Connie Okasaki
AMATH 584
Assignment 4
16 Nov 2020

Problem 1:

(a) Consider the matrix $2I$. Determine its eigenvalues and the algebraic and geometric multiplicity of each.

(b) Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

.

(a) We can first construct the characteristic polynomial $f(\lambda) = |2I - \lambda I| = (2-\lambda)^3$. Thus we find that this matrix has eigenvalue $\lambda = 2$ with algebraic multiplicity 3. Then we can verify that any vector is an eigenvector so we can arbitrarily choose the three linearly independent unit vectors e_1, e_2, and e_3 and we find that this eigenvalue also has geometric multiplicity 3.

(b) Now we can construct the characteristic polynomial

$$f(\lambda) = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^3$$

and we can see that again this matrix has eigenvalue $\lambda = 2$ with algebraic multiplicity 3. However now, when we plug in $\lambda = 2$ and attempt to solve

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

we find that $y = z = 0$ and we may only choose x. Thus the nullspace of $M - 2I$ is spanned by e_1 and we find that this eigenvalue has geometric multiplicity 1.

Problem 2: For each of the given statements, prove that it is true or give an example to show iut is false. Here $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ unless otherwise indicated

(a) This is true: if $Av = \lambda v$ then

$$(A - \mu I)v = Av - \mu v = \lambda v - \mu v = (\lambda - \mu)v$$

so indeed $\lambda - \mu$ is an eigenvalue of $A - \mu I$

(b) This is false: consider the matrix $2I$ from problem 1 which has 2 but not -2 as an eigenvalue.

(c) This is true. The characteristic polynomial of a real matrix must have real coefficients. Therefore, it can be factored into a product of linear factors (corresponding to real eigenvalues, which are their own complex conjugate) and irreducible quadratic factors (corresponding to pairs of complex eigenvalues $\lambda, \overline{\lambda}$).

(d) This is true. If A is nonsingular and $Av = \lambda v$ then $v = A^{-1}\lambda v$ and $A^{-1}v = \lambda^{-1}v$.

(e) This is false. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This matrix has characteristic polynomial $f(\lambda) = \lambda^2$ and therefore has only zero eigenvalues.

(f) This is true. We showed in HW 2 that the nonzero singular values are the square roots of the nonzero eignevalues of AA^*. If $A = A*$ and $AV = V\Lambda$ then $AA^*V = A(AV) = A(V\Lambda) = V\Lambda^2$ so we see that indeed the singular values of a hermitian matrix A are the square roots of its eigenvalues squared, or $|\lambda|$.

(g) This is true. If A is diagonalizable then $A = P^{-1}DP$ for some invertible matrix P. Furthermore if $AV = V(\lambda I) = \lambda V$ then $P^{-1}DPV = \lambda V$ and $DPV = \lambda PV$ and we see that $D = \lambda I$. However, since constant multiples of I commute, we then see that $A = P^{-1}(\lambda I)P = (\lambda I)P^{-1}P = \lambda I$.

Problem 3: Let $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ be tridiagonal and Hermitian, with all its sub- and super-diagonal entries nonzero. Prove that the eigenvalues of $\boldsymbol{A}$ are distinct (Hint: Show that for any $\lambda \in \mathbb{C}$, $\boldsymbol{A} - \lambda\boldsymbol{I}$ has rank at least $m-1$.)

For any $\lambda \in \mathbb{C}$, the first $m-1$ columns of $A - \lambda I$ are linearly independent since any linear combination of columns contains a nonzero entry corresponding to the right-most column included. Since $A - \lambda I$ has rank at least $m-1$, therefore the nullspace has rank at most 1. Therefore each eigenvalue has geometric multiplicity at most 1, and therefore algebraic multiplicity at most 1. Since there are no repeated eigenvalues, each eigenvalue is distinct.