

# Categories in Context - Ch. 2 Solutions

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## 1 Representable Functors

**problem 1.1.** (2.1.i) For each of the three functors

$$\mathbb{1} \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{!} \xrightarrow{1} \end{array} \mathbb{2}$$

Between the categories  $\mathbb{1}$  and  $\mathbb{2}$  describe the corresponding natural transformations between the covariant functors  $\mathbf{Cat} \rightrightarrows \mathbb{2}$  represented by the categories by the categories  $\mathbb{1}$  and  $\mathbb{2}$ .

*proof (2.1.i).* Consider the covariant representable functors  $\mathbf{Cat}(\mathbb{1}, -), \mathbf{Cat}(\mathbb{2}, -) : \mathbf{Cat} \rightarrow \mathbf{Set}$  and let  $F : \mathbf{C} \rightarrow \mathbf{D}$ . Consider the following diagram for the components of the transformations:

$$\begin{array}{ccc} \mathbf{Cat}(\mathbb{1}, \mathbf{C}) & \begin{array}{c} \xleftarrow{\mathbf{Cat}(0, \mathbf{C})} \xrightarrow{\mathbf{Cat}(!, \mathbf{C})} \\ \xleftarrow{\mathbf{Cat}(1, \mathbf{C})} \end{array} & \mathbf{Cat}(\mathbb{2}, \mathbf{C}) \\ \downarrow \mathbf{Cat}(\mathbb{1}, F) & & \downarrow \mathbf{Cat}(\mathbb{2}, F) \\ \mathbf{Cat}(\mathbb{1}, \mathbf{D}) & \begin{array}{c} \xleftarrow{\mathbf{Cat}(0, \mathbf{D})} \xrightarrow{\mathbf{Cat}(!, \mathbf{D})} \\ \xleftarrow{\mathbf{Cat}(1, \mathbf{D})} \end{array} & \mathbf{Cat}(\mathbb{2}, \mathbf{D}) \end{array}$$

The transformations may be described as follows:

- $\mathbf{Cat}(0, -)$  maps  $*$  to the 0 object of  $\mathbb{2}$ , and given any functor  $F : \mathbb{2} \rightarrow \mathbf{C}$ , will map the domain of the unique nontrivial arrow  $f : 0 \rightarrow 1$  in  $\mathbb{2}$  to the domain of the arrow  $Ff : F0 \rightarrow F1$ .
- Likewise, for  $\mathbf{Cat}(1, -)$ , given any functor, the transformation will choose a codomain for the unique arrow in  $\mathbb{2}$ , and given any functor  $F : \mathbb{2} \rightarrow \mathbf{C}$ , will correspond to the codomain object of the chosen arrow in  $\mathbf{C}$ .
- The transformation  $\mathbf{Cat}(!, -)$  takes any choice of arrow in  $\mathbb{2} \rightarrow \mathbf{C}$  and maps it to an object of  $\mathbf{C}$ .

□

**problem 1.2.** (2.1.ii) Prove that if  $F : \mathcal{C} \rightarrow \mathbf{Set}$  is representable, then  $F$  preserves monomorphisms, i.e., sends every monomorphism in  $\mathcal{C}$  to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite concrete category which is not representable.

*Proof.* Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a representable functor with representing object  $c \in \mathcal{C}$ , and let  $f : x \rightarrow y$  be a monomorphism in  $\mathcal{C}$ . Consider the set function  $Ff : Fx \rightarrow Fy$ . Since  $F$  is representable it is naturally isomorphic to  $\mathcal{C}(c, -)$ , and  $Ff$  is then isomorphic to a set function  $\mathcal{C}(c, f) : \mathcal{C}(c, x) \rightarrow \mathcal{C}(c, y)$ . Consider the parallel morphisms  $h, k : w \rightrightarrows x$ . Since  $f$  is monic, we have that  $fh = fk$  implies that  $h = k$ . Hence,  $\mathcal{C}(c, fh) = \mathcal{C}(c, fk)$  implies that  $\mathcal{C}(c, k) = \mathcal{C}(c, h)$ . Let  $w, w' \in \mathcal{C}(c, x)$ . Then, we recover the usual notion of injective function in set:  $fw = fw' \Rightarrow w = w'$ . Hence, representable functors preserve monomorphisms.

For the contrapositive, let the functor  $\pi_0 : \mathbf{Top} \rightarrow \mathbf{Set}$  be the functor taking a topological space to its set of connected components. Then the monomorphism  $\{0, 1\} \rightarrow [0, 1]$  is mapped to  $\{*\}$ .  $\square$

**problem 1.3.** (2.1.iii) Suppose  $F : \mathcal{C} \rightarrow \mathbf{Set}$  is equivalent to  $G : \mathcal{D} \rightarrow \mathbf{Set}$  in the sense that there is an equivalence of categories  $H : \mathcal{C} \rightarrow \mathcal{D}$  so that  $GH$  and  $F$  are naturally isomorphic.

- (i) If  $G$  is representable, then is  $F$  representable?
- (ii) If  $F$  is representable, then is  $G$  representable?

*Proof.* Let  $G$  be representable and let  $K : \mathcal{D} \rightarrow \mathcal{C}$  be the inverse equivalence to  $H$ . Note now that we have the following:

$$\begin{aligned}
 F &\cong GH \\
 &\cong \mathcal{D}(d, -)H && \text{(representability of } G) \\
 &\cong \mathcal{D}(d, H-) \\
 &\cong \mathcal{D}(HKd, H-) && \text{(equivalence } HK \cong 1_{\mathcal{D}}) \\
 &\cong \mathcal{C}(Kd, -) && \text{(H is f.f. due to equivalence)} \\
 &\cong \mathcal{C}(c, -) && \text{(K is e.s.o due to equivalence)}
 \end{aligned} \tag{1}$$

Hence,  $F$  is representable. Verbatim proof for the opposite direction.  $\square$

**problem 1.4.** (2.1.iv) A functor  $F$  defines a **subfunctor** of  $G$  if there is a natural transformation  $\alpha : F \Rightarrow G$  whose components are monomorphisms. In the case of  $G : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ , a subfunctor is given by a collection of subsets  $Fc \subset Gc$  so that each  $Gf : Gc \rightarrow Gc'$  restricts to a function  $Ff : Fc \rightarrow Fc'$ . Characterize those subsets that assemble into a subfunctor of the representable functor  $\mathcal{C}(-, c)$ .

*Proof.* For the functor  $F$  to be a subfunctor of  $\mathcal{C}(-, c)$ , we must build a natural transformation  $\alpha : F \Rightarrow \mathcal{C}(-, c)$  such that, given  $f : d' \rightarrow d \in \mathcal{C}$ , the components  $\alpha_d : Fd \rightarrow \mathcal{C}(d, c)$  restrict to  $Fd$  and widen to  $Fd'$  when precomposed with  $f$ . Thus, to completely characterize the subsets of a subfunctor, we need that the family  $\bigcup_d Fd$  is closed under precomposition by arbitrary morphisms  $d' \rightarrow d$  so that we have a sieve on  $d$ .  $\square$

**problem 1.5.** (2.1.v) *The functor of Example 2.1.5(xi) that sends a category to its collection of isomorphisms is a subfunctor of the functor of example 2.1.5(x) that sends a category to its collection of morphisms. Define a functor between the representing categories  $\mathbb{I}$  and  $2$  that induces the corresponding monic natural transformation between these representable functors.*

*Proof.* Define the functor  $F : \mathbb{I} \rightarrow 2$  as follows: let  $F$  be the identity on objects, but for  $f : 0 \rightarrow 1$ ,  $g : 1 \rightarrow 0$ , let them both map to the unique arrow in  $2$ . Then, the components  $\alpha_C : \text{Cat}(\mathbb{C}, \mathbb{I}) \rightarrow \text{Cat}(\mathbb{C}, 2)$  may be identified with  $\text{Cat}(\mathbb{C}, F)$ , which is an injection of sets, picking out two elements of  $\text{Cat}(\mathbb{C}, 2)$  for each non-trivial arrow in  $\mathbb{I}$ .

□