Ingram Algebraic Number Theory Course Solutions (Appendix)

Emily Pillmore

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Exercise 1 (A.9). Let R be a (commutative) ring (with identity), and let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals. Show that

$$\mathfrak{a} + \mathfrak{b} =_{def} \{ a + b : a \in \mathfrak{a}, b \in \mathfrak{b} \}$$

Is an ideal of R.

Proof. By definition, $\mathfrak{a} + \mathfrak{b}$ is an ideal of R if it forms an additive subgroup of R, (i.e. if $(\mathfrak{a} + \mathfrak{b}) \pm (\mathfrak{a} + \mathfrak{b}) \subseteq \mathfrak{a} + \mathfrak{b}$) and if it is closed under (left) multiplicative actions $r\mathfrak{a} \subseteq \mathfrak{a}$ for all $r \in R$. The former is proven by noting that distributivity inherited by R yields the following for all $r \in R$: $r(\mathfrak{a} + \mathfrak{b}) = r\mathfrak{a} + r\mathfrak{b} = \mathfrak{a} + \mathfrak{b}$. Hence $\mathfrak{a} + \mathfrak{b}$ is closed under (left) multiplicative actions. We must now show the former requirement holds.

Let $k, k' \in \mathfrak{a} + \mathfrak{b}$. Note that k has the form k = a + b as defined above. Therefore, $k + k' = (a+b) + (a'+b') = (a+a') + (b+b') \in \mathfrak{a} + \mathfrak{b}$ using associativity inherited by additivity in R, with 0 = 0 + 0. A similar proof is given for subtraction, hence $\mathfrak{a} + \mathfrak{b}$ is closed under the additive group operation of R, and is therefore an ideal of R.

Exercise 2 (A.11). Show that if $a, b \subseteq R$ are ideals, then so is \mathfrak{ab} , defined as the set of all finite sums of elements of the form ab with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ (including the "empty sum" 0). Show also that this is the smallest ideal containing all elements of the form ab (with $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$).

Proof. Note that each element $ab \in \mathfrak{ab}$ takes the form $\sum_{i,j=0}^{n-1} a_i b_j$ where $a \in \mathfrak{a}, b \in \mathfrak{b}$. Must check closure under left multiplicative actions by elements in R, and that these elements are closed under the additive group action of R. Let's first check multiplicativity:

$$\begin{split} rab &= r(\sum_{i,j} a_i b_j) \\ &= \sum_{i,j} ra_i b_j \\ &= \sum_{i,j} a_i b_j \qquad (ra \in \mathfrak{a} \text{ for all } r \in R) \end{split}$$

Hence, finite formal sums are closed under left multiplicative actions by elements of R. Now, we check that it is closed under addition:

$$ab + a'b' = \sum_{i,j}^{n} a_i b_j + \sum_{k,l}^{m} a'_k b'_l$$
$$= \sum_{t=i+k, u=j+l}^{n+m} a_t b_u$$

Where t < n enumerates the indices i with t >= n enumerates k, likewise for u. Hence, the sum of finite formal sums of elements $a, a' \in \mathfrak{a}, b, b' \in \mathfrak{b}$ is again a finite formal sum of elements in \mathfrak{a} and \mathfrak{b} . The proof is similar for subtraction, with extra steps noting that \mathfrak{a} and \mathfrak{b} are closed under subtraction themselves:

$$\begin{split} ab - a'b' &= \sum_{i,j}^{n} a_i b_j - \sum_{k,l}^{m} a_k' b_l' \\ &= \sum_{i,j}^{n} a_i b_j + (-1) \sum_{k,l}^{m} a_k' b_l' \\ &= \sum_{i,j}^{n} a_i b_j + \sum_{k,l}^{m} (-1) a_k' b_l' \\ &= \sum_{i,j}^{n} a_i b_j + \sum_{k,l}^{m} a_k'' b_l' \\ &= \sum_{t-i+k}^{n} a_t b_u \end{split}$$

Hence, \mathfrak{ab} is an ideal of R.

Exercise 3 (A.12). Let $a, b \in R$. Show that (a)(b) = (ab) (i.e., the product of two ideals means what you think it does for principal ideals). Note again that the product operation does not turn the ideals of R (or even the non-zero ideals of R) into a group.

Proof. Let (a) = aR, (b) = bR be principal ideals of R. We must show that (a)(b) = (ab) is again a principle of R. Consider (a)(b):

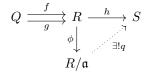
$$(a)(b) = aRbR$$

= $abRR$ (commutativity of multiplication in R)
= abR (closure under \times)
= (ab)

Hence, (ab) is again a principal ideal of R. This is not a group in general for obvious reasons when considering (0), but also in the case of non-zero ideals. Let a and b be zero elements such that ab = 0. Then $(b) = (1)(b) = (a^{-1}a)(b) = (a^{-1})(a)(b) = (a^{-1})(ab) = (a^{-1})(0) = (0)$ yields a contradiction. This holds, in fact, even for arbitrary ideals \mathfrak{ab} (via a similar proof). Hence ideals can't form a group under multiplication in the presence of zero elements which may not be 0.

Exercise 4 (A.15 (optional)). If R is a commutative ring with identity and $\mathfrak{a} \subseteq R$ is an ideal, then R/\mathfrak{a} is a commutative ring with multiplicative identity $1 + \mathfrak{a}$ and additive identity a = 0 + a.

Proof. This is equivalent to noting that for any other ring Q, R/\mathfrak{a} is the coequalizer of the parallel pair $Q \rightrightarrows R$:



Hence, R/\mathfrak{a} is a quotient object in CRng. (this is a mechanical proof)

Exercise 5 (A.18). Let F be a field, and let R = F[X]. Prove that every non-zero ideal in R is principal. You may use the division algorithm for polynomials, which says that if $a, b \in F[X]$, with $b \neq 0$, then there exist $q, r \in F[X]$ such that a = bq + r, and $0 \leq deg(r) < deg(b)$.

Proof.