The Algebra and Topology of $\mathbb{R}^n$

Emily Pillmore

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1 Review of Linear Algebra

This section was a quick review of linear algebra from Munkres' perspective. It delved into Vector spaces, linear transformations/isomorphisms, rank, transposition, matrices, inner-products and norms.

problem 1.1 (Cauchy-Schwarz Inequality)**.** *Let V be a vector space with inner product $\langle x, y\rangle$ and norm $\|x\| = \langle x, y\rangle^{1/2}$.*

*a) Prove the **Cauchy-Schwarz inequality**: $\langle x, y\rangle \leq \|x\|\|y\|$. Hint: If $x, y \neq 0$, set $c = 1/\|x\|$ and $d = 1/\|y\|$ and use the fact that $\|cx \pm dy\| \geq 0$.*

b) Prove that $\|x + y\| \leq \|x\| + \|y\|$. Hint: Compute $\langle x + y, x + y\rangle$ and apply a).

c) Prove that $\|x - y\| \geq \|x\| + \|y\|$

Proof. (Cauchy-Schwarz Inequality.)

a) We'll prove this using the hint. Consider the following:

$$\begin{aligned} \langle x, y\rangle &\leq \|x\|\|y\| \\ \langle x, y\rangle^2 &\leq \langle x, x\rangle\langle y, y\rangle \\ 0 &\leq \langle x, x\rangle - \frac{\langle x, y\rangle^2}{\langle y, y\rangle} \end{aligned} \tag{1}$$

Let $\eta = \frac{\langle x,y\rangle}{\langle y,y\rangle}$. Then, we have

$$\begin{aligned} 0 &\leq \langle x, x\rangle - \eta\langle x, y\rangle \\ &\leq \langle x, x\rangle - \eta\langle y, x\rangle \quad \textit{symmetry of the symmetric bilinear form} \\ &\leq \langle x - \eta y, x\rangle \end{aligned} \tag{2}$$

Now, what the hint tells us is that if $\|ax-by\| \geq 0$ then certainly $\langle ax-by, ax-by\rangle = \|ax-by\|^2 \geq 0$. So in order to make use of the hint, we need to relate $\langle x - \eta y, x\rangle$ to something of that form. We will show that $\langle x - \eta y, x\rangle = \langle x - \eta y, x - \eta y\rangle$.

$$\begin{aligned}
\langle x - \eta y, x - \eta y \rangle &= \langle x - \eta y, x \rangle - \langle x - \eta y, -\eta y \rangle \\
&= \langle x - \eta y, x \rangle + \eta \langle x - \eta y, y \rangle \\
&= \langle x - \eta y, x \rangle + \eta(\langle x, y \rangle - \eta \langle y, y \rangle) \\
&= \langle x - \eta y, x \rangle + \eta(\langle x, y \rangle - \langle x, y \rangle) \\
&= \langle x - \eta y, x \rangle
\end{aligned} \tag{3}$$

Hence, $\langle x - \eta y, x \rangle$ is equal to $\langle x - \eta y, x - \eta y \rangle$, which is gauranteed to be ≥ 0. It is 0 in the case where either x or y is 0, since $\langle x, y \rangle \leq \|x\| \|y\|$ forces the equation to 0.

b) Consider the following:

$$\begin{aligned}
\|x + y\| &\leq \|x\| + \|y\| \\
\langle x + y, x + y \rangle &\leq (\|x\| + \|y\|)^2 \\
\langle x + y, x + y \rangle &\leq \langle x, x \rangle + 2\|x\|\|y\| + \langle y, y \rangle \\
\langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle &\leq \langle x, x \rangle + 2\|x\|\|y\| + \langle y, y \rangle \\
\langle x, y \rangle &\leq \|x\|\|y\|
\end{aligned} \tag{4}$$

The final clause of the proof is precisely the Cauchy-Schwarz inequality.

c) Let's begin by considering the following:

$$\begin{aligned}
\|x - y\| &\geq \|x\| - \|y\| \\
\|x - y\|^2 &\geq \langle x, x \rangle - 2\|x\|\|y\| + \langle y, y \rangle \\
\langle x - y, x - y \rangle &\geq \langle x, x \rangle - 2\|x\|\|y\| + \langle y, y \rangle \\
\langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle &\geq \langle x, x \rangle - 2\|x\|\|y\| + \langle y, y \rangle \\
\|x\|\|y\| &\geq \langle x, y \rangle
\end{aligned} \tag{5}$$

The last clause is again the Cauchy-Schwarz inequality.

□

problem 1.2 (Matrix norm)**.** *If A is an n by m matrix, and B is an m by p matrix, show that*

$$|A \cdot B| \leq m|A||B| \tag{6}$$

Proof. Note first that by definition, $A \cdot B = \{c_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}\}$. Since each entry $a_{ik} \leq |A|$ and $b_{kj} \leq |B|$ by definition, we have the following:

$$
\begin{aligned}
|A \cdot B| &= \max\{|c_{ij}| : c_{ij} \sum_{k=1}^{m} a_{ik} \cdot b_{kj}\} \\
&\leq \max\{|c_{ij}| : c_{ij} = \sum_{k=1}^{m} |A| \cdot |B|\} \\
&\leq \max\{|c_{ij}| : c_{ij} = m|A| \cdot |B|\} \\
&\leq m|A| \cdot |B|
\end{aligned} \tag{7}
$$

Hence, $|A \cdot B| \leq m|A| \cdot |B|$.

$\square$

problem 1.3 (Sup norm)**.** *Show that the sup norm on $\mathbb{R}^2$ is not derived from the inner product on $\mathbb{R}^2$. Hint: Suppose $\langle x, y\rangle$ is an inner product on $\mathbb{R}^2$ (not the dot product) having the property that $|x| = \langle x, x\rangle^{1/2}$. Compute $\langle x \pm y, x \pm y\rangle$ and apply the case $x = e_1$ and $y = e_2$.*

Proof. To prove this, we will show a violation of the parallelogram law. Recall that a norm associated with an inner product must satisfy the following law for all $x, y \in \mathbb{R}^2$:

$$
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) \tag{8}
$$

Consider the sup-norm. Using the parallelogram and plugging in the basis vectors e_1 and e_2 for x and y, we have

$$
\begin{gathered}
|x + y|^2 + |x - y|^2 = 2(|x|^2 + |y|^2) \\
|e_1 + e_2|^2 + |e_1 - e_2|^2 = 2(|e_1|^2 + |e_2|^2) \\
|(1, 1)|^2 + |(1, -1)^2 = 2(1 + 1) \\
1 + 1 = 2(2) \\
2 \neq 4
\end{gathered} \tag{9}
$$

A contradiction. Hence, the sup-norm cannot be derived from an inner product.

$\square$