

# Adjunctions

This section is optional, but a wonderful insight into the nature of duality and the previously discussed theory.

# Goals

Here's a sneak peek:

## Proposition (LAPC, RAPL)

*Any category  $\mathbf{C}$  admits all limits and colimits indexed by a small category  $\mathbf{J}$  if and only if the constant diagram functor  $\Delta : \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{J}}$  has a right and left adjoint  $\text{colim} \dashv \Delta \dashv \text{lim}$ :*

$$\begin{array}{ccc} & \xleftarrow{\text{lim}} & \\ \mathbf{C} & \xrightarrow{\Delta} & \mathbf{C}^{\mathbf{J}} \\ & \xleftarrow{\text{colim}} & \end{array}$$

When these functors exist, we can prove our arithmetic!

# Adjunctions

*"Adjoint Functors arise everywhere" - Saunders MacLane*

# Adjunctions

In fact, they are more than pertinent to the study of functional programming, as it is a classic theorem of adjoint functors (also called adjunctions) that every adjunction gives rise to a monad/comonad pair.

# Adjunctions

Lets begin with some definitions.

# Adjunctions

## Definition (Adjunction)

Let  $L : \mathbf{C} \rightarrow \mathbf{D}$  and  $R : \mathbf{D} \rightarrow \mathbf{C}$  be functors.  $L$  and  $R$  are called *adjoint* functors if there exists a natural isomorphism  $\Phi : \mathbf{D}(L-, =) \cong \mathbf{C}(-, R=)$ .

The components of  $\Phi$  are those morphisms

$$\Phi_{x,y} : \mathbf{D}(Lx, y) \cong \mathbf{C}(x, Ry).$$

### Definition (Unit/Co-unit of an Adjunction)

Two functors  $L : \mathbf{C} \rightarrow \mathbf{D}$  and  $R : \mathbf{D} \rightarrow \mathbf{C}$  are adjoint if there exist natural transformations  $\eta : 1_{\mathbf{C}} \Rightarrow RL$  and  $\epsilon : LR \Rightarrow 1_{\mathbf{D}}$  which satisfy the following triangle identity:

$$1_{Lx} : Lx \xrightarrow{L\eta_x} LRLx \xrightarrow{\epsilon_{Lx}} Lx$$

and

$$1_{Ry} : Ry \xrightarrow{\eta_{Ry}} RL Ry \xrightarrow{R\epsilon_y} Ry$$

Let  $U : \mathbf{D} \rightarrow \mathbf{C}$  be a forgetful functor, and let  $c \in \mathbf{C}$ . A **free  $\mathbf{D}$ -object** on  $c$  with respect to  $U$  is an object of  $\mathbf{D}$  satisfying the universal property that  $F$  would have if  $F$  would have if it were left-adjoint to  $U$ . More precisely: a free  $\mathbf{D}$ -object on  $c$  consists of an object  $d \in \mathbf{D}$  together with a morphism  $f : c \rightarrow Ud$  in  $\mathbf{C}$  such that for any other  $d' \in \mathbf{D}$  and morphism  $g : c \rightarrow Ud'$ , there exists a unique  $h : d \rightarrow d'$  in  $\mathbf{D}$  such that  $Uh \circ f = g$ . Diagrammatically:

$$\begin{array}{ccc}
 c & \xrightarrow{f} & Ud \\
 & \searrow g & \downarrow Uh \\
 & & Ud'
 \end{array}$$

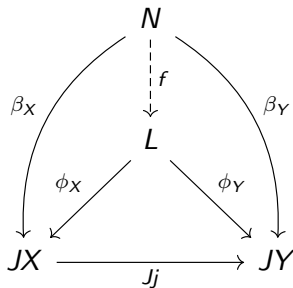


# Examples

Let  $U : \mathbf{Mon} \rightarrow \mathbf{Set}$  be a forgetful functor, and let  $MS$  be the free object in  $\mathbf{Mon}$  generated by set  $S$ . Any set function from  $S$  to the underlying set  $UN$  of another monoid  $N \in \mathbf{Mon}$  extends to a unique monoid homomorphism  $MS \rightarrow N$  per the free construction described in the previous slide. One can check that this construction forms an adjunction  $M \dashv U$  - that is, the following hom-sets are in bijection:

$$\mathbf{Mon}(MS, N) \cong \mathbf{Set}(S, UN)$$

Let  $J : \mathbf{J} \rightarrow \mathbf{C}$  be a diagram of shape  $\mathbf{J}$  in  $\mathbf{C}$ . A **cone** in  $\mathbf{C}$  is an object  $N \in \mathbf{C}$  together with a family  $\beta_X : N \rightarrow JX$  such that for every morphism  $Jj : JX \rightarrow JY$  we have  $\beta_Y = Jj \circ \beta_X$ . A **limit** is a terminal cone  $(L, \phi)$  inducing a unique  $f : N \rightarrow L$  such that all  $\psi_{(-)}$  factor through  $(L, \phi)$ . I.e.  $\beta_X = \phi_X \circ f$ . Diagrammatically,



# Limits and Colimits

A **colimit** is defined dually, in a similar way, with the arrows reversed.

# Limits and Colimits

Notice how familiar those diagrams look?

# Limits and Colimits

This is because products and coproducts as previously defined are limits and colimits themselves!

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This is because products and coproducts as previously defined are limits and colimits themselves! In fact, limits and colimits define functors  $\mathbf{C}^J \rightarrow \mathbf{C}$ .

# Limits and Colimits

Define the diagonal functor  $\Delta : \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{J}}$  taking objects  $c \in \mathbf{C}$  to the constant **J**-shaped diagram. If  $\mathbf{C}$  is a locally small category, then  $\mathbf{C}$  preserves all limits (colimits, resp.) if and only if  $\Delta$  admits a right adjoint (left adjoint, resp.).

# Examples

Let **C** be **Set** or its subcategory **Fin**. Note that there exists an adjunction  $A \times - \dashv (-)^A$  for any  $A$ . For any sets  $A, B, C, D$ , the following natural isomorphisms hold:

$$A \times (B + C) \cong (A \times B) + (A \times C) \quad (B \times C)^A \cong B^A \times C^A$$

$$A^{B+C} \cong A^B \times A^C$$



# Limits and Colimits

## Proof.

The left adjoint  $A \times -$  preserves the coproduct  $B + C$ , the right adjoint  $(-)^A$  preserves the product  $B \times C$ , and the functor  $A^{(-)} : \mathbf{Set}^{op} \rightarrow \mathbf{Set}$  is mutually right-adjoint to itself, and so carries coproducts in  $\mathbf{Set}$  to products in  $\mathbf{Set}$ . The laws of cardinal arithmetic follow by applying the cardinality functor

$$|-| : \mathbf{Fin}_{\mathbf{Iso}} \rightarrow \mathbf{Fin}$$



# Bonus

Bonus: Functors of the form  $\mathbf{D}^{op} \times \mathbf{C} \rightarrow \mathbf{Set}$  are known as *profunctors*. Profunctors define proof-relevant relations between objects in (possibly) heterogeneous categories and the sets of morphisms between objects in  $\mathbf{C}^{op} \times \mathbf{D}$  are called *heteromorphisms*. Adjunctions are natural isomorphisms of profunctors which establishes a correspondence between heteromorphisms in an opposite category, and heteromorphisms in a standard category. Taking this perspective that profunctors denote relations between objects implies that adjunctions encode an isomorphism between dual relations! Very cool. Thanks.