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AMATH 584
Assignment 2
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Problem 1: Show that for a matrix A

(a) The nonzero singular values of A are the square roots of the nonzero eigenvalues of AA^* or A^*A

(b) If $A = A^*$, then the singular values are the absolute values of the eigenvalues of A

(c) Given that the determinant of a matrix U is unity, show $|\det(A)| = \prod_{j=1}^m \sigma_j$

(a) Suppose that the SVD of $A = U\Sigma V^*$, with singular values in descending order as usual. Then:

$$\begin{aligned} AA^* &= U\Sigma V^* V\Sigma^* U^* \\ &= U\Sigma\Sigma^* U^* \\ AA^*U &= U\Sigma\Sigma^*. \end{aligned}$$

This is an eigenvalue problem. Since the singular values are real, we know that $\Sigma\Sigma^* = \Sigma^2$ so that the singular values are in fact the positive square roots of the eigenvalues of AA^*. Likewise

$$\begin{aligned} A^*A &= V\Sigma U^* U\Sigma V^* \\ &= V\Sigma^2 V^* \\ A^*AV &= V\Sigma^2 \end{aligned}$$

(b) If $A = A^*$ then let V be the eigenvalues of A. We know that $AV = V\Lambda$. But then $A^*AV = V\Lambda^2 = V\Sigma^2$. Thus, since the singular values are the positive square roots of λ_i^2 we see that they are the absolute values of the eigenvalues of A.

(c) Presumably the problem is referencing the property that $|\det(U)| = 1$ for a unitary matrix. Therefore,

$$|\det(A)| = |\det(U)| \times |\det(\Sigma)| \times |\det(V)| = |\det(\Sigma)| = \prod_{j=1}^m \sigma_j.$$