

**Machine Learning Course - CS-433**

# **Neural Nets – Representation Power**

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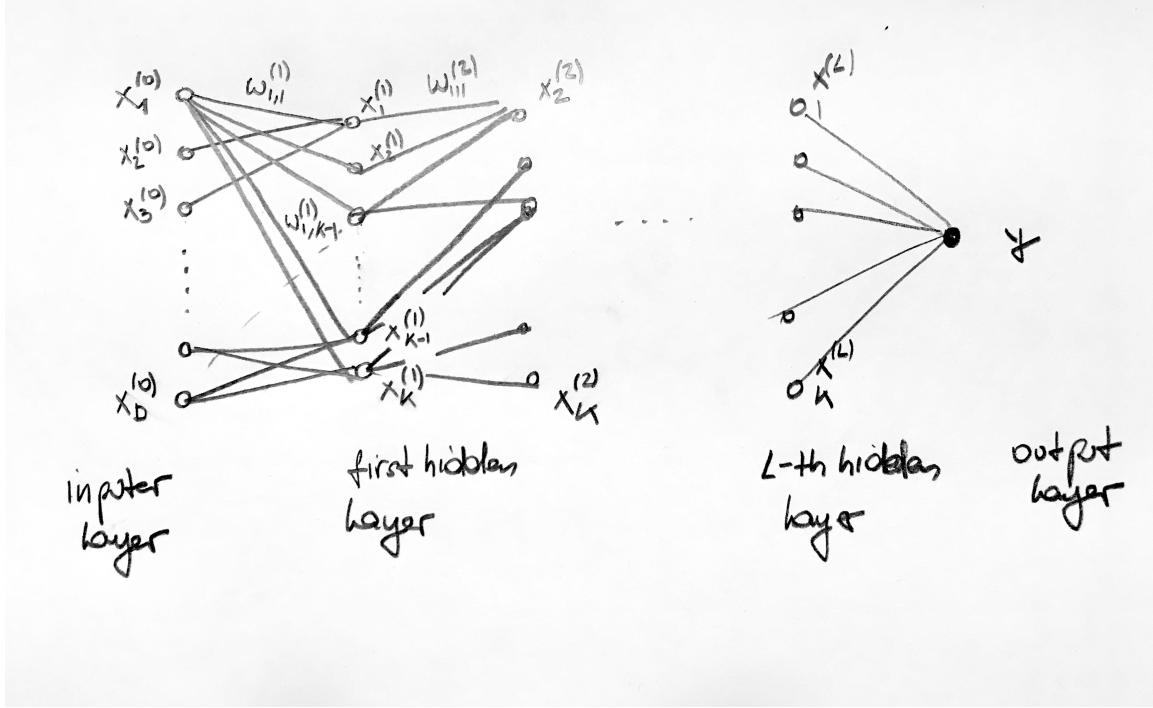


Figure 1: A neural network with one input layer,  $L$  hidden layers, and one output layer.

## Motivation

In the last lecture we introduced the basic structure of a neural net. It is shown in Figure 1. Recall that we can split this network into two parts. The first part comprises the input layer and all the  $L$  hidden layers. The task of this part of the net is to transform the original input into a more suitable representation. In other words, this part of the net represents a *function* from  $\mathbb{R}^D$  to  $\mathbb{R}^K$ . The second part, represented by the final layer performs the actual ML task (regression or classification).

We will now focus on the first part. We will ask: How “powerful” are neural nets? More precisely, what functions  $f(\mathbf{x})$  can they represent, or better, what functions can they approximate? We will see that even relatively simple nets (with at most two hidden layers) are capable of approximating any

continuous function arbitrarily closely on a bounded domain, assuming only that we allow a large number of nodes and arbitrary weights and biases.

We will not take a rigorous approach. Rather, we will follow the lead of Michael Nielsen and give a heuristic but rather convincing argument which shows why neural nets are so expressive.

If you are interested in a rigorous approach we recommend that you read either “Approximation by superposition of a sigmoidal function” by Cybenko (1989), or “Universal approximation bounds for superpositions of a sigmoidal function” by Barron (1993). Both of these papers deal with networks with a single layer and sigmoids as approximation functions. Since then, many further approximation results for various network structures, activation functions, and approximation measures have been derived.

Before we get into our heuristic argument let us state the main theorem of the paper by Barron. This gives you a flavor of what kind of results can be proved.

**Lemma.** Let  $f : \mathbb{R}^D \rightarrow \mathbb{R}$  be a function such that

$$\int_{\mathbb{R}^D} |\omega| |\tilde{f}(\omega)| d\omega \leq C,$$

where

$$\tilde{f}(\omega) = \int_{\mathbb{R}^D} f(\mathbf{x}) e^{-j\omega^\top \mathbf{x}} d\mathbf{x}$$

is the Fourier transform of  $f(\mathbf{x})$ .

Then for all  $n \geq 1$ , there exists a function  $f_n$  of the form

$$f_n(\mathbf{x}) = \sum_{j=1}^n c_j \phi(\mathbf{x}^\top \mathbf{w}_j + b_j) + c_0,$$

i.e., a function that is representable by a NN with one hidden layer with  $n$  nodes and “sigmoid-like” activation functions so that

$$\int_{|\mathbf{x}| \leq r} (f(\mathbf{x}) - f_n(\mathbf{x}))^2 d\mathbf{x} \leq \frac{(2Cr)^2}{n}.$$

*Discussion:* First note that the condition on the Fourier transform is a “smoothness condition.” E.g., functions so that  $\int_{\mathbb{R}^D} |\omega| |\tilde{f}(\omega)| d\omega < \infty$  can be shown to be continuously differentiable.

Second note that the lemma only guarantees a good approximation in a bounded domain. The larger the domain, the more nodes we need in order to approximate a function to the same level (see the term  $r^2$ , where  $r$  is the radius of the ball where we want the approximation to be good, in the upper bound).

Third, this is an approximation “in average”, more precisely in  $L_2$ -norm. We will mostly discuss approximations in this sense but come back to this point at the end.

Fourth, the approximation  $f_n$  with  $n$  terms corresponds exactly to our model of a neural net with one hidden layer containing  $n$  nodes and sigmoids as activation functions.

Fifth, the theorem applies to all activation functions that are “sigmoid-like,” i.e., all activation functions whose left limit is 0, whose right limit is 1, and that are sufficiently smooth. In words the lemma says that a sufficiently “smooth” function can be approximated by a neural net with one hidden layer and the approximation error goes down like one over the number of nodes in the hidden layer. Note that this is a very fast convergence.

## Approximation in Average

We will now give a very simple and intuitive explanation why neural nets with a sigmoid as activation function and at most two hidden layers have already a large expressive power. This explanation will not be rigorous and it will fall far short of the stated lemma by Barron which proves that in fact we only need one hidden layer and not too many hidden nodes. We will search for an approximation “in average”, i.e., an approximation so that the integral over the absolute value of the difference is arbitrarily small.

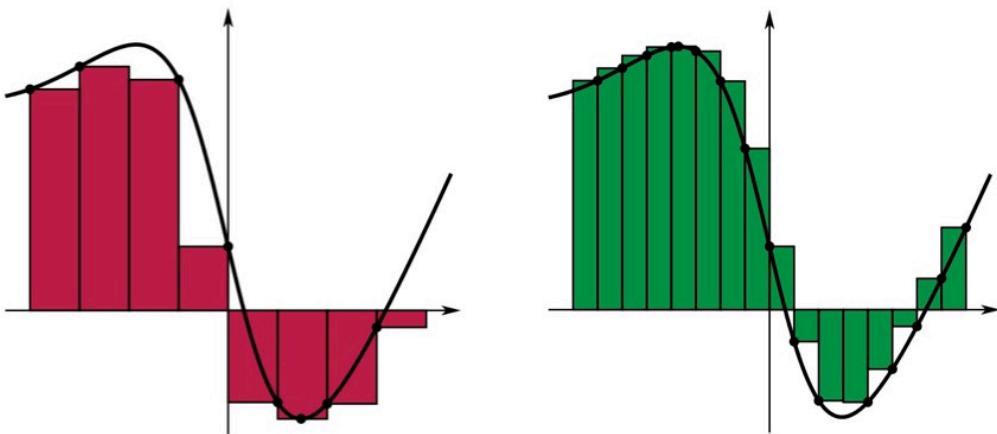


Figure 2: A lower and an upper Riemann sum.

## Functions $\mathbb{R} \rightarrow \mathbb{R}$

We start with a scalar function  $f(x)$  on a bounded domain. Recall that if this function is continuous then it is Riemann integrable, i.e., it can be approximated arbitrarily closely by “upper” and “lower” sums of rectangles, see Figure 2. Of course, we might need a lot of such rectangles to approximate the area with an error of at most  $\epsilon$ , but for every  $\epsilon > 0$  we can find such an approximation.

We will now show that if we do not limit the weights, then with two hidden nodes (of a neural network with one hidden layer) we can construct a function which is arbitrarily close to a given rectangle. But since, as we have just seen, a finite number of rectangles suffices to approximate a bounded continuous function arbitrarily closely, it follows that with a finite number of hidden nodes of a neural network with one hidden layer we can approximate any such function arbitrarily closely (in the sense that the integral of the absolute value

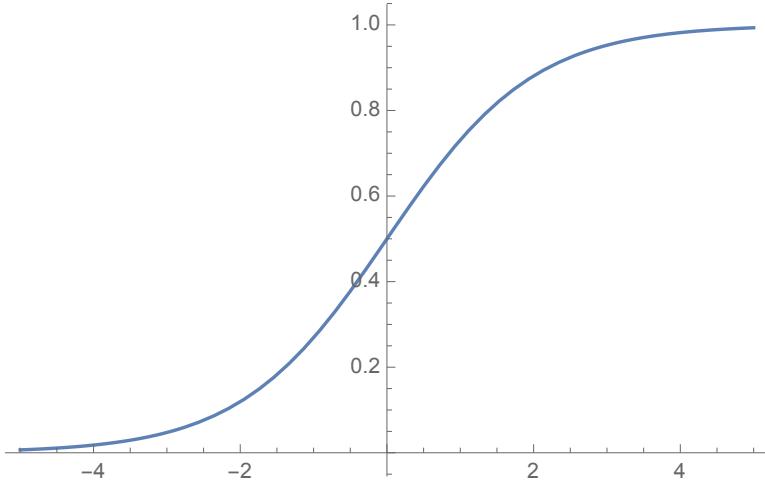


Figure 3: The sigmoid function.

of their difference is arbitrarily small).

The following idea is taken from the tutorial by Michael Nielsen, <http://neuralnetworksanddeeplearning.com>.

Let  $\phi(x) = \frac{1}{1+e^{-x}}$  be the sigmoid function. We have encountered it already in the last lecture. But here it is again in Figure 3.

Consider the function  $f(x) = \phi(w(x - b))$ , where  $w$  is the *weight* of a particular edge and  $-wb$  is the *bias* term.

Note that if  $w \geq 0$  then  $f(x)$  is an increasing function that increases smoothly from 0 to 1. Further, the “transition” happens at the spot  $x = b$ , i.e., at this value of  $x$  the function has the value  $\frac{1}{2}$ . The transition is the faster the larger we choose the weight  $w$ . In fact, if we set  $b = 0$  so that the transition from 0 to 1 happens at  $x = 0$ , then the derivative of the function at 0 is  $w/4$ . In other words, the *width of the transition* is of order  $4/w$ .

Therefore, if we want to create a rectangle that jumps from 0 to 1 at  $x = a$  and jumps back to 0 at  $x = b$ ,  $a < b$ , then

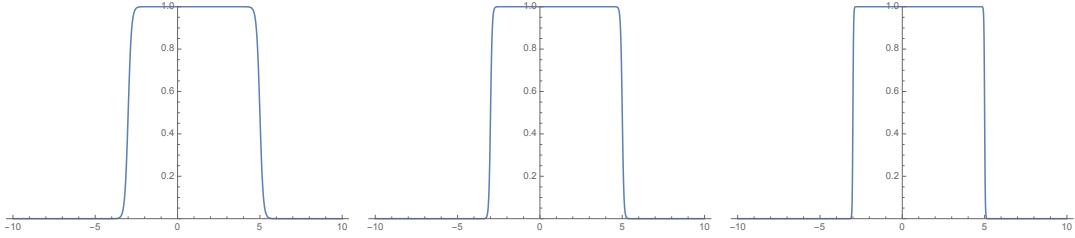


Figure 4: An approximate rectangle of the form  $\phi(w(x - a)) - \phi(w(x - b))$  with  $w = 10, 20$ , and  $50$ , respectively.

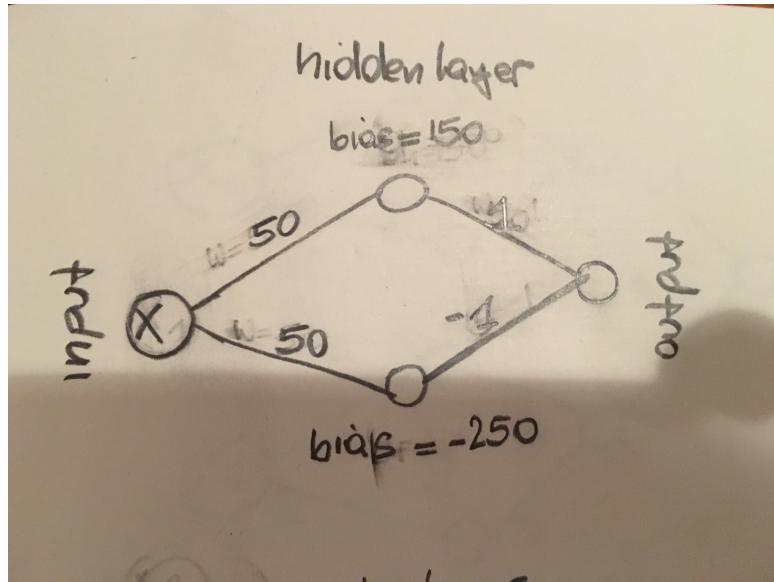


Figure 5: A simple NN implementation of equation (1).

we can accomplish this by taking

$$\phi(w(x - a)) - \phi(w(x - b)), \quad (1)$$

and taking  $w$  to be very large. This is shown in Figure 4 where the three figures correspond to  $w = 10, 20$  and  $50$ , respectively and  $a = -3$  and  $b = 5$ . We see that for large values of  $w$  the result is barely distinguishable from a true rectangle.

Note that equation (1) has a very simple representation in form of a neural net. This is shown in Figure 5. There is one input node which contains the value  $x$ . This value is

multiplied by some large weight (in the figure it is 50) and it is then forwarded to the two hidden nodes. One of these hidden nodes has a bias of 150 the other one has a bias of  $-250$ , so that the sums at these two hidden nodes are  $50(x + 3)$  and  $50(x - 5)$ , respectively. Each node applies the sigmoid function and forwards the result to the output layer. The edge from the top hidden node to the output has weight 1 and the one from the bottom hidden node to the output has weight  $-1$ . The output node adds the two inputs. The result is  $\phi(50(x + 3)) - \phi(50(x - 5))$ , which is approximately a unit-height rectangle from  $-3$  to  $5$ . If we want a rectangle of height  $h$ , use the weights  $h$  and  $-h$  in the second layer instead of the weights 1 and  $-1$ .

It is hopefully clear at this point why any continuous function on a bounded domain can be approximated via a neural network with one hidden layer. Let us summarize in telegram style: Take the function. Approximate it in the Riemann sense. Approximate each of the rectangles in the Riemann sum by means of two nodes in the hidden layer of a neural net. Compute the sum (with appropriate sign) of all the hidden layers at the output node. If we are using a Riemann sum with  $K$  rectangles we get therefore a neural network approximation with one hidden layer containing  $2K$  nodes

A few remarks are in order:

1. The same intuition applies to many activation functions. All we have used is that the activation function has left limit 0 and right limit 1, just like for Barron's result.
2. Whereas Barron's result gave us a strong bound on

the number of required nodes, our intuitive explanation gave us no bound.

3. The above approximation only works if we allow the weights to become arbitrarily large. Of course, very large weights would likely cause problems in practice. It is also not clear why we should need them. There is no fundamental reason why we should first approximate rectangles very precisely which then in turn approximate a continuous function.

## Functions $\mathbb{R}^D \rightarrow \mathbb{R}$

So far we have talked about a real function of a single variable. What about functions over  $\mathbb{R}^D$ ? We will see that the same principle applies but we will need one more layer. The extra idea that is needed to extend the above scheme from one to several dimensions is already present in  $\mathbb{R}^2$ . We will therefore stick to this case. You should have no trouble figuring out how to extend it to  $\mathbb{R}^D$ .

As before, it will suffice if we can show how to approximate any two-dimensional rectangle. In fact, all we need are two-dimensional rectangles that are parallel to the two axes.

Let the two inputs/axes be  $x_1$  and  $x_2$ , represented by two nodes in the input layer. Assume that we want to represent a rectangle of height 1 that extends from  $a_1$  to  $b_1$ ,  $a_1 < b_1$ , on the  $x_1$  axis and from  $a_2$  to  $b_2$ ,  $a_2 < b_2$ , on the  $x_2$  axis.

Consider the neural net in Figure 6. It represents a rectangle that goes from  $a_1$  to  $b_1$  in the direction of the  $x_1$  axis but is unbounded in the  $x_2$  direction. The function it represents is

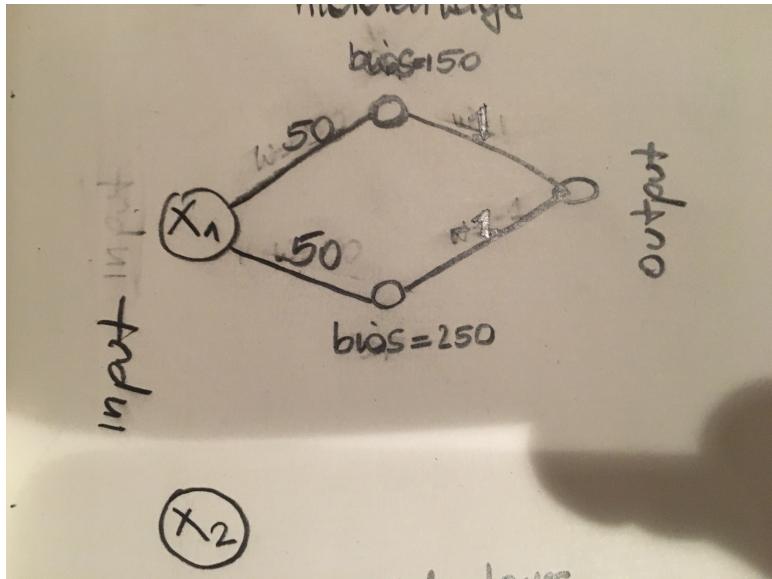


Figure 6: A 2-D rectangle bounded in one dimension and unbounded in the other.

shown Figure 7.

Let us add to this rectangle another one that is unbounded in the  $x_1$  direction but extends from  $a_2$  to  $b_2$  in the direction of the  $x_2$  axis. The corresponding neural net is shown in Figure 8 and the plot of the corresponding function is shown in Figure 9. This is close to what we want. In the region where we want the function to be 1 it is in fact close to 2 since we have there the sum of two functions, each of height 1. A trivial scaling by a factor 1/2 would bring it to the desired value in this region. But unfortunately we still have in addition the two “arms” that extend to infinity along each axis. Those have height 1.

But those additional unwanted “arms” are easily suppressed. Just apply the sigmoid function with some large weight and bias, lets say 3/2, to the node that sums up the two rectangles. A plot of the resulting function is shown in Figure 10. So instead of being the output node, as it was before, this

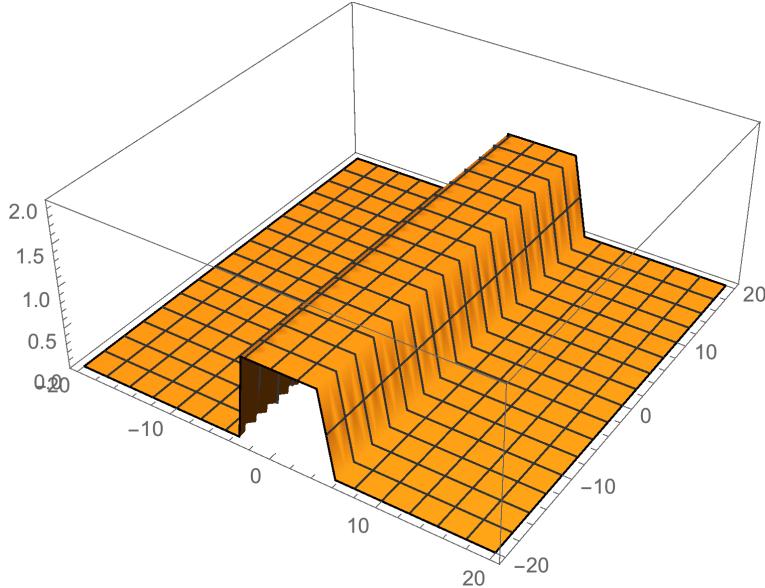


Figure 7: The function  $\phi(w(x_1 + a_1)) - \phi(w(x_1 - b_1))$  corresponding to the NN shown in Figure 6. Here,  $w = 50$ ,  $a_1 = -3$ , and  $b_1 = 5$ .

node now becomes a hidden node, forming one extra layer of hidden nodes. With this we have created the desired 2-dimensional rectangle. We can approximate any sufficiently smooth 2-dimensional function on a bounded domain by using a suitable number of those and adding them at the output node.

## Point-wise Approximations and other Activation Functions

So far we have considered approximations “in average”. I.e., we have seen for Barron’s result that the  $L_2$  norm can be made arbitrarily small. And for our intuitive derivation we took the Riemann integral as a starting point, i.e., we considered the  $L_1$  norm.

We can also ask if a point-wise approximation with vanishing

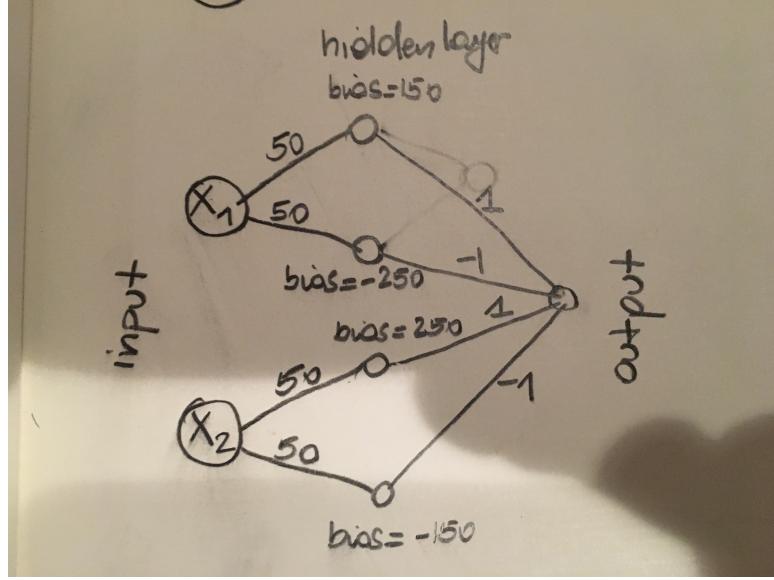


Figure 8: Sum of two rectangles, each bounded in one dimension and unbounded in the other.

error is possible, i.e., we can consider the  $L_\infty$  norm. Further, we have limited our discussion so far to “sigmoid-type” activation functions, i.e., activation functions whose left limit is 0 and whose right limit is 1.

So let us now look at point-wise approximations with an activation function that is the *rectified linear function*,

$$(x)_+ = \max\{0, x\}.$$

Many other combinations (approximation criterion and activation function) are of course possible and have been considered.

Let  $f(x)$  be a continuous function on a bounded domain. Without loss of generality we can assume that this domain is  $[0, 1]$ , rescaling and shifting the  $x$ -axis if necessary. The classical *Stone-Weierstrass* theorem says that for every  $\epsilon > 0$ , there exist a polynomial  $p(x)$  so that for all  $x \in [0, 1]$ ,

$$|f(x) - p(x)| < \epsilon.$$

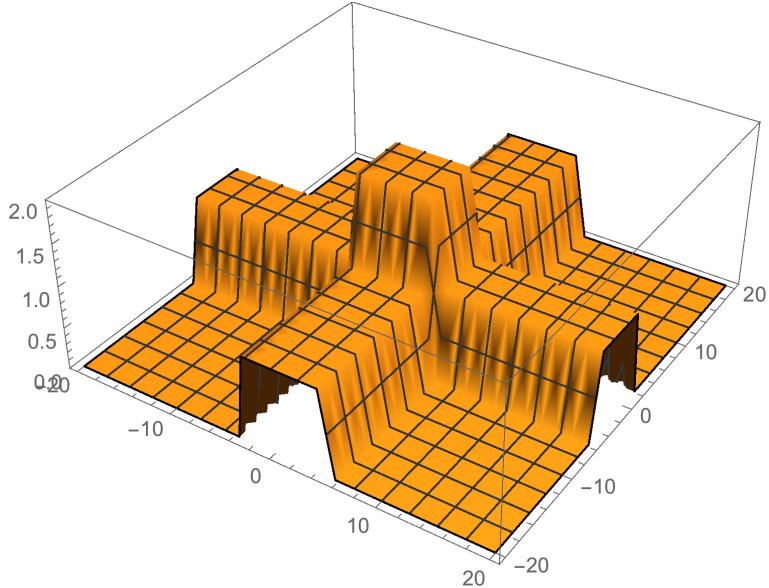


Figure 9: The function  $g(x_1, x_2) = \phi(w(x_1 + a_1)) - \phi(w(x_1 - b_1)) + \phi(w(x_2 + a_2)) - \phi(w(x_2 - b_2))$  corresponding to the NN shown in Figure 8. Here,  $w = 50$ ,  $a_1 = -3$ ,  $b_1 = 5$ ,  $a_2 = -5$ ,  $b_2 = 3$ .

But such a function  $f(x)$  can also be approximated in  $L_\infty$  norm by even simpler functions, namely continuous piecewise-linear functions, see Shektman, 1982. Let  $q(x)$  be a continuous piecewise-linear function. Then it has the form

$$q(x) = \sum_{i=1}^m (a_i x + b_i) \mathbb{1}_{\{r_{i-1} \leq x < r_i\}},$$

where  $0 = r_0 < r_1 < \dots < r_m = 1$  is a suitable partition of  $[0, 1]$ . Note that continuity imposes the constraints

$$a_i r_i + b_i = a_{i+1} r_i + b_{i+1}, \quad i = 1, \dots, m-1.$$

For our purpose it is more convenient to write it in the al-

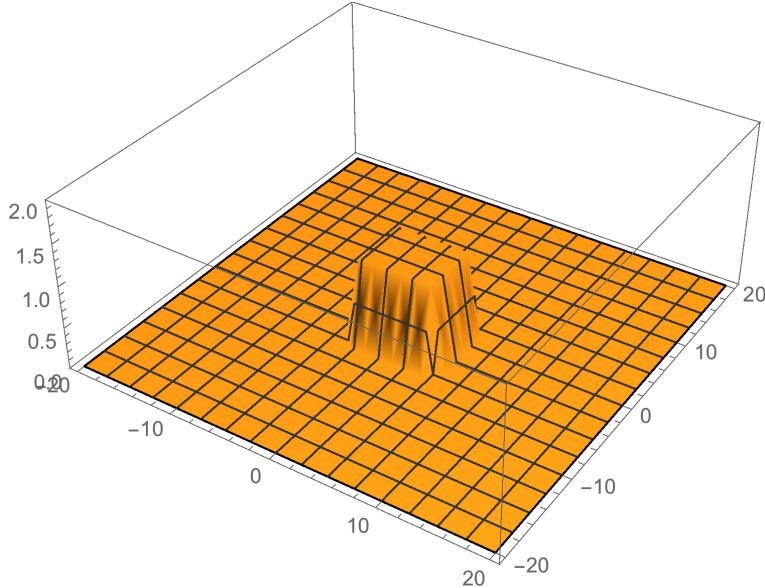


Figure 10: The function  $\phi(w(g(x_1, x_2) - 3/2))$  where  $g(x_1, x_2)$  is the function shown in Figure 9. This is now a good approximation of the desired rectangle.

ternative form

$$q(x) = \tilde{a}_1 x + \tilde{b}_1 + \sum_{i=2}^m \tilde{a}_i (x - \tilde{b}_i)_+.$$

Here,  $\tilde{a}_1 = a_1$  and  $\tilde{b}_1 = b_1$  and for  $i = 2, \dots, m$ , the remaining parameters can be computed via the relations

$$a_i = \sum_{j=1}^i \tilde{a}_i,$$

$$\tilde{b}_i = r_{i-1}.$$

Each term in the sum on the right corresponds to one node in a hidden layer with input  $x$ , bias  $-\tilde{b}_i$ , and activation function  $(x)_+$ . The bias term  $\tilde{b}_1$  can be absorbed into the bias term of the output node. This leaves the term  $\tilde{a}_1 x$ . This

term can also be represented by a node in the hidden layer with activation function  $(x)_+$  by choosing  $x_0 = 0$ , since we only need the this representation to be correct in the range  $[0, 1]$ . So this shows that we can approximate any continuous function on a bounded domain by a neural net with one hidden layer in the  $L_\infty$  norm to arbitrary precision.

We have only considered the one-dimensional case. But it turns out that a similar scheme works also for higher dimensions. We skip the details.