Assignment 7

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7.2

- $P^0_{0,0} = 1$
- $$\begin{aligned} P^t_{0,0} &= p \times P^{t-1}_{0,0} + (1-p) \times P^{t-1}_{0,1} \\ &= p \times P^{t-1}_{0,0} + (1-p) \times \left(1 - P^{t-1}_{0,0}\right) \\ &= (2p-1)P^{t-1}_{0,0} + (1-p) \end{aligned}$$
- $$\begin{aligned} \left(P^t_{0,0} - \frac{1}{2}\right) &= (2p-1)\left(P^{t-1}_{0,0} - \frac{1}{2}\right) \\ &= (2p-1)^t\left(P^0_{0,0} - \frac{1}{2}\right) = \frac{1}{2}(2p-1)^t \end{aligned}$$
- $$\therefore P^t_{0,0} = \frac{1}{2}\left(1 + (2p-1)^t\right)$$

7.6

- Let:
  - Y_j: The position after moving j times
  - Z_k: The number of moves to reach n starting from position k
- Conditional probability for Y_j
  - $Y_0 = i$
  - $\Pr\left(Y_{j+1} = 0 \mid Y_j = 0\right) = \Pr\left(Y_{j+1} = 1 \mid Y_j = 0\right) = \frac{1}{2}$
  - $\Pr\left(Y_{j+1} = k+1 \mid Y_j = k\right) = \Pr\left(Y_{j+1} = k-1 \mid Y_j = k\right) = \frac{1}{2} \ \ (k \neq 0)$
- Recurrence relation for $\mathbf{E}[Z_k]$
  - $\mathbf{E}[Z_n] = 0$
  - $\mathbf{E}[Z_k] = \frac{1}{2}\mathbf{E}[Z_{k-1}+1] + \frac{1}{2}\mathbf{E}[Z_{k+1}+1] = \frac{1}{2}\left(\mathbf{E}[Z_{k-1}] + \mathbf{E}[Z_{k+1}]\right) + 1 \ \ (k \neq 0)$
    - $\frac{1}{2}$ chance to move to $k-1$, from which Z_{k-1} more moves are required
    - $\frac{1}{2}$ chance to move to $k+1$, from which Z_{k+1} more moves are required
  - $\mathbf{E}[Z_0] = \frac{1}{2}\mathbf{E}[Z_0+1] + \frac{1}{2}\mathbf{E}[Z_1+1] \Leftrightarrow \mathbf{E}[Z_0] = \mathbf{E}[Z_1] + 2$
    - $\frac{1}{2}$ chance to move to 0, from which Z_0 more moves are required
    - $\frac{1}{2}$ chance to move to 1, from which Z_1 more moves are required
- From the second equation, we can get

  $$\begin{aligned} \mathbf{E}[Z_k] - \mathbf{E}[Z_{k+1}] &= \mathbf{E}[Z_{k-1}] - \mathbf{E}[Z_k] + 2 \\ &= \mathbf{E}[Z_0] - \mathbf{E}[Z_1] + 2k = 2k + 2 \end{aligned}$$

- $$\begin{aligned}\mathbf{E}[Z_k] &= \mathbf{E}[Z_n] + \sum_{t=n}^{k+1} (\mathbf{E}[Z_{t-1}] - \mathbf{E}[Z_t]) \\ &= \mathbf{E}[Z_n] + 2\sum_{t=n}^{k+1} t \\ &= 0 + n(n+1) - k(k+1) \\ &= (n-k)(n+k+1)\end{aligned}$$
- $\therefore \mathbf{E}[Z_i] = (n-i)(n+i-1)$

7.12

The Markov Chain

- Let $Y_n \coloneqq X_n \bmod k$
  - $X_0 = 0, Y_0 = 0$
- $Y_0, Y_1, ..., Y_n$ is a Markov chain
  - $\because \Pr(Y_t = a_t \mid Y_{t-1} = a_t, ..., Y_0 = a_0) = \Pr(Y_t = a_t \mid Y_{t-1} = a_{t-1})$
    $$= \sum_{\substack{1 \leq l \leq 6 \\ l \equiv a_t - a_{t-1} (\bmod k)}} \frac{1}{6}$$
    - Rolling l makes state a_{t-1} transition to a_t if $l \equiv a_t - a_{t-1} (\bmod k)$
  - The value of Y_t is only dependent on Y_{t-1} and not the sequence $Y_0, ..., Y_{t-1}$

Existence of Stationary Distribution

- The Markov chain is:
  - Finite: $Y_t \in \{0, 1, ..., k-1\}$
  - Irreducible: For any two states a and b, without loss of generality, if $a > b$
    - $P_{b,a}^{a-b} \geq \left(\frac{1}{6}\right)^{a-b}$ (Getting $(a-b)$ 1s in a row)
    - $P_{a,b}^{k-a+b} \geq \left(\frac{1}{6}\right)^{k-a+b}$ (Getting $(k-a+b)$ 1s in a row)
  - Ergodic:
    - Aperiodic:
      - If there exists a state a that is periodic with cycle Δ
        - $P_{a,a}^{k} \geq \left(\frac{1}{6}\right)^{k}$ (Getting k 1s in a row)
        - $P_{a,a}^{k-1} \geq \left(\frac{1}{6}\right)^{k-1}$ (Getting $k-2$ 1s in a row, then 2)
        - $\Delta \mid k$ and $\Delta \mid (k-1) \Rightarrow \Delta \mid 1$
        - $\perp$
      - $\therefore$ The chain is aperiodic
    - The chain is finite, irreducible, and aperiodic.
      - $\therefore$ The chain is also ergodic.
- Since the chain is finite, irreducible, and ergodic, the chain has a stationary distribution $\overline{\pi}$

Calculating the Stationary Distribution

- Proposition:

  $$\overline{\pi} = \begin{bmatrix}\pi_0 & \pi_1 & \dots & \pi_{k-1}\end{bmatrix} = \begin{bmatrix}\frac{1}{k} & \frac{1}{k} & \dots & \frac{1}{k}\end{bmatrix}$$

- Let $\mathbf{P}$: The probability matrix for the chain
  - $$P_{ij} = \sum_{\substack{1 \le l \le 6 \\ l \equiv j-i (\operatorname{mod} k)}} \frac{1}{6} = \frac{1}{6} \sum_{\substack{1 \le l \le 6 \\ l \equiv j-i (\operatorname{mod} k)}} 1$$
- $\overline{\pi} P = \overline{\pi}$
  - $$\begin{aligned} \because (\overline{\pi}\mathbf{P})_j &= \frac{1}{k} \sum_{i=0}^{k-1} P_{ij} \\ &= \frac{1}{6k} \sum_{i=1}^{n} \sum_{\substack{1 \le l \le 6 \\ l \equiv j-i (\operatorname{mod} k)}} 1 \\ &= \frac{1}{6k} \sum_{l=1}^{6} \sum_{\substack{0 \le i \le k-1 \\ l \equiv j-i (\operatorname{mod} k)}} 1 \\ &= \frac{1}{6k} \sum_{l=1}^{6} 1 (\because \text{There is exactly 1 } i \text{ that satisfies the condition}) \\ &= \frac{1}{k} \end{aligned}$$
  - $\therefore \overline{\pi}$ is a stationary distribution

The Answer

- $$\therefore \lim_{n \to \infty} \Pr(X_n \text{ is divisible by } k) = \lim_{n \to \infty} \Pr(Y_n = 0) = \lim_{n \to \infty} P^n_{0,0} = \pi_0 = \frac{1}{k}$$

7.13

(a)

- $\Pr(X_k = a_k \mid X_{k+1} = a_{k+1}, X_{k+2} = a_{k+2}, \dots, X_m = a_m) = \Pr(X_k = a_k \mid X_{k+1} = a_{k+1})$

  $$= \frac{\Pr(X_{k+1} = a_{k+1} \mid X_k = a_k) \Pr(X_k = a_k)}{\Pr(X_{k+1} = a_{k+1})}$$

  $$= P_{a_k, a_{k+1}} \times \frac{\Pr(X_k = a_k)}{\Pr(X_{k+1} = a_{k+1})}$$

  - The second term of the multiplication is constant
- $\therefore$ The value of X_k is only dependent on X_{k+1} and not the sequence $X_m, \dots, X_{k+1}, X_k$

(b)

- Assuming that the chain started from a stationary distribution $\overline{\pi}$ on time 0
  - Otherwise, the reverse chain is not time-homogeneous, and $Q_{i,j}$ cannot be defined independent of k

- $\forall k, \Pr(X_k = a_k) = \pi_{a_k}$
- From the equation derived in (a), we get
  - $$Q_{i,j} = \Pr(X_k = j \mid X_{k+1} = i) = P_{j,i} \times \frac{\Pr(X_k = j)}{\Pr(X_{k+1} = i)} = \frac{\pi_j P_{j,i}}{\pi_i}$$

(c)

- $$Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i} = P_{i,j}$$

7.17

- Assuming $p > 0$
  - If $p = 0$, states cannot be positive recurrent, since the chain can only move up
- Let $X_0, ..., X_t$: The given Markov chain
- Starting from i, the chain can either go down or go up in the first step
  - $r^t_{X,i,i} = (1-p) r^{t-1}_{X,i-1,i} + p r^{t-1}_{X,i+1,i}$
    $h_{X,i,i} = 1 + (1-p) h_{X,i-1,i} + p h_{X,i+1,i}$

If the chain goes down in the first step

- Let $Y_0, Y_1, ..., Y_t$: A Markov chain with the following properties
  - $Y_t \in \{0, 1, ..., k\}$
  - $\Pr(Y_{t+1} = 1 \mid Y_t = 0) = 1$
  - $\Pr(Y_{t+1} = i+1 \mid Y_t = i) = p, \Pr(Y_{t+1} = i-1 \mid Y_t = i) = 1-p \;\; (0 < i < k)$
  - $\Pr(Y_{t+1} = k-1 \mid Y_t = k) = 1$
- The chain Y is equivalent to the lower part of given chain X except for the state k, which is reflective
- $r^t_{Y,i-1,i} = r^t_{X,i-1,i}$
  $h_{Y,i-1,i} = h_{X,i-1,i}$
  - $\because$ The chains X and Y cannot differ before reaching i
- The chain is:
  - Finite: $Y_t \in \{0, 1, ..., k\}$
  - Irreducible: For any two states a and b, without loss of generality, if $a > b$
    - $P^{a-b}_{a,b} = (1-p)^{a-b}$ (The chain goes down $a-b$ times)
    - $P^{a-b}_{b,a} = p^{a-b}$ (The chain goes up $a-b$ times)
  - Since the chain is finite and irreducible, every state in the chain is positive recurrent
- $\sum_{t \geq 1} r^{t-1}_{Y,i-1,i} = \sum_{t \geq 1} r^t_{Y,i,i} = 1$
  $h_{Y,i-1,i} = h_{Y,i,i} - 1 < \infty$

If the chain goes up in the first step

- Let $Z_0, ..., Z_t$: A Markov chain with the following properties
  - $Z_t \in \{k, k+1, ...\}$
  - $\Pr(Z_{t+1} = i+1 \mid Z_t = i) = p, \Pr(Z_{t+1} = i-1 \mid Z_t = i) = 1-p \;\; (i > k)$
  - $\Pr(Z_{t+1} = k+1 \mid Z_t = k) = 1$
- $r^t_{Z,i+1,i} = r^t_{Z,i+1,i}$

  $h_{Z,i+1,i} = h_{Z,i+1,i}$

Catalan number

- Let C_n: The number of ways to arrange n ups and n downs so that the number of ups is greater than or equal to the number of down at any given point in the sequence
  - The Catalan number
- Recurrence relation for C_n
  - If $2(i+1)$ is the first point in the sequence where there are equal number of ups and downs
    - The first move should be up
    - There are C_i ways to arrange the $2i$ moves from the 2nd to $2i+1$-th move, since these moves should have the following properties to ensure that $2(i+1)$ is the following properties
      - Among these $2i$ moves, there should be i ups and i downs, because there should be equal number of ups and downs by the $2(i+1)$-th move
      - Among these $2i$ moves, the number of ups is greater than or equal to the number of downs at any given point, because otherwise there $2(i+1)$ is not the first point where there are equal number of ups and downs
    - The $2(i+1)$-tg move should be down
    - There are C_{n-i} moves to arrange the remaining $2(n-i)$ moves, since these moves should also have the following properties
      - Among these $2(n-i)$ moves, there should be $n-i$ ups and $n-i$ downs since there should be equal number of ups and downs by the end
      - Among these $2(n-i)$ moves, the number of ups is greater than or equal to the number of downs at any given point.
    - There are $C_i C_{n-i}$ ways to arrange moves in a way to follow the condition
  - $$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$
- The generating function for C_n
  - Let $c(x) := \sum_{n=0}^{\infty} C_n x^n$
  - $$\begin{aligned} c(x) &= \sum_{n=0}^{\infty} C_n x^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} C_i C_{n-i} x^n \\ &= 1 + x(c(x))^2 \end{aligned}$$
    - Solving this, we get

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

- Since $C_0 = \lim_{x \to 0} c(x) = 1$,

  $$c(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

  - The function converges when $0 < x \leq \frac{1}{4}$

$r^t_{Z,i,i}$

- $$r^t_{Z,i,i} = \begin{cases} 0 & (t = 0) \\ p^{k-1}(1-p)^k C_{k-1} & (t = 2k, k \geq 1) \\ 0 & (t = 2k+1) \end{cases}$$

- $$\begin{aligned} \sum_{t \geq 1} r^{t-1}_{Z,i+1,i} = \sum_{t \geq 1} r^t_{Z,i,i} \\ &= \sum_{k \geq 1} p^{k-1}(1-p)^k C_{k-1} = (1-p) \sum_{k \geq 1} C_{k-1}(p(1-p))^{k-1} \\ &= (1-p) \times c(p(1-p)) \\ &= \frac{1 - \sqrt{1-4p(1-p)}}{2p} = \frac{1 - |1-2p|}{2p} \\ &= \begin{cases} 1 & \left(p \leq \frac{1}{2}\right) \\ \frac{1-p}{p} < 1 & \left(p > \frac{1}{2}\right) \end{cases} \end{aligned}$$

- For $p \leq \frac{1}{2}$,

  $$\begin{aligned} h_{Z,i+1,i} &= h_{Z,i,i} - 1 \\ &= \sum_{t \geq 1} t r^t_{Z,i,i} - 1 = \sum_{k \geq 1} 2k r^t_{Z,i,i} - 1 \\ &= 2 \sum_{k \geq 1} k p^{k-1}(1-p)^k C_{k-1} = 2(1-p) \sum_{k \geq 1} (k-1+1) C_{k-1}(p(1-p))^{k-1} - 1 \\ &= 2(1-p)(c'(p(1-p)) + c(p(1-p))) - 1 \end{aligned}$$

  - $c'(x) = \frac{-2x - \sqrt{1-4x} + 1}{2x^2\sqrt{1-4x}}$
  - The value converges when $x < \frac{1}{4}\left(p \neq \frac{1}{2}\right)$, and diverges when $x = \frac{1}{4}\left(p = \frac{1}{2}\right)$

Conclusion

- When $p < \frac{1}{2}$
  - $$\begin{aligned} \sum_{t \geq 1} r^t_{X,i,i} &= (1-p) \sum_{t \geq 1} r^{t-1}_{X,i-1,i} + p \sum_{t \geq 1} r^{t-1}_{X,i+1,i} \\ &= (1-p) \sum_{t \geq 1} r^{t-1}_{Y,i-1,i} + p \sum_{t \geq 1} r^{t-1}_{Z,i+1,i} \\ &< 1 \end{aligned}$$
  - Every state is transient
- When $p = \frac{1}{2}$

- $\displaystyle\sum_{t\geq 1} r^t_{X,i,i} = (1-p)\sum_{t\geq 1} r^{t-1}_{X,i-1,i} + p\sum_{t\geq 1} r^{t-1}_{X,i+1,i}$
  $\displaystyle\quad = (1-p)\sum_{t\geq 1} r^{t-1}_{Y,i-1,i} + p\sum_{t\geq 1} r^{t-1}_{Z,i+1,i}$
  $\quad = 1$
- $h_{X,i,i} = 1 + (1-p)h_{X,i-1,i} + ph_{X,i+1,i}$
  $\quad = 1 + (1-p)h_{Y,i-1,i} + ph_{Z,i+1,i}$
  $\quad = \infty$
- Every state is null recurrent

- When $p > \frac{1}{2}$
  - $\displaystyle\sum_{t\geq 1} r^t_{X,i,i} = (1-p)\sum_{t\geq 1} r^{t-1}_{X,i-1,i} + p\sum_{t\geq 1} r^{t-1}_{X,i+1,i}$
    $\displaystyle\quad = (1-p)\sum_{t\geq 1} r^{t-1}_{Y,i-1,i} + p\sum_{t\geq 1} r^{t-1}_{Z,i+1,i}$
    $\quad = 1$
  - $h_{X,i,i} = 1 + (1-p)h_{X,i-1,i} + ph_{X,i+1,i}$
    $\quad = 1 + (1-p)h_{Y,i-1,i} + ph_{Z,i+1,i}$
    $\quad < \infty$
  - Every state is postiive recurrent

7.21

Existence of Stationary Distribution

- Let $X_0, \ldots, X_t$: The given Markov chain
- The Markov chain is
  - Finite: $X_t \in \{0, \ldots, n\}$
  - Irreducible: For any two states a and b
    - $P^{a+1}_{b,a} = \left(\frac{1}{2}\right)^{a+1}$ (Go to 0, then go up a times)
    - $P^{b+1}_{a,b} = \left(\frac{1}{2}\right)^{b+1}$ (Go to 0, then go up b times)
  - Ergodic:
    - Aperiodic:
      - If there exists a state a that is periodic with cycle Δ
        - $P^{a+1}_{a,a} = \left(\frac{1}{2}\right)^{a+1}$ (Go to 0, then go up a times)
        - $P^{a+2}_{a,a} = \left(\frac{1}{2}\right)^{a+2}$ (Go to 0 twice, then go up a times)
        - $\Delta \mid a+1$ and $\Delta \mid a+2 \Rightarrow \Delta \mid 1$
        - $\perp$
      - $\therefore$ The chain is aperiodic
    - The chain is finite, irreducible, and aperiodic.
      - $\therefore$ The chain is also ergodic.

- Since the chain is finite, irreducible, and ergodic, the chain has a stationary distribution $\overline{\pi}$

Calculating the Stationary Distribution

- Proposition:

$$\overline{\pi} = \begin{bmatrix}\pi_0 & \pi_1 & \dots & \pi_{k-1}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & \frac{1}{2^{n-1}} & \frac{1}{2^{n-1}}\end{bmatrix}$$

  - $$\pi_i = \begin{cases} \frac{1}{2^{i+1}} & (i < n) \\ \frac{1}{2^n} & (i = n) \end{cases}$$

- The probability matrix for the chain:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 0 & 0 & 0 & \dots & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix}$$

- $\overline{\pi}\mathbf{P} = \overline{\pi}$
  - $$\because (\overline{\pi}P)_j = \sum_{i=0}^{n} \pi_i P_{ij} = \begin{cases} \sum_{j=0}^{n} \pi_i \times \frac{1}{2} = \frac{1}{2} & (j = 0) \\ \pi_{j-1} \times \frac{1}{2} = \frac{1}{2^{j+1}} & (0 < j < n) \\ \pi_{n-1} \times \frac{1}{2} + \pi_n \times \frac{1}{2} = \frac{1}{2^n} & (j = n) \end{cases}$$
  - $\therefore \overline{\pi}$ is a stationary distribution

The Answer

- $\overline{\pi} = \begin{bmatrix}\frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \cdots & \frac{1}{2^n} & \frac{1}{2^n}\end{bmatrix}$