Categories in Context - Ch. 2 Solutions

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1 Representable Functors

problem 1.1. *(2.1.i) For each of the three functors*

$$\mathbb{1} \underset{1}{\overset{0}{\underset{\longleftarrow ! \longrightarrow}{\longrightarrow}}} \mathbb{2}$$

Between the categories $\mathbb{1}$ *and* $\mathbb{2}$ *describe the corresponding natural transformations between the covariant functors* $\mathsf{Cat} \rightrightarrows \mathbb{2}$ *represented by the categories by the categories* $\mathbb{1}$ *and* $\mathbb{2}$.

proof (2.1.i). Consider the covariant represented functors $\mathsf{Cat}(\mathbb{1}, -), \mathsf{Cat}(\mathbb{2}, -) : \mathsf{Cat} \to \mathsf{Set}$ and let $F : \mathsf{C} \to \mathsf{D}$. Consider the following diagram for the components of the transformations:

$$\begin{tikzcd}
\mathsf{Cat}(\mathbb{1}, \mathsf{C}) \arrow[d, "\mathsf{Cat}(\mathbb{1}, F)"'] & \mathsf{Cat}(\mathbb{2}, \mathsf{C}) \arrow[l, shift right=2, "\mathsf{Cat}(0,\mathsf{C})"'] \arrow[l, "\mathsf{Cat}(!,\mathsf{C})"] \arrow[l, shift left=2, "\mathsf{Cat}(1,\mathsf{C})"] \arrow[d, "\mathsf{Cat}(\mathbb{2}, F)"] \\
\mathsf{Cat}(\mathbb{1}, \mathsf{D}) & \mathsf{Cat}(\mathbb{2}, \mathsf{D}) \arrow[l, shift right=2, "\mathsf{Cat}(0,\mathsf{D})"'] \arrow[l, "\mathsf{Cat}(!,\mathsf{D})"] \arrow[l, shift left=2, "\mathsf{Cat}(1,\mathsf{D})"]
\end{tikzcd}$$

The transformations may be described as follows:

- $\mathsf{Cat}(0, -)$ maps $*$ to the 0 object of $\mathbb{2}$, and given any functor $F : \mathbb{2} \to \mathsf{C}$, will map the domain of the unique nontrivial arrow $f : 0 \to 1$ in $\mathbb{2}$ to the domain of the arrow $Ff : F0 \to F1$.
- Likewise,, for $\mathsf{Cat}(1, -)$, given any functor, the transformation will choose a codomain for the unique arrow in $\mathbb{2}$, and given any functor $F : \mathbb{2} \to \mathsf{C}$, will correspond to the codomain object of the chosen arrow in C.
- The transformation $\mathsf{Cat}(!, -)$ takes any choice of arrow in $\mathbb{2} \to \mathsf{C}$ and maps it to an object of C.

$\square$

problem 1.2. *(2.1.ii)Prove that if $F : \mathsf{C} \to \mathsf{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in C to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite concrete category which is not representable.*

Proof. Let $F : \mathsf{C} \to \mathsf{Set}$ be a representable functor with representing object $c \in \mathsf{C}$, and let $f : x \rightarrowtail y$ be a monomorphism in C. Consider the set function $Ff : Fx \to Fy$. Since F is representable it is naturally isomorphic to $\mathsf{C}(c, -)$, and Ff is then isomorphic to a set function $\mathsf{C}(c, f) : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$. Consider the parallel morphisms $h, k : w \rightrightarrows x$. Since f is monic, we have that $fh = fk$ implies that $h = k$. Hence, $\mathsf{C}(c, fh) = \mathsf{C}(c, fk)$ implies that $\mathsf{C}(c, k) = \mathsf{C}(c, h)$. Let $w, w' \in \mathsf{C}(c, x)$. Then, we recover the usual statement for set injection: for any two w, w', $f1_w = f1_{w'}$ implies that $1_w = 1_{w'}$, and therefore $w = w'$. Hence, we recover the usual notion of injective function in set: $fw = fw' \Rightarrow w = w'$. Therefore representable functors preserve monomorphisms.

For the contrapositive, let the functor $\pi_0 : \mathsf{Top} \to \mathsf{Set}$ be the functor taking a topological space to its set of connected components. Then the monomorphism $\{0, 1\} \to [0, 1]$ is mapped to $\{*\}$. ☐

problem 1.3. *(2.1.iii) Suppose $F : \mathsf{C} \to \mathsf{Set}$ is equivalent to $G : \mathsf{D} \to \mathsf{Set}$ in the sense that there is an equivalence of categories $H : \mathsf{C} \to \mathsf{D}$ so that GH and F are naturally isomorphic.*

(i) If G is representable, then is F representable?

(ii) If F is representable, then is G representable?

Proof. Let G be representable and let $K : D \to C$ be the inverse equivalence to H. Note now that we have the following:

$$\begin{aligned}
F &\cong GH \\
&\cong \mathsf{D}(d, -)H && \text{(representability of } G\text{)} \\
&\cong \mathsf{D}(d, H-) \\
&\cong \mathsf{D}(HKd, H-) && \text{(equivalence } HK \cong 1_{\mathsf{D}}\text{)} \\
&\cong \mathsf{C}(Kd, -) && \text{(H is f.f. due to equivalence)} \\
&\cong \mathsf{C}(c, -) && \text{(K is e.s.o due to equivalence)}
\end{aligned} \tag{1}$$

Hence, F is representable. Verbatim proof for the opposite direction. ☐

problem 1.4. *(2.1.iv) A functor F defines a **subfunctor** of G if there is a natural transformation $\alpha : F \Rightarrow G$ whose components are monomorphisms. In the case of $G : \mathsf{C}^{op} \to \mathsf{Set}$, a subfunctor is given by a collection of subsets $Fc \subset Gc$ so that each $Gf : Gc \to Gc'$ restricts to a function $Ff : Fc \to Fc'$. Characterize those subsets that assemble into a subfunctor of the representable functor $\mathsf{C}(-, c)$.*

Proof. For the functor F to be a subfunctor of $\mathsf{C}(-, c)$, we must build a natural transformation $\alpha : F \Rightarrow \mathsf{C}(-, c)$ such that, given $f : d' \to d \in \mathsf{C}$, the components $\alpha_d : Fd \to \mathsf{C}(d, c)$ restrict Fd and widen Fd' when precomposed with f. Thus, to completely characterize the subsets of a subfunctor, we need that the family $\bigcup_d Fd$ is closed precomposition by arbitrary morphisms $d' \to d$ so that we have a sieve on d. $\square$