AMATH 584 Midterm

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Problem 1: Prove that the LU decomposition of a matrix $\boldsymbol{A}$ is unique.

Suppose that the matrix A has two different decompositions, LU and $L'U'$. Then since $LU = L'U'$ we have that $L'^{-1}L = U'U^{-1}$. Since the inverse of an upper (lower) triangular matrix is also an upper (lower) triangular matrix, and the product of two upper (lower) triangular matrices is also an upper (lower) triangular matrix, we see that $U'U^{-1}$ is upper triangular amd $L'^{-1}L$ is lower triangular. Therefore both are diagonal: $L'^{-1}L = D = U'U^{-1}$. Therefore $L = L'D$ and $U = D^{-1}U'$ and the LU decomposition is indeed unique up to multiplication by a diagonal matrix and its inverse.

Problem 2: Show that the largest singular value of a matrix $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ is given by

$$\sigma_{\max}(\boldsymbol{A}) = \max_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{y}\in\mathbb{R}^m} \frac{\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x}}{\|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2}$$

Let $\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^*$ be the singular value decomposition of $\boldsymbol{A}$. Then

$$\begin{aligned}
\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x} &= (\boldsymbol{y}^T\boldsymbol{U})\boldsymbol{\Sigma}(\boldsymbol{V}^*\boldsymbol{x}) \\
&= \boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{v} \\
&= \sum_{i=1}^{n} \sigma_{ii} u_i v_i
\end{aligned}$$

Since U and V are unitary, they preserve the 2-norm so if we require that x and y be unit vectors, then so are u and v. However for a diagonal matrix D the maximum inner product $u^T Dv$ for unit vectors u and v is $D_{\max}$, achieved when $u = v$ points in the direction of the largest entry of D. In any other direction:

$$u^T Dv = \|u\|\|Dv\|\cos(\theta) \leq \|Dv\| = \sum_{i=1}^{n} D_i^2 v_i^2 \leq \sum_{i=1}^{n} D_{\max}^2 v_i^2 = D_{\max}.$$

Therefore the maximum of $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x}$ is the maximum singular value.

Problem 3: What are the singular values of an orthogonal projection?

We know that all singular values are non-negative real numbers. We also know that orthgonal projections are defined by $AA^* = I$. Let $A = U\Sigma V^*$. Then $AA^* = U\Sigma V^* V \Sigma U^* = \Sigma^2 =$ I. Since the singular values are real we can rule out complex units and since the singular values are non-negative we can rule out -1 so the singular values are all 1.

Problem 4: Show that for a given norm $\kappa(\boldsymbol{AB}) \leq \kappa(\boldsymbol{A})\kappa(\boldsymbol{B})$ and that $\kappa(\alpha\boldsymbol{A}) = \kappa(\boldsymbol{A})$ for a given (nonzero) constant α.

For any induced matrix norm (such as the p-norms) it is defined that $\|A\| = \max_{u\in\mathbb{R}^n} \frac{\|Au\|}{\|u\|}$. Using this definition we can see that:

$$\begin{aligned}
\|AB\| &= \max_{u\in\mathbb{R}^n} \frac{\|ABu\|}{\|u\|} \\
&= \max_{u\in\mathbb{R}^n} \frac{\|A(Bu)\|}{\|Bu\|}\frac{\|Bu\|}{\|u\|} \\
&\leq \max_{v\in\mathbb{R}^n} \frac{\|Av\|}{\|v\|} \max_{u\in\mathbb{R}^n} \frac{\|Bu\|}{\|u\|} \\
&= \|A\|\|B\|.
\end{aligned}$$

In essence this is true because in the latter expression we are free to maximize over both u and v rather than requiring that $v = Bu$. Furthermore,

$$\begin{aligned}
\|\alpha A\| &= \max_{u\in\mathbb{R}^n} \frac{\|\alpha Au\|}{\|u\|} \\
&= \max_{u\in\mathbb{R}^n} \frac{\alpha\|Au\|}{\|u\|} \\
&= \alpha \max_{u\in\mathbb{R}^n} \frac{\|Au\|}{\|u\|} \\
&= \alpha\|A\|
\end{aligned}$$

This is different from the stated problem so either I'm wrong or there's a typo.

Problem 5: Write a python or matlab script that does an LU decomposition (including pivoting).

See attached python script.