Categories in Context - Ch. 1 Solutions

NYC Categories Theory Meetup

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1 Categories, Functors, Natural Transformations

This chapter describes the fundamental data associated with categorical structure and its structure-preserving maps called **Functors**. Additionally, we learn about the maps between functors, **Natural Transformations**, and their coherence data. Other topics include sizing constraints, additional structure of morphisms within a category, and begin to understand arguments from duality and via the art of the diagram chase.

1.1 Abstract + Concrete Categories

In this section, we discuss the data associated with categories.

1.1.1

i. (i) *Show that a morphism can have at most one inverse isomorphism.*

We proceed by routine proof of uniqueness. Let $f : x \to y$ be a morphism with $g, h : y \rightrightarrows x$ be inverse isomorphisms for f. By definition, if g or h are isomorphisms, then $fg = fh = 1_y$ and $gf = hf = 1_x$. Thus we have that $g(fh) = g(fg) = g1_y = g$ and $(gf)g = (gf)h = 1_x h = h$ (parentheses added for emphasis). Hence, $g = h$.

(ii) *Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.*

This shows that we can weaken the above to understand that for a given $f : x \to y$, *any* pair of parallel morphisms $g, h : y \rightrightarrows x$ such that $gf = 1_x$ and $fh = 1_y$ implies $g = h$ and f is an isomorphism. The notion being described here is that an isomorphism is "built", so to speak, by the data of a right and left inverse. If these inverses coincide, then one has an isomorphism!

As before, note that if $gf = 1_x$, then $(gf)h = 1_x h = h$. Likewise, we have that $fh = 1_y$, and therefore $g(fh) = g1_y = g$. Hence, $fgfh = fg = fh$, which implies $fg = 1_y$. Hence, g is a two-sided inverse, and h is as well, and hence, are isomorphisms. According to the previous exercise, these must coincide.

(iii) *Let C be a category. Show that the collection of isomorphisms in C defines a subcategory, the **maximal groupoid** of C.*

First, let's show that the collection of isomorphisms of $\mathbf{C}$ defines a subcategory, $\mathbf{Iso}_{\mathbf{C}}$ which has a groupoid structure. Then, we will prove it is maximal in a precise sense.

Note that for any object $c \in \mathbf{C}$, we require that $c \in \mathbf{Iso}_{\mathbf{C}}$, since the identity morphism is an isomorphism. Thus the objects of $\mathbf{Iso}_{\mathbf{C}}$ must be those of $\mathbf{C}$. Additionally, composition in $\mathbf{Iso}_{\mathbf{C}}$ is that of $\mathbf{C}$. Indeed, it is a groupoid by definition, since all isomorphisms are invertible, and this subcategory $\mathbf{Iso}_{\mathbf{C}}$.

Next, we must show that it is maximal. To make this precise, we simply show that for any other groupoid $\mathbf{G}$, then $\mathbf{G}$ is a subcategory of $\mathbf{Iso}_{\mathbf{C}}$, or is $\mathbf{Iso}_{\mathbf{C}}$ itself. Let $\mathbf{G}$ be another maximal groupoid. Since $\mathbf{G}$ is the collection of all isomorphisms for all objects in $\mathbf{C}$, we must have that every object and every isomorphism of $\mathbf{Iso}_{\mathbf{C}}$ is in $\mathbf{G}$, hence, $\mathbf{Iso}_{\mathbf{C}}$ is a subcategory of $\mathbf{G}$. Likewise, $\mathbf{G}$ is a subcategory of $\mathbf{Iso}_{\mathbf{C}}$.There are no differences between these categories because isomorphisms are unique, and since they define the same objects and morphisms, they are identical.

ii. For any category, show that:

(i) There is a category $c/\mathbf{C}$, whose objects are morphisms $f : c \to x$ with domain c and in which a morphisms from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "g"] & \\
x \arrow[rr, "h"'] & & y
\end{tikzcd}$$

commutes, i.e., so that $g = hf$.

We must show that the above forms a category $c/\mathbf{C}$ with objects that are morphisms $f : c \to x$, $g : c \to y$ in $\mathbf{C}$, and morphisms $h : x \to y$. Consider the following triangles, which are the identity for f and composite arrow for $h'h$ for $h' : y \to z$, respectively:

$$\begin{tikzcd}
& c \arrow[dl, "f"'] \arrow[dr, "f"] & \\
x \arrow[rr, "1_x"'] & & x
\end{tikzcd}
\qquad
\begin{tikzcd}
& c \arrow[rr, equal] \arrow[dl, "f"'] \arrow[dr, "g"] & & c \arrow[dl, "g"'] \arrow[dr, "k"] & \\
x \arrow[rr, "h"'] \arrow[rrrr, bend right, "h'h"'] & & y \arrow[rr, "h'"'] & & z
\end{tikzcd}$$

Note that composition of the latter triangles is associative, inheriting associativity from the underlying category $\mathbf{C}$, and the identity triangle is indeed an identity as a result. Hence, $c/\mathbf{C}$ is a category, and dually, $\mathbf{C}/c$ is a category, satisfying (ii).

1.2 Duality

In this section, we discuss additional structure of morphisms, and introduce the notion of "duality" and opposite categories $\mathbf{C}^{op}$.

1.2.1

(i) Show that $\mathsf{C}/c \cong (c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$. Defining C/c to be $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$, deduce the last 2 exercises from section 1.1.

Note that $c/\mathsf{C}^{\mathrm{op}}$ is simply c/C with the morphisms reversed. Morphisms in the original category are simply arrows $h : x \to y$ between the codomains of objects $f : c \to x$ and $g : c \to y$, hence, morphisms in $c/\mathsf{C}^{\mathrm{op}}$ simply reverse direction, so that h becomes $h^{\mathrm{op}} : y \to x$. Indeed, if we consider $(c/\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$, then the opposite functor takes f, g to $f^{\mathrm{op}}, g^{\mathrm{op}}$ and h^{op} to $(h^{\mathrm{op}})^{\mathrm{op}} = h$. Thus we can see that the direction of f and g reverse, but h is as it was defined - a morphism in C again. However, the direction of f and g has reversed, we now have objects $f^{\mathrm{op}} : x \to c$, and $g^{\mathrm{op}} : y \to c$. This is precisely the data of C/c.

1.2.2

(i) Show that a morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, post-composition $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ defines a surjective function.

Lets show the original implication ($\Rightarrow$) first. Let $f : x \to y$ be a split epimorphism. By definition, a morphism is a split-epimorphism if there exists a monomorphism $s : y \to x$ which is a right inverse of f - i.e. $fs = 1_y$. Note that since s is monic, by definition 1.2.7, post-composition by s defines an injection $s_* : \mathsf{C}(c, y) \to \mathsf{C}(c, x)$. Now consider post-composition by f. $f_* s_* = 1_{\mathsf{C}(c,x)}$, hence the induced f_* and s_* is also a split monic and epic pair, and hence, f_* is a surjection.

Alternatively, suppose s and f are as above. Let $g : c \to y$ be a morphism. Consider $t = sg : c \to x$. Then, $g = 1_y g = (fs)g = f(sg) = t$, whence, $fg = t$. This ranges over all c, x, y, hence, f_* is a surjection.

Now lets show ($\Leftarrow$). Since f_* is surjective on all points $c \in \mathsf{C}$, let $c = y$. Thus, we have a function we have a function $1_y \in \mathsf{C}(y, y)$ and a surjective function f_* : **C**(y,x) →**C**(y,y), such that there exists some g with $fg = 1_y$. Hence, f splits.

(ii) Argue by duality...

Left up to meetup. This one isn't bad, just tedious.

1.2.3

Lets do (i) and (ii') for flavor, but we should probably do the others in the meetup.

(i) If $f : x \rightarrowtail y$ is monic and $g : y \rightarrowtail$ is monic, then $gf : x \rightarrowtail z$ is also monic.

Let $h, k : w \rightrightarrows x$ and suppose $gfh = gfk$. We will show that gf is monic. Note that by associativity, $g(fh) = g(fk)$ implies that $fh = fk$ since g is mono. Likewise, $fh = fk$ implies $h = k$ since f is mono. Hence, $(gf)h = (gf)k$ implies $h = k$. Thus, gf is monic.

(ii) If $f : x \to y$ and $g : y \to z$ are morphisms, then if gf is epic, g is epic.

Suppose gf is epic. Let $h, k : z \to w$, and suppose $hgf = kgf$. Since gf is epic, this implies $h = k$. However, note that if $hg = kg$, then $hgf = kgf$ implies $h = k$ by epicness of gf. Hence, g must be epic.

Now, note that monomorphisms (resp. epimorphisms) are closed under composition and the identity for defines a trivial monomorphism (resp. epimorphism) for all for every $c \in \mathbf{C}$. Thus, the class of mono- and epimorphisms define subcategories of $\mathbf{C}$.

1.2.4

Jumping to exercise 1.2.*iv*.

(i) Prove that a morphism that is both a monomorphism and a split epimorphism is necessarily an isomorphism. Prove its dual statement.

Let $f : x \to y$ be a monomorphism and a retract for $s : y \to x$. To prove f is an isomorphism, we must prove that f has a two-sided inverse. Note that s is probably a good candidate, since it is already a one-sided inverse for f - $fs = 1_y$. Note that s is the unique right inverse of f, since if there were another $s' : y \to x$, then if $fs = fs'$, then $s = s'$ because f is monic. Note also, by f's monic properties that $fsf = f1_x$ implies $sf = 1_x$. Hence s is a unique two-sided inverse for f.

Flip the arrows for the dual statement. Lets do this in the meetup.

1.3 Functoriality

(i) What is a functor between groups, regarded as one-object categories?

The category structure of a group seen as a one-object category $\mathbf{BG}$ contains the following data: there is a single object, $*$, and morphisms the whole of the group G. Thus, each element of G can be seen as an element collection of endomorphisms $\mathbf{BG}(*, *)$. A functor of groups $F : \mathbf{BG} \to \mathbf{BG}'$ takes objects to objects and morphisms to morphisms. Hence, $F : * \mapsto *$ and $F(gh) = FgFh = g'h'$ in $\mathbf{BG}'$. Note also that the identity element in G maps to the 1_*, hence, $F1_* = 1_{F*=*}$, which tells us that identities map to identities. This is precisely the structure of the group homomorphism with which we are familiar.

(ii) What is a functor between preorders regarded as categories?

Monotonic functions.

(iii) Verify that the constructions introduced in 1.3.11 are functorial

Lets do this in meetup

(iv) Find an example to show that the objects and morphisms in the image of a functor $F : \mathbf{C} \to \mathbf{D}$ do not necessarily define a subcategory of $\mathbf{D}$.

https://math.stackexchange.com/questions/413138/can-it-happen-that-the-image-of-a-functor-is-not-a-category

(v) What is the difference between a functor $\mathbf{C}^{\text{op}} \to \mathbf{D}$ and a functor $\mathbf{C} \to \mathbf{D}^{\text{op}}$? What is the difference between a functor $\mathbf{C} \to \mathbf{D}$ and $\mathbf{C}^{\text{op}} \to \mathbf{D}^{\text{op}}$?

A functor $F : \mathbf{C}^{\text{op}} \to \mathbf{D}$ maps arrows $d \to c$ to arrows $Fc \to Fd$, while $F : \mathbf{C} \to \mathbf{D}^{\text{op}}$ maps $c \to d$ to $Fd \to Fc$. The outcome? roughly the same. It depends on the intent. Note that the two constructions are dual, but F is always a contravariant funtor. Likewise, next case, F is always a covariant functor. There is a kind of polarity thing going on here, as well, where one can view variance as positive or negative polarity. The composition of two co- or contravariant functors is a covariant functor, and the composition of contra with covariant is contravariant no matter what.

(vi) Given functors $F : \mathsf{D} \to \mathsf{C}$ and $G : \mathsf{E} \to \mathsf{C}$, show that there is a category, called the **comma category** $F \downarrow G$ which has

- as objects, triples $(d \in \mathsf{D}, e \in \mathsf{E}, f : Fd \to Ge \in \mathsf{C})$, and
- as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $(h : d \to d', k : e \to e')$ so that the square

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$

commutes in C, i.e., so that $f' \cdot Fh = Gk \cdot f$.

Define a pair of projection functors $dom : F \downarrow G \to \mathsf{D}$ and $cod : F \downarrow G \to \mathsf{E}$.

First, note the target categories are D and E respectively for these projection functors. Define *dom* and *cod* as follows: For each object (triple) in $F \downarrow G$, take first projections to be *dom*, and for *cod*, second projections.

(vii) Define functors to construct the slice categories c/C and C/c as special cases of comma categories. What are the projection functors?

Let $F : \mathbb{1} \to \mathsf{C}$ be the functor from the terminal category $\mathbb{1}$ to the object $c \in \mathsf{C}$. Let $G \cong 1_{\mathsf{C}}$. Then, the data of the comma category construction gives us objects $(* \in \mathbb{1}, x \in \mathsf{C}, f : c \to x)$, and morphisms $(*, x, f : c \to x) \to (*, y, g : c \to y)$ given by a pair $(1_*, k : x \to y)$.

Thus, we have the square

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, equal] & x \arrow[d, "h"] \\ c \arrow[r, "g"'] & y \end{tikzcd}$$

Dually, we, swap F and G to achieve the over slice category. The projection functors for this data remain the same.

(viii) Lets do this one in the Meetup.

1.4 Naturality

This section begins to discuss naturality, natural transformations, and (de)categorification.

(i) Suppose $\alpha : F \Rightarrow G$ is a natural isomorphism. Show that the inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$

Let α be a natural isomorphism. By definition, this means that every component α_c is an isomorphism. Hence, every component α_c has a unique two-sided inverse - call it α_c^{-1} :

$Gc \to Fc$. Since α_c ranges over all $c \in \mathbf{C}$, so too must α_c^{-1}, defining precisely the data of a transformation $G \Rightarrow F$. Naturality follows from the naturality of α_c. For if each component α_c^{-1} is an inverse for α_c, then the following rectangle must commute for some $f : c \to d$:

$$\begin{tikzcd}
Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "Gf"'] \arrow[rr, bend left, "1_{Gc}"] & Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"] & Gc \arrow[d, "Gf"] \\
Gd \arrow[r, "\alpha_d^{-1}"'] \arrow[rr, bend right, "1_{Gd}"'] & Fd \arrow[r, "\alpha_d"'] & Gd
\end{tikzcd}$$

(ii) What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

Let $\alpha : F \Rightarrow G$ be a natural transformation between functors $F, G : \mathbf{BG} \to \mathbf{BG}^{'}$. The data of α consists of components α_g for each $g \in G$ such that $Fg \to Gg$. From the previous exercises, we know that functors F, G of these categories translate into group homomorphisms $G \to G'$ mapping unital elements to unital elements, and preserving the group structure. There is precisely one component of α: α_*. The picture being revealed is that of a composition between permutation of the group $\mathbf{BG}$ - an inner automorphism - and a homomorphism of groups.

(iii) If you're into it, go for it.

(iv) In the notation of Example 1.4.7, prove that distinct parallel morphisms $f, g : c \rightrightarrows d$ define distinct natural transformations

$$f_*, g_* : \mathbf{C}(-, c) \Rightarrow \mathbf{C}(-, d) \qquad \text{and} \qquad f^*, g^* : \mathbf{C}(d, -) \Rightarrow \mathbf{C}(c, -)$$

Lets suppose that f, g were distinct, but defined indistinct natural transformations. This yields a contradiction via the naturality condition. lets write this one out.

(v) Recall the construction of the comma category for any pair of functors $F : \mathbf{D} \to \mathbf{C}$ and $G : \mathbf{E} \to \mathbf{C}$ described in the previous section's exercises. From this data, construct a canonical natural transformation $\alpha : Fdom \Rightarrow Gcod$ between the functors that form the boundary of the square

$$\begin{tikzcd}
F \downarrow G \arrow[r, "cod"] \arrow[d, "dom"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & \mathbf{E} \arrow[d, "G"] \\
\mathbf{D} \arrow[r, "F"'] & \mathbf{C}
\end{tikzcd}$$

The components α_d must translate $(Fdom)d \Rightarrow (Gcod)d$ for $d = domc$. Because $c \in F \downarrow G$, it is a triple, $(d, e, f : Fd \to Ge)$, $(Fdom)c$ is Fd for $d \in \mathbf{D}$. In order to translate $Fd \to Gd$, we must factor through $F \downarrow G$. Lets do this one in meetup.

(vi) TODO