Assignment 5

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6.2

(a)

- Let:
  - S: A sample space consisting of all possible colorings of the edges of K_n
  - X: A random variable on S that denotes a number of monochromatic copies of K_4
  - C_i: The i-th 4-vertex clique of K_n
  - $$A_i = \begin{cases} 1 & (\text{if } C_i \text{ is monochromatic}) \\ 0 & (\text{otherwise}) \end{cases}$$
    - $$X = \sum_{i=1}^{\binom{n}{4}} A_i$$
- If we color each edge of the K_n independently, with each edge taking each of the two colors with probability $\frac{1}{2}$, we obtain a random coloring chosen uniformly from S
  - $$\mathbf{E}[A_i] = \frac{2}{2^{\binom{4}{2}}} = \frac{1}{32}$$
  - $$\mathbf{E}[X] = \sum_{i=1}^{\binom{n}{4}} \mathbf{E}[A_i] = \binom{n}{4} 2^{-5}$$
  - $$\therefore \Pr\left(X \leq \binom{n}{4} 2^{-5}\right) > 0$$
- Since there is a probability strictly greater than zero to select a coloring for K_n with at most $\binom{n}{4}2^{-5}$ monochromatic K_4,
  - There exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$

(b)

- Color each edge of the K_n independently, with each edge taking each of the two colors with probability $\frac{1}{2}$
  - From (a), there is a nonzero chance that this randomized algorithm results in a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic copies of K_4
  - This algorithm requires $\binom{n}{2} = O(n^2)$ time to run ($\binom{n}{4}\binom{4}{2} = O(n^4)$ for checking if the result is correct)
- Success probability:

- $$\binom{n}{4}2^{-5} = \mathbf{E}[X]$$
  $$= \sum_{i \le \binom{n}{4}2^{-5}-1} i \Pr(X = i) + \sum_{i \ge \binom{n}{4}2^{-5}} i \Pr(X = i)$$
  $$\le \sum_{i \le \binom{n}{4}2^{-5}-1} \left(\binom{n}{4}2^{-5} - 1\right) \Pr(X = i) + \sum_{i \ge \binom{n}{4}2^{-5}} \binom{n}{4} \Pr(X = i)$$
  $$= (1-p)\left(\binom{n}{4}2^{-5} - 1\right) + p\binom{n}{4}$$
  - $\therefore p \ge O(n^{-4})$
- $$\text{Expected runtime} = \text{Single runtime} \times \frac{1}{\text{Success probability}} = O(n^8)$$

(c)

- Let:
  - $e_1, ..., e_{\binom{n}{2}}$: The edges of K_n
  - $C_{i_1}, ..., C_{i_{\binom{n-2}{2}}}$: The 4-vertex clique of K_n containing e_j
    - There are $\binom{n-2}{2}$ because 2 additional vertices must be selected along with two that is connected by e_j
  - x_j: The coloring of e_j
  - The colors used: Red & Blue

i. The algorithm

- For $j = 1$ to $\binom{n}{2}$
  - $w \leftarrow 0$
  - For $k = 1$ to $\binom{n-2}{2}$
    - If edges in C_{i_k} are only colored in red so far, $w \leftarrow w - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right]$
    - If edges in C_{i_k} are only colored in blue so far, $w \leftarrow w + \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right]$
    - Otherwise (If edges in C_{i_k} are not colored yet, or are colored using both red and blue), $w \leftarrow w$
  - If $w \ge 0$, color e_j red, oherwise ($w < 0$), color e_j blue
- The algorithm takes $\binom{n}{2} \times \binom{n-2}{2} \times 6 = O(n^4)$ time

ii. Justification

- Proposition: $\mathbf{E}\left[X \mid x_1, ..., x_j\right] \le \mathbf{E}\left[X \mid x_1, ..., x_{j-1}\right]$
  - Every clique in $\left\{C_1, ..., C_{\binom{n}{4}}\right\} - \left\{C_{i_1}, ..., C_{i_{\binom{n-2}{2}}}\right\}$ (Cliques that doesn't contain e_j) is unaffected by coloring of e_j.
  - For clique C_{i_k} in $\left\{C_{i_1}, ..., C_{i_{\binom{n-2}{2}}}\right\}$
    - If edges in C_{i_k} not colored yet:
      - $$\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = \frac{2}{64} = \frac{1}{32}$$

- $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] = \dfrac{1}{32}$ $\left(\text{Regardless of the color of } e_j\right)$
- $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = 0$

- If edges in C_{i_k} are previously colored using both red and blue:
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = 0$
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] = 0$ $\left(\text{Regardless of the color of } e_j\right)$
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = 0$
- If edges in C_{i_k} are previously colored only in red:
  - Let: t the number of edges in C_{i_k} not already colored
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = \dfrac{1}{2^t}$
    - $\because t$ more edges should be colored in red in order for A_{i_k} to be 1
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] = \begin{cases} \frac{1}{2^{t-1}} & (\text{If we color } e_j \text{ in red}) \\ 0 & (\text{If we color } e_j \text{ in blue}) \end{cases}$
    - $\because t - 1$ more edges should be colored in red in order for A_{i_k} to be 1
  - $\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = \begin{cases} \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] & (\text{If we color } e_j \text{ in red}) \\ -\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] & (\text{If we color } e_j \text{ in blue}) \end{cases}$
- If edges in C_{i_k} are previously colored only in blue:
  - In the same way as previous case,

$$\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = \begin{cases} -\mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] & (\text{If we color } e_j \text{ in red}) \\ \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] & (\text{If we color } e_j \text{ in blue}) \end{cases}$$

- $\therefore \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_j\right] - \mathbf{E}\left[A_{i_k} \mid x_1, ..., x_{j-1}\right] = \begin{cases} -w & (\text{If we color } e_j \text{ in red}) \\ w & (\text{If we color } e_j \text{ in blue}) \end{cases}$
  - Since the algorithm selects red if $w \geq 0$ and blue if $w < 0$, this difference is always less than or equal to 0

- $\mathbf{E}[X \mid x_1, ..., x_n] \leq \mathbf{E}[X] = \binom{n}{4}2^{-5}$
  - The total number of monochromatic copies of K_4 in the result is at most $\binom{n}{4}2^{-5}$.

6.6

Existence

- Let:
  - S: The sample space consisting of every possible k-cut of graph G

- $C(A_1, ..., A_k)$: A random variable on S which denotes the value of k-cut corresponding to sets $A_1, ..., A_k$
- $e_1, ..., e_m$: The edges of G
- $$X_i = \begin{cases} 1 & (\text{if } e_i \text{ connects different sets}) \\ 0 & (\text{otherwise}) \end{cases}$$
  - $$C(A_1, ..., A_k) = \sum_{i=1}^{m} X_i$$
- If we assign each vertex to $A_1, ..., A_k$ with equal probabilities $\frac{1}{k}$, we get obtain a k-cut of G chosen uniformly from S
  - $$\mathbf{E}[X_i] = 1 - \frac{1}{k} = \frac{k-1}{k}$$
    - $\because \Pr(X_i = 0) = \frac{1}{k}$ (Both vertices must be assigned in the same set)
  - $$\mathbf{E}[C(A_1, ..., A_k)] = \sum_{i=1}^{m} \mathbf{E}[X_i] = \frac{m(k-1)}{k}$$
  - $\Pr\left(C(A_1, ..., A_k) \geq \frac{m(k-1)}{k}\right) > 0$
- Since there is a probability strictly greater than zero to select a k-cut with value higher than $\frac{m(k-1)}{k}$
  - There exists a k-cut of G with a value higher than $\frac{m(k-1)}{k}$

Deterministic algorithm

- Let:
  - $v_1, ..., v_n$: The vertices of G
  - x_j: The set v_j is assigned to

i. The algorithm

- For $j = 1$ to n
  - Assign v_j to the set with the least vertices connected to v_j

ii. Justification

- Let:
  - c_l be the number of vertices in A_l connected to v_j
- Without loss of generality, v_j is assigned to A_1
- Proposition: $\mathbf{E}\left[C(A_1, ..., A_k) \mid x_1, ..., x_j\right] = \mathbf{E}\left[C(A_1, ..., A_k) \mid x_1, ..., x_{j-1}\right]$
  - For an edge e_i in $\{e_1, ..., e_m\}$
    - If e_i connects two vertex from $v_1, ..., v_{j-1}$
      - $\mathbf{E}\left[X_i \mid x_1, ..., x_j\right] - \mathbf{E}\left[X_i \mid x_1, ..., x_{j-1}\right] = 0$ (Already determined)
    - If e_i includes $v_{j+1}, ..., v_m$
      - $\mathbf{E}\left[X_i \mid x_1, ..., x_j\right] - \mathbf{E}\left[X_i \mid x_1, ..., x_{j-1}\right] = \frac{k-1}{k} - \frac{k-1}{k} = 0$
    - If e_i connects a vertex from $v_1, ..., v_{j-1}$ to v_j

- $\mathbf{E}[X_i \mid x_1, ..., x_j] - \mathbf{E}[X_i \mid x_1, ..., x_{j-1}]$
  $= \begin{cases} 0 - \frac{k-1}{k} = -\frac{k-1}{k} & \text{(If } v_j \text{ is assigned to the same set as the other vertex)} \\ 1 - \frac{k-1}{k} = \frac{1}{k} & \text{(Otherwise)} \end{cases}$
- $\mathbf{E}[C(A_1, ..., A_k) \mid x_1, ..., x_j] - \mathbf{E}[C(A_1, ..., A_k) \mid x_1, ..., x_{j-1}]$
  $$= \sum_{i=1}^{m} (\mathbf{E}[X_l \mid x_1, ..., x_j] - \mathbf{E}[X_l \mid x_1, ..., x_{j-1}])$$
  $$= \sum_{v_j \in e_i} (\mathbf{E}[X_i \mid x_1, ..., x_j] - \mathbf{E}[X_i \mid x_1, ..., x_{j-1}])$$
  $$= -\frac{k-1}{k} \times c_1 + \sum_{l=2}^{k} \frac{1}{k} \times c_l$$
  $$= \sum_{l=1}^{k} \left(\frac{1}{k} \times (c_l - c_1) \right)$$
- Since we assign v_j to the set with least vertices connected to v_j
  - $\forall l, c_l - c_1 > 0$
  - This difference is always greater than equal to 0
- $\mathbf{E}[X \mid x_1, ..., x_n] \geq \mathbf{E}[X] \frac{m(k-1)}{k}$
  - The value of the resulting cut is greater than or equal to $\frac{m(k-1)}{k}$

6.10

(a)

- The set of subsets of $\{1, 2, ..., n\}$ with $\lfloor \frac{n}{2} \rfloor$ elements
  - Each set cannot contain other set because they have equal number of elements.

(b)

- Let:
  - $P_1, P_2, ..., P_N$: An arbitrary ordering of every possible permutation of $\{1, ..., n\}$
    - $N = n!$
  - $$X_{i,k} := \begin{cases} 1 & \text{(If the first } k \text{ numbers in } P_i \text{ yields a set in } \mathcal{F}) \\ 0 & \text{(Otherwise)} \end{cases}$$
    - $$X_i := \sum_{k=0}^{n} X_{i,k}$$
    - $$X := \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} \sum_{k=0}^{n} X_{i,k}$$
- For an antichain $\mathcal{F}$
  - $X_i \leq 1$

- $\because \mathcal{F}$ cannot contain two sets formed by taking the first k elements from C_i, as one is always subset of another.
- $X = \sum_{i=1}^{N} X_i \leq n!$
- For every set $A \in \mathcal{F}$, if $|A| = k$
  - There are $k!(n-k)!$ permutations from which A can be formed by taking first k numbers
    - $\because k!$ ways to order the k elements in A in front, and $(n-k)!$ ways to order the remaining $n-k$ elements in the back
-
$$
\begin{aligned}
\therefore X &= \sum_{i=0}^{N} \sum_{k=0}^{n} X_{i,k} \\
&= \sum_{A \in \mathcal{F}} |A|!(n-|A|)! \\
&= \sum_{k=0}^{n} \sum_{\substack{A \in \mathcal{F} \\ |A| = k}} |A| \ (n-|A|)! \\
&= \sum_{k=0}^{n} \left(k!(n-k)! \cdot \sum_{\substack{A \in \mathcal{F} \\ |A| = k}} 1 \right) \\
&= \sum_{k=0}^{n} f_k \cdot k!(n-k)!
\end{aligned}
$$
  - Dividing both sides by $n!$, we get
  $$\sum_{k=0}^{n} f_k \cdot \frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} = \frac{X}{n!} \leq 1$$
- $\therefore$ For an antichain $\mathcal{F}$,
$$\sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq 1$$

(c)

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$$|\mathcal{F}| = \sum_{k=0}^{n} f_k \leq \sum_{k=0}^{n} f_k \times \frac{\binom{n}{\lfloor \frac{n}{2} \rfloor}}{\binom{n}{k}} = \binom{n}{\lfloor \frac{n}{2} \rfloor} \times \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$
  -
  $$\because \forall k; 0 \leq k \leq n, \binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \Leftrightarrow 1 \leq \frac{\binom{n}{\lfloor \frac{n}{2} \rfloor}}{\binom{n}{k}}$$

6.13

- Threshold function: $n^{-\frac{2}{k-1}}$
  - If $p = f(n)$ and $f(n) = o\left(n^{-\frac{2}{k-1}}\right)$
    - For any ε and for sufficiently large n,

- The probability that a random graph chosen from $G_{n,p}$ has a clique with k or more vertices is less than ε
- If $p = f(n)$ and $f(n) = \omega\left(n^{-\frac{2}{k-1}}\right)$
  - For any ε and for sufficiently large n,
  - The probability that a random graph chosen from $G_{n,p}$ doesn't have a clique with k or more vertices is less than ε
- Let:
  - $C_1, ..., C_N$: An arbitrary ordering of subset of k vertices from G
    - $$N = \binom{n}{k}$$
  - $$X_i := \begin{cases} 1 & \text{(If } C_i \text{ is a } k\text{-clique)} \\ 0 & \text{(Otherwise)} \end{cases}$$
    - $$X := \sum_{i=1}^{N} X_i$$
- $\mathbf{E}[X]$
  - $\mathbf{E}[X_i] = p^{\binom{k}{2}}$
  - $$\mathbf{E}[X] = \sum_{i=1}^{N} \mathbf{E}[X_i] = \binom{n}{k} p^{\binom{k}{2}} \leq \Theta\left(n^k p^{\binom{k}{2}}\right)$$
- If $p = f(n)$ and $f(n) = o\left(n^{-\frac{2}{k-1}}\right)$
  - $$\lim_{n \to \infty} \left(pn^{\frac{2}{k-1}}\right)^{\binom{k}{2}} = 0$$
    - $\therefore \mathbf{E}[X] = o(1)$
  - Sicne X can only have nonnegative integer value,
    - $\Pr(X \geq 1) \leq \mathbf{E}[X] = o(1)$
  - $\therefore$ The probability that a random graph G chosen from $G_{n,p}$ has a clique with k or more vertices is less than ε for a sufficiently large n
- If $p = f(n)$ and $f(n) = \omega\left(n^{-\frac{2}{k-1}}\right)$
  - $$\Pr(X > 0) \geq \sum_{i=1}^{N} \frac{\Pr(X_i = 1)}{\mathbf{E}[X \mid X_i = 1]}$$
  - When $k = 5$:
    - $\Pr(X_i = 1) = p^{10}$

- $$\Pr\left(X_j = 1 \mid X_i = 1\right) = \begin{cases} 1 & \left(C_i \text{ and } C_j \text{ share 5 vertices} (i = j)\right) & (1 \text{ case}) \\ p^4 & \left(C_i \text{ and } C_j \text{ share 4 vertices}\right) & \left(\binom{5}{4}\binom{n-5}{1} \text{ cases}\right) \\ p^7 & \left(C_i \text{ and } C_j \text{ share 3 vertices}\right) & \left(\binom{5}{3}\binom{n-5}{2} \text{ cases}\right) \\ p^9 & \left(C_i \text{ and } C_j \text{ share 2 vertices}\right) & \left(\binom{5}{2}\binom{n-5}{3} \text{ cases}\right) \\ p^{10} & \left(C_i \text{ and } C_j \text{ share 1 vertex}\right) & \left(\binom{5}{1}\binom{n-5}{4} \text{ cases}\right) \\ p^{10} & \left(C_i \text{ and } C_j \text{ share no vertex}\right) & \left(\binom{5}{0}\binom{n-5}{5} \text{ cases}\right) \end{cases}$$
    - $$\begin{aligned} \mathbf{E}[X \mid X_i = 1] &= \sum_{j=1}^{N} \mathbf{E}\left[X_j \mid X_i = 1\right] \\ &= \binom{5}{0}\binom{n-5}{5}p^{10} + \binom{5}{1}\binom{n-5}{4}p^{10} + \binom{5}{2}\binom{n-5}{3}p^9 + \binom{5}{3}\binom{n-5}{2}p^7 \\ &+ \binom{5}{4}\binom{n-5}{1}p^4 + 1 \end{aligned}$$
- If $p = \omega\left(n^{-\frac{2}{k-1}}\right) = \omega\left(n^{-\frac{1}{2}}\right)$, the term $\binom{n-5}{5}p^{10}$ dominates, so

  $$\mathbf{E}[X] \sim \binom{n-5}{5}p^{10}$$
- $$\Pr(X > 0) \geq \sum_{i=1}^{N} \frac{\Pr(X_i = 1)}{\mathbf{E}[X \mid X_i = 1]} \sim \frac{\binom{n}{5}}{\binom{n-5}{5}}$$
    - As n approaches ∞, this value reaches 1
- The probability that a random graph G chosen from $G_{n,p}$ doesn't have a clique with 5 or more vertices is less than ε for a sufficiently large n

6.15

- Let:
    - $C_1, C_2, ..., C_N$: An arbitrary ordering of subset of 3 vertices from G
        - $N = \binom{n}{3}$
    - $$X_i \coloneqq \begin{cases} 1 & (\text{If } C_i \text{ is a 3-clique}) \\ 0 & (\text{Otherwise}) \end{cases}$$
        - $$X = \sum_{i=1}^{N} X_i$$

$\Pr(X \geq 1) \geq \frac{1}{6}$

- $$\Pr(X \geq 1) = \Pr\left(\bigcup_{i=1}^{N} (X_i = 1)\right) \leq \sum_{i=1}^{N} \Pr(X_i = 1) = \binom{n}{3}p^3 \leq \frac{1}{6}$$

$$\lim_{n\to\infty} \Pr(X \geq 1) \leq \frac{1}{7}$$

- $$\Pr(X \geq 1) = \Pr(X > 0) \geq \sum_{i=1}^{N} \frac{\Pr(X_i = 1)}{\mathbf{E}[X \mid X_i = 1]}$$
- $\Pr(X_i = 1) = p^3$
- $$\Pr\left(X_j = 1 \mid X_i = 1\right) = \begin{cases} 1 & \left(C_i \text{ and } C_j \text{ share 3 vertices} (i = j)\right) & (1 \text{ case}) \\ p^2 & \left(C_i \text{ and } C_j \text{ share 2 vertices}\right) & \left(\binom{3}{2}\binom{n-3}{1} \text{ cases}\right) \\ p^3 & \left(C_i \text{ and } C_j \text{ share 1 vertex}\right) & \left(\binom{3}{1}\binom{n-3}{2} \text{ cases}\right) \\ p^3 & \left(C_i \text{ and } C_j \text{ share no vertex}\right) & \left(\binom{3}{0}\binom{n-3}{3} \text{ cases}\right) \end{cases}$$
  - $$\begin{aligned} \mathbf{E}[X \mid X_i = 1] &= \sum_{i=1}^{N} \mathbf{E}\left[X_j \mid X_i = 1\right] \\ &= \binom{3}{0}\binom{n-3}{3}p^3 + \binom{3}{1}\binom{n-3}{2}p^3 + \binom{3}{2}\binom{n-3}{1}p^2 + 1 \end{aligned}$$
- $$\begin{aligned} \Pr(X \geq 1) &\geq \sum_{i=1}^{N} \frac{\Pr(X_i = 1)}{\mathbf{E}[X \mid X_i = 1]} \\ &= \frac{\binom{n}{3}p^3}{\binom{3}{0}\binom{n-3}{3}p^3 + \binom{3}{1}\binom{n-3}{2}p^3 + \binom{3}{2}\binom{n-3}{1}p^2 + 1} \\ &= \frac{\binom{n}{3}\left(\frac{1}{n}\right)^3}{\binom{3}{0}\binom{n-3}{3}\left(\frac{1}{n}\right)^3 + \binom{3}{1}\binom{n-3}{2}\left(\frac{1}{n}\right)^3 + \binom{3}{2}\binom{n-3}{1}\left(\frac{1}{n}\right)^2 + 1} \\ &= \frac{\frac{1}{6} + O\left(\frac{1}{n}\right)}{\frac{7}{6} + O\left(\frac{1}{n}\right)} \end{aligned}$$
  - $$\therefore \lim_{n\to\infty} \Pr(X \geq 1) \geq \frac{1}{7}$$

6.17

- Let:
  - S: A sample space consisting of all possible colorings of the edges of K_n
  - $C_1, \ldots, C_N$: An arbitrary ordering of k-cliques from K_n
    - $N = \binom{n}{k}$
  - E_i: The event that C_i is colored monochromatic
- If we color each edge of the K_n independently, with each edge taking each of the two colors with probability $\frac{1}{2}$, we obtain a random coloring chosen uniformly from S
  - $$\Pr(E_i) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}}$$
  - In the dependency graph for events $E_1, \ldots, E_N$:

- $\deg(i) \leq \binom{k}{2}\binom{n-2}{k-2} \leq \binom{k}{2}\binom{n}{k-2}$
  - E_i is only dependent to E_j if C_i and C_j shares edges (Shares more than two vertices)
  - Choose two vertices from C_i, and other $k-2$ from the remaining $n-2$ vertices
    - Some graphs will be counted multiple times, but the product is still greater than deg(i)
- $4 \times \max_i \deg(i) \times \max_j \Pr(E_j) \leq 4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$
  - $\therefore \Pr\left(\bigcap_{i=1}^{N} \overline{E_i}\right) > 0 \text{ (Lovász local lemma)}$

- Since there is a probability strictly greater than zero to select a coloring for K_n such that there is no monochromatic K_k
  - It is possible to color edges of K_n with two colors so that it has no monochromatic K_k subgraph