**Stochastic Models for Finance and Insurance**
**Take Home Final, May 31, 2020**

**Problem 1.**
Consider two step ($N = 2$) binomial model with $u = 2$, $d = \frac{1}{2}$ and $X_Y(0) = 4$. Find the price and the hedging portfolio for a contract with a payoff

$$V(2) = \mathbb{I}(X_Y(2) \neq 4) \cdot Y(2).$$

More specifically, find $V(0)$, $V(1,H)$, $V(1,T)$ and for the hedge the corresponding positions in the asset $X$ ($\Delta^X(0)$, $\Delta^X(1,H)$ and $\Delta^X(1,T)$).

**Problem 2.**
Solve

$$u^Y(t,x) = \mathbb{P}^Y\left[(L \leq X_Y(T) \leq U)|X_Y(t) = x\right],$$

for some $L < U$, where

$$dX_Y(t) = \sigma X_Y(t) dW^Y(t).$$

This corresponds to a contract $V$ with the payoff

$$V(T) = \mathbb{I}(L \leq X_Y(T) \leq U) \cdot Y(T).$$

Find the hedging portfolio for this contract (give both $\Delta^X(t)$ and $\Delta^X(t)$). Write down the partial equation for the function $u^Y$ and check that your solution satisfies this equation.

**Problem 3.**
Compute $\mathbb{E}[X(T)^2]$ for a process

$$dX(t) = \sqrt{X(t)(1 - X(t))} \cdot dW(t).$$

**Hint:** Use Ito's formula for the process $X^2(t)$ and take the expectation. Also note that

$$d(e^t X^2(t)) = e^t X^2(t) dt + e^t dX^2(t).$$

**Problem 4 – Python implementations.**

Write a Python code in a Jupyter notebook that does the following:

(a) Simulate Brownian motion evolution and the stock price evolution that follows stochastic differential equation

$$dX_Y(t) = \sigma X_Y(t) dW^Y(t).$$

Make plots of the Brownian increments, the corresponding Brownian motion and the resulting stock price.

(b) Write a function that computes the price of a European call option with a payoff $(X_Y(T) - K)^+$ units of $Y$ as a function of $X_Y(t), K, t, T, \sigma$. This corresponds to the Black-Scholes formula. Write a function that computes $\Delta^X(t)$, the hedging position for the option. Construct a hedging portfolio for this option using the data from a simulated path in part (a). Plot the corresponding evolution of the European call option and its hedging portfolio in one graph. Plot the hedging position $\Delta^X(t)$ in a separate graph.

(c) Modify your code from part (b) to allow for a continuous dividend yield in the stock $S$ (represented by an asset $X$) and a continuous interest rate $r$ (represented by an asset $Y$). Price the European stock option with a payoff $(S_\$(T) - K)^+$. Get 2019 data for SP500 (from 2019-01-01 to 2019-12-31), choose $K = S_\$(2019\text{-}01\text{-}01)$ as the strike and plot the evolution of this European call option together with its hedging portfolio. Use $\sigma$ as estimated from SP500 2018 values.

(d) Write the pricing and hedging functions for your solution in Problem 2. Show the simulated data of the stock price in one graph, the corresponding option price evolution together with the evolution of the hedging portfolio in another graph, and finally the evolution of the hedging position in the last separate graph. Show two scenarios, one that ends up in the money, and one that ends up out of the money.