > 0 \end{aligned}$$

By integration, we have

$$\begin{aligned} \Delta \dot{x}(t) &= \dot{x}(t) - \dot{x}_0(t) \\ \dot{x}(t) &= \dot{x}_0(t) + \int_{x^0}^{x} J(s)ds \text{ (s is a trajectory from } x^0 \text{ to } x) \\ &= \dot{x}_0(t) + J(x^*)\Delta x \text{ (mean value theorem, } x^* \text{ is in the cone).} \end{aligned}$$

Therefore, in the i^{th} piece, we have

$$
\begin{aligned}
\sum_{i\in\phi} \Delta x_i \Delta \dot{x}_i &= \Delta x_\phi^T \cdot J[\phi, :] \cdot \Delta x \\
&= \Delta x_\phi^T \cdot J[\phi,\phi] \cdot \Delta x_\phi + \Delta x_\phi^T \cdot J[\phi,\psi] \cdot \Delta x_\psi \\
&\geq \Delta x_\phi^T \cdot J[\phi,\phi] \cdot \Delta x_\phi + \min(\Delta x_\phi^T \cdot J[\phi,\psi] \cdot \kappa \begin{bmatrix} \pm\Delta x_i \\ \cdots \\ \pm\Delta x_i \end{bmatrix}) \quad (|\Delta x_\psi| < \kappa\Delta|x_i|) \\
&= \Delta x_\phi^T \cdot J[\phi,\phi] \cdot \Delta x_\phi + \kappa \min(\Delta x_\phi^T \cdot \pm J[\phi,\psi] \cdot \begin{bmatrix} \Delta x_i \\ \cdots \\ \Delta x_i \end{bmatrix}) \text{ (same sign in a column of J)} \\
&= \Delta x_\phi^T \cdot J[\phi,\phi] \cdot \Delta x_\phi + \kappa \min(\Delta x_\phi^T \cdot \sum_{q\in\psi} \pm J[\phi,q] \cdot \Delta x_i) \\
&= \min(\Delta x_\phi^T \cdot (J[\phi,\phi] + \kappa \sum_{q\in\psi} \pm J[\phi,q]) \cdot \Delta x_\phi) \\
&= \min(\Delta x_\phi^T \cdot \hat{J}_i[\phi,\phi] \cdot \Delta x_\phi) \\
\hat{J}_i[\phi,p] &= \begin{cases} J[\phi,p] & p \neq i \\ J[\phi,p] + \kappa \sum_{q\in\psi} \pm J[\phi,q] & p = i \end{cases}
\end{aligned}
$$

Therefore, if $\forall i \in \phi$, $\hat{J}_i$ is positive definite, then the distance between two trajectories increases.

Now, we should show the trajectory remain in the cone. Which is similar with the previous proof. In the i^{th} piece, we should show that $\forall j \in \psi$,

$$
\frac{\Delta \dot{x}_j}{\Delta x_j} \leq 0.
$$

on the face

$$
\begin{aligned}
\frac{|\Delta x_q|}{|\Delta x_i|} &\leq \kappa \quad \forall q \in \psi \\
\frac{|\Delta x_j|}{|\Delta x_i|} &= \kappa.
\end{aligned}
$$

Let us compute

$$
\begin{aligned}
\dot{x}_j &= \dot{x}_j^0 + J[j,:]\Delta x \\
\Delta\dot{x}_j &= J[j,\phi]\Delta x_\phi + J[j,\psi]\Delta x_\psi \\
&= J[j,j]\Delta x_j + J[j,i]\Delta x_i + J[j,\phi-\{i\}]\Delta x_{\phi-\{i\}} + J[j,\psi-\{j\}]\Delta x_{\psi-\{j\}} \\
&\leq J[j,j]\Delta x_j + J[j,i]\Delta x_i + |J[j,\phi-\{i\}]|\cdot|\Delta x_{\phi-\{i\}}| + |J[j,\psi-\{j\}]|\cdot|\Delta x_{\psi-\{j\}}| \\
&\leq J[j,j]\Delta x_j + J[j,i]\Delta x_i + \sum_{l\in\phi-\{i\}}|J[j,l]|\cdot|\Delta x_i| + \sum_{k\in\psi-\{j\}}|J[j,k]|\cdot\kappa|\Delta x_i| \quad (\frac{|\Delta x_\phi|}{|\Delta x_i|}\leq 1, \frac{|\Delta x_\psi|}{|\Delta x_i|}\leq\kappa) \\
&= J[j,j]\Delta x_j + J[j,i]\frac{1}{\kappa}|\Delta x_j| + \sum_{l\in\phi-\{i\}}|J[j,l]|\cdot\frac{1}{\kappa}|\Delta x_j| + \sum_{k\in\psi-\{j\}}|J[j,k]|\cdot|\Delta x_j| \quad (\frac{|\Delta x_j|}{|\Delta x_i|}=\kappa) \\
\frac{\Delta\dot{x}_j}{\Delta x_j} &= J[j,j] + (\ \frac{1}{\kappa}(J[j,i] + \sum_{l\in\phi-\{i\}}|J[j,l]|)\ + \sum_{k\in\psi-\{j\}}|J[j,k]|\ )\cdot\frac{|\Delta x_j|}{\Delta x_j} \\
&\leq J[j,j] + \left|\frac{1}{\kappa}(J[j,i] + \sum_{l\in\phi-\{i\}}|J[j,l]|)\ + \sum_{k\in\psi-\{j\}}|J[j,k]|\right| \\
&\leq J[j,j] + \frac{1}{\kappa}\sum_{l\in\phi}|J[j,l]| + \sum_{k\in\psi-\{j\}}|J[j,k]|
\end{aligned}
$$

To make $\frac{\Delta\dot{x}_j}{\Delta x_j}$ negative, it requires (indepent of i, thus same for all pieces)

$$
\begin{aligned}
J[j,j] &< 0 \\
\kappa &> -\frac{\sum_{l\in\phi}|J[j,l]|}{J[j,j]+\sum_{k\in\psi-\{j\}}|J[j,k]|} \\
0 &> J[j,j] + \sum_{k\in\psi-\{j\}}|J[j,k]|
\end{aligned}
$$

Therefore, the sufficient condition is

$$
\begin{aligned}
J[q,q] &\leq 0 \quad \forall q\in\psi \\
|J[q,q]| &> \sum_{j\in\psi-\{q\}}|J[q,j]| \quad \forall q\in\psi \\
\kappa &= \max_{q\in\psi}(\frac{\sum_{i\in\phi}|J[q,i]|}{|J[q,q]|-\sum_{j\in\psi-\{q\}}|J[q,j]|}) \\
\hat{J}_p \quad is &\quad \text{positive definite} \quad \forall p\in\phi
\end{aligned}
$$

We know that

$$
\begin{aligned}
& J \text{ is positive definite} \\
\Leftrightarrow\ & \frac{1}{2}(J+J^T) \text{ is positive definite} \\
\Leftrightarrow\ & |J[1:i,1:i]| > 0, \quad \forall i\in\{1,\cdots,n\}
\end{aligned}
$$

Note x^* can be any point in the region. Therefore J is an interval matrix. Using interval methods during the computation.

6 Jacobian Matrix of Ring Oscillator

The Jacobean matrix of a n-stage ring oscillator has special structure. All nodes are indexed in the form of

$$\begin{array}{ccccc} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n-1} & x_{2n} \end{array}$$

For each node i, its current (derivative) only depends node $(i-1)\%2n$, node $(i+n)\%2n$ and its self. Therefore, the Jacobian matrix is in the form of

$$\begin{aligned} J(x) &= \left(\begin{array}{ccccc|ccccc} a_1 & 0 & \cdots & \cdots & 0 & c_1 & 0 & \cdots & 0 & b_1 \\ b_2 & a_2 & 0 & \cdots & 0 & 0 & c_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & b_{n-1} & a_{n-1} & 0 & 0 & \cdots & 0 & c_{n-1} & 0 \\ 0 & \cdots & 0 & b_n & a_n & 0 & \cdots & \cdots & 0 & c_n \\ \hline c_{n+1} & 0 & \cdots & 0 & b_{n+1} & a_{n+1} & 0 & \cdots & \cdots & 0 \\ 0 & c_{n+2} & 0 & \cdots & 0 & b_{n+2} & a_{n+2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_{2n-1} & 0 & 0 & \cdots & b_{2n-1} & a_{2n-1} & 0 \\ 0 & \cdots & \cdots & 0 & c_{2n} & 0 & \cdots & 0 & b_{2n} & a_{2n} \end{array}\right) \\ &= \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \end{aligned}$$

Now, let's transfor to another u-corrdinate

$$\begin{aligned} u &= M^{-1}x = M'x \\ M &= \frac{\sqrt{2}}{2}\left[\begin{array}{r|r} eye(n) & eye(n) \\ \hline -eye(n) & eye(n) \end{array}\right] \end{aligned}$$

Therefore, the Jacobian matrix in the new coordinate is

$$\begin{array}{lll} \Sigma a_i = a_i + a_{n+1} & \Sigma b_i = b_i + b_{n+1} & \Sigma c_i = c_i + c_{n+1} \\ \Delta a_i = a_i - a_{n+1} & \Delta b_i = b_i - b_{n+1} & \Delta c_i = c_i - c_{n+1} \end{array}$$

$$
\begin{aligned}
J(u) &= M'J(x)M \\
&= \left(\begin{array}{ccc|ccc}
\Sigma a_1 & 0\cdots0 & 0\cdots0 & \Delta a_1 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & 0\cdots0 & \Sigma a_n & 0\cdots0 & 0\cdots0 & \Delta a_n\\ \hline
\Delta a_1 & 0\cdots0 & 0\cdots0 & \Sigma a_1 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & 0\cdots0 & \Delta a_n & 0\cdots0 & 0\cdots0 & \Sigma a_n
\end{array}\right)\\
&+ \left(\begin{array}{ccc|ccc}
0\cdots0 & 0\cdots0 & -\Sigma b_1 & 0\cdots0 & 0\cdots0 & \Delta b_1\\
\Sigma b_2 & 0\cdots0 & 0\cdots0 & \Delta b_2 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & \Sigma b_n & 0\cdots0 & 0\cdots0 & \Delta b_n & 0\cdots0\\ \hline
0\cdots0 & 0\cdots0 & -\Delta b_1 & 0\cdots0 & 0\cdots0 & \Sigma b_1\\
\Delta b_2 & 0\cdots0 & 0\cdots0 & \Sigma b_2 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & \Delta b_n & 0\cdots0 & 0\cdots0 & \Sigma b_n & 0\cdots0
\end{array}\right)\\
&+ \left(\begin{array}{ccc|ccc}
-\Sigma c_1 & 0\cdots0 & 0\cdots0 & \Delta c_1 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & 0\cdots0 & -\Sigma c_n & 0\cdots0 & 0\cdots0 & \Delta c_n\\ \hline
-\Delta c_1 & 0\cdots0 & 0\cdots0 & \Sigma c_1 & 0\cdots0 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & 0\cdots0 & -\Delta c_n & 0\cdots0 & 0\cdots0 & \Sigma c_n
\end{array}\right)\\
&= \left(\begin{array}{ccc|ccc}
\Sigma a_1-\Sigma c_1 & 0\cdots0 & -\Sigma b_1 & \Delta a_1+\Delta c_1 & 0\cdots0 & \Delta b_1\\
\Sigma b_2 & \Sigma a_2-\Sigma c_2 & 0\cdots0 & \Delta b_2 & \Delta a_2+\Delta c_2 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & \Sigma b_n & \Sigma a_n-\Sigma c_n & 0\cdots0 & \Delta b_n & \Delta a_n+\Delta c_n\\ \hline
\Delta a_1-\Delta c_1 & 0\cdots0 & -\Delta b_1 & \Sigma a_1+\Sigma c_1 & 0\cdots0 & \Sigma b_1\\
\Delta b_2 & \Delta a_2-\Delta c_2 & 0\cdots0 & \Sigma b_2 & \Sigma a_2+\Sigma c_2 & 0\cdots0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0\cdots0 & \Delta b_n & \Delta a_n-\Delta c_n & 0\cdots0 & \Sigma b_n & \Sigma a_n+\Sigma c_n
\end{array}\right)\\
&= \left(\begin{array}{c|c}
(j_{11}+j_{22})-(j_{12}+j_{21}) & (j_{11}-j_{22})+(j_{12}-j_{21})\\ \hline
(j_{11}-j_{22})-(j_{12}-j_{21}) & (j_{11}+j_{22})+(j_{12}+j_{21})
\end{array}\right)
\end{aligned}
$$

When $n=2$,

$$
J(u) = \left(\begin{array}{cc|cc}
\Sigma a_1-\Sigma c_1 & -\Sigma b_1 & \Delta a_1+\Delta c_1 & \Delta b_1\\
\Sigma b_2 & \Sigma a_2-\Sigma c_2 & \Delta b_2 & \Delta a_2+\Delta c_2\\ \hline
\Delta a_1-\Delta c_1 & -\Delta b_1 & \Sigma a_1+\Sigma c_1 & \Sigma b_1\\
\Delta b_2 & \Delta a_2-\Delta c_2 & \Sigma b_2 & \Sigma a_2+\Sigma c_2
\end{array}\right)
$$