Assignment 5

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5.4

- Count the number of configurations where at most two people share the same birthday
- Let B_k: The number of configurations where k pair of people shares the same birthday, and the other $100 - 2k$ people have distinct birthdays
  - Pick $k + 100 - 2k = 100 - k$ days: $\binom{365}{100-k}$
  - Pick k pairs of people $\frac{1}{k!} \prod_{i=0}^{k-1} \binom{365-2i}{2}$
  - Assign $100 - k$ birthdays: $(100 - k)!$
  - $\therefore B_k = \frac{1}{k!} \binom{365}{100-k} (100-k)! \prod_{i=0}^{k-1} \binom{365-2i}{2}$
- The answer:

$$\frac{1}{365^{100}} \sum_{k=0}^{50} B_k = \frac{1}{365^{100}} \sum_{k=0}^{50} \left(\frac{1}{k!} \binom{365}{100-k} (100-k)! \prod_{i=0}^{k-1} \binom{365-2i}{2} \right)$$

5.9

- The expected time of the first step, where each elements are placed in the appropriate bucket, remains the same as in the case of uniform distribution at $O(n)$
  - Assuming the time to place each element to the appropriate bucket is $O(1)$,
  - Iterating through the entire array and placing each element to the appropriate bucket trivially takes $O(n)$ time
- Let X_j be the number of elements in the j-th bucket
  - $X_j \sim B(n, p_j)$
  - When p_j is the probability that any number that would be placed in the j-th bin is chosen as an element of the initial array
  - $$p_j \leq \sum_{\substack{0 \leq t < 2^k \\ t \equiv j \pmod{n}}} q_t \leq \sum_{\substack{0 \leq t < 2^k \\ t \equiv j \pmod{n}}} \frac{a}{2^k} \approx \frac{2^k}{n} \times \frac{a}{2^k} = \frac{a}{n}$$
  - When q_t is a probability that t is chosen as an element of the initial array
- The expected time of the second step, where elements in each bucket are sorted, is as follows
  - Assuming the quadratic sorting algorithm used to sort elements in each bucket takes cn^2 expected time to sort an array of n elements for a constant c
  - The expected time of the second step:

$$
\begin{aligned}
\mathbf{E}\left[\sum_{j=1}^{n} cX_j^2\right] &= c\sum_{j=1}^{n}\mathbf{E}\left[X_j^2\right] \\
&= c\sum_{j=1}^{n}\left(n(n-1)p_j^2 + np_j\right) \\
&\le c\sum_{j=1}^{n}\left(n(n-1)\left(\frac{a}{n}\right)^2 + n\left(\frac{a}{n}\right)\right) \\
&= cn\left(a^2 - \frac{a}{n} + a\right) \\
&= O(n)
\end{aligned}
$$

- The expected time of concatenating the bucket is also trivially $O(n)$
- $\therefore$ The overall algorithm required a linear expected time

5.10

- Let:
  - X_i: The number of balls in the i-th bin
  - Y_i: $Y_i \sim \text{Poisson}(1)$
    - Y_i are mutually independent

(a)

- $$\Pr((Y_1, ..., Y_n) = (1, ..., 1)) = \prod_{i=1}^{n}\Pr(Y_i = 1) = \prod_{i=1}^{n}\frac{e^{-1} \times 1^1}{1!} = e^{-n}$$

(b)

- Number of configurations where n bins receives exactly one ball: $n!$
  - $\because$ First ball can land in any bin (n configurations), second ball can land in any bin that didn't receive the first ball ($n-1$ configurations), ..., last ball can land in any bin that didn't receive the first ball (1 configuration)
- Number of every possible configuration: n^n
- $$\therefore \Pr((X_1, ..., X_n) = (1, ..., 1)) = \frac{n!}{n^n}$$

(c)

- Let $Y := \sum_{i=1}^{n} Y_i$
  - $Y \sim \text{Poisson}(n)$
- From theorem 5.6:

$$
\begin{aligned}
\Pr((X_1, ..., X_n) = (1, ..., 1)) &= \Pr((Y_1, ..., Y_n) = (1, ..., 1) \mid Y = n) \\
&= \frac{\Pr((Y_1, ..., Y_n) = (1, ..., 1), Y = n)}{\Pr(Y = n)} \\
&= \frac{\Pr((Y_1, ..., Y_n) = (1, ..., 1))}{\Pr(Y = n)}
\end{aligned}
$$

- $\therefore \dfrac{\Pr((Y_1, ..., Y_n) = (1, ..., 1))}{\Pr((X_1, ..., X_n) = (1, ..., 1))} = \Pr(Y = n)$
- The two probabilities differ by a multiplicative factor of the probability that a Poisson random variable with parameter n takes on the value n

- Dividing the answer to the two previous problems yields a result consistent with this:

  $\dfrac{\text{(a)}}{\text{(b)}} = \dfrac{e^{-n} \times n^n}{n!}$

5.12

(a)

- Let:
  - $Y_i := \begin{cases} 0 \text{ if ball } i \text{ is removed} \\ 1 \text{ if ball } i \text{ remains} \end{cases}$
  - Y: The number of balls at the start of next round
    - $Y = \sum_{i=1}^{n} Y_i$
- For ball i to be removed, $b - 1$ other balls should land in a different bin
  - $\Pr(Y_i = 0) = \left(1 - \frac{1}{n}\right)^{b-1}, \Pr(Y_i = 1) = 1 - \left(1 - \frac{1}{n}\right)^{b-1}$
  - $\mathbf{E}[Y_i] = 1 - \left(1 - \frac{1}{n}\right)^{b-1}$
- $\mathbf{E}[Y] = \sum_{i=1}^{b} \mathbf{E}[Y_i] = b\left(1 - \left(1 - \frac{1}{n}\right)^{b-1}\right)$

(b)

- Let:
  - x_j: The number of balls after the j-th round, given exactly expected number of balls are removed each round

- $$
\begin{aligned}
x_{j+1} &= x_j \left(1 - \left(1 - \frac{1}{n}\right)^{x_j - 1}\right) \\
&\leq x_j \left(1 - \left(1 - \frac{x_j - 1}{n}\right)\right) \\
&= \frac{x_j (x_j - 1)}{n} \leq \frac{x_j^2}{n}
\end{aligned}
$$

- Special cases:

- $x_1 = n\Big(1 - \big(1 - \frac{1}{n}\big)^{n-1}\Big) \approx n\Big(1 - e^{-\frac{1}{n}(n-1)}\Big) \le n\big(1 - \frac{1}{e}\big)$
- $$0 \le x_k \le 1 \Rightarrow x_{k+1} \le \max_{0 \le a \le 1} \frac{a(a-1)}{n} = 0 \Rightarrow x_{k+1} = 0$$

- From this equation, we get:

$$\frac{x_{j+1}}{n} \le \left(\frac{x_j}{n}\right)^2$$

$$\frac{x_k}{n} \le \left(\frac{x_{k-1}}{n}\right)^2 \le \dots \le \left(\frac{x_1}{n}\right)^{2^{k-1}} = \left(1 - \frac{1}{e}\right)^{2^{k-1}}$$

- If $k - 1 \ge \log_2 \log_{\frac{e}{e-1}} n = \frac{1}{\ln 2 \ln\left(\frac{e}{e-1}\right)} \ln \ln n$
  - $\dfrac{x_k}{n} \le \dfrac{1}{n} \Rightarrow x_k \le 1$
  - $x_{k+1} = 0$
- $\therefore$ All balls would be served after a maximum of $\frac{1}{\ln 2 \ln\left(\frac{e}{e-1}\right)} \ln \ln n + 2 = O(\log \log n)$ rounds

5.14

(a)

- For an integer $k \le \mu - 1$, $\Pr(Z = k) \le \Pr(Z = 2\mu - k - 1)$
  - Proof:

$$\begin{aligned}
\frac{(2\mu - k - 1)!}{k!} &= (k+1)(k+2)\dots(2\mu - k - 1) \\
&= \mu \times (\mu - 1)(\mu + 1) \times (\mu - 2)(\mu + 2)\dots \times (\mu + (\mu - k - 1))(\mu - (\mu - k - 1)) \\
&= \mu \times \prod_{i=1}^{\mu - k - 1} (\mu - i)(\mu + i) \\
&\le \mu \times \prod_{i=1}^{\mu - k - 1} \mu^2 \\
&= \mu^{2\mu - 2k - 1}
\end{aligned}$$

$$\therefore \Pr(Z = k) = \frac{e^{-\mu}\mu^k}{k!} \le \frac{e^{-\mu}\mu^{2\mu - k - 1}}{(2\mu - k - 1)!} = \Pr(Z = 2\mu - k - 1)$$

- From this formula, when $k = \mu - h - 1$:
  - $\Pr(Z = \mu + h) \ge \Pr(Z = \mu - h - 1)$

(b)

- $$\begin{aligned}\Pr(Z \le \mu - h - 1) &= \sum_{k=0}^{\mu-h-1} \Pr(Z = k) \\ &\le \sum_{k=0}^{\mu-h-1} \Pr(Z = 2\mu - k - 1) \\ &= \sum_{k=\mu+h}^{2\mu-1} \Pr(Z = k) \\ &\le \sum_{k=\mu+h}^{\infty} \Pr(Z = k) \\ &= \Pr(Z \ge \mu + h)\end{aligned}$$
  - $\therefore \forall h; 0 \le h \le \mu - 1, \Pr(Z \ge \mu + h) \ge \Pr(Z \le \mu - h - 1)$
- From this formula, when $h = 0$:
  - $\Pr(Z \ge \mu) \ge \Pr(Z \le \mu - 1)$
- Since $\Pr(Z \ge \mu) + \Pr(Z \le \mu - 1) = 1$
  - $\Pr(Z \ge \mu) \ge \frac{1}{2}$

5.16

(a)

- Proposition: $\forall i_1, ..., i_k \subset \{1, ..., n\}$ (allowing repeat), $\mathbf{E}\left[\prod_{m=1}^{k} X_{i_m}\right] \le \mathbf{E}\left[\prod_{m=1}^{k} Y_{i_m}\right]$
  - Let $\{j_1, ..., j_t\} = \{i_1, ..., i_k\}$ (set of unique elements in $i_1, ..., i_k$)
  - Proof:

$$\mathbf{E}\left[\prod_{m=1}^{k} X_{i_m} = 1\right] = \Pr\left(\prod_{m=1}^{k} X_{i_m} = 1\right) = \Pr\left(\bigcap_{m=1}^{t} X_{j_t} = 1\right) = \left(1 - \frac{t}{n}\right)^n$$

$$\mathbf{E}\left[\prod_{m=1}^{k} Y_{i_m} = 1\right] = \Pr\left(\prod_{m=1}^{k} Y_{i_m} = 1\right) = \Pr\left(\bigcap_{m=1}^{t} Y_{j_m} = 1\right) = \prod_{m=1}^{t} \Pr\left(Y_{j_m} = 1\right) = \left(1 - \frac{1}{n}\right)^{tn}$$

$$\therefore \mathbf{E}\left[\prod_{t=1}^{k} X_{i_t} = 1\right] \le \mathbf{E}\left[\prod_{t=1}^{k} Y_{i_t} = 1\right] \left(\because \forall t \ge 1,\ 1 - \frac{t}{n} \le \left(1 - \frac{1}{n}\right)^t\right)$$

- The given inequality naturally holds from this when $i_1 = 1, ..., i_k = k$
  - $$\mathbf{E}\left[\prod_{m=1}^{k} X_m\right] \le \mathbf{E}\left[\prod_{m=1}^{k} Y_m\right]$$

(b)

- Proposition: $\mathbf{E}[X^k] \le \mathbf{E}[Y^k]$ for all k
  - Proof:

$$
\begin{aligned}
\mathbf{E}[X^k] &= \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^k\right] \\
&= \sum_{i_1,\dots,i_n \in \{1,\dots,n\}} \mathbf{E}\left[\prod_{m=1}^{n} X_{i_m}\right] \\
&\le \sum_{i_1,\dots,i_n \in \{1,\dots,n\}} \mathbf{E}\left[\prod_{m=1}^{n} Y_{i_m}\right] \quad (\because \text{inequality proved in (a)}) \\
&= \mathbf{E}\left[\left(\sum_{i=1}^{n} Y_i\right)^k\right] \\
&= \mathbf{E}[Y^k]
\end{aligned}
$$

- $$\mathbf{E}[e^{tX}] = \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!}(tX)^k\right] = \sum_{k=0}^{n} \frac{1}{k!} t^k \mathbf{E}[X^k] \le \sum_{k=0}^{n} \frac{1}{k!} t^k \mathbf{E}[Y^k] = \mathbf{E}\left[\sum_{k=0}^{\infty} \frac{1}{k!}(tY)^k\right] = \mathbf{E}[e^{tY}]$$
- $\therefore \mathbf{E}[e^{tX}] \le \mathbf{E}[e^{tY}]$

(c)

- $$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = n\left(1 - \frac{1}{n}\right)^n$$
  $$\mathbf{E}[Y] = \mathbf{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbf{E}[Y_i] = n\left(1 - \frac{1}{n}\right)^n$$
  - $\therefore \mathbf{E}[X] = \mathbf{E}[Y]$
- $$
\begin{aligned}
\Pr(X \ge (1+\delta)\mathbf{E}[X]) &\le \min_{t>0} \frac{\mathbf{E}[e^{tX}]}{e^{t(1+\delta)\mathbf{E}[X]}} \\
&\le \min_{t>0} \frac{\mathbf{E}[e^{tY}]}{e^{t(1+\delta)\mathbf{E}[X]}} \\
&= \min_{t>0} \frac{\mathbf{E}[e^{tY}]}{e^{t(1+\delta)\mathbf{E}[Y]}} \\
&\le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mathbf{E}[Y]} \qquad \text{(Chernoff bound for some of Poisson trials)} \\
&= \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{n\left(1-\frac{1}{n}\right)^n}
\end{aligned}
$$
- $\therefore$ The Chernoff bound:

  $$\Pr(X \ge (1+\delta)\mathbf{E}[X]) \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{n\left(1-\frac{1}{n}\right)^n}$$