Take Home Final

Stochastic Models for Finance and Insurance

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Problem 1 – Solution

Firstly we calculate the prices $X(t,\cdot)$ at time $t = \{1,2\}$ using binomial tree with the root $X_Y(0) = 4$ and relation

$$X(t,H) = u \cdot X(t-1), \quad X(t,T) = d \cdot X(t-1) \tag{1}$$

for $u = 2$ and $d = \frac{1}{2}$:

$$\begin{array}{ccc} & & X_Y(2,HH) = 16 \\ & X_Y(1,H) = 8 & \\ X_Y(0) = 4 & & X_Y(2,HT) = X_Y(2,TH) = 4 \\ & X_Y(1,T) = 2 & \\ & & X_Y(2,TT) = 1 \end{array} \tag{2}$$

Then we calculate the probability measures

$$\begin{aligned} \mathbb{P}^Y(H) &= \frac{1-d}{u-d} = \frac{1-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{3}, \\ \mathbb{P}^Y(T) &= 1 - P^Y(H) = \frac{2}{3}. \end{aligned} \tag{3}$$

Applying (3) to the following formula of the price of contingent claim V using Y as a reference asset for $t = \{0,1\}$

$$V_Y(t) = V_Y(t+1,H) \cdot \mathbb{P}^Y(H) + V_Y(t+1,T) \cdot \mathbb{P}^Y(T) \tag{4}$$

together with the results in (2) and facts

$$V(2) = \mathbb{1}(X_Y(2) \neq 4) \cdot Y(2), \tag{5}$$

$$Y_X(t) = \frac{1}{X_Y(t)}, \tag{6}$$

we obtain the following binomial tree for the prices $V(t,\cdot)$, $t = \{0,1,2\}$:

$$\begin{array}{ccc} & & V_Y(2,HH) = 1 \\ & V_Y(1,H) = \frac{1}{3} & \\ V(0) = \frac{5}{9} & & V_Y(2,HT) = V(2,TH) = 0 \\ & V_Y(1,T) = \frac{2}{3} & \\ & & V_Y(2,TT) = 1 \end{array} \tag{7}$$

Using the results in (7) and (2), the hedging positions are as follows:

$$\begin{aligned} \Delta^X(0) &= \frac{V_Y(1,H) - V_Y(1,T)}{X_Y(1,H) - X_Y(1,T)} = \frac{\frac{1}{3} - \frac{2}{3}}{8-2} = -\frac{1}{18}, \\ \Delta^X(1,H) &= \frac{V_Y(2,HH) - V_Y(1,HT)}{X_Y(2,HH) - X_Y(2,HT)} = \frac{1-0}{16-4} = \frac{1}{12}, \\ \Delta^X(1,T) &= \frac{V_Y(2,TH) - V_Y(2,TT)}{X_Y(2,TH) - X_Y(2,TT)} = \frac{0-1}{4-1} = -\frac{1}{3}. \end{aligned} \tag{8}$$

Problem 2 – Solution

At time T the contract's payoff is

$$V(T) = \mathbb{1}(L \leq X_Y(T) \leq U) \cdot Y(T) \ . \tag{9}$$

The price $V_Y(t)$ is a $\mathbb{P}^Y$ martingale and thus it holds

$$V_Y(t) = \mathbb{E}_t^Y[\mathbb{1}(L \leq X_Y(T) \leq U)] = \mathbb{P}_t^Y[L \leq X_Y(T) \leq U] \tag{10}$$

in which the lower index t represents conditional expectation or probability, under condition $X_Y(t) = x$. Using the theory in sub-chapter **3.3 Price as an Expectation** in the book Večeř [2011], we get

$$\begin{aligned} V_Y(t) &= \mathbb{E}_t^Y[V_Y(T)] = \mathbb{E}_t^Y[f^Y(X_Y(T))] \\ &= \mathbb{E}_t^Y[\mathbb{1}(L \leq X_Y(T) \leq U) \cdot Y(T)] = \mathbb{E}_t^Y[\mathbb{1}(L \leq X_Y(T) \leq U)] \end{aligned} \tag{11}$$

Expressing $V^Y(t)$ in terms of price function u^Y, we obtain the following representation

$$\begin{aligned} u^Y(t,x) = V_Y(t) &= \mathbb{E}_t^Y[f^Y(X_Y(T))] \\ &= \mathbb{E}_t^Y[\mathbb{1}(L \leq X_Y(T) \leq U)] \\ &= \mathbb{P}_t^Y[L \leq X_Y(T) \leq U] \\ &= \mathbb{P}_t^Y[X_Y(T) \leq U] - \mathbb{P}_t^Y[X_Y(T) \leq L] \\ &= 1 - \mathbb{P}_t^Y[X_Y(T) \geq U] - 1 + \mathbb{P}_t^Y[X_Y(T) \geq L] \\ &= N(d_L-) - N(d_U-) \end{aligned} \tag{12}$$

where in the last equation we use the relation

$$\mathbb{P}_t^X[X_Y(T) \geq K] = N\Big(\frac{1}{\sigma\sqrt{T-t}} \cdot log\Big(\frac{X_Y(T)}{K}\Big) - \frac{1}{2}\sigma\sqrt{T-t}\Big) =: N(d_K-) \tag{13}$$

in which $N(\cdot)$ represents a cumulative distribution function of standard normal distribution

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \quad \sim \quad \mathcal{N}(0,1) \ . \tag{14}$$

The hedging portfolio for this contract is of form

$$P(t) = \Delta^X(t) \cdot X(t) + \Delta^Y(t) \cdot Y(t) \tag{15}$$

with the following hedging positions

$$\begin{aligned} \Delta^X(t) &= u_x^Y(t, X_Y(t)) \\ &= \frac{1}{X_Y(t)\sqrt{T-t}} \cdot (f(d_L-) - f(d_U-)) \\ \Delta^Y(t) &= u^Y(t, X_Y(t)) - u_x^Y(t, X_Y(t)) \cdot X_Y(t) \\ &= N(d_L-) - N(d_U-) - \frac{1}{\sigma\sqrt{T-t}} \cdot (f(d_L-) - f(d_U-)) \end{aligned} \tag{16}$$

where $f(\cdot)$ is a density of standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R} . \tag{17}$$

Generally, based on the theorem 3.3. in Večeř [2011], the price function $u^Y(t,x)$ satisfies the partial differential equation

$$u_t^Y(t,x) + \frac{1}{2}\sigma^2 \cdot x^2 \cdot u_{xx}^2(t,x) = 0 \ , \tag{18}$$

where for this time the lower index t represents a partial derivative. In our case this was verified by the software Wolfram Mathematica, see **Attachment A.1**.

Bibliography

J. Večeř. *Stochastic Finance: A Numeraire Approach.* Matfyzpress, 2011. ISBN 1-439-81252-7.

A. Attachments

A.1 Verification of Partial Differential Equation

- Partial differential equation - verification

In[1]:=
```
d[K_] := 1/(σ * √(T - t)) * Log[x / K] -
   1/2 * σ * √(T - t)   (*value corresponding to denotation d- *)
```

In[2]:= CumFunc = $\int_{-\infty}^{c} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-y^{\wedge}2}{2}}$

```
  dy (*definition of N(.) - cummulative d.f. of standard normal distribution*)
```

```
(*derivative by x of N(.) is density of N(0,1)*)
```

In[3]:=
```
DerivationofN[x_] = 1/√(2 π) * e^(-(x^2)/2)
```

Out[3]= $\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$

In[4]:=
```
uDerByt = DerivationofN[d[L]] * D[d[L], t] - DerivationofN[d[U]] * D[d[U], t]
    (*derivative of u^Y(t,x) by t applying derivative rules for difference*)
```

Out[4]= $$\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{L}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}\left(\frac{\sigma}{4\sqrt{-t+T}}+\frac{\text{Log}\left[\frac{x}{L}\right]}{2\,(-t+T)^{3/2}\,\sigma}\right)}{\sqrt{2\pi}}-\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{U}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}\left(\frac{\sigma}{4\sqrt{-t+T}}+\frac{\text{Log}\left[\frac{x}{U}\right]}{2\,(-t+T)^{3/2}\,\sigma}\right)}{\sqrt{2\pi}}$$

```
(*first derivative of u^Y(t,x) by x*)
```

In[5]:=
```
uDerByx = DerivationofN[d[L]] * D[d[L], x] - DerivationofN[d[U]] * D[d[U], x]
```

Out[5]= $$\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{L}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{-t+T}\,x\,\sigma}-\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{U}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{-t+T}\,x\,\sigma}$$

```
(*second derivative of u^Y(t,x) by x*)
```

In[6]:=
```
uDerByxx = D[uDerByx, x]
```

Out[6]= $$-\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{L}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{-t+T}\,x^2\,\sigma}+\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{U}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}}{\sqrt{2\pi}\,\sqrt{-t+T}\,x^2\,\sigma}-$$
$$\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{L}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{L}\right]}{\sqrt{-t+T}\,\sigma}\right)}{\sqrt{2\pi}\,(-t+T)\,x^2\,\sigma^2}+\frac{e^{-\frac{1}{2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{U}\right]}{\sqrt{-t+T}\,\sigma}\right)^2}\left(-\frac{1}{2}\sqrt{-t+T}\,\sigma+\frac{\text{Log}\left[\frac{x}{U}\right]}{\sqrt{-t+T}\,\sigma}\right)}{\sqrt{2\pi}\,(-t+T)\,x^2\,\sigma^2}$$

In[8]:=
```
FullSimplify[uDerByt + 1/2 * σ^2 * x^2 * uDerByxx]
(*partial differential equation = 0 --> IT'S GOOD!*)
```

Out[8]= 0