

Categories in Context - Ch. 2 Solutions

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1 Representable Functors

problem 1.1. (2.1.i) For each of the three functors

$$\mathbb{1} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{!} \xrightarrow{1} \end{array} \mathbb{2}$$

Between the categories $\mathbb{1}$ and $\mathbb{2}$ describe the corresponding natural transformations between the covariant functors $\mathbf{Cat} \rightrightarrows \mathbb{2}$ represented by the categories by the categories $\mathbb{1}$ and $\mathbb{2}$.

proof (2.1.i). Consider the covariant representable functors $\mathbf{Cat}(\mathbb{1}, -), \mathbf{Cat}(\mathbb{2}, -) : \mathbf{Cat} \rightarrow \mathbf{Set}$ and let $F : \mathbf{C} \rightarrow \mathbf{D}$. Consider the following diagram for the components of the transformations:

$$\begin{array}{ccc} \mathbf{Cat}(\mathbb{1}, \mathbf{C}) & \begin{array}{c} \xleftarrow{\mathbf{Cat}(0, \mathbf{C})} \xrightarrow{\mathbf{Cat}(!, \mathbf{C})} \\ \xleftarrow{\mathbf{Cat}(1, \mathbf{C})} \end{array} & \mathbf{Cat}(\mathbb{2}, \mathbf{C}) \\ \downarrow \mathbf{Cat}(\mathbb{1}, F) & & \downarrow \mathbf{Cat}(\mathbb{2}, F) \\ \mathbf{Cat}(\mathbb{1}, \mathbf{D}) & \begin{array}{c} \xleftarrow{\mathbf{Cat}(0, \mathbf{D})} \xrightarrow{\mathbf{Cat}(!, \mathbf{D})} \\ \xleftarrow{\mathbf{Cat}(1, \mathbf{D})} \end{array} & \mathbf{Cat}(\mathbb{2}, \mathbf{D}) \end{array}$$

The transformations may be described as follows:

- $\mathbf{Cat}(0, -)$ maps $*$ to the 0 object of $\mathbb{2}$, and given any functor $F : \mathbb{2} \rightarrow \mathbf{C}$, will map the domain of the unique nontrivial arrow $f : 0 \rightarrow 1$ in $\mathbb{2}$ to the domain of the arrow $Ff : F0 \rightarrow F1$.
- Likewise, for $\mathbf{Cat}(1, -)$, given any functor, the transformation will choose a codomain for the unique arrow in $\mathbb{2}$, and given any functor $F : \mathbb{2} \rightarrow \mathbf{C}$, will correspond to the codomain object of the chosen arrow in \mathbf{C} .
- The transformation $\mathbf{Cat}(!, -)$ takes any choice of arrow in $\mathbb{2} \rightarrow \mathbf{C}$ and maps it to an object of \mathbf{C} .

□

problem 1.2. (2.1.ii) Prove that if $F : \mathbf{C} \rightarrow \mathbf{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in \mathbf{C} to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite concrete category which is not representable.

Proof. Let $F : \mathbf{C} \rightarrow \mathbf{Set}$ be a representable functor with representing object $c \in \mathbf{C}$, and let $f : x \rightarrow y$ be a monomorphism in \mathbf{C} . Consider the set function $Ff : Fx \rightarrow Fy$. Since F is representable it is naturally isomorphic to $\mathbf{C}(c, -)$, and Ff is then isomorphic to a set function $\mathbf{C}(c, f) : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, y)$. Consider the parallel morphisms $h, k : w \rightrightarrows x$. Since f is monic, we have that $fh = fk$ implies that $h = k$. Hence, $\mathbf{C}(c, fh) = \mathbf{C}(c, fk)$ implies that $\mathbf{C}(c, k) = \mathbf{C}(c, h)$. Let $w, w' \in \mathbf{C}(c, x)$. Hence, we recover the usual notion of injective function in set: $fw = fw' \Rightarrow w = w'$. Therefore representable functors preserve monomorphisms.

For the contrapositive, let the functor $\pi_0 : \mathbf{Top} \rightarrow \mathbf{Set}$ be the functor taking a topological space to its set of connected components. Then the monomorphism $\{0, 1\} \rightarrow [0, 1]$ is mapped to $\{*\}$. \square

problem 1.3. (2.1.iii) Suppose $F : \mathbf{C} \rightarrow \mathbf{Set}$ is equivalent to $G : \mathbf{D} \rightarrow \mathbf{Set}$ in the sense that there is an equivalence of categories $H : \mathbf{C} \rightarrow \mathbf{D}$ so that GH and F are naturally isomorphic.

- (i) If G is representable, then is F representable?
- (ii) If F is representable, then is G representable?

Proof. Let G be representable and let $K : \mathbf{D} \rightarrow \mathbf{C}$ be the inverse equivalence to H . Note now that we have the following:

$$\begin{aligned}
F &\cong GH \\
&\cong \mathbf{D}(d, -)H && \text{(representability of } G) \\
&\cong \mathbf{D}(d, H-) \\
&\cong \mathbf{D}(HKd, H-) && \text{(equivalence } HK \cong 1_{\mathbf{D}}) \\
&\cong \mathbf{C}(Kd, -) && \text{(H is f.f. due to equivalence)} \\
&\cong \mathbf{C}(c, -) && \text{(K is e.s.o due to equivalence)}
\end{aligned} \tag{1}$$

Hence, F is representable. Verbatim proof for the opposite direction. \square

problem 1.4. (2.1.iv) A functor F defines a **subfunctor** of G if there is a natural transformation $\alpha : F \Rightarrow G$ whose components are monomorphisms. In the case of $G : \mathbf{C}^{op} \rightarrow \mathbf{Set}$, a subfunctor is given by a collection of subsets $Fc \subset Gc$ so that each $Gf : Gc \rightarrow Gc'$ restricts to a function $Ff : Fc \rightarrow Fc'$. Characterize those subsets that assemble into a subfunctor of the representable functor $\mathbf{C}(-, c)$.

Proof. For the functor F to be a subfunctor of $\mathbf{C}(-, c)$, we must build a natural transformation $\alpha : F \Rightarrow \mathbf{C}(-, c)$ such that, given $f : d' \rightarrow d \in \mathbf{C}$, the components $\alpha_d : Fd \rightarrow \mathbf{C}(d, c)$ restrict Fd and widen Fd' when precomposed with f . Thus, to completely characterize the subsets of a subfunctor, we need that the family $\bigcup_d Fd$ is closed precomposition by arbitrary morphisms $d' \rightarrow d$ so that we have a sieve on d . \square