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AMATH 584
Assignment 1
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**Problem 1:** Show that if matrix $A$ is triangular and unitary, then it is diagonal.

Suppose that $A$ is upper triangular. Since it is unitary its columns are orthogonal and have unit length. The first column must therefore be of the form $(a_{11}, 0, 0, ...., 0)$ where $\|a_{11}\| = 1$. Now suppose by induction that the first $k$ columns are diagonal, i.e. each consists of a single norm-1 complex entry on the diagonal. The $k+1$st column must be orthogonal to the first $k$ columns. In order to achieve this, each inner product $\langle a_{k+1}, a_i \rangle = a_{k+1,i} a_{ii} = 0$. Therefore, since $a_{ii}$ is non-zero $a_{k+1,i} = 0$, and $a_{k+1}$ is of the same diagonal form. Therefore the whole matrix $A$ is diagonal. The same holds for lower-triangular matrices, where we induct from $a_{nn}$ instead of $a_{11}$.

**Problem 2:** Consider that the matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{n\times m}$ are Hermitian (self-adjoint)

(a) Prove that all eigenvalues $\lambda_k$ of $A$ are real

(b) Prove that if $x_k$ is the $k$th eigenvector, then eigenvectors with distinct eigenvalues are orthogonal

(c) Prove the sum of two Hermitian matrices is Hermitian

(d) Prove the inverse of an invertible Hermitian matrix is Hermitian as well

(e) Prove the produce of two Hermitian matrices is Hermitian if and only if $AB = BA$.

(a) By definition of the adjoint has the property $\langle Ax_k, x_k\rangle = \langle x_k, A^* x_k\rangle$. Similarly, for scalars $\langle cx_k, x_k\rangle = \langle x_k, \bar{c}x_k\rangle$. Therefore,

$$\langle \lambda x_k, x_k\rangle = \langle Ax_k, x_k\rangle = \langle x_k, A^* x_k\rangle = \langle x_k, Ax_k\rangle = \langle x_k, \lambda x_k\rangle = \langle \overline{\lambda} x_k, x_k\rangle.$$

Therefore, $\lambda = \overline{\lambda}$ is a real number.

(b) Suppose $\lambda_k \neq \lambda_i$. Since these eigenvalues are distinct at least one must be nonzero, and without loss of generality let that be $\lambda_k$. Then

$$\langle x_k, x_i\rangle = \lambda_k^{-1}\langle \lambda_k x_k, x_i\rangle = \lambda_k^{-1}\langle Ax_k, x_i\rangle = \lambda_k^{-1}\langle x_k, Ax_i\rangle = \lambda_k^{-1}\langle x_k, \lambda_i x_i\rangle = \frac{\lambda_i}{\lambda_k}\langle x_k, x_i\rangle.$$

If $\lambda_i = 0$ then $\langle x_k, x_i\rangle = 0$. Otherwise, nevertheless $\frac{\lambda_i}{\lambda_k} \neq 1$ so $\langle x_k, x_i\rangle = 0$.

(c) The adjoint operation is distributive so therefore $(A+B)^* = A^* + B^* = A + B$.

(d) Suppose here that $n = m$. Then $I = (AA^{-1}) = (A^{-1})^* A^* = (A^{-1})^* A$. Therefore since the inverse of $A$ is unique, $(A^{-1})^* = A^{-1}$.

(e) Suppose here that $B \in \mathbb{C}^{m\times ell}$. Then $(AB)^* = B^* A^* = BA$. Therefore, $(AB)^* = AB$ if and only if $BA = AB$.

**Problem 3:** Consider the matrix $U \in \mathbb{C}^{n \times m}$ which is unitary

(a) Prove that the matrix is diagonalizable

(b) Prove that the inverse is $U^{-1} = U^*$.

(c) Prove it is isometric with respect to the $\ell_2$ norm, i.e. $\|Ux\| = \|x\|$

(d) Prove that all eigenvalues have modulus unity.

We will assume here that $n = m$.

(a) The spectral theorem states that complex normal matrices have an orthonormal basis consisting of eigenvectors of $U$. Since $U$ is unitary, $U$ is normal so simply choose $V$ to transform into the given orthonormal basis and $U = VDV^{-1}$.

(b) Since the columns are orthogonal and have norm 1, $U^*U = I$. Therefore $U^{-1} = U^*$.

(c) $\|Ux\| = \langle Ux, Ux \rangle = \langle x, U^*Ux \rangle = \langle x, x \rangle = \|x\|$

(d) $\|x\| = \|Ux\| = \|\lambda x\| = \|\lambda\| \|x\|$. Therefore $\|\lambda\| = 1$.