Assignment 3

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3.3

- Let X_i be the result of the i-th roll.
  - All X_i are mutually independent, as the result of the dice roll do not affect one another.
  - $$\mathbf{E}[X_i] = \sum_{k=1}^{6} k \times \frac{1}{6} = \frac{7}{2}$$
  - $$\mathbf{Var}[X_i] = \sum_{k=1}^{6} (k - \mathbf{E}[X_i])^2 \times \frac{1}{6} = \frac{35}{12}$$
- $$X = \sum_{i=1}^{100} X_i$$
  - $$\mathbf{E}[X] = \sum_{i=1}^{100} \mathbf{E}[X_i] = 350$$
  - $$\mathbf{Var}[X] = \sum_{i=1}^{100} \mathbf{Var}[X_i] = \frac{875}{3}$$
- $$\Pr(|X - 350| \geq 50) \leq \frac{\mathbf{Var}[X]}{50^2} = \frac{7}{60} \text{ (Chevyshev's inequality)}$$

3.6

- Let X be the number of flips until the k-th head appears
- Let X_i be the number of flips after the $(i-1)$-th head until the i-th head appears (Excluding the $(i-1)$-th flips and including the i-th flip)
  - All X_i are mutually independent, as the number of flips until one head does not affect the number of flip until another head.
  - $$\mathbf{Var}[X_i] = \frac{1-p}{p^2}$$
- $$X = \sum_{i=1}^{k} X_i$$
  - $$\mathbf{Var}[X] = \sum_{i=1}^{k} \mathbf{Var}[X_i] = \frac{k(1-p)}{p^2}$$
- $$\mathbf{Var}[X] = \frac{k(1-p)}{p^2}$$

3.7

- Let X_d be the price after d days

The expected value of X_d

- $$\mathbf{E}[X_d] = p \times r\mathbf{E}[X_{d-1}] + (1-p) \times \frac{1}{r}\mathbf{E}[X_{d-1}] = \frac{pr^2 + 1 - p}{r}\mathbf{E}[X_{d-1}]$$
- $\mathbf{E}[X_0] = 1$
- $$\mathbf{E}[X_d] = \left(\frac{pr^2 + 1 - p}{r}\right)^d$$

The variance of X_d

- $$\mathbf{E}[{X_d}^2] = p \times r^2\mathbf{E}[{X_{d-1}}^2] + (1-p) \times \left(\frac{1}{r}\right)^2\mathbf{E}[{X_{d-1}}^2] = \frac{pr^4 + 1 - p}{r^2}\mathbf{E}[{X_{d-1}}^2]$$
- $\mathbf{E}[{X_0}^2] = 1$
- $$\mathbf{E}[{X_d}^2] = \left(\frac{pr^4 + 1 - p}{r^2}\right)^d$$
- $\mathbf{Var}[X_d] = \mathbf{E}[{X_d}^2] - (\mathbf{E}[X_d])^2 = \frac{(pr^4+1-p)^d - (pr^2+1-p)^{2d}}{r^{2d}}$

3.15

- Proposition: for two random variables X and Y, if $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$, then $\mathbf{Var}[X+Y] = \mathbf{Var}[X] + \mathbf{Var}[Y]$
  - $$\begin{aligned}\mathbf{Var}[X+Y] &= \mathbf{E}\left[(X+Y)^2\right] - (\mathbf{E}[X+Y])^2 \\ &= \mathbf{E}[X^2 + 2XY + Y^2] - (\mathbf{E}[X] + \mathbf{E}[Y])^2 \\ &= \mathbf{E}[X^2] + 2\mathbf{E}[XY] + \mathbf{E}[Y^2] - \left((\mathbf{E}[X])^2 + 2\mathbf{E}[X]\mathbf{E}[Y] + (\mathbf{E}[Y])^2\right) \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 + \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 + 2\mathbf{E}[XY] - 2\mathbf{E}[X]\mathbf{E}[Y] \\ &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 + \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 \\ &= \mathbf{Var}[X] + \mathbf{Var}[Y]\end{aligned}$$
- For n random variables X_i that satisfies $\mathbf{E}\left[X_i X_j\right] = \mathbf{E}[X_i]\mathbf{E}\left[X_j\right]$ with $1 \leq i < j \leq n$
  - For all k such that $1 \leq k \leq n$,

$$
\begin{aligned}
\mathbf{E}\left[\left(\sum_{i=1}^{k} X_i\right) X_{k+1}\right] &= \mathbf{E}\left[\sum_{i=1}^{k} X_i X_k\right] \\
&= \sum_{i=1}^{k} \mathbf{E}[X_i]\mathbf{E}[X_k] \\
&= \left(\sum_{i=1}^{k} \mathbf{E}[X_i]\right)\mathbf{E}[X_k] \\
&= \mathbf{E}\left[\left(\sum_{i=1}^{k} X_i\right)\right]\mathbf{E}[X_k]
\end{aligned}
$$

- $$
\begin{aligned}
\mathbf{Var}[X] = \mathbf{Var}\left[\left(\sum_{i=1}^{n-1} X_i\right) + X_n\right] &= \mathbf{Var}\left[\left(\sum_{i=1}^{n-1} X_i\right)\right] + \mathbf{Var}[X_n] \\
&= \mathbf{Var}\left[\left(\sum_{i=1}^{n-2} X_i\right) + X_{n-1}\right] + \mathbf{Var}[X_n] = \mathbf{Var}\left[\left(\sum_{i=1}^{n-2} X_i\right)\right] + \mathbf{Var}[X_{n-1}] + \mathbf{Var}[X_n] \\
&\dots \\
&= \mathbf{Var}[X_1 + X_2] + \sum_{i=3}^{n} \mathbf{Var}[X_i] = \mathbf{Var}[X_1] + \mathbf{Var}[X_2] + \sum_{i=3}^{n} \mathbf{Var}[X_i] \\
&= \sum_{i=1}^{n} \mathbf{Var}[X_i]
\end{aligned}
$$

3.20

Upper bound: $\Pr[Y \neq 0] \leq \mathbf{E}[Y]$

- $$
\begin{aligned}
\Pr[Y \neq 0] &= \Pr[Y \geq 1] && \text{(Y has nonnegative integer-value)} \\
&\leq \frac{E[Y]}{1} = E[Y] && \text{(Markov's Inequality)}
\end{aligned}
$$

Lower bound: $\Pr[Y \neq 0] \geq \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]}$

- Let Z be a positive-valued integer random variable
  - That satisfies $\Pr(Z = k) = \frac{\Pr(Y=k)}{\Pr(Y \neq 0)}$ for all positive integer k
- $$
\begin{aligned}
\mathbf{E}[Z] &= \sum_{k=1}^{n} k \cdot \frac{\Pr(Y = k)}{\Pr(Y \neq 0)} \\
&= \frac{1}{\Pr(Y \neq 0)} \sum_{k=1}^{n} k \cdot \Pr(Y = k) \\
&= \frac{1}{\Pr(Y \neq 0)} \sum_{k=0}^{n} k \cdot \Pr(Y = k) \\
&= \frac{1}{\Pr(Y \neq 0)} \times \mathbf{E}[Y]
\end{aligned}
$$

- $$\begin{aligned}\mathbf{E}[Z^2] &= \sum_{k=1}^{n} k^2 \cdot \frac{\Pr(Y=k)}{\Pr(Y\neq 0)} \\ &= \frac{1}{\Pr(Y\neq 0)} \sum_{k=1}^{n} k^2 \cdot \Pr(Y=k) \\ &= \frac{1}{\Pr(Y\neq 0)} \sum_{k=0}^{n} k^2 \cdot \Pr(Y=k) \\ &= \frac{1}{\Pr(Y\neq 0)} \times \mathbf{E}[Y^2]\end{aligned}$$
- $$\begin{aligned}\mathbf{Var}[Z] \geq 0 \Leftrightarrow \mathbf{E}[Z^2] &\geq (\mathbf{E}[Z])^2 \\ &\Leftrightarrow \frac{1}{\Pr(Y\neq 0)} \times \mathbf{E}[Y^2] \geq \left(\frac{1}{\Pr(Y\neq 0)} \times \mathbf{E}[Y]\right)^2 \\ &\Leftrightarrow \Pr(Y\neq 0) \geq \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]}\end{aligned}$$

3.26

- Let $X = \frac{1}{n}\sum_{i=1}^{n} X_i$
  - $$\mathbf{E}[X] = \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}[X_i] = \frac{n\mu}{n} = \mu$$
  - $$\mathbf{Var}[X] = \frac{1}{n^2}\sum_{i=1} \mathbf{E}[X_i] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$
- $$\Pr(|X-\mu| > \varepsilon) \leq \frac{\mathbf{Var}(X)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \text{ (Chebyshev's inequality)}$$

  $$\lim_{n\to\infty} \Pr(|X-\mu| > \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0$$