Adjunctions

This section is optional, but a wonderful insight into the nature of duality and the previously discussed theory.

Goals

Here's a sneak peek:

Proposition (LAPC, RAPL)

Any category $\mathbf{C}$ *admits all limits and colimits indexed by a small category* $\mathbf{J}$ *if and only if the constant diagram functor* $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ *has a right and left adjoint* $\mathit{colim} \dashv \Delta \dashv \mathit{lim}$*:*

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\Delta" description] & \mathbf{C}^{\mathbf{J}} \arrow[l, bend right, "lim"'] \arrow[l, bend left, "colim"] \end{tikzcd}$$

When these functors exist, we can prove our arithmetic!

Adjunctions

"Adjoint Functors arise everywhere" - Saunders MacLane

Adjunctions

In fact, they are more than pertinent to the study of functional programming, as it is a classic theorem of adjoint functors (also called adjunctions) that every adjunction gives rise to a monad/comonad pair.

Adjunctions

Lets begin with some definitions.

Adjunctions

Definition (Adjunction)

Let $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ be functors. L and R are called *adjoint* functors if there exists a natural isomorphism $\Phi : \mathbf{D}(L-, =) \cong \mathbf{C}(-, R =)$.

The components of Φ are those morphisms

$$\Phi_{x,y} : \mathbf{D}(Lx, y) \cong \mathbf{C}(x, Ry).$$

Definition (Unit/Co-unit of an Adjunction)

Two functors $L : \mathbf{C} \to \mathbf{D}$ and $R : \mathbf{D} \to \mathbf{C}$ are adjoint if there exist natural transformations $\eta : 1_C \Rightarrow RL$ and $\epsilon : LR \Rightarrow 1_D$ which satisfy the following triangle identity:

$$1_{Lx} : \quad Lx \xrightarrow{L\eta_x} LRLx \xrightarrow{\epsilon_{Lx}} Lx$$

and

$$1_{Ry} : \quad Ry \xrightarrow{\eta_{Ry}} RLRy \xrightarrow{R\epsilon_y} Ry$$

Let $U : \mathbf{D} \to \mathbf{C}$ be a forgetful functor, and let $c \in \mathbf{C}$. A **free D-object** on c with respect to U is an object of $\mathbf{D}$ satisfying the universal property that F would have if F would have if it were left-adjoint to U. More precisely: a free $\mathbf{D}$-object on c consists of an object $c \in C$ together with a morphism $f : c \to Ud$ in C such that for any other $d' \in D$ and morphism $g : c \to Ud'$, there exists a unique $h : d \to d'$ in $\mathbf{D}$ such that $Uh \circ f = g$. Diagrammatically:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[dr, "g"'] & Ud \arrow[d, dashed, "Uh"] \\ & Ud' \end{tikzcd}$$

Examples

Let $U : \mathbf{Mon} \to \mathbf{Set}$ be a forgetful functor, nad let MS be the free object in **Mon** generated by set S. Any set function from S to the underlying set UN of another monoid $N \in \mathbf{Mon}$ extends to a unique monoid homomorphism $MS \to N$ per the free construction described in the previous slide. One can check that this construction forms an adjunction $M \dashv U$ - that is, the following hom-sets are in bijection:

$$\mathbf{Mon}(MS, N) \cong \mathbf{Set}(S, UN)$$

Let $J : \mathbf{J} \to \mathbf{C}$ be a diagram of shape $\mathbf{J}$ in $\mathbf{C}$. A **cone** in $\mathbf{C}$ is an object $N \in \mathbf{C}$ together with a family $\beta_X : N \to JX$ such that for every morphism $Jj : JX \to JY$ we have $\beta_Y = Jj \circ \beta_X$. A **limit** is a terminal cone (L, ϕ) inducing a unique $f : N \to L$ such that all $\psi_{(-)}$ factor through (L, ϕ). I.e. $\beta_X = \phi_X \circ f$. Diagrammatically,

$$
\begin{tikzcd}
& N \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\beta_X"'] \arrow[ddr, bend left, "\beta_Y"] & \\
& L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
JX \arrow[rr, "Jj"'] & & JY
\end{tikzcd}
$$

Limits and Colimits

A **colimit** is defined dually, in a similar way, with the arrows reversed.

Limits and Colimits

Notice how familiar those diagrams look?

Limits and Colimits

This is because products and coproducts as previously defined are limits and colimits themselves!

Limits and Colimits

This is because products and coproducts as previously defined are limits and colimits themselves! In fact, limits and colimits define functors $\mathbf{C}^{\mathbf{J}} \to \mathbf{C}$.

Limits and Colimits

Define the diagonal functor$\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ taking objects $c \in \mathbf{C}$ to the constant $\mathbf{J}$-shaped diagram. If $\mathbf{C}$ is a locally small category, then $\mathbf{C}$ preserves all limits (colimits, resp.) if and only if Δ admits a right adjoint (left adjoint, resp.).

Examples

Let **C** be **Set** or its subcategory **Fin**. Note that there exists an adjunction $A \times - \dashv (-)^A$ for any A. For any sets A, B, C, D, the following natural isomorphisms hold:

$$A \times (B + C) \cong (A \times B) + (A \times C) \qquad (B \times C)^A \cong B^A \times C^A$$

$$A^{B+C} \cong A^B \times A^C$$

Limits and Colimits

Proof.

The left adjoint $A \times -$ preserves the coproduct $B + C$, the right adjoint $(-)^A$ preserves the product $B \times C$, and the functor $A^{(-)} : \mathbf{Set}^{op} \to \mathbf{Set}$ is mutually right-adjoint to itself, and so carries coproducts in **Set** to products in **Set**. The laws of cardinal arithmetic follow by applying the cardinality functor $|-| : \mathbf{Fin}_{\mathbf{Iso}} \to \mathbf{Fin}$

$\square$

Bonus

Bonus: Functors of the form $\mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$ are known as *profunctors*. Profunctors define proof-relevant relations between objects in (possibly) heterogeneous categories and the sets of morphisms between objects in $\mathbf{C}^{op} \times \mathbf{D}$ are called *heteromorphisms*. Adjunctions are natural isomorphisms of profunctors which establishes a correspondence between heteromorphisms in an opposite category, and heteromorphisms in a standard category. Taking this perspective that profunctors denote relations between objects imples that adjunctions encode an isomorphism between dual relations! Very cool. Thanks.