

# Categories in Context - Ch. 1 Solutions

NYC Categories Theory Meetup

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## 1 Categories, Functors, Natural Transformations

This chapter describes the fundamental data associated with categorical structure and its structure-preserving maps called **Functors**. Additionally, we learn about the maps between functors, **Natural Transformations**, and their coherence data. Other topics include sizing constraints, additional structure of morphisms within a category, and begin to understand arguments from duality and via the art of the diagram chase.

### 1.1 Abstract + Concrete Categories

In this section, we discuss the data associated with categories.

#### 1.1.1

- i. (i) *Show that a morphism can have at most one inverse isomorphism.*

We proceed by routine proof of uniqueness. Let  $f : x \rightarrow y$  be a morphism with  $g, h : y \rightrightarrows x$  be an inverse isomorphisms for  $f$ . By definition, if  $g$  or  $h$  are isomorphisms, then  $fg = fh = 1_y$  and  $gf = hf = 1_x$ . Thus we have that  $g(fh) = g(fg) = g1_y = g$  and  $(gf)g = (gf)h = 1_xh = h$  (parentheses added for emphasis). Hence,  $g = h$ .

- (ii) *Consider a morphism  $f : x \rightarrow y$ . Show that if there exists a pair of morphisms  $g, h : y \rightrightarrows x$  so that  $gf = 1_x$  and  $fh = 1_y$ , then  $g = h$  and  $f$  is an isomorphism.*

This shows that we can weaken the above to understand that for a given  $f : x \rightarrow y$ , any pair of parallel morphisms  $g, h : y \rightrightarrows x$  such that if  $gf = 1_x$  and  $fh = 1_y$  implies  $g = h$  and  $f$  is an isomorphism. The notion being described here is that an isomorphism is "built", so to speak, by the data of a right and left inverse. If these inverses coincide, then one has an isomorphism!

As before, note that if  $gf = 1_x$ , then  $(gf)h = 1_xh = h$ . Likewise, we have that  $fh = 1_y$ , and therefore  $g(fh) = g1_y = g$ . Hence,  $fgfh = fg = fh$ , which implies  $fg = 1_y$ . Hence,  $g$  is a two-sided inverse, and  $h$  is as well, and hence, are isomorphisms. According to the previous exercise, these must coincide.

- (iii) *Let  $\mathbf{C}$  be a category. Show that the collection of isomorphisms in  $\mathbf{C}$  defines a subcategory, the **maximal groupoid** of  $\mathbf{C}$ .*

First, let's show that the collection of isomorphisms of  $\mathbf{C}$  defines a subcategory,  $\mathbf{Iso}_{\mathbf{C}}$  which has a groupoid structure. Then, we will prove it is maximal in a precise sense.

Note that for any object  $c \in \mathbf{C}$ , we require that  $c \in \mathbf{Iso}_{\mathbf{C}}$ , since the identity morphism is an isomorphism. Thus the objects of  $\mathbf{Iso}_{\mathbf{C}}$  must be those of  $\mathbf{C}$ . Additionally, composition in  $\mathbf{Iso}_{\mathbf{C}}$  is that of  $\mathbf{C}$ . Indeed, it is a groupoid by definition, since all isomorphisms are invertible, and this subcategory  $\mathbf{Iso}_{\mathbf{C}}$ .

Next, we must show that it is maximal. To make this precise, we simply show that for any other groupoid  $\mathbf{G}$ , then  $\mathbf{G}$  is a subcategory of  $\mathbf{Iso}_{\mathbf{C}}$ , or is  $\mathbf{Iso}_{\mathbf{C}}$  itself. Let  $\mathbf{G}$  be another maximal groupoid. Since  $\mathbf{G}$  is the collection of all isomorphisms for all objects in  $\mathbf{C}$ , we must have that every object and every isomorphism of  $\mathbf{Iso}_{\mathbf{C}}$  is in  $\mathbf{G}$ , hence,  $\mathbf{Iso}_{\mathbf{C}}$  is a subcategory of  $\mathbf{G}$ . Likewise,  $\mathbf{G}$  is a subcategory of  $\mathbf{Iso}_{\mathbf{C}}$ . There are no differences between these categories because isomorphisms are unique, and since they define the same objects and morphisms, they are identical.

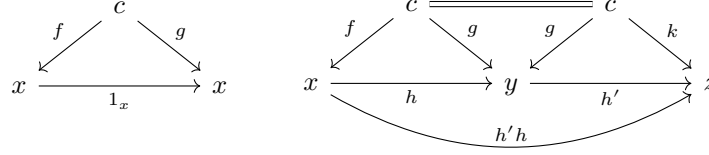
ii. For any category, show that:

- (i) There is a category  $c/\mathbf{C}$ , whose objects are morphisms  $f : c \rightarrow x$  with domain  $c$  and in which a morphism from  $f : c \rightarrow x$  to  $g : c \rightarrow y$  is a map  $h : x \rightarrow y$  between the codomains so that the triangle

$$\begin{array}{ccc} & c & \\ f \swarrow & & \searrow g \\ x & \xrightarrow{h} & y \end{array}$$

commutes, i.e., so that  $g = hf$ .

We must show that the above forms a category  $c/\mathbf{C}$  with objects that are morphisms  $f : c \rightarrow x$ ,  $g : c \rightarrow y$  in  $\mathbf{C}$ , and morphisms  $h : x \rightarrow y$ . Consider the following triangles, which are the identity for  $f$  and composite arrow for  $h'h$  for  $h' : y \rightarrow z$ , respectively:



Note that composition of the latter triangles is associative, inheriting associativity from the underlying category  $\mathbf{C}$ , and the identity triangle is indeed an identity as a result. Hence,  $c/\mathbf{C}$  is a category, and dually,  $\mathbf{C}/c$  is a category, satisfying (ii).

## 1.2 Duality

In this section, we discuss additional structure of morphisms, and introduce the notion of "duality" and opposite categories  $\mathbf{C}^{\text{op}}$ .

### 1.2.1

- (i) Show that  $\mathbf{C}/c \cong (c/\mathbf{C}^{\text{op}})^{\text{op}}$ . Defining  $\mathbf{C}/c$  to be  $(c/\mathbf{C}^{\text{op}})^{\text{op}}$ , deduce the last 2 exercises from section 1.1.

Note that  $c/\mathbf{C}^{\text{op}}$  is simply  $c/\mathbf{C}$  with the morphisms reversed. Morphisms in the original category are simply arrows  $h : x \rightarrow y$  between the codomains of objects  $f : c \rightarrow x$  and  $g : c \rightarrow y$ , hence, morphisms in  $c/\mathbf{C}^{\text{op}}$  simply reverse direction, so that  $h$  becomes  $h^{\text{op}} : y \rightarrow x$ . Indeed, if we consider  $(c/\mathbf{C}^{\text{op}})^{\text{op}}$ , then the opposite functor takes  $f, g$  to  $f^{\text{op}}, g^{\text{op}}$  and  $h^{\text{op}}$  to  $(h^{\text{op}})^{\text{op}} = h$ . Thus we can see that the direction of  $f$  and  $g$  reverse, but  $h$  is as it was defined - a morphism in  $\mathbf{C}$  again. However, the direction of  $f$  and  $g$  has reversed, we now have objects  $f^{\text{op}} : x \rightarrow c$ , and  $g^{\text{op}} : y \rightarrow c$ . This is precisely the data of  $\mathbf{C}/c$ .

### 1.2.2

- (i) Show that a morphism  $f : x \rightarrow y$  is a split epimorphism in a category  $\mathbf{C}$  if and only if for all  $c \in \mathbf{C}$ , post-composition  $f_* : \mathbf{C}(c, x) \rightarrow \mathbf{C}(c, y)$  defines a surjective function.

Lets show the original implication ( $\Rightarrow$ ) first. Let  $f : x \rightarrow y$  be a split epimorphism. By definition, a morphism is a split-epimorphism if there exists a monomorphism  $s : y \rightarrow x$  which is a right inverse of  $f$  - i.e.  $fs = 1_y$ . Note that since  $s$  is monic, by definition 1.2.7, post-composition by  $s$  defines an injection  $s_* : \mathbf{C}(c, y) \rightarrow \mathbf{C}(c, x)$ . Now consider post-composition by  $f$ .  $f_*s_* = 1_{\mathbf{C}(c, x)}$ , hence the induced  $f_*$  and  $s_*$  is also a split monic and epic pair, and hence,  $f_*$  is a surjection.

Now lets show ( $\Leftarrow$ ). Let  $f_*$  be a surjection for all  $c \in \mathbf{C}$ . Note that in  $\mathbf{C}^{\text{op}}$ ,  $f_*^{\text{op}} : \mathbf{C}^{\text{op}}(y, c) \rightarrow \mathbf{C}^{\text{op}}(x, c)$  is an injection. By 1.2.7, this defines an epimorphism in  $f : x \rightarrow y$  in  $\mathbf{C}$ .

- (ii) Argue by duality...

Left up to meetup. This one isn't bad, just tedious.

### 1.2.3

Lets do (i) and (ii') for flavor, but we should probably do the others in the meetup.

- (i) If  $f : x \rightarrowtail y$  is monic and  $g : y \rightarrowtail z$  is monic, then  $gf : x \rightarrowtail z$  is also monic.

Let  $h, k : w \rightarrowtail x$  and suppose  $gh = gk$ . We will show that  $gf$  is monic. Note that by associativity,  $g(fh) = g(fk)$  implies that  $fh = fk$  since  $g$  is mono. Likewise,  $fh = fk$  implies  $h = k$  since  $f$  is mono. Hence,  $(gf)h = (gf)k$  implies  $h = k$ . Thus,  $gf$  is monic.

- (ii) If  $f : x \rightarrow y$  and  $g : y \rightarrow z$  are morphisms, then if  $gf$  is epic,  $g$  is epic.

Suppose  $gf$  is epic. Let  $h, k : z \rightarrow w$ , and suppose  $hg = kg$ . Since  $gf$  is epic, this implies  $h = k$ . However, note that  $hg = kg$ , then  $hg = kg$  implies  $h = k$  by epicness of  $g$ . Hence,  $g$  must be epic.

Now, note that monomorphisms (resp. epimorphisms) are closed under composition and defines a trivial monomorphism (resp. epimorphism) for all for every  $c \in \mathbf{C}$ . Thus, the class of mono- and epimorphisms define subcategories of  $\mathbf{C}$ .

### 1.2.4

Jumping to exercise 1.2.iv.

- (i) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Prove its dual statement.

Let  $f : x \rightarrow y$  be a monomorphism and a retract for  $s : y \rightarrow x$ . To prove  $f$  is an isomorphism, we must prove that  $f$  has a two-sided inverse. Note that  $s$  is probably a good candidate, since it is already a one-sided inverse for  $f$  -  $fs = 1_y$ . Note that  $s$  is the unique right inverse of  $f$ , since if there were another  $s' : y \rightarrow x$ , then if  $fs = fs'$ , then  $s = s'$  because  $f$  is monic. Note also, by  $f$ 's monic properties that  $fsf = f1_x$  implies  $sf = 1_x$ . Hence  $s$  is a unique two-sided inverse for  $f$ .

Flip the arrows for the dual statement. Lets do this in the meetup.

## 1.3 Functoriality

- (i) What is a functor between groups, regarded as one-object categories?

The category structure of a group seen as a one-object category  $\mathbf{BG}$  contains the following data: there is a single object,  $*$ , and morphisms the whole of the group  $G$ . Thus, each element of  $G$  can be seen as an element collection of endomorphisms  $\mathbf{BG}(*, *)$ . A functor of groups  $F : \mathbf{BG} \rightarrow \mathbf{catBG}'$  takes objects to objects and morphisms to morphisms. Hence,  $F : * \mapsto *$  and  $F(gh) = FgFh = g'h'$  in  $\mathbf{BG}'$ . Note also that the identity element in  $G$  maps to the  $1_*$ , hence,  $F1_* = 1_{F*} = 1_{F*}$ , which tells us that identities map to identities. This is precisely the structure of the group homomorphism with which we are familiar.

- (ii) What is a functor between preorders regarded as categories?

Monotonic functions.

- (iii) Verify that the constructions introduced in 1.3.11 are functorial

Lets do this in meetup

- (iv) Find an example to show that the objects and morphisms in the image of a functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  do not necessarily define a subcategory of  $\mathbf{D}$ .

<https://math.stackexchange.com/questions/413138/can-it-happen-that-the-image-of-a-functor-is-not-a-category>

- (v) What is the difference between a functor  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$  and a functor  $\mathbf{C} \rightarrow \mathbf{D}^{\text{op}}$ ? What is the difference between a functor  $\mathbf{C} \rightarrow \mathbf{D}$  and  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$ ?

A functor  $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$  maps arrows  $d \rightarrow c$  to arrows  $Fc \rightarrow Fd$ , while  $F : \mathbf{C} \rightarrow \mathbf{D}^{\text{op}}$  maps  $c \rightarrow d$  to  $Fd \rightarrow Fc$ . The outcome? roughly the same. It depends on the intent. Note that the two constructions are dual, but  $F$  is always a contravariant functor. Likewise, next case,  $F$  is always a covariant functor. There is a kind of polarity thing going on here, as well, where one can view variance as positive or negative polarity. The composition of two co- or contravariant functors is a covariant functor, and the composition of contra with covariant is contravariant no matter what.

- (vi) Given functors  $F : \mathbf{D} \rightarrow \mathbf{C}$  and  $G : \mathbf{E} \rightarrow \mathbf{C}$ , show that there is a category, called the **comma category**  $F \downarrow G$  which has

- as objects, triples  $(d \in \mathbf{D}, e \in \mathbf{E}, f : Fd \rightarrow Ge \in \mathbf{C})$ , and
- as morphisms  $(d, e, f) \rightarrow (d', e', f')$ , a pair of morphisms  $(h : d \rightarrow d', k : e \rightarrow e')$  so that the square

$$\begin{array}{ccc} Fd & \xrightarrow{f} & Ge \\ Fh \downarrow & & \downarrow Gk \\ Fd' & \xrightarrow{f'} & Ge' \end{array}$$

commutes in  $\mathbf{C}$ , i.e., so that  $f' \cdot Fh = Gk \cdot f$ .

Define a pair of projection functors  $dom : F \downarrow G \rightarrow \mathbf{D}$  and  $cod : F \downarrow G \rightarrow \mathbf{E}$ .

First, note the target categories are  $\mathbf{D}$  and  $\mathbf{E}$  respectively for these projection functors. Define  $dom$  and  $cod$  as follows: For each object (triple) in  $F \downarrow G$ , take first projections to be  $dom$ , and for  $cod$ , second projections.

- (vii) Define functors to construct the slice categories  $c/\mathbf{C}$  and  $\mathbf{C}/c$  as special cases of comma categories. What are the projection functors?

Let  $F : \mathbf{1} \rightarrow \mathbf{C}$  be the functor from the terminal category  $\mathbf{1}$  to the object  $c \in \mathbf{C}$ . Let  $G \cong 1_{\mathbf{C}}$ . Then, the data of the comma category construction gives us objects  $(* \in \mathbf{1}, x \in \mathbf{C}, f : c \rightarrow x)$ , and morphisms  $(*, x, f : c \rightarrow x) \rightarrow (*, y, g : c \rightarrow y)$  given by a pair  $(1_*, k : x \rightarrow y)$ .

Thus, we have the square

$$\begin{array}{ccc} c & \xrightarrow{f} & x \\ \parallel & & \downarrow h \\ c & \xrightarrow{g} & y \end{array}$$

Dually, we, swap  $F$  and  $G$  to achieve the over slice category. The projection functors for this data remain the same.

(viii) Lets do this one in the Meetup.

## 1.4 Naturality

This section begins to discuss naturality, natural transformations, and (de)categorification.

(i)